Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 51 (2010), No. 2, 477-491.

Generalized Gergonne and Nagel Points

Boris Odehnal

Institut für Diskrete Mathematik und Geometrie, TU Wien,
Wiedner Hauptstraße 8–10, A-1040 Wien, Austria
e-mail: boris@geometrie.tuwien.ac.at

Abstract. In this paper we show that the Gergonne point G of a triangle Δ in the Euclidean plane can in fact be seen from a more general point of view, i.e., from the viewpoint of projective geometry. So it turns out that there are up to four Gergonne points associated with Δ. The Gergonne and Nagel point are isotomic conjugates of each other, and thus we find up to four Nagel points associated with a generic triangle. We reformulate the problems in a more general setting and illustrate the different appearances of Gergonne points in different affine geometries. Darboux's cubic can also be found in the more general setting, and finally a projective version of Feuerbach's circle appears.

MSC 2000: 51M04, 51M05, 51B20
Keywords: Brianchon's theorem, Darboux's cubic, excenters, Feuerbach's nine point circle, Gergonne point, incenter, isotomic conjugate, Nagel point, triangle

1. Introduction

Assume $\Delta = \{A, B, C\}$ is a triangle with vertices A, B, C and the incircle i. Gergonne's point is the locus of concurrency of the three cevians connecting the contact points of i and Δ with the opposite vertices. The Gergonne point, which can also be labeled with X_7 according to [10], [11] has frequently attracted mathematician's interest. Some generalizations have been given: In [8] the incircle i is replaced with a concentric circle i'. The scaling that maps i to i' is also applied to the contact points of i with the sides of Δ. The cevians through the scaled contact points are concurrent. The generalization given in [3] by Boyd and Raychowdhury is a subcase of that given in [8]. Many interesting references on elementary geometric properties of the triangle and generalizations can also be found in [13]. For further information on triangle geometry we refer to [1, chapters X, XII] and [9, §6] as well as to Kimberling's monograph [11].

Gergonne points of tetrahedra are investigated in [6], [7]. There it turned out that a tetrahedron has to fulfill some conditions in order to have a Gergonne point and so the classical definition of this particular point cannot be carried over to the tetrahedron and therefore its definition has to be modified. The Gergonne and Nagel points for n-simplices are also studied in [12].

In this paper we shall study the planar case in a more general setting, i.e., we formulate everything in terms of projective geometry. This allows to state theorems in some Cayley-Klein geometries, e.g., Euclidean or Minkowskian geometry. There is, to the best of the author's knowledge, only one paper dealing with Gergonne's point and Nagel's point (together with some other phenomena of triangle geometry) in a Minkowskian plane. In [2] the authors translated things in an elementary way without leaving affine geometry.

This paper discloses the relation of Gergonne's point and Brianchon's theorem which directly leads to three more Gergonne points, see Section 2. Thereby further Nagel points appear in a natural way as the isotomic conjugates of Gergonne points. Furthermore, it is possible to reformulate a lot of results in a projectively invariant way which is the content of Section 3. Then we pay our attention to Darboux's cubic in Section 4. Finally, we give the trilinear coordinates of the three additional Gergonne and Nagel points in Section 5.

2. Gergonne points and Brianchon's theorem

Let there be given a triangle $\Delta := \{A, B, C\}$ in the Euclidean plane $\mathbb{R}^2$. The incircle is labeled with i and its center shall be denoted by I. This circle is one of those touching the sides of Δ considered as straight lines. There are three further circles sharing this property with the incircle. These are called excircles and will be denoted by i_1, i_2, and i_3 and their respective centers by I_1, I_2, and I_3.

While the incircle touches Δ's sides from the inside and is thus entirely contained in Δ, the excircles touch from the outside. In order to avoid confusions we denote the excircle touching the side BC in between B and C, by i_1. So i_1 lies opposite to A, i_2 is opposite to B, and i_3 is opposite to C, cf. Figure 1.

