The Development of Students' Algebraic Thinking in Earlier Grades: A Cross-Cultural Comparative Perspective[1]

Jinfa Cai, Newark, DE (USA)
Hee Chan Lew, Cheongwon, Chungbuk (South Korea)
Anne Morris, Newark, DE (USA)
John C. Moyer, Milwaukee, WI (USA)
Swee Fong Ng, Nanyang (Singapore)
Jean Schmittau, Binghamton, NY (USA)

Abstract: We analyzed how algebraic concepts and representations are introduced and developed in the Chinese, South Korean, and Singaporean elementary curricula, and in selected Russian and U.S. elementary curricula. In all five curricula, the main goal for learning algebraic concepts is to deepen students' understanding of quantitative relationships, but the emphases and approaches to helping students deepen their understanding of quantitative relationships are very different. Based on the analyses of the five curricula, we discuss four issues related to the development of algebraic thinking in earlier grades: (1) To what extent do curricula expect students in early grades to think algebraically? (2) What level of formalism should we expect of students in the early grades? (3) How can we help students make a smooth transition from arithmetic to algebraic thinking? and (4) Are authentic applications necessary for students in early grades?

Kurzreferat: In dem Beitrag wird analysiert, wie algebraische Begriffe und Repräsentationen in Grundschulcurricula aus China, Russland, Südkorea, Singapur und USA eingeführt und entwickelt werden. In allen fünf Curricula besteht das Hauptziel des Lernens von algebraischen Begriffen darin, das Verständnis von Lernenden hinsichtlich quantitativer Beziehungen zu vertiefen. Jedoch sind die Schwerpunkte und die Ansätze der fünf Curricula zur Vertiefung des Verständnisses der Schülerinnen und Schüler von quantitativen Beziehungen sehr unterschiedlich. Auf der Basis der Analysen dieser fünf genannten Curricula diskutieren wir vier Themen in Zusammenhang mit der Entwicklung von algebraischem Denken in unteren Jahrgangsstufen: (1) Inwieweit erwarten Curricula von Lernenden in unteren Klassenstufen, algebraisch zu denken? (2) Welchen Grad an Formalismus dürfen wir bei Lernenden der unteren Klassenstufen erwarten? (3) Wie können wir Lernenden zu einem angemessenen Übergang von arithmetischem zu algebraischem Denken verhelfen? und (4) Sind authentische Anwendungen für Lernende der unteren Jahrgangsstufen erforderlich?

ZDM-Classification: D30, H20

1. Introduction

Algebra has been characterized as the most important "gatekeeper" in mathematics. It is widely accepted that to achieve the goal of "algebra for all", students in elementary school should have experiences that prepare them for the more formal study of algebra in the later grades (National Council of Teachers of Mathematics [NCTM] 2000). However, curriculum developers, educational researchers, and policy makers are just beginning to explore the kinds of mathematical experiences elementary students need to prepare them for the formal study of algebra at the later grades (Carpenter, Franke, & Levi 2003, Kaput 1999, Mathematical Sciences Education Board 1998, NCTM 2000, Schifter 1999, Stacey, Chick, & Kenda 2004).

Results from both national and international assessments (e.g., U.S. National Assessment of Educational Progress [NAEP], Third International Mathematics and Science Study [TIMSS]) suggest that many young children enjoy school mathematics and are good at it, but that starting at fourth or fifth grade, significant numbers of students start to have difficulty with the subject and dislike it (National Academy of Education 1999). In particular, there is evidence that U.S. students are ill-prepared for the study of algebra (Silver & Kenney 2001). One of the challenges teachers in the United States face is the lack of a coherent K-8 curriculum that can provide students with algebraic experiences that are both early and rich (Schmidt et al. 1996). Often, algebra has not been treated explicitly in the school curriculum until the traditional algebra course offered in middle school or high school (NCTM 2000). Moreover, according to a rigorous academic analysis by the American Association for the Advancement of Sciences (AAAS 2000), the majority of textbooks used for algebra in the United States have serious weaknesses. In addition, there is evidence that most elementary school teachers in the United States are not adequately prepared to integrate algebraic reasoning into their instructional practices (e.g., van Dooren, Verschaffel, & Onghena 2002).

Most U.S. students do not start the formal study of algebra until eighth or ninth grade, and many of them experience difficulties making the transition from arithmetic to algebra because they have little or no prior experience with the subject (Silver & Kenney 2001). The unsatisfactory findings from the national and international assessments (e.g., NAEP, TIMSS) indicate a need to emphasize and develop U.S. students' algebraic thinking in the early grades. In several countries (e.g., China, Russia, Singapore, and South Korea), students begin the formal study of algebra much earlier. However, it is likely that, across these settings, curriculum developers, educational researchers, and policy makers have faced challenges similar to those of their counterparts in the United States as they attempted to develop early and appropriate algebraic experiences for younger children (Kieran 2004, Stacey et al. 2004).

The purpose of this study is to analyze how algebraic concepts and representations are developed and introduced in the Chinese, Singaporean, and South Korean elementary curricula, and in selected Russian and

1 An earlier version of this paper was presented at the Annual Meeting of the American Educational Research Association in San Diego, CA, April 12-17, 2004. All authors have made significant contribution to the paper. The preparation of this paper was supported, in part, by a grant from the National Science Foundation (ESI-0114768). Any opinions expressed herein are those of the authors and do not necessarily represent the views of the National Science Foundation.

U.S. elementary curricula. We selected these five nations to reflect diverse views about the development of algebraic thinking in earlier grades (Cai 2004a). Knowledge of the curricula and instructional practices of different nations can increase educators' and teachers' abilities to meet the challenges associated with the development of students' algebraic thinking.

This study is not intended to evaluate the curricula in these countries. Instead, our focus is on studying and understanding how the design of curricula in these countries contributes to the development of students' algebraic thinking. We believe that such understanding will increase our knowledge and our ability to address the issues related to the development of students' algebraic thinking in elementary school from an international perspective. We need to support students' development of algebraic thinking in the early grades as a way to help them make a smooth transition between arithmetic and algebra, and as a way to help them appreciate the usefulness of algebraic approaches in solving various problems. Therefore, of special interest will be the way the various curricula prepare students to make smooth transitions from informal to formal algebraic thinking.

