Adaptive Interpretation: Building Continuity Between Students' Experiences Solving Problems in Arithmetic and in Algebra[1]

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Abstract: We discuss opportunities to build better continuity between students' experiences in arithmetic and in algebra by examining ways that external representations can be used to solve problems. We use examples from our research on algebra learning to illustrate often overlooked complexities that arise when using a single representation to analyze relationships and patterns of change between two covarying quantities. We use the term *adaptive interpretation* to describe ways in which, in the course of solving problems about situations that contain covarying quantities, students must shift their perspective on a representation as they shift their thinking about the situation. One set of examples demonstrates students' difficulties shifting their perspective on equations when shifting their attention from a varying quantity in a situation to a specific unknown value of that quantity. A second demonstrates students' difficulties shifting their perspective on tables and graphs when shifting their attention from initial quantities in a situation to changes in those quantities. Finally, we describe possible antecedent experiences with adaptive interpretation in arithmetic problem solving that could better prepare students for solving problems about situations containing covarying quantities.

Kurzreferat: Wir diskutieren Möglichkeiten zur Entwicklung eines besseren, kontinuierlichen Übergangs von arithmetischen zu algebraischen Erfahrungen der Lernenden, indem wir die Rolle externer Repräsentationen beim Problemlösen untersuchen. Wir verwenden Beispiele aus unseren Untersuchungen über das Lernen von Algebra, um häufig übersehene Schwierigkeiten aufzuzeigen, die entstehen, wenn nur eine Repräsentation verwendet wird, um Beziehungen zwischen zwei kovariierenden Größen und deren Veränderungen zu analysieren. Wir verwenden den Terminus adaptive Interpretation um die Wege zu beschreiben, in denen im Problemlöseprozess mit kovariierenden Größen Lernende ihre Perspektive auf Repräsentationen ändern müssen, wenn sich ihr Denken über die zugehörige Situation ändert. Einige ausgewählte Beispiele demonstrieren die Schwierigkeiten von Lernenden, ihre Perspektive in Zusammenhang mit Gleichungen zu wechseln, wenn ihre Aufmerksamkeit von einer variierenden Größe in einer Situation zu einem unbekannten Wert dieser Größe übergeht. Weitere Beispiele demonstrieren die Schwierigkeiten von Lernenden, ihre Perspektive auf Tabellen und Graphen zu wechseln, wenn ihre Aufmerksamkeit von den ursprünglichen Größen in einer Situation zu den Veränderungen bei diesen Größen wechselt. Abschließend beschreiben wir mögliche frühe Erfahrungen mit adaptiven Interpretationen beim Lösen von arithmetischen Problemen, die Lernende besser auf Problemlösungen mit kovariierenden Größen vorbereiten.

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ZDM-Classification: C30, H20

1. Introduction

In international studies, U.S. middle school students' achievement in algebra lags behind that of their peers in high-achieving countries (Peak 1996). In an effort to raise student achievement in algebra, and mathematics more generally, many U.S. middle schools now offer algebra courses that were previously reserved for high school students and that emphasize "symbol pushing." At the same time, a number of researchers have been investigating opportunities to build better continuity between students' earlier experiences with arithmetic and their initial experiences with algebra. This article contributes to such research by examining ways that external representations can be used to solve problems in arithmetic and in algebra. When discussing algebra, we will focus on problems about situations that contain covarying quantities.

Past research on teaching and learning algebra has often highlighted not continuities but discontinuities between students' problem-solving experiences in arithmetic and in algebra courses. Kieran (1992) illustrated some of these discontinuities by comparing solutions to two word problems:

"Daniel went to visit his grandmother, who gave him \$1.50. Then he bought a book costing \$3.20. If he has \$2.30 left, how much money did he have before visiting his grandmother?" (p. 393)

"The Westmount Video Shop offers two rental plans. The first plan costs \$22.50 per year plus \$2.00 per video rented. The second plan offers a free membership for one year but charges \$3.25 per video rented. For what number of rental videos will these two plans cost exactly the same?" (p. 393)