The incircle is tangent to Δ's sides at the points I_{AB}, I_{BC}, and I_{CA}. Now the following statement is well known:

Theorem 2.1. *The three lines* $[A, I_{BC}]$, $[B, I_{CA}]$, *and* $[C, I_{AB}]$ *are concurrent.*

The common point G of $[A, I_{BC}]$, $[B, I_{CA}]$, and $[C, I_{AB}]$ is usually referred to as Gergonne's point. Figure 2 shows a triangle with its Gergonne point G.

One can easily give a proof of Theorem 2.1, using Brianchon's theorem.

Let k be a conic section in a projective plane (which is henceforth assumed to be Pappian) and let further $t_1, \ldots, t_6$ be tangents of k. If their points of

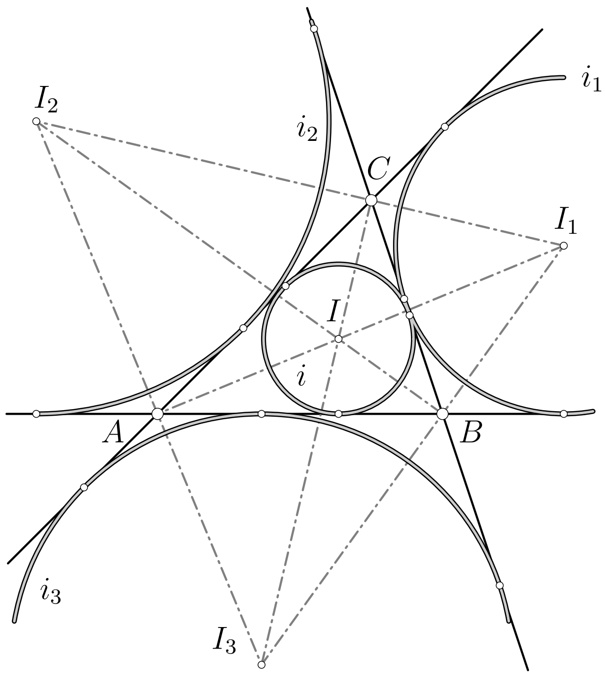

Figure 1. A triangle $\Delta = \{A, B, C\}$ with its incircle i and excircles i_1, i_2, i_3

intersection are denoted by $T_{ij} := [t_i, t_j]$ then we can build the three lines $[T_{12}, T_{45}]$, $[T_{23}, T_{56}]$, and $[T_{34}, T_{61}]$. Brianchon's theorem says:

Theorem 2.2. *The three lines* $[T_{12}, T_{45}]$, $[T_{23}, T_{56}]$, *and* $[T_{34}, T_{61}]$ *are concurrent.*

Theorem 2.2 is valid, even if two consecutive lines, for instance the lines t_1 and t_2, coincide. (It is always possible to number five lines with integers $1, \ldots, 6$ such that one line is labeled twice, and with a pair of consecutive numbers.) In case of $t_1 = t_2$ the point T_{12} is replaced by the contact point $T_1 = T_2$ of $t_1 = t_2$ with the conic section k. Even if two pairs of consecutive tangents coincide, e.g., $t_1 = t_2$ and $t_3 = t_4$, Theorem 2.2 holds true. Surprisingly, Brianchon's theorem is also