In the next four sections, we first present the rationale for selecting the curricula and the methods of analysis. Second, we discuss the unique features of the five curricula we studied. Then we offer some insights regarding the development of students' algebraic thinking in earlier grades based on five case studies.

2. Curriculum selection and analysis

For each of the five nations in our study, we selected either the national curriculum or a widely used curriculum (Curriculum Planning & Development Division, 2000, Davydov, Gorbov, Mikulina, & Saveleva 1999, Division of Elementary Mathematics, 1999, Russell, Economopoulos, Murray, Mokros, & Goodrow 1998). In the case of the U.S. curriculum, we chose to study the *Investigations in Number, Data, and Space* curriculum. We did this for a number of reasons: (1) It is one of the three elementary mathematics reform curricula developed with National Science Foundation funding; (2) It provides a comprehensive mathematics program for grades K-5; (3) It teaches a broad range of mathematical concepts; (4) It is a widely used series in the United States; and (5) It has an identifiable algebra strand. As we indicated elsewhere (Moyer, Huinker, & Cai 2004), a revision of the algebra strand in the *Investigations in Number, Data and Space* curriculum began in 2001. One of the intents of the revision is to strengthen the integration of algebraic thinking throughout the curriculum. Since the revised version was not yet available, our analysis was based on the 1998 edition of the curriculum.

Each of the five case studies focuses on how the chosen elementary curriculum for each country has been designed to develop students' algebraic thinking. We were interested in the algebraic concepts included in the curriculum and how these concepts are developed and represented. In particular, each curriculum was analyzed along three dimensions: (1) goal specification, (2) content coverage, and (3) process coverage.

2.1 Goal specification

In our analysis, we identified the goals for the development of algebraic thinking in each of the mathematics curricula. We also selected a corresponding set of mathematical problems whose solutions addressed each of the goals. The algebra goals in each curriculum were also compared with the four algebra goals specified in NCTM's *Principles and Standards for School Mathematics* (NCTM 2000). These four goals are to: (1) understand patterns, relations, and functions; (2) represent and analyze mathematical situations and structures using algebraic symbols; (3) use mathematical models to represent and understand quantitative relationships; and (4) analyze change in various contexts.

2.2 Content coverage

For the second dimension, we identified the *big ideas* of algebraic thinking in each of the mathematics curricula. A big idea of algebraic thinking is an essential concept or technique for reasoning about quantitative conditions and relationships. We focused on the following commonly identified algebraic ideas, widely accepted as important in algebraic understanding and reasoning: variables, proportional reasoning, patterns and relationships, equivalence of expressions, equation and equation solving, change, and representation and modeling. We also examined how the big ideas were developed throughout the curriculum.

2.3 Process coverage

Algebra is much more than just solving for x and y; instead, algebra is a way of thinking. Success in algebra depends on the ability to think in a variety of powerful ways that foster productive algebraic performance. When people think algebraically to solve problems, various habits of thinking come into play, such as doing-undoing, building rules to represent functions, and abstracting from computation (Driscoll 1999). Curricula can serve to demystify algebra by providing activities that foster these sorts of thinking in students. In this third dimension of our analysis, we examined how the design of each curriculum fosters the development of algebraic thinking habits. We also analyzed how the curricula help students connect the way they employ habits of thinking in their early mathematical experience to the way they should employ habits of thinking when doing formal algebra.

Results from each case study are published in a special issue of *The Mathematics Educator* (Cai 2004a). The focus of this paper is to identify the unique features of each curriculum, and then to illustrate how we can use an international perspective to tackle difficult issues related to the development of algebraic thinking in early grades.

3. Features of the five curricula

All the curricula explicitly or implicitly indicate that their main goal in teaching algebraic concepts is to deepen students' understanding of quantitative relationships. However, the approach to helping students deepen their understanding of quantitative relationships differs significantly across the five curricula, as do the algebraic concepts emphasized in each curriculum. In this section, we highlight distinct features of the five curricula.

3.1 The Chinese curriculum

The overarching algebra-related goal in the Chinese elementary curriculum is to help students to better represent and understand quantitative relationships, numerically and symbolically. The main focus is on equations and equation solving. Variables, equations, equation solving, and function sense permeate the curriculum in grades 1 to 4. Equations and equation solving are formally introduced in the first half of grade 5. Once equation solving is introduced, it is applied to the learning of mathematical topics, such as fractions, percents, statistics, and proportional reasoning, in the last half of grades 5 and 6.

The term "variable" is not defined in Chinese elementary school mathematics. However, in the teacher's guide for the national curriculum, teachers are reminded that variables can represent many numbers simultaneously, that they have no place value, and that representations of variables can be selected arbitrarily. In Chinese elementary school mathematics, variable ideas are used in three different ways. First, they are used as place holders for unknowns in equation solving. In grades 1-3, for example, a question mark, a picture, a word, a blanket, or a box is used to represent the unknowns in equations. Second, variables are viewed as pattern generalizers or as representatives of a range of values. Variables are not always represented with letters. In the third grade, in order to help students understand the meanings of formulas, words are used, rather than letters, to represent variables. For example, after examining several specific examples, the formulas for the areas of rectangles and squares are represented as: Area of a rectangle = its length □ its width and Area of a square = its side □ its side. However, teachers are counseled to emphasize the generalizable nature of the formulas. That is, for any rectangle, its area can be found by multiplying its length and width. In grades 5 and 6, letters are used to represent formulas for finding areas of squares, triangles, rectangles, trapezoids, and circles. The third use of variables is to represent relationships, such as direct proportionality ($y/x = k$) and inverse proportionality ($xy = k$). This way of using variables is discussed more thoroughly in the next section.

In the Chinese elementary curriculum, the function concept is not formally introduced. However, function ideas permeate the curriculum so that students' function sense can be informally developed. According to the curriculum guide, the pervasive use of function ideas in various content areas not only fosters students' learning of the content topics, but also provides a solid foundation for the future learning of advanced mathematical topics in middle and high schools. In the early grades, the Chinese curriculum provides students with many opportunities to develop function sense at the concrete and intuitive level. Function sense is first introduced in the context of comparing and operating with whole numbers at grade 1 using a one-to-one mapping. In grade 6, multiple representations (pictures, diagrams, tables, graphs, and equations) are used to represent functional relationships between two quantities. These functional relationships are embedded in the curricular treatment of circles, statistics, and proportional reasoning.