According to Kieran, elementary students will often solve the first problem by (1) starting with the amount that Daniel had at the end and working backward through the sequence of events, (2) using the equal sign to compute 2.30 + 3.20 = 5.50 before writing "2.30 + 3.20 = 5.50 – 1.50 = 4," and (3) using the expressions 2.30 + 3.20 and 5.50 – 1.50 to describe Daniel's money at two specific times. In traditional algebra courses, students are to solve the video rental problem by generating and solving $22.50 + 2.00x = 3.25x$. Comparing the following three characteristics with those just described for the problem about Daniel highlights some discontinuities in problem-solving approach. Students must (1') work forward from the initial conditions, reversing familiar ways of looking at problem contexts and the mathematical operations used when representing those contexts, (2') use the equal sign to express not the results of intermediate calculations but a relationship between the expressions $22.50 + 2.00x$ and $3.25x$ that holds for a particular value of x, and (3') express costs for arbitrary numbers of videos with $22.50 + 2.00x$ and $3.25x$. Moreover, thinking of a letter, such as x, standing not for a fixed unknown value, but for a variable can be particularly challenging for high school students (Küchemann 1981).

Shifting the focus of research from discontinuities to continuities between students' experiences in arithmetic and in algebra is complex, in part, because algebra is a multifaceted branch of mathematics. Mathematics education researchers have discussed algebra as generalized arithmetic, as a problem solving tool, as the study of functions, and as a means for modeling the physical and social world (Bednarz, Kieran, Lee 1996). More recently, the National Research Council (Kilpatrick, Swafford, Findell 2001) characterized school algebra as including *representational* activities (generating algebraic expressions and equations), *transformational* activities (manipulating symbols and solving equations), and *generalizing and justifying* activities (noting mathematical structure, predicting, and proving). Researchers have emphasized different combinations of these facets when investigating opportunities to build better continuity in students' experiences. For instance, Carpenter, Franke, and Levi (2003) described an approach to whole-number arithmetic designed to help young children develop understandings of the equal sign and properties of numbers that are central to generating and solving algebraic equations. At the center of their approach are activities where students make and justify conjectures about properties of numbers. Meanwhile, Schliemann, Carraher, Brizuela, and Earnest (2003) worked with elementary grade students on tasks that emphasize describing and representing relations among sets of numbers and measures, including variables. The tasks include work with tabular, algebraic, and graphic representations of functions.

We focus on representations of functions as well. Past research and curriculum development efforts that emphasize the functions perspective on algebra have often emphasized multiple representations, including the coordination of graphs, tables, symbols, and/or narrative descriptions of situations to analyze relationships and patterns of change between two varying quantities (e.g., Confrey, Smith 1995; Heid 1996; Janvier 1987; Kieran, Boileau, Garançon 1996; Leinhardt, Zaslavsky, Stein 1990; Romberg, Fennema, Carpenter 1993; Thompson 1994). We focus instead on subtleties that arise when using a single representation. In particular, we will use examples from our research on algebra learning to illustrate challenges that students can face when coordinating aspects of situations with representational features to which they attend. The common thread running through the examples is that, in the course of a given solution, students must coordinate shifts in interpretations of problem situations with shifts in interpretations of representations. We refer to this as *adaptive interpretation* and organize our examples into two categories. The first arises when students need to shift attention from a varying quantity in a situation to a specific (possibly unknown) value of that quantity. (In contrast to the results of Küchemann (1981) discussed above, we focus on cases where students struggle with algebraic representations even when they can interpret letters both as variables and as specific unknowns.) The second arises when students need to shift attention from initial quantities in a situation to changes in those quantities. We will close with a discussion of possible antecedent experiences in arithmetic problem solving that could better prepare students for adaptive interpretation.

2. Early Experiences Using Inscriptions to Solve Problems

We begin with a whole-number addition example to illustrate early experiences that students commonly have using inscriptions to solve problems. Consider the following:

Sarah has 7 books. A friend gives her 5 more. How many books does Sarah now have?