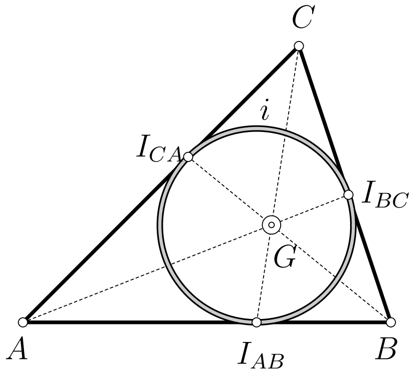

Figure 2. Gergonne's point as the intersection of three lines

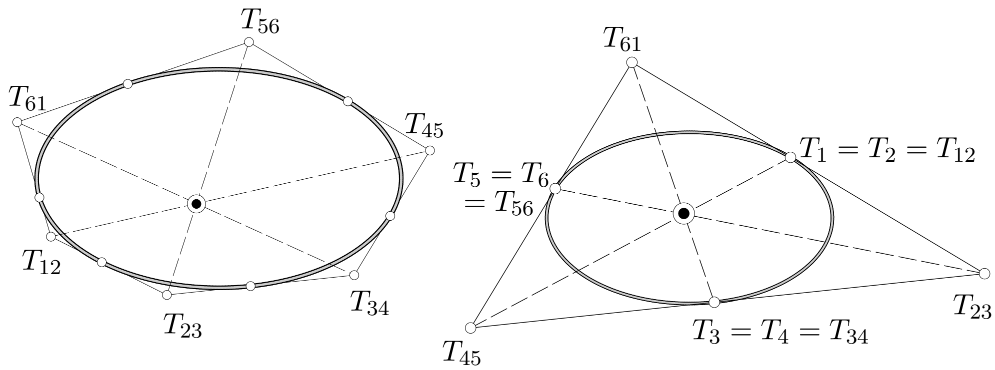

Figure 3. Brianchon figures of different kinds: six tangents of a conic section (left), three line elements of a conic section (right)

valid in the case of $t_1 = t_2$, $t_3 = t_4$, and $t_5 = t_6$. This is illustrated in Figure 3. Consequently, one can check, if three line elements are line elements of one conic section.

Now it is obvious that the lines mentioned in Theorem 2.1 are concurrent and that the point G exists.

Proof. (Proof of Theorem 2.1) Apply Brianchon's theorem 2.2 to the three line elements $([B,C], I_{BC})$, $([C,A], I_{CA})$, and $([A,B], I_{AB})$. □

The contact points of the excircles with the sides of Δ shall be labeled in the following way: The i-th excircle i_i touches the lines $[A,B]$, $[B,C]$, and $[C,A]$ at $I_{i,AB}$, $I_{i,BC}$, and $I_{i,CA}$, respectively. Now it is easily seen that each excircle determines its own Gergonne point with respect to Δ:

Theorem 2.3. *Any triangle Δ has four Gergonne points.*

The three lines $[A, I_{BC}]$, $[B, I_{CA}]$, and $[C, I_{AB}]$ are concurrent.

The three lines $[A, I_{i,BC}]$, $[B, I_{i,CA}]$, and $[C, I_{i,AB}]$ are concurrent. This holds true for all $i \in \{1,2,3\}$.

Proof. We immediately recognize three Brianchon figures consisting of the tangents $[A,B]$, $[B,C]$, $[C,A]$ and the respective contact points $I_{i,AB}$, $I_{i,BC}$, $I_{i,CA}$ of the excircles. The fourth Brianchon figure is formed by the lines $[A,B]$, $[B,C]$, $[C,A]$ and the contact points I_{AB}, I_{BC}, and I_{CA} of the incircle. □

The Gergonne point associated with the i-th excircle shall be denoted by G_i, according to Figure 4.

Note that Nagel's point N – or X_8 according to [10], [11] – is not a generalized Gergonne point since it is found as the intersection of lines connecting Δ's vertices with contact points of different conic section, namely the three different excircles. Nevertheless, Nagel's point is closely related to Gergonne's point since X_7 and X_8 are isotomic conjugates of each other. (The isotomic mapping is defined for all points of the triangle Δ as follows: The projections of any point $P \notin \{[A,B] \cup$