The Chinese elementary school curriculum is intended to develop at least three thinking habits in students. The first thinking habit is to examine quantitative relationships from different perspectives. Students are consistently encouraged, and provided with opportunities, to represent a quantitative relationship both arithmetically and algebraically. Furthermore, students are asked to make comparisons between arithmetic and algebraic ways of representing quantitative relationships. Throughout the Chinese elementary school curriculum, there are numerous examples and problems in which students identify quantitative relationships and represent them in multiple ways (Cai 2004b). For example, in the following problem, the quantitative relationship involves the amount of money paid to the cashier, the change, and the cost of the two batteries: *Xiao Qing purchased two batteries. She gave the cashier 6 yuans and got .4 yuan in change back. How much does each battery cost?* The teacher's reference book recommends that teachers allow students to represent the quantitative relationship in different ways, such as the following:

The amount of money paid to the cashier – the cost of the two batteries = the change.

The cost of the two batteries + the change = the amount of money paid to the cashier.

The amount of money paid to the cashier – the change = the cost of the two batteries.

The second thinking habit is to use reverse operations to solve equations. Starting in the first grade, subtraction is defined as the reverse of addition. Although the term "solve" is not used at grade 1, students learn to solve equations starting at grade 1 and they continue solving equations throughout the entire curriculum. The third thinking habit is generalizing from specific examples. By examining specific examples, students are guided to create generalized expressions. Students develop this habit of mind at a variety of points in the curriculum, but especially when formulas for finding perimeters, areas, and volumes are introduced, when operational laws are presented, or when the averaging algorithm is discussed.

3.2 The Russian curriculum

In the Russian elementary mathematics curriculum of V.V. Davydov, which spans grades one through three, children develop algebraic thinking by exploring and comparing quantities. This is done before the study of arithmetic. Children use letters to express the relationships between quantities, learn to express part-whole relationships between quantities, transform inequalities into equalities, and find missing wholes and parts using addition and subtraction. They proceed to the measurement of quantities, out of which number itself is defined. They learn to distinguish actions with numbers from actions with quantities when solving equations arising from word problems, and to model both using special representations and algebraic symbols.

By the second grade students are required to solve equations and two-step word problems involving all of the four fundamental operations. Equations increase in complexity to reflect the problems they model. Children are expected to represent schematically the internal quantitative relationships in word problems, and to write equations that express these relationships symbolically.

In grade three, proportional reasoning is developed. The proportional reasoning part of the curriculum builds on a carefully developed foundation of concepts of quantity, relationships between quantities, and measuring units, as well as concepts of number, variable, and uniform motion. The curriculum also draws upon children's skill in solving equations as well as their ability to solve multi-step problems, to use models to analyze and express quantitative relationships, and to manipulate these relationships symbolically.

A deep understanding of proportional reasoning is fostered, with much attention given to the organization of data in story texts, the identification of the question that problematizes the text, the study of various uniform and non-uniform processes, and the study of the speed of such processes. The word problems that involve proportional reasoning are multi-step in nature, and represent situations of considerable complexity. They are comparable to problems that U.S. students first confront at the secondary level, such as problems involving rate, time and distance, and which even high school students typically find challenging.

3.3 The South Korean curriculum

South Korean students begin the formal study of algebra in the seventh grade. Based on the principle that learning algebra is much more than mastering techniques and knowledge, the elementary school curriculum provides many activities to help students establish a good foundation for the formal learning of algebra. To reduce the cognitive gap between algebra and arithmetic, components of algebraic thinking such as generalization, abstraction, analytic thinking, dynamic thinking, modeling, and organization are emphasized in the early grades.

The symbol " " to represent an unknown value is introduced in the first grade. First, the process to solve for the unknown value " " is introduced intuitively. Various modeling activities to relate a given expression to a situation are provided. Activities to identify mathematical structures like the commutative law of addition are introduced in the first grade. Some problem solving strategies, such as working backward and making a table, are introduced in first grade as well.

In grades 3-4, activities that deal with two or more objects simultaneously are introduced. For example, there are activities that require students to determine some digits represented as " " or as an alphabetic symbol in vertical multiplication. Problem-solving strategies such as simplification, and trial and error are introduced in the fourth grade. The process of formally solving equations by using inverse operations is also introduced. The functional thinking needed to deal with the relation between two variables and to represent it in terms of symbols like " " and ( ) is introduced at these grade levels.

A form of the "geometric" modeling process is introduced through activities in which students use diagrams and figures to solve problems. This means that the range of representation tools becomes broader than numbers and symbols. The curriculum also introduces direct proportionality as a problem-solving tool in fourth grade, thus signaling an underlying conviction that fourth-grade students' cognitive ability can support hypothetic-deductive level activities.

In grades 5-6, problem-solving strategies such as simplification, working backward, and trial and error are dealt with frequently. Formal algebraic thinking is promoted in problem solving—e.g., by emphasizing the use of proportional expressions. Activities are provided that use logical deduction strategies to solve problems involving two ratios.

Dynamic thinking is particularly emphasized in grades 5-6 (Lew 2004). This might be helpful for teaching the concepts of variables and functions in the junior high school. However, materials used in the textbooks tend to be quite artificial in nature. As a result, students generally consider mathematics to be uninteresting. If more realistic situations were used in South Korean textbooks, perhaps students would become more interested in learning mathematics.

3.4 The Singaporean curriculum

In Singapore, some algebraic concepts are formally introduced in grade primary six (age 12+). At this level children are taught how to construct, simplify, and evaluate algebraic expressions of one variable. The notion of letters as variables is introduced at this level. The concept of equations and other structural aspects of algebra are developed in lower secondary years (age 13+ onwards). However, the Singaporean primary mathematics curriculum provides a wide variety of experiences to help children develop algebraic thinking, and this development is made possible by the use of three approaches—a problem solving approach, a generalizing approach, and functional approach. Three algebraic thinking processes support these approaches, namely, analyzing parts and wholes, generalizing and specializing, and doing and undoing. The fundamental aim of these activities is to sharpen students' attendant thinking processes and thus to help develop their problem solving skills.

One of the big ideas related to algebraic thinking in the Singaporean elementary curriculum is the concept of unknowns, which is developed by using rectangles rather than abstract letters, to represent unknowns in word problems. Children solve word problems using the "model method" to construct pictorial equations that represent all the information in word problems as a cohesive whole, rather than as distinct parts. To solve for the unknown, children undo each operation. This approach helps further enhance their knowledge of the properties of the four operations.