Extensive research exists on the strategies (e.g., counting all and counting on) and underlying conceptual structures that children use to solve problems like this (see Fuson 1992, for a review). Young children would likely model the situation directly by drawing a rectangle, circle, or similar icon for each book in the situation. After drawing the icons, students could then determine the total number using their available counting strategies. The strategy used might direct students' attention to each icon individually, if they counted all, or to a combination of grouped and individual icons, if they counted on. We point out three characteristics of such solutions: (1) the one-to-one correspondence between icons and books is the key feature of the representation that affords counting, (2) students always attend to the same representational feature, (possibly grouped) icons, and (3) once students have represented all the books, they need not modify their perspective on the representation in order to execute their counting strategies. These solution characteristics contrast with those we describe in the next section.

3. Adaptive Interpretations of Algebraic Expressions and Equations

The thesis at the center of this article is that as students move from solving arithmetic problems to problems about covarying quantities they have not only to construct new concepts but also to coordinate further ways they attend to external representations and problem situations. Traditional approaches to algebra, at least in the United States, tend not support such coordination. Instead, they often use a balance scale metaphor in which the left- and right-hand side of equations represent weights. The metaphor is intended to emphasize for students that the initial quantities on either side of the equal sign are the same and that any transformation preserving the balance is admissible. The metaphor is then used to justify rules for "doing the same thing to both sides." Once the standard rules are established, subsequent activities often focus narrowly on symbolic manipulation. In terms of the NRC's classification, the balance scale emphasizes transformational activities but ignores representational activities of algebra as well as the equation-solving possibilities provided by graphs and tables.

From our perspective, the balance-scale to symbol-pushing trajectory often overlooks important relations between external representations, in this case equations, and situations containing covarying quantities that afford solutions to problems. Izsák (2000; 2003; 2004) has reported some of these issues through a study in which 12 eighth-grade U.S. students (age 14) developed algebraic representations of a physical device called a winch. Nine of the students were taking an introductory algebra course, and three were taking a pre-algebra course that included some work with variables and functions. The winch (see Fig. 1) exemplifies situations, similar to the video rental problem above, that can be modeled by pairs of simultaneous linear functions. The device stands 4 feet tall and at the top has a rod with a handle for turning two spools, one 3 and one 5 inches in circumference. Fishing line attaches one weight to each spool. For convenience, we refer to these as the 3- and 5-inch weights, respectively. Turning the handle moves the weights up and down a yardstick, allowing measurements of heights, displacements, and distances between the two weights. Izsák configured the initial heights of the weights in various ways and asked three types of questions:

(1) Predict the distance between the weights after an arbitrary number of cranks.
(2) Determine whether and, if so, when one weight will ever be twice as high as the other.
(3) Determine whether and, if so, when the weights will meet at the same height.

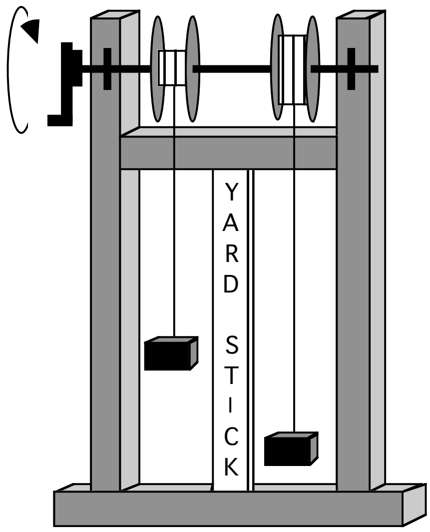


Figure 1: The winch. From "Inscribing the Winch: Mechanisms by Which Students Develop Knowledge Structures for Representing the Physical World with Algebra" by A. Izsák (2000), *The Journal of the Learning Sciences*, 9, p. 33. Copyright 2000 by Lawrence Erlbaum Associates, Inc. Reprinted with permission.

For the second and third types of questions, Izsák first used initial conditions that allowed students to answer by simply turning the crank (for instance, the two weights might meet at 28 inches). He then asked students to imagine a larger 100-inch winch and changed the initial conditions so that students could not simply turn the crank (for instance, the two weights might meet at 50 inches). This interview strategy focused students first on the physical phenomena and then on ways of representing those phenomena that afforded solutions to the problems.