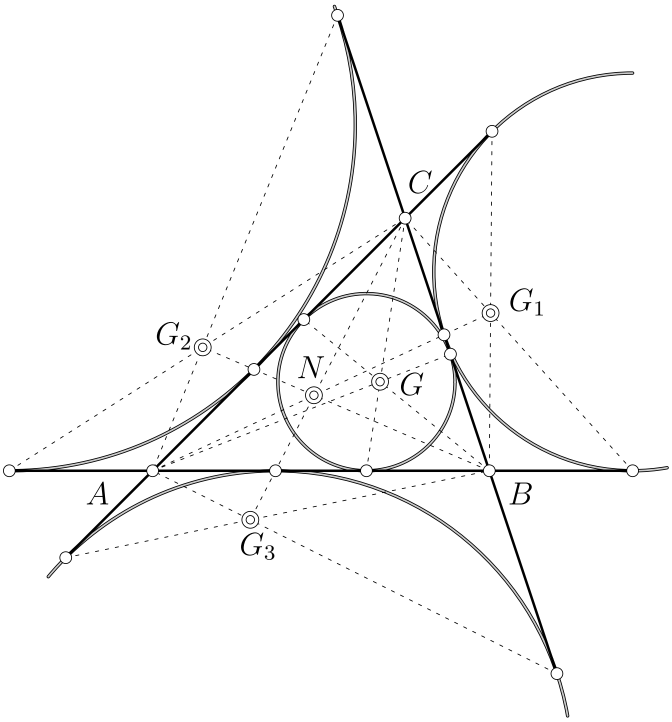

Figure 4. The Gergonne points G, G_1, G_2, G_3 and the Nagel point N of Δ

$[B, C] \cup [C, A]\}$ from Δ's vertices onto Δ's sides are reflected about the midpoints of the respective sides. The lines connecting these reflections with the opposite vertices of Δ are concurrent in the isotomic conjugate of P, cf. [11].) Moreover we observe:

Lemma 2.1. *The three lines $[G_1, A]$, $[G_2, B]$, and $[G_3, C]$ are concurrent at the Nagel's point.*

Proof. It is well known that Nagel's point is the common point of the cevians $[I_{1,BC}, A]$, $[I_{2,CA}, B]$, and $[I_{3,AB}, C]$. Since $G_1 \in [I_{1,BC}, A]$, $G_2 \in [I_{2,CA}, B]$, and $G_3 \in [I_{3,AB}, C]$ we are done. □

Since Gergonne's point and Nagel's point are isotomic conjugates of each other we find further Nagel points:

Theorem 2.4. *Any triangle $\Delta = \{A, B, C\}$ has four Nagel points.*
The lines $[A, G]$, $[B, G_3]$, $[C, G_2]$ share the point N_1.
The lines $[B, G]$, $[C, G_1]$, $[A, G_3]$ share the point N_2.
The lines $[C, G]$, $[A, G_2]$, $[B, G_1]$ share the point N_3.
The points N_i are the isotomic conjugates of G_i for all $i \in \{1, 2, 3\}$.

Proof. We construct the isotomic conjugate of G_1 in order to show the existence of the point N_1: The cevians through G_1 meet AB, BC, and CA at $I_{1,AB}$, $I_{1,BC}$, and $I_{1,CA}$. The latter points are reflected at the midpoints of AB, BC, and