The next big idea in the curriculum is developed by exploring the structures that underpin number patterns. Children explore repeating and growing number patterns and develop the complementary thinking processes of generalizing and specializing. Through such activities, children's knowledge of numbers and algorithms are further enhanced. Also, children explore geometric patterns that eventually lead to generalizations about attributes of geometric shapes.

Another big idea is the notion of letters as variables, and this is developed using a functional approach. Through this approach, children are provided with activities that help them develop the point-wise notion of function by looking for the relation inherent in two coordinate points—the input and output of a problem situation (e.g. (1,2), (2,4), (3,6)…(x,2x)). Tasks move from simple activities where children are engaged in doing and undoing with numerical entities to using letters to generalize the forward operation that produces the output from a given input. Children address the question, "What's the rule for this pattern?"

Furthermore children are provided with a variety of activities that foster the development of algebraic thinking habits—doing and undoing, building rules to represent functions and also to abstract from computations. It should be pointed out that the "model method" described above could provide a smooth transition from working with unknowns in less abstract form to the more abstract use of letters in formal algebra in secondary school.

3.5 The U.S. Investigations curriculum

Investigations in Number, Data, and Space (hereafter, *Investigations*) is one of three elementary school reform-mathematics curricula developed with funding from the National Science Foundation. It was designed specifically to help to reform mathematics instruction by implementing the recommendations of the *Curriculum and Evaluation Standards for School Mathematics* developed by the NCTM in 1989.

As a result, it is not surprising that the algebra strand of *Investigations* exhibits many non-traditional features. However, there are two non-traditional features that stand out as being unique, even when compared to other reform curricula in the United States. The first unique feature is *Investigations'* heavy emphasis on mathematical change. The second unique feature is *Investigations'* uncompromising commitment to establishing informal algebra-related intuitions and to avoiding (or at least postponing for a very long time) formal, symbolic representations and procedures.

We analyzed the *Investigations* algebra strand for its emphasis on change. We found not only that mathematical change unifies the activities in the algebra strand, but also that mathematical change motivates the development of other big ideas of algebra. Specifically, our analysis uncovered a progression of concepts across grades K-5 (repeating patterns, additive change, time, net change, change over time, growing patterns, and rate of change) that build an increasingly robust notion of the big idea of change. An essential part of the establishment of this robust notion of change is *Investigations'* development of conceptions and connections to other big ideas of algebra, namely patterns and relationships, representation, and modeling.

Our analysis also uncovered numerous instances in which the curriculum delays formal treatment of mathematical content until it has enriched and refined a foundation of basic mathematical intuitions that children informally learn through their daily interactions. For example, students do extensive work combining and separating everyday items in first grade, but formal introduction to the operations of addition and subtraction is postponed until grade 2. Students develop their own strategies to solve start unknown problems during extensive problem solving sessions in grades 3 and 4, but formal equation solving is not taught anywhere in the curriculum. A variety of tasks in the third grade unit use integers to model net change, but formal procedures for adding and subtracting integers are not taught in *Investigations*. In fifth grade, an entire series of activities is designed to informally characterize a runner's speed as being the same thing as the runner's step size (rather than formally characterize speed as the ratio of distance to time) despite the obvious limitations such a characterization imposes.

Similarly we found that *Investigations* postpones symbolic representation until the students have had significant experience using and discussing the informal representations they have developed on their own. For example, in first grade, students develop their own ways of representing numbers and of finding combinations, but writing formal addition equations (e.g., 3 + 7 = 10) is postponed until grade 2. In second grade, students design their own timelines before they are introduced to standard methods of representing time along a number line. In third grade, students label the floors of a fantasy skyscraper, B2, B1, 0, 1, 2, (rather than –1, –2, –1, 0, 1, 2) even though they use negative numbers to represent change in the downward direction. Also in third grade, despite the fact that students invent their own methods to solve start unknown problems, formally representing the problems with equations is postponed until grade 4. Even then, a question mark is used as a variable to represent the missing start in the equation. In fourth grade, students spend considerable time devising and analyzing their own ways to represent change over time, even though those methods are eventually abandoned in favor of more standard methods. In fifth grade, students come up with rules that describe the generalized change in various growing patterns, but they do not use variables to express their rules. Also in fifth grade, students work extensively to analyze the relation between distance, speed, and time, but they are not introduced to the formula $d = rt$, nor is speed represented as a ratio.

3.6 Algebra emphases in the five curricula

In the *Principles and Standards for School Mathematics* (NCTM, 2000), four goals are listed related to the algebra strand: Goal 1—Understand patterns, relations, and functions; Goal 2—Represent and analyze mathematical situations and structures using algebraic symbols; Goal 3—Use mathematical models to represent and understand quantitative relationships; and Goal 4—Analyze change in various contexts. We examined the emphases of each curriculum by comparing them with the four NCTM goals based on our case studies of these curricula (Cai 2004b, Lew 2004, Moyer et al. 2004, Ng 2004, Schmittau & Morris 2004). To determine if a curriculum has the algebra emphasis related to a certain NCTM goal, the curriculum has to specify similar learning goals or include extensive instructional activities related to the NCTM goal. Table 1 shows how the emphases of each curriculum compare to the four algebra goals listed in *Principles and Standards.*

The algebraic emphasis in Chinese elementary school mathematics is consonant with the first three NCTM goals. For the fourth goal, only qualitative analysis of change is included. The fourth goal is addressed fully when the concept of function is formally introduced in Chinese junior high schools. Like the Chinese curriculum, the Singaporean and South Korean elementary mathematics curricula provide ample activities for students to achieve the first three goals. The last goal is developed in secondary mathematics. It is interesting to note the consistency of the curricular emphases among the three Asian curricula.

Table 1: Algebra Emphases in the Five Curricula.