In general, students in the study could represent linear patterns with algebraic expressions—and thus could write expressions for the height of each weight in terms of cranks—and some students could solve equations to answer questions of the third type. Students who answered fluently questions of the third type by generating expressions for the height of each weight, setting the two expressions equal, and solving the resulting equation, however, often experienced significant difficulty with questions of the second type. Students would generate algebraic representations that expressed correct relationships among winch attributes, but could not use their resulting representations to solve the problem. For example, Izsák (2003) reported that one such pair of students examined the winch set up so that the 3-inch weight started by the 28-inch mark and the 5-inch weight started by the 0-inch mark. The students described one weight being twice as high as the other, correctly, by equating the height of the lower weight with the distance between the two weights. They then generated a set of three equations:

$$0 + 5n = h$$
$$28 - 2n = d$$
$$d = h.$$

The first equation expressed the height of the 5-inch weight for any number of cranks, the second expressed the distance between the two weights for any number of cranks (the initial distance decreased by 2 inches with each crank), and the third expressed the specific moment when the lower height would equal the distance between the two weights. Although the students interpreted letters as fixed unknown values in the third equation and as variables in the first two equations, they initially resisted equating $0 + 5n$ and $28 - 2n$ because these expressions were true for any number of cranks but the resulting equation was not. Thus, the students had yet to develop adaptive interpretations that would allow them to examine algebraic representations for all n in some contexts and for unique n in others. The students finally accepted $0 + 5n = 28 - 2n$ after discovering they could solve for n and answer the problem. Matz (1982) also reported students who had trouble distinguishing between equations that are true for any value of the independent variable from those that constrain the independent variable to a unique value.

The winch example occurred in the context of representational activities, but adaptive interpretation is important for generalizing and justifying activities as well. We give an example from Findell's (2001) study of undergraduates learning in an abstract algebra course. Wendy was convinced that the sum of two multiples of 4 is also a multiple of 4 but struggled to use algebraic expressions to support a sound argument. She used the calculation $4x + 4x = 8x$ and explained that $8x$ is a multiple of 4. At different points in her argument, she used the expression $4x$ to denote a particular multiple of 4, any multiple of 4, all multiples of 4, and the set of all multiples. She demonstrated the same range of interpretations of the letter x. Wendy's difficulties with the task appeared rooted, in large part, in understanding when to employ the different interpretations. For instance, she asserted that the expression $4x + 4x$ could express a sum such as 8 + 12, as each term could represent any multiple of 4. Moreover, she did not distinguish between a set and an element of that set. After the interviewer pointed to the expression $4x + 4x$, asked Wendy what she meant by each of the terms, and then emphasized the indefinite article in her response, "*a* multiple of 4," she changed the second term to $4y$. In the new expression, $4x + 4y$, Wendy factored out a 4, renamed $x + y$ as z, and noted that $4z$ is also a multiple of 4, thereby coordinating more fully her use of symbols with her understandings of multiples. In reflecting on her solution, Wendy displayed the beginnings of adaptive interpretations: She acknowledged that it helped her to imagine that the x and y were momentarily fixed, but noted immediately that "any integer you put in there it will work," and then interpreted x and y as "all the integers" and also "as the whole set." She remained uncertain about employing the different interpretations and still did not distinguish between a set and an element. Wendy's comments suggest an important psychological shift: first imagining the x and y as particular fixed values in order reason through the algebraic manipulation; and then allowing x and y to vary to conclude that the proof works for any two multiples of 4. Like the students in the winch example, Wendy was apparently learning to examine algebraic representations for particular values in some contexts and for all such values in others.

4. Adaptive Interpretations of Tables and Graphs

Our second category of examples comes from our current National Science Foundation funded project, Coordinating Students' and Teachers' Algebraic Reasoning. The project is developing case-study methods for investigating ways that teachers and students understand and learn from classroom interactions in which they participate together. We conduct the project in a U.S. middle school that uses the Connected Mathematics Project (CMP) materials (Lappan, Fey, Fitzgerald, Friel, Phillips 2002a). These materials were developed with support from the National Science Foundation in response to the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics 1989) and *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics 2000). The CMP materials introduce mathematical concepts through problem situations that students explore individually, in groups, or as a class. The example we discuss comes from the seventh-grade unit called *Variables and Patterns* (Lappan, Fey, Fitzgerald, Friel, Phillips 2002b) in which students (age 13) use tables and graphs to solve problems about situations that contain covarying quantities.