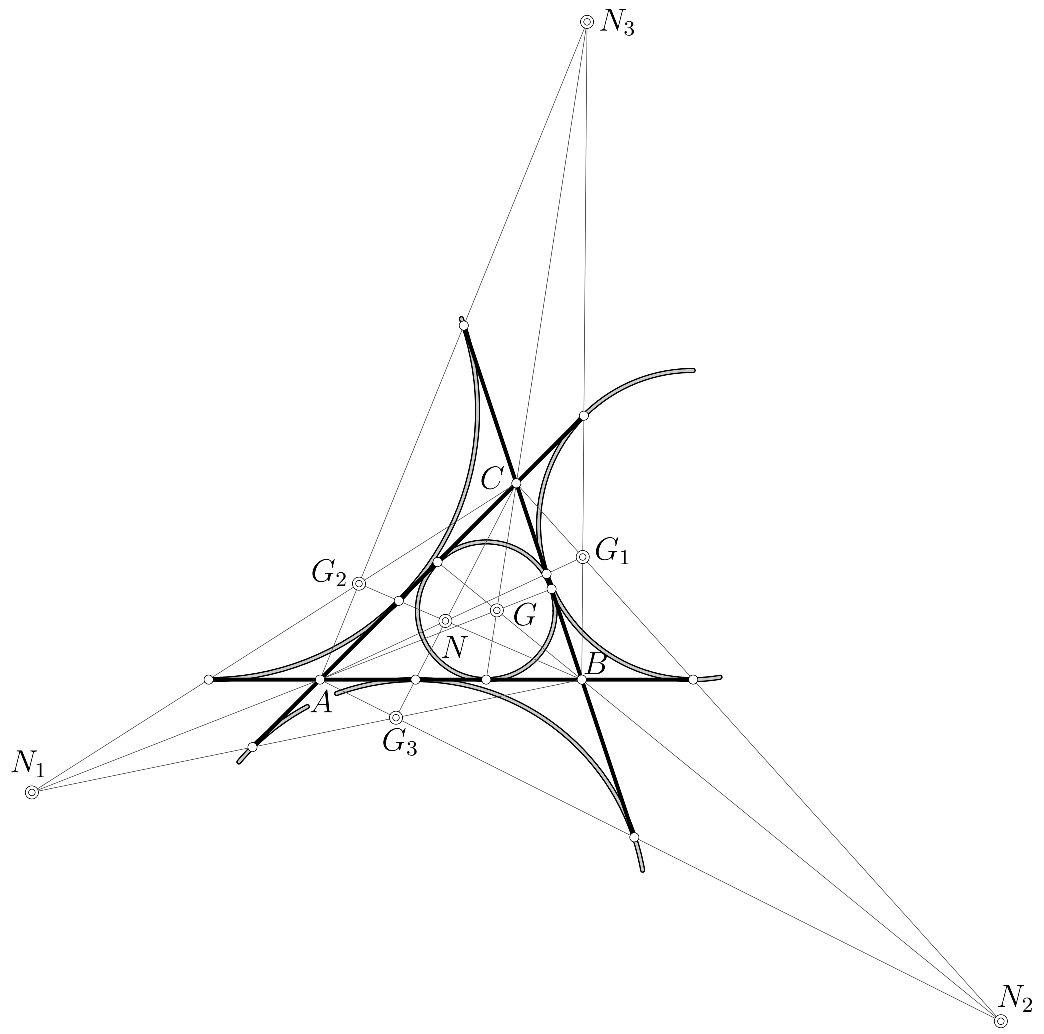

Figure 5. Gergonne points G, G_1, G_2, G_3 and Nagel points N, N_1, N_2, N_3

CA and map thus to $I_{2,AB}$, I_{BC}, and $I_{3,CA}$. Since the isotomic mapping is well defined off the triangle, the cevians through $I_{2,AB}$, I_{BC}, and $I_{3,CA}$ are concurrent in the isotomic conjugate N_1 of G_1. Furthermore we have $G_2 \in [C, I_{2,AB}]$, $G_3 \in [B, I_{3,CA}]$, and $G \in [A, I_{BC}]$.
Analogously we can show that the remaining triplets of lines belong to pencils of lines and that the points N_2 and N_3 are the isotomic conjugates of G_2 and G_3, respectively. □

Figure 5 shows the four Nagel points of a given triangle together with the four Gergonne points.

So far we discovered that there is not a single Gergonne point. To each Gergonne point G_i there exists exactly one isotomic conjugate point N_i. We call the points N_i also Nagel points since they can be found in a similar way compared to the point X_8.