	CHINA	RUSSIA	SINGAPORE	S.KOREA	U.S.
GOAL 1 (Understand patterns)	÷		÷	÷	÷
GOAL 2 (Use algebra symbols)	÷	÷	÷	÷	
GOAL 3 (Use math models)	÷	÷	÷	÷	÷
GOAL 4 (Analyze change)		÷			÷

The Russian elementary curriculum of Davydov does not include work with patterns and functions, but deals extensively with topics nominally related to the other three NCTM goals. However, there is not necessarily congruence between the meaning these topics have in the goal statements of NCTM and those intended by Davydov and his colleagues. For example, the Russian curriculum does not build on children's experiences with number, nor does it follow the NCTM recommendation that children discover patterns through induction. Instead they analyze the relationships between physical quantities (e.g., amounts of water), model these, and express them symbolically with algebraic symbols. In the case of the NCTM *Principles and Standards*, the early mathematical experiences of children are numerical. In Davydov's curriculum, numerical experiences follow rather than precede the development of algebra.

"Analyze change in its various contexts" is the central goal of the *Investigations* algebra strand. This curriculum accomplishes the teaching of "change" from an informal perspective. It builds on basic intuitions that children express from a very early age, enriching and refining them. By design, therefore, it does not progress to symbolic or formal work.

4. Some insights from case studies of the five curricula

In the sections that follow we present insights from our curricular analyses. In particular, we address the following issues: the extent to which curricula expect students in early grades to think algebraically, the level of formalism expected of students, the nature of support for helping students make a smooth transition from arithmetic to algebraic thinking, and the role authentic applications play in fostering algebraic thinking.

4.1 To what extent do curricula expect students in early grades to think algebraically?

We realize that the question of whether we should expect students in early grades to think algebraically is not an issue these days. Based on recent research on learning, there are many obvious and widely accepted reasons for maintaining this expectation. Nonetheless, we raise the question in order to offer a less obvious reason for developing algebraic ideas in the earlier grades, namely that resistance to algebra would be reduced if we could remove the misconception that arithmetic and algebra are disjoint subjects. Currently, the need to learn algebraic ideas is not as accepted as the need to learn arithmetic, history, or writing (Usiskin 1995). Even those who have taken an algebra course and have done well can live productive lives without ever using it. Therefore, many middle and high school students are not motivated to learn algebra.

Although one can make an eloquent argument in favor of studying algebra at the secondary level (e.g., Usiskin 1995), we believe that resistance to algebra can be more effectively addressed by helping students form algebraic habits of thinking in elementary school. If students and teachers routinely spent the first six years of elementary school simultaneously developing arithmetic and algebraic thinking (with differing emphases on both at different stages of learning), arithmetic and algebra would come to be viewed as being inextricably interconnected. We believe an important outcome would be that the study of algebra in secondary school would become a natural and non-threatening extension of the mathematics of the elementary school curriculum.

Although it is widely accepted that we should expect students in early grades to think algebraically, the real question is how we can prepare students in earlier grades to think algebraically. The case studies of the five curricula provide somewhat different responses to this question. In the next several sections, we provide some insights from the case studies of the five curricula.

4.2 What level of formal algebraic representation should we expect of students in the early grades?

Formal algebraic representation, one of the important characteristics of algebraic thinking, is related to generalization and symbolism. Symbols are used not only as placeholders for unknowns (e.g., ! + 5 = 10, what is the value of !?), but also as representatives of a range of values (e.g., $s = 2t + 3$). For elementary school students, what level of formal algebraic representation should we expect? Our case studies show that the five curricula offer quite different answers to the question, with the Russian and U.S. curricula providing very different perspectives.

According to Schmittau and Morris (2004), the study of algebra precedes the study of arithmetic in the Davydov curriculum. The arithmetic of real numbers follows as a concrete application of the algebraic generalizations.

The algebraic understandings of children studying Davydov's curriculum begin with children's analysis of a concrete or real world situation. They do a great deal of work with quantities such as length, area, volume, and weight, through a series of problems emanating from various contexts. Children model the relations between quantities first schematically, then using letter symbolism. Prior to the introduction of measurement (from which number is defined), they have no numerical values to attach to these quantities. Even after the introduction of number, however, these algebraic thinking habits persist and are extended as children continue to analyze mathematical relationships at the most abstract and generalized level within elementary grades, and to model them both schematically and symbolically.

The ability to analyze a concept and apprehend it at its most general and abstract level reflects the overriding aim of the Davydov curriculum, which is the development of the ability to think theoretically. Unlike the perspective taken by the U.S. *Investigations,* Russian psychologists in the Vygotskian tradition believe that keeping a child at the level of empirical thinking for too long hinders the development of theoretical thinking.

Investigations takes a different approach to both formal algebraic representation and empirical thinking. According to Moyer, Huinker, and Cai (2004), *Investigations* was designed on the premise that a rush to symbolism is counterproductive to the learning process. Consequently, *Investigations* delays formal treatment of algebraic content until it can enrich and refine a foundation of basic algebraic intuitions that children have informally learned through their daily interactions. Accordingly, *Investigations* postpones symbolic representation until students have had significant experiences using and discussing the informal representations that they develop for their own use. For example, in first grade, students develop their own ways of representing numbers and of finding combinations, but writing formal addition equations (3 + 7 = 10) is postponed until grade 2. In third grade, despite the fact that students invent their own methods to solve start unknown problems, formally representing the problems with equations is postponed until grade 4. In fifth grade. students come up with rules that describe the generalized change in various growing patterns, but they are not asked to use variables to express the rules.

4.3 How can we help students to think arithmetically and algebraically?

According to Kieran (2004), in the transition from arithmetic to algebra, students need to make many adjustments in the way they think, even those students who are quite proficient in arithmetic. Kieran particularly suggested the following five types of adjustments in developing an algebraic way of thinking: (1) Focus on relationships and not merely on the calculation of a numerical answer, (2) Focus on inverses of operations, not merely on the operations themselves, and on the related idea of doing/undoing, (3) Focus on both representing and solving a problem rather than on merely solving it, (4) Focus on both numbers and letters, rather than on numbers alone, (5) Refocus on the meaning of the equal sign. Helping students make a smooth transition from arithmetic to algebraic thinking is a common goal in the three Asian curricula. There are at least three ideas in the Asian curricula that help students make the adjustments needed to develop algebraic ways of thinking.