Our discussion centers on a homework problem called the Popcorn Problem (abbreviated in Fig. 2). Prior to working on the this problem, students in Ms. Bishop's[2] class had completed an activity in which they performed jumping jacks and recorded the total every 10 seconds. As the students tired, the total increased more slowly. The Popcorn Problem was a second task in which students were to reason about non-linear data. Although the word "change" in part b could refer either to changes each hour or to increases over the course of the day, the sample answer in the teacher's guide makes the intended meaning clear:

"Very few bags were sold before 7 a.m., perhaps because many people do not eat popcorn so early in the morning. But the number jumped by 12 bags between 7 a.m. and 8 a.m., when perhaps people were stopping for a snack on their way to school. The number goes up at a rate of about 5 bags per hour between 8 a.m. and 11 a.m. From 11 a.m. until noon it jumps to 15 bags, and 13 bags from noon to 1 p.m.; during these two hours, perhaps people are buying lunch." (Lappan et al. 2002b, p. 10)

Note that the intended task requires adaptive interpretations because students must be able to look at the fourth line of the table, for example, and interpret it as 20 bags sold by 9 a.m. but, when turning their attention to sales each hour, be able to determine by subtraction that 5 bags were sold between 8 a.m. and 9 a.m., and so on. Thus, students must coordinate a shift in attention from total sales to sales each hour with a shift in attention from rows in the table to differences between those rows.

Adaptive interpretations may be unproblematic for those with experience using tables and graphs to reason about covarying quantities, but the instruction we observed and the student interviews we conducted made clear that coordinating shifting perspectives on situations with shifting perspectives on representations was challenging for many students in Ms. Bishop's class. During the initial class discussion of part b to the Popcorn Problem, some students gave answers that did not make clear distinctions between the total number of bags sold for the entire day and the number of bags sold each hour. For instance, Ashley may have focused on increases in both total bags sold and bags sold each hour when she offered, "The changes increased probably because as the day went on more people wanted popcorn, and most people don't want popcorn in the morning." Other students apparently misinterpreted the table in Fig. 2 as bags sold per hour: Rachel said, "I put from 6 a.m. to

1. The convenience store across the street from Metropolis School has been keeping track of their popcorn sales. The table to the right shows the total number of bags sold beginning at 6:00 A.M. on a particular day.
   a. Make a coordinate graph of these data. Which variable did you put on the x-axis? Why?
   b. Describe how the number of bags of popcorn sold changed during the day. Explain why these changes may have occurred.

Time	Total bags sold
6:00 A.M.	0
7:00 A.M.	3
8:00 A.M.	15
9:00 A.M.	20
10:00 A.M.	26
11:00 A.M.	30
Noon	45
1:00 P.M.	58

Figure 2: The Popcorn Problem. From *Connected Mathematics Variables and Patterns: Introducing Algebra Teacher's Guide* © 2002 by Michigan State University, Glenda Lappan, James T. Fey, William M. Fitzgerald, Susan N. Friel, & Elizabeth D. Phillips. Published by Pearson Education, Inc., publishing as Pearson Prentice Hall. Used by Permission.

[2] All names are pseudonyms.

noon they only sold 139 bags." As the class discussion continued, students apparently attended to both total sales and sales each hour but many seemed not to coordinate the situation and the table appropriately. To make clearer the distinction between total sales and sales each hour, Ms. Bishop introduced the phrase "cumulative graph" to describe the popcorn data and defined *cumulative* as "successive addition" or "continuously adding to." Later during the same lesson, she moved to the next problem in the materials which used discrete dots to show soda sales each hour over a day. Ms. Bishop called this a "rate graph." Absent from the lesson was an explicit explanation of how the table in Fig. 2 could convey both cumulative and rate data.

Over the next several lessons, the students often began work on a problem by considering whether the included table or graph presented cumulative or rate data but continued to have difficulties with the distinction. To address the persistent confusion, Ms. Bishop returned to the Popcorn Problem 10 days later and added a third column to record differences between successive rows. Fig. 3 reproduces her written work. Although Ms. Bishop had verbally explained the relationship between total sales and sales each hour during previous lessons, this was the first time that she used inscriptions to explicitly demonstrate how to see both in either a table or graph.