The construction of the three Gergonne points G_1, G_2, G_3, uses a theorem from projective geometry. This motivates the question: Is it possible to generalize the notion of Gergonne points in a way that it becomes a matter of projective geometry? A positive answer to this question would imply that Gergonne points also exist in other geometries, for example in pseudo-Euclidean (Minkowskian) geometry.

3. Generalized Gergonne points

In order to find Gergonne points in a more general setting we perform the projective closure of Δ's plane. We add the ideal line ω as the set of all ideal points. When ever necessary we extend the concept of real geometry by adding complex elements. Note that a pair of conjugate complex points is connected by a real line. So in the complex extension of the projectively closed Euclidean plane each circle has a pair of conjugate complex ideal points $(I, \overline{I})$ and any conic section through these points is a circle.

The points I, $\overline{I}$ are frequently called *absolute points* of the Euclidean plane, see [5]. They are the fixed points of the absolute polarity ι on ω. Any pair of points $(G, H) \subset \omega$ with $\iota(G) = H$ are the ideal points of orthogonal lines and the quadruple $(I, \overline{I}, G, H)$ is harmonic.

When considering Euclidean geometry as a Cayley-Klein geometry, the line ω together with the quadric $(I, \overline{I}) \subset \omega$ is the absolute figure of this geometry, see [5].

Now we replace the above absolute figure $(\omega, (I, \overline{I}))$ by another one $(\omega, (U, \overline{U}))$ and obtain different geometries, with their own notion of circles and their own notion of Gergonne points. Independent of the choice of the absolute points, we define a circle as a conic section passing through U and $\overline{U}$.

Since any (real) conic section (in the projectively closed and complex extended plane) that contains U and $\overline{U}$ is a circle in the respective Cayley-Klein geometry (and since any circle goes through U and $\overline{U}$), it makes no difference, if we assign the Gergonne points to Δ and its in- and excircles or to Δ and the pair $(U, \overline{U})$ of points. Therefore one possible generalization of the Gergonne points reads:

Theorem 3.1. *Any triangle $\Delta = \{A, B, C\}$ and any pair of distinct points $(U, \overline{U})$ (neither U nor $\overline{U}$ is located on the sides of Δ) determine a quadruple of Gergonne points.*

Proof. In general there are four conic sections on three given tangents and two given points.

A conic section tangent to the lines $[A, B]$, $[B, C]$, and $[C, A]$ and passing through U and $\overline{U}$ touches the given lines at I_{AB}, I_{BC}, and I_{CA}. By completing the Brianchon figure, we find the associated Gergonne point. □

Remark. Note that degenerate cases, i.e., situations may occur where two or more Gergonne points coincide. This depends on the choice of $(U, \overline{U})$ and needs a separate treatment. In the following we exclude special cases. We do neither allow $[U, \overline{U}]$ to pass through a vertex of Δ nor U or $\overline{U}$ to be contained in any side of Δ. We shall not discuss special cases since this leads to far.

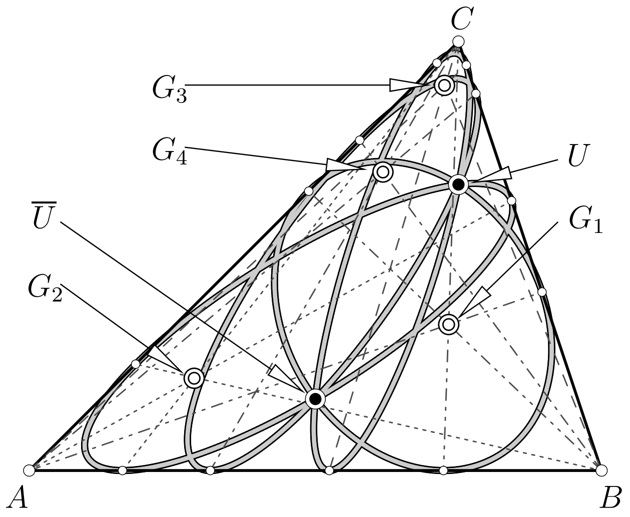

Figure 6. Four conic sections on three tangents and two points determine four Gergonne points