The first is that all three Asian curricula relate reverse operations to equation solving. For example, in Chinese elementary schools, addition and subtraction are introduced simultaneously at the first grade, and subtraction is introduced as the reverse operation of addition (Cai 2004). Students are guided to think about the following question: "If 1 + ( ) = 3, what is the number in ( )?" In order to find the number in ( ), the subtraction is introduced: 3 – 1 = 2. Throughout the first grade, students are consistently asked to solve similar problems. For example, they are asked to find the number in ( ) so that 7 + ( ) = 13. In the second grade, multiplication and division with whole numbers are introduced in Chinese elementary school mathematics. Division is first introduced using equal sharing. Division is also introduced as the reverse operation of multiplication: "What multiplied by 2 = 8?" That is, "If ( ) × 2 = 8, what is the number in ( )?" The idea of equation and equation solving permeates the introduction of both subtraction and division. Similar approaches are taken in both the Singaporean and South Korean curricula (Lew 2004, Ng 2004). There is no doubt that this reverse operation approach to subtraction and division can help students make two of the adjustments suggested by Kieran: (1) Focus on relationships and not merely on the calculation of a numerical answer, and (2) Focus on inverses of operations, not merely on the operations themselves, and on the related idea of doing/undoing.

The second idea is the pictorial equation solving in Singaporean curriculum (Ng 2004). It is common that pictures are initially used to model problem situations and then pictures are replaced by the more abstract rectangles.

Raju and Samy shared \$410 between them. Raju received \$100 more than Samy. How much money did Samy receive?
2 units = \$410 - \$100
= \$310
1 unit = \$
Samy received \$ _________

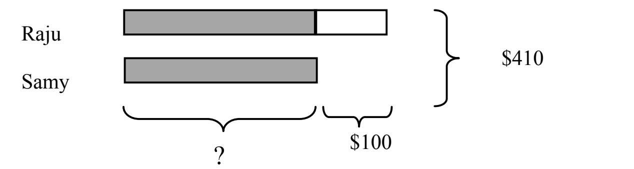

Figure 1: Pictorial equation solving.

Fig. 1 is an example from Grade 5. Samy's rectangle or unit is the generator of all the relationships presented in the problem. Raju's rectangle is dependent upon Samy's, with Raju's share represented by a unit identical to

Samy's plus another rectangle representing the relational portion of \$100 more. Using the model drawing, a pictorial equation representing the problem is formed and if the letter x replaces Samy's unit, then the algebraic equation $x + x + \$100 = \$410$ is produced. The use of the rectangle as a unit representing the unknown provides a pictorial link to the more abstract idea of letters representing unknowns. The entire structure of the model can be described as a pictorial equation. Pictorial equation solving clearly can help students to "focus on both representing and solving a problem rather than on merely solving it", as suggested by Kieran.

The third idea is that solving problems is done using both arithmetic and algebraic approaches. This idea can clearly be can be seen in the Chinese curriculum (Cai 2004). Solving a problem in multiple ways is a common practice in Chinese classrooms. For example, teachers can talk about four ways to solve the following problem: *Liming Elementary school has funds to buy 12 basketballs at 24 Yuan each. Before buying the basketballs, they decided to spend 144 Yuan of the fund for some soccer balls. How many basketballs can they buy*?

Solution 1: $(24 \times 12 - 144) \div 24 = 144 \div 24 = 6$ basketballs.

Solution 2: $12 - 144 \div 24 = 6$ basketballs.

Solution 3: Assume that the school can buy x basketballs. $24x + 144 = 24 \times 12$. Therefore, $x = 6$ basketballs.

Solution 4: Assume that the school can buy x basketballs. $24x = 24 \times 12 - 144$. Therefore, $x = 6$ basketballs.

According to the Chinese curriculum, solving a problem using both an arithmetic approach and an algebraic approach helps students build arithmetic and algebraic ways of thinking about the problem. At the elementary school level, Chinese students solve problems like the examples given in this section, all of which can be solved arithmetically. As might be expected, it is common at the beginning of the transition period for students to not see why they need to learn an equation-solving approach to solve problems. However, after a period of time using both approaches, students come to see the advantages of using equations to solve these types of problems.

There are three objectives in teaching students to solve problems both arithmetically and algebraically: (1) to help students attain an in-depth understanding of quantitative relationships by representing them both arithmetically and algebraically; (2) to guide students to discover the similarities and differences between arithmetic and algebraic approaches, so they can understand the power of a more general, algebraic approach; and (3) to develop students' thinking skills as well as flexibility in using appropriate approaches to solve problems. Post, Behr, and Lesh (1988) indicated that "first-describing-and-then-calculating" is one of the key features that make algebra different from arithmetic. Comparisons between the arithmetic and algebraic approaches can highlight this unique feature.

The Russian curriculum employs all five of Kieran's imperatives from the very beginning of grade one. It is not necessary to make an adjustment from arithmetic to algebraic thinking, since algebraic thinking is developed from the earliest days of formal schooling, and arithmetic is developed as a concrete application of algebraic generalizations. Kieran's imperatives are only "adjustments" when school mathematics begins with the study of calculations and builds toward the generality of structure and relationships.

There are many reversals built into the Russian curriculum, such as setting up an equation to solve a given word problem, or writing a word problem for a given equation. Formulating an equation for a given schematic or writing a word problem to fit the schematic are also common requirements. Not only are addition and subtraction taught simultaneously, but both are modeled by a part-whole schematic "/\". The whole appears at the top of the "/\" shape, and the parts appear at the other ends of the line segments that comprise it. When children model a problem situation using this schematic, if a part is missing they know that to find it, the other part(s) must be subtracted from the whole. If they have all the parts, these must be added to find the whole. Thus, children must analyze every problem situation and detect and identify a part-whole structure wherever it exists. The schematic model serves an intermediate representation for the children, enabling them to set up and solve the appropriate equation. In this sense, the Asian and Russian curricula are similar; all employ a pictorial or schematic model of the relationship between quantities, which the children use to help them formulate an equation.

4.4 Are authentic applications necessary for students in early grades?

Some researchers and educators believe that the learning of algebraic ideas should always be anchored in real-world situations that the students are familiar with. Others believe algebra does not have to be learned using real-world situations because the essence of algebra is not applied (Kieran 1992, Usiskin 1995). Rather, at its core, algebraic knowledge is an understanding of mathematical structures and relationships. So the work of algebra should be to abstract properties of operations and structures, and the goal should be to learn the abstract structures themselves, rather than to learn how the structures can be used to describe the real world. These two rather different views are clearly reflected in the Russian and the U.S. *Investigations* curricula.