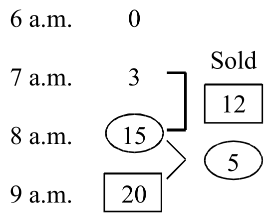


Figure 3: Ms. Bishop Revisits the Popcorn Problem.

To demonstrate that adaptive interpretations of the popcorn table remained problematic for some students, even after Ms. Bishop revisited the problem, we present data from an interview with one pair, Nikki and Jennifer. Nikki was a mid-achieving student, and Jennifer was a low- to mid-achieving student. The first author conducted the interview with these students four days after Ms. Bishop generated the table reproduced in Fig. 3. After watching a video clip of the class discussion, Nikki and Jennifer could describe the subtraction that Ms. Bishop used to generate each of the numbers in the augmented table. Nikki also commented, "I really understood what she did that day for the first time 'cause she actually broke it down from hour to hour." That the students continued to have difficulty with adaptive interpretations became clear in the following exchange (Nikki's language did not distinguish between graphs and tables):

Int: So what do you think about the 12 and the 5 (referring to the right hand column in Fig. 3)? Are they, are those cumulative data? Are they rate data? Are they some other kind of data?

Nikki: Now, with those two numbers, I'm not exactly sure because she subtracted them.

Jennifer: I think rate.

Nikki: Maybe. But it's a cumulative graph. So why would it have two different sets of information on there if it's cumulative?

As the interview progressed, Nikki asked her question about "two different sets of information" in graphs as well. The interviewer gave the students a graph showing new popcorn sales data as discrete dots, similar to other discrete graphs with which the students had worked. The interviewer emphasized that each point on the graph indicated how many bags had been sold all together and then asked the students how many were sold each hour. Nikki and Jennifer answered by calculating differences. The interviewer then pointed out that calculated differences were "rate data" in the sense discussed in class because they represented how many bags sold in each hour. As with the table in Fig. 3, the interviewer was trying to get the students to see that one graph could convey both cumulative and rate data. He then reminded Nikki of her comment that a graph can show only one type of information:

Nikki: Yeah. That's what I figured. Since it's being a cumulative graph, why would it be one kind of graph and have another type of information on it?

Int: Okay. What do you think now?

Nikki: I'm still unsure.

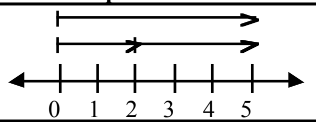
Int: Really?

Nikki: I guess I won't be clear about it unless I ask Ms. Bishop personally. That's definitely something I can't answer myself. Because it's confusing: You thinking about a graph being cumulative and having rate information, or you think about a rate graph having cumulative information, or you think about another graph having another kind of information. It doesn't ring a bell to me.

In further discussion, Nikki continued to demonstrate understanding of the relationship between total sales and sales each hour and was apparently learning that graphs and tables of total sales could convey both types of information. That Nikki struggled with her question about "two different sets of information" when working with both tables and graphs indicates that she was not asking about one particular form of representation, tables or graphs, but about representations more generally. Finally, we point out that classroom discussions in which Ms. Bishop and her students identified particular tables and graphs as cumulative or rate appeared to contribute unintentionally to Nikki's difficulties.

5. Addition and Subtraction on the Number Line

In this section, we return to opportunities for building better continuity in students' experiences using external representations to solve problems in arithmetic and in algebra. In particular, we ask, Are there experiences that might better prepare students for the adaptive interpretations required when using equations, tables, and graphs to solve problems about covarying quantities? We now examine experiences that could be developed when adding and subtracting on the number line. The experiences we describe are direct antecedents because they focus on initial quantities and changes in those

	Representation	Addition (2 + 3 = 5)	Subtraction (5 – 2 = 3)
(a)		Begin at 0, go a length of 2, then go 3 more. How long is the combined length?	Begin at 0, go a length of 2. How much further should you go to create a combined length of 5?