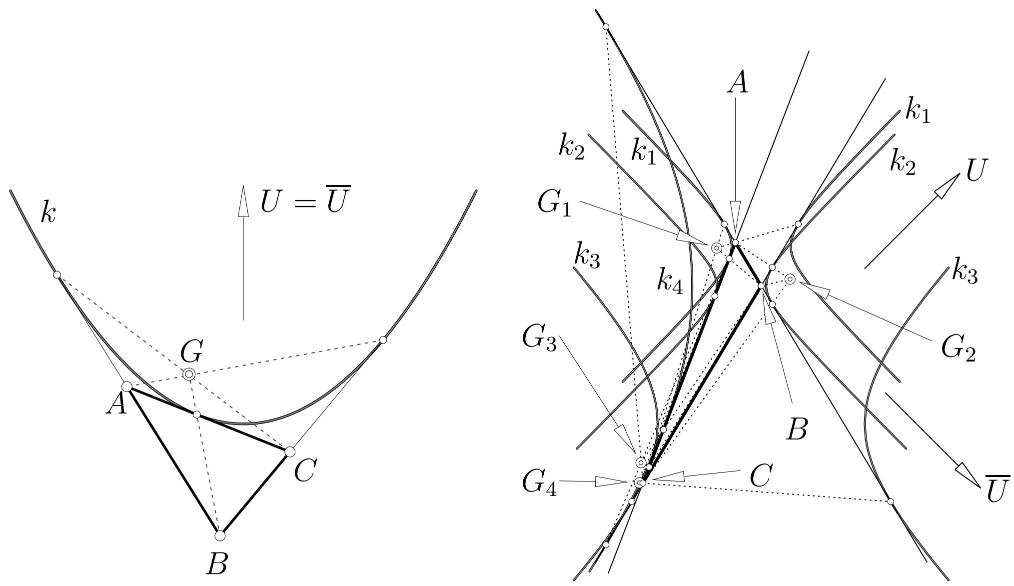

Figure 7. The unique Gergonne point in an isotropic plane (left), the four Gergonne points in a Minkowskian plane (right)

Since any three line elements of a conic section determine a Brianchon figure and a unique Brianchon point, we can say:

Theorem 3.2. *Any conic tangent to three lines determines a unique Gergonne point. The two parameter-family of conic sections tangent to three lines determines a two-parameter family of Gergonne points.*

The set of Gergonne points of the two-parameter family of conic section mentioned in Theorem 3.2 lies dense in the plane of the triangle, when we remove the three

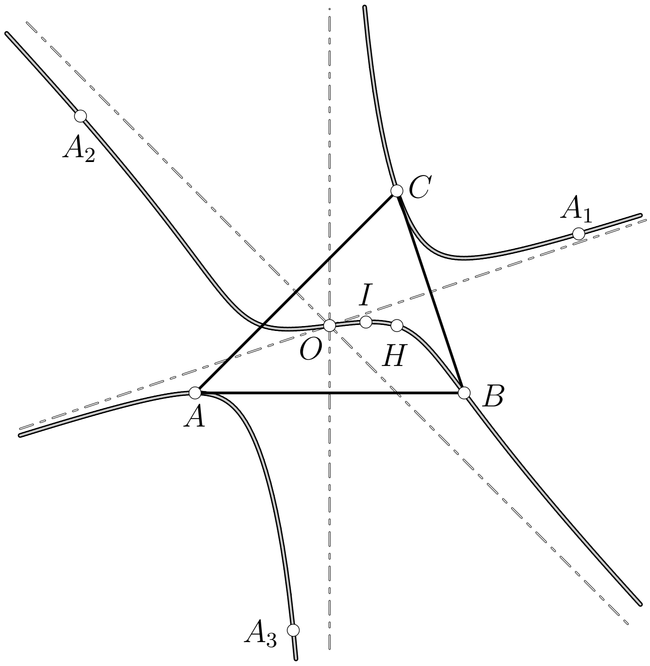

Figure 8. Darboux's cubic associated with Δ

given lines. This holds true as long as the lines are not concurrent and as long as the conic section does not degenerate.