The Russian curriculum is a series of very deliberately sequenced problems to help students develop a complete theoretical understanding of mathematical concepts. More importantly, students' consistent engagement with these problems develops their ability to analyze problem situations at a theoretical level, rather than an empirical or concrete level. Thus, Russian students learn to form *theoretical* rather than *empirical* generalizations. Students engage with problems from many different authentic contexts. Problems that require comparing and measuring quantities and representing their relative sizes, for example, are illustrative of authentic problems that arose in and were important to the cultural historical development of mathematics. The Davydov curriculum is unique, however, in developing in students the ability to transcend the empirical features of problem contexts, and requiring them to analyze such problems at the most general mathematical (i.e., theoretical) level, to model these understandings schematically, and express them symbolically. They must also be able to *ascend from the abstract to the concrete* (i.e., to see the mathematical essence they previously discerned embodied in new and varied contexts). Rather than proceeding inductively from the concrete to the abstract as *Investigations* does, the Davydov curriculum develops students' ability to see the same abstract relationships across different concrete contexts. *Investigations* also engages U.S. students in mathematical problems embedded in authentic contexts. These applied problem solving activities require U.S. students to explore contextualized problems in depth, construct strategies and approaches based on their understanding of mathematical relationships, utilize a variety of tools (e.g., manipulatives, computers, calculators), and communicate their mathematical reasoning through drawing, writing, and talking.

The South Korean seventh-grade curriculum that we analyzed includes much more modeling and application activities than earlier curricula in South Korea (Lew 2004). However, Lew pointed out that the modeling and application problems of the South Korean curriculum are embedded in very artificial (unrealistic) contexts. Because of the artificial nature of the contexts, South Korean students generally believe that mathematics is neither interesting nor fun. Inclusion of authentic situations in the South Korean curriculum would perhaps make mathematics more interesting and meaningful to students.

While applications are important in both the Chinese and Singaporean curricula, the contexts of application problems are not as authentic and natural as those used in *Investigations*. In the case of the Chinese curriculum, for example, the main focus is on equations and the process of equation solving itself, rather than on the use of applications to provide insight into the equation solving process. As a result, the development of equation and equation solving ideas in the Chinese elementary mathematics curriculum is done in three interrelated stages: (1) the intuitive stage, (2) the introduction stage, and (3) the application stage. After Chinese students have been formally introduced to equations and equation solving, there are opportunities to use an equation-solving approach to solve application problems as they learn statistics, percents, fractions, and ratios and proportions (Cai 2004). This arrangement is desirable in order to deepen the students' understanding of quantitative relationships and to help students appreciate the equation-solving approach. However, it is very different than *Investigations'* approach to equations, which begins with applications, and develops intuitive equation-solving skills based upon U.S. students' insights into the applications themselves.

5. Conclusion

This study analyzed and compared how algebraic concepts and representations are introduced and developed throughout the Chinese, South Korean, and Singaporean curricula, and the selected Russian and U.S. elementary curricula. It provides an international perspective about the kinds of algebraic experiences elementary school students should have. In particular, the study identified unique features of the five curricula analyzed. In all five curricula, the main goal for learning algebraic concepts is to deepen students' understanding of quantitative relationships, but the emphases and approaches to helping students deepen their understanding of quantitative relationships are very different across the curricula.

The Chinese elementary school curriculum emphasizes the examination of quantitative relationships from various perspectives. Students are consistently encouraged and provided with opportunities to represent quantitative relationships both arithmetically and algebraically. Furthermore, students are asked to make comparisons between arithmetic and algebraic ways of representing a quantitative relationship. The Russian curriculum of Davydov engages children in a focused analysis and description of the quantitative world as the starting point for developing algebraic thought. The curriculum emphasizes the development of algebraic understandings through direct work with quantities, and the representational modeling of mathematical actions and relationships.

In South Korea, many concrete operational activities are used to reduce the cognitive gap between algebra and arithmetic. The symbol " " is introduced in the first grade to represent an unknown value, as is an intuitive process to solve for the unknown value " ". In Singapore, students are provided ample opportunities to make generalizations through number pattern activities. Equations are not introduced symbolically; instead, they are introduced through pictures. Such "pictorial equations" are used extensively to represent quantitative relationships. The "pictorial equations" not only provide a tool for students to solve mathematical problems, but they also provide a means for developing students' algebraic ideas. "Analyze change in its various contexts" is the central goal of the U.S. *Investigations* algebra strand. This curriculum accomplishes the teaching of "change" from an informal perspective. The curriculum builds on basic intuitions that children express from a very early age, enriching and refining them. By design, therefore, *Investigations* does not progress to symbolic or formal work.

In this paper, we addressed four questions related to the development of algebraic thinking in earlier grades: (1) To what extent do curricula expect students in early grades to think algebraically? (2) What level of formalism should we expect of students in the early grades? (3) How can we help students make a smooth transition from arithmetic to algebraic thinking? (4) Are authentic applications necessary for students in early grades? Results from the analyses of the five curricula clearly indicate that students who use these curricula are expected to think algebraically in all five countries. The earlier emphasis of algebraic ideas may indeed help students develop arithmetic and algebraic ways of thinking about problems.

Regarding the use of authentic applications and the level of formal algebraic symbolism in elementary school, the results from the analyses of the five curricula are not uniform. The Russian curriculum uses letter symbols to denote the results of investigations of quantities, such as length, area, volume, and weight, arising from authentic contextualized problems. *Investigations* also uses exclusively authentic and contextualized problems, but does not employ algebraic symbolism. The three Asian curricula also include many application problems, but the contexts are not necessarily authentic. Only the Chinese and South Korean elementary curricula use formal algebraic symbolism.

The Russian curriculum begins with a study of quantity in authentic contexts, the comparisons and measurements of which are expressed using letter symbols. These denote theoretical generalizations that are then representative of entire classes of applied problems. In fact, the study of algebra precedes the study of arithmetic in the Russian curriculum. On the other hand, the U.S. *Investigations* involves very little formal algebraic symbolism.

It is important to indicate that any curriculum has a complex relationship to what actually occurs in classrooms. Our study analyzed the intended approaches to algebraic thinking of five curricula in five nations.Although research shows that a curriculum is an important predictor of students' learning (Schmidt et al. 2002), it does not always dictate the content of mathematics instruction (Freeman & Porter 1989). A natural extension of this study is to explore how teachers in the five nations actually develop their students' algebraic thinking in elementary school classrooms using these curricula.