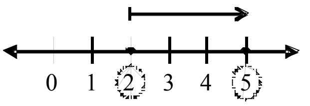
	Representation	Addition (2 + 3 = 5)	Subtraction (5 – 2 = 3)
(b)		Go to the point 2, and draw to the right a length of 3. At what point does that length end?	Locate the points 2 and 5 and draw a segment between them. How long is that segment?

Figure 4: (a) Addition and Subtraction on the Number Line Emphasizing Parts of a Whole. (b) Addition and Subtraction as Distance on the Number Line.

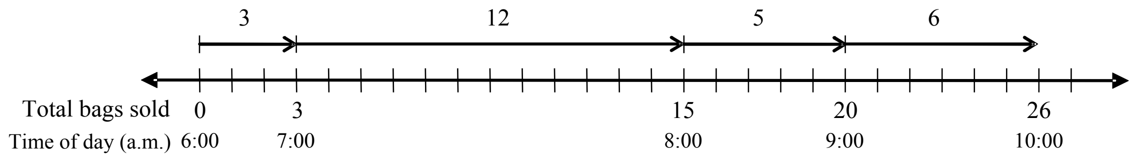


Figure 5: The Popcorn Data on a Number Line.

quantities and because one can think of axes in Cartesian graphs as orthogonal number lines.

Rather than describing typical approaches to addition and subtraction on the number line, we focus below on two less common representations that we believe are particularly useful in supporting algebraic thinking. We illustrate both addition and subtraction in each representation in order to demonstrate an instance of adaptive interpretation: a shift from thinking about addition to a related subtraction problem coordinated with a shift in perspective on the representation. Our examples use whole numbers, but the same perspectives can be extended to real numbers. Fig. 4 shows how number lines can be drawn to help students initially coordinate addition and subtraction with lengths. When considering the sum of two and three, for example, students can think of lengths of two and three being combined to form a length of five. We intentionally use arrows to support a motion metaphor where students can start at 0, go a length of 2, and then go 3 more units. Although it would be more conventional to represent subtraction with arrows in the opposite direction, we propose representing subtraction via a "missing addend" approach. In this case, 5 – 2 is the solution to 2 + __ = 5, and the representation supports the observation that the solution is 3, because a length of 3 must be added to a length of 2 to yield a length of 5.

To support reasoning about both actual values and changes in those values, we need a more efficient representation than that in Fig. 4a. Consider Fig. 4b as an abbreviation, where one of the numbers is a length or distance and other numbers are points or positions. An advantage is that this representation calls attention to the interpretation of subtraction as the distance between two points on the number line. Furthermore, in an algebra context in which a variable changes from 2 to 5, this representation can support a distinction between actual values (i.e., 2 and 5 represented as points) and changes in those values (i.e., the arrow of length 3). This distinction requires adaptive interpretation and, we conjecture, could support students' reasoning with graphs and tables in algebra.

Fig. 5 shows a number line representing the first few lines of popcorn data from Fig. 2. This number line inscribes total sales as points and changes in sales as lengths. By interpreting Fig. 5 as multiple instances of Fig. 4b, students could attend to the different representational features that support adaptive interpretations. Finally, Fig. 6 indicates how coordinating initial quantities with points and changes with distances, as shown in Fig. 5, might better prepare students for adaptive interpretations of graphs. In particular, the number line representation becomes vertical distances, spread out horizontally (over time). Note that this emphasis on vertical differences might better prepare students to understand slope as a ratio of differences at a later point in time.