6. References

AAAS (2000): Algebra for all: Not with today's textbooks. See www.project2061.org/newsinfo/press/r1000426.htm

Cai, J. (2004a): Introduction to the special issue on developing algebraic thinking in the earlier grades from an international perspective. – In: The Mathematics Educator (Singapore), 8(No. 1), p. 1-5

Cai, J. (2004b): Developing algebraic thinking in the earlier grades: A case study of the Chinese mathematics curriculum. - In: The Mathematics Educator (Singapore), 8(No. 1), p. 107-130

Carpenter, T. P.; Franke, M. L.; Levi, L. (2003): Thinking mathematically: Integrating arithmetic and algebra in elementary school. - Portsmouth, NH: Heinemann

Curriculum Planning & Development Division (2000): Mathematics syllabus primary. - Singapore: Ministry of Education

Davydov, V. V.; Gorbov, S. F.; Mikulina, G. G.; Saveleva, O. V. (1999): Mathematics: class 1. J. Schmittau (Ed.) - Binghamton, NY: State University of New York

Division of Elementary Mathematics (1999): Mathematics: Elementary school textbook (number 1). - Beijing, China: People's Education Press

Driscoll, M. (1999): Fostering algebraic thinking: A guide for teachers grades 6-10. - Portsmouth, NH: Heinemann

Freeman, D. J.; Porter, A. C. (1989): Do textbooks dictate the content of mathematics instruction in elementary school? - In: American Educational Research Journal, Vol. 26, p. 403-421

Kaput, J. (1999): Teaching and learning a new algebra. In: E. Fennema; T. Romberg (Eds.), Mathematics classrooms that promote understanding. Mahwah, NJ: Erlbaum, p. 133–155

Kieran, C. (2004): Algebraic thinking in the early grades: What is it? – In: The Mathematics Educator (Singapore) 8(No. 1), p. 139-151

Lew, H. C. (2004): Developing algebraic thinking in the earlier grades: A case study of the South Korean elementary school mathematics curriculum. - In: The Mathematics Educator (Singapore) 8(No. 1), p. 88-106

Mathematical Sciences Education Board (1998): The nature and role of algebra in the K-14 curriculum: Proceedings of a national symposium. - Washington D.C.: National Research Council

Moyer, J. C.; Huinker, D.; Cai, J. (2004): Developing algebraic thinking in the earlier grades: A case study of the U.S. Investigation series. – In: The Mathematics Educator (Singapore) 8(No. 1), p. 6-38

National Council of Teachers of Mathematics. (1989): Curriculum and evaluation standards for school mathematics. - Reston, VA: Author

National Council of Teachers of Mathematics (2000): Principles and standards for school mathematics.- Reston, VA: Author

Ng, S. F. (2004): Developing algebraic thinking: A case study of the Singaporean primary school curriculum. – In: The Mathematics Educator (Singapore) 8(No. 1), p. 39-59

Post, T. R.; Behr, M. J.; Lesh, R. (1988): Proportionality and the development of prealgebra understandings. – In: A. Coxford; A. Shulte (Eds.), The ideas of algebra, K-12. Reston, VA: NCTM (1988 Yearbook), p. 78-90.

Russell, S. J.; Economopoulos, K.; Murray, M.; Mokros, J.; Goodrow, A. (1998): Implementing the *Investigations in Number, Data, and Space* curriculum: Grades K, 1, and 2. - Menlo Park: CA: Dale Seymour Publications

Schifter, D. (1999): Reasoning about operations: Early algebraic thinking in grades K-6. – In: L. V. Stiff; F. R. Curcio (Eds.), Developing mathematical reasoning in grades K-12. Reston, VA: NCTM (1999 Yearbook), p. 62-81

Schmidt, W. H.; McKnight, C. C.; & Raizen, S. A.: (1996): Characterizing pedagogical flow: An investigation of mathematics and science teaching in six countries. - The Netherlands: Kluwer Academic Publishers

Schmidt, W. H.; McKnight, C. C.; Houang, R. T.; Wang, H.; Wiley, D. E.; Cogan, L. S.; Wolfe, R. G. (2002): Why schools matter: A cross-national comparison of curriculum and learning. - San Francisco, CA: Jossey-Bass

Schmittau, J.; Morris, A. (2004): The development of algebra in the elementary mathematics curriculum of V. V. Davydov. - In: The Mathematics Educator (Singapore) 8(No. 1), p. 60-87

Silver, E. A.; Kenney, P. A. (2001): Results from the sixth mathematics assessment of the National Assessment of Educational Progress. - Reston, VA: NCTM

Stacey, K.; Chick H.; Kendal M. (2004): The Future of the Teaching and Learning of Algebra: The 12th ICMI Study. - The Netherlands: Kluwer Academic Publishers

Usiskin, Z. (1995): Why is algebra important to learn? – In: B. Moses (Ed.), Algebraic thinking in grades K-12: Readings from NCTM's school-based journals and other publications. Reston, VA: NCTM, p. 16-21.

van Dooren, W.; Verschaffel, L.; Onghena, P. (2002): The impact of preservice teachers' content knowledge on their evaluation of students' strategies for solving arithmetic and algebra word problems. - In: Journal for Research in Mathematics Education, Vol. 33, p. 319-351

Authors

Cai, Jinfa, Prof., Dr., 523 Ewing Hall, Department of Mathematical Sciences, University of Delaware, Newark, De 19716, USA.
E-mail: jcai@math.udel.edu

Lew, Hee Chan, Prof., Dr., Dept. of Math Education, Korea National University of Education, Cheongwon, Chungbuk, Korea, 363-791
Email: hclew@knue.ac.kr

Morris, Anne, Associate Prof., Dr., School of Education, University of Delaware, Newark, De 19716, USA.
E-mail: amorris@udel.edu.

Moyer, John C., Prof., Dr., Marquette University, Dept. of Mathematics, Statistics and Computer Science, P.O. Box 1881, Milwaukee, WI 53201-1881
Email: johnm@mscs.mu.edu

Ng, Swee Fong, Associate Prof., Dr., Mathematics and Mathematics Education Academic Group, National Institute of Education, 1 Nanyang Walk, Singapore 637 616
Email: sfng@nie.edu.sg

Schmittau, Jean, Associate Prof., Dr., SUNY At Binghamton, School of Education and Human Development, P. O. Box 6000, Binghamton, NY 13902-6000
Email: jschmitt@binghamton.edu