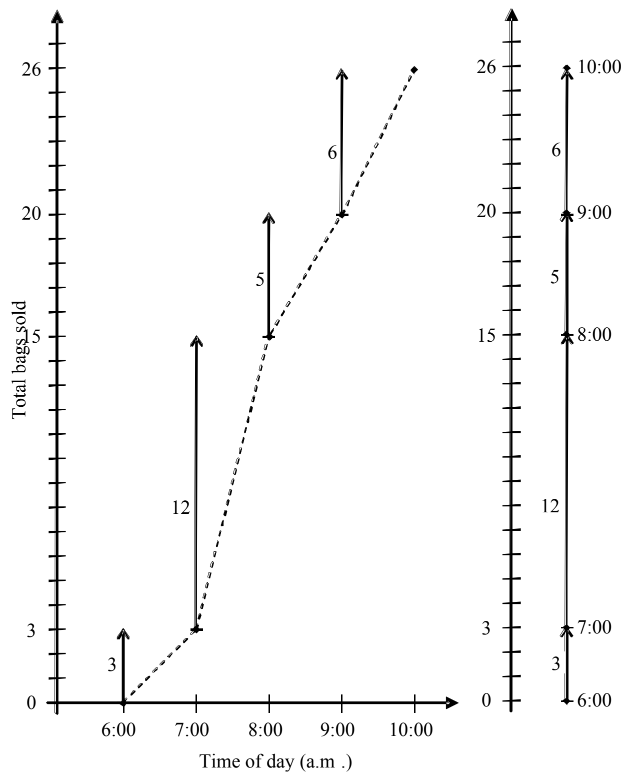


Figure 6: Moving from a Number Line to a Graph.

6. Conclusion

Our work contributes to current research that examines opportunities to better connect students' experiences in algebra with earlier experiences in arithmetic. In contrast to past research that has focused on connections across multiple representations, we have discussed challenges that arise when students interpret a single representation of a situation that contains covarying quantities. To describe ways in which students must coordinate their thinking about the situation with their perspective on a representation, we have introduced the term *adaptive interpretation* and have presented examples that show (1) adaptive interpretation is an important, yet usually implicit, aspect of algebra learning and (2) early arithmetic experiences with adaptive interpretation might support subsequent algebra learning. In the first examples, students had to develop adaptive interpretation so as to examine algebraic representations for all n in some contexts and for unique n in others. In the second example, students had to coordinate initial quantities and changes in those quantities with two perspectives on tables and graphs. These examples demonstrate that adaptive interpretation is necessary in representational activities as well as in generalizing and justifying activities that occur in algebra.

Working backwards from the algebra learning phenomena described above, we conjecture that if students focused more explicitly on adaptive interpretations when representing and solving problems in arithmetic, they might be better prepared for this aspect of representing and solving problems in algebra. The problem about Sarah's books illustrates that some elementary mathematics may not afford opportunities to develop adaptive interpretation of representations, but our discussion of addition and subtraction on the number line demonstrates that some opportunities do exist. Furthermore, such opportunities are not limited to representations of situations containing covariation. For instance, Izsák (in press) presented a case study in which three of four pairs of fifth-grade students (age 10) from one classroom struggled to determine areas and perimeters of rectangles correctly. A key difficulty the students faced was that in their past experiences with discrete arrays they had always attended to one representational feature, dots. Now they had to attend to two different representational features, unit line segments when determining perimeters and unit squares when determining areas. For these students, developing understandings of rectangular areas and perimeters also involved developing adaptive interpretations of the external representations at hand.

Developing curricular materials that explicitly attend to adaptive interpretation could help create important learning opportunities for students in classrooms. The Popcorn Problem illustrates that when curricular materials and instruction leave relationships between representations and problem situations implicit, classroom discussions can become confused. Ms. Bishop focused on coordinating shifts in attention between initial quantities and changes in those quantities after seeing her students struggle with such shifts for several days. To highlight the shifts, she augmented the table included in the Popcorn Problem materials, as shown in Fig. 3. We described augmentations to number line representations that highlight analogous distinctions. These examples point to some ways that curricular materials could make appropriate distinctions and coordination more explicit. Furthermore, developing the distinction between initial quantities and changes in those quantities in earlier grades may better prepare students in later grades to understand slope and is a discrete precursor to the distinction in calculus between a function and its derivative. Thus, we believe that more explicit attention to adaptive interpretation could better prepare students for algebra and for calculus.

Finally, Thompson (1994) critiqued work on multiple representations of functions arguing that the concept of function is not represented by tables, graphs, or expressions, but rather is built by establishing connections between inscriptions and problem situations as understood by the student. The ideas presented in this paper and, in particular, the notation of adaptive interpretation fit with Thompson's perspective. We believe that explicit attention to adaptive interpretation may be fruitful for other researchers describing challenges in learning algebra, as well as those seeking connections between algebra and arithmetic and articulating the very notion of representation in mathematical learning.

7. References

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