The two apparently different cases of a pair of real points $(U, \overline{U})$ and a pair of conjugate complex cases allow a unifying generalization. The points U and $\overline{U}$ can be seen as the fixed points of a hyperbolic or an elliptic involutive projective mapping $\iota: \ \omega \to \omega$. We call a line g and an involutive projective mapping $\iota: \ g \to g$ admissible, if g does not pass through any vertex of Δ and no fixed point of ι is contained in any side of Δ. Thus the notion of Gergonne points allows a further generalization:

Theorem 3.3. *Any triangle $\Delta = \{A, B, C\}$ and any admissible involutive projective mapping $\iota: \ g \to g$ on a line g in admissible position determine four Gergonne points.*

Proof. The involutive mapping $\iota: \ g \to g$ together with the lines $[A, B]$, $[B, C]$, and $[C, A]$ defines four conics such that they touch the given lines and their respective polar systems induce ι on g. The four Gergonne points are the Brianchon points of the four resulting Brianchon figures. □

Remark. If we choose $\omega = [U, \overline{U}]$ as the ideal line and the points U and $\overline{U}$ is the absolute quadric in the sense of a Cayley-Klein geometry, we can find two different (affine) versions of Gergonne points:

In case of a pair of conjugate complex points we rediscover the well known Euclidean version. It is illustrated in Figure 4 and Figure 6.

If we choose a pair of real points $(U, \overline{U})$, we obtain the Gergonne points to a triangle in a pseudo-Euclidean or Minkowskian plane. An example is illustrated in Figure 7.

Remark. An isotropic plane has a double point $U = \overline{U} \in \omega$ at infinity for its absolute quadric. There appears to be only one isotropic circle tangent to three lines, since three tangents and one line element (ω, U) determine exactly one conic section, assumed that the four lines form a quadrilateral and the given point is located at exactly one of the lines.

From the view point of affine geometry the isotropic circle is a parabola. The isotropic normals of the parabola are parallel and concurrent in U. Since there is only one isotropic circle tangent to three lines, there is also only one Gergonne point. Figure 7 gives an idea how the isotropic situation looks like.

4. Darboux's cubic

Now we return to Euclidean plane. Assume a triangle $\Delta = \{A, B, C\}$ is given. We have seen that the incenter I and the excenters I_i have the following property: The cevians through the orthogonal projections of I onto Δ's sides are concurrent (Theorem 2.3). The orthogonal projections of I and I_i are the contact points of the incircle and the excircles with Δ's sides, respectively.

The incenter and excenters share this property with the orthocenter H and the circumcenter O.

Consider now a point X in Δ's plane. Denote the orthogonal projections of X to the sides $[A, B]$, $[B, C]$, and $[C, A]$ of Δ by X_3, X_1, and X_2, respectively. The set of all points X in Δ's plane with the property that the three lines $[A, X_1]$, $[B, X_2]$, and $[C, X_3]$ are concurrent is an elliptic cubic curve c provided that Δ is neither isosceles, nor equilateral, nor right angled. The curve c is called Darboux's cubic (cf. [4]). It is symmetric with respect to the circumcenter O. The bisectors of Δ's edges are c's asymptotes, which are concurrent in O. The latter one is an inflection point of c, see Figure 8.

So far it seems that Darboux's cubic is a term of Euclidean geometry. In the following we will show that this cubic can be found in arbitrary projective planes. For that purpose we return to the assumption we have made earlier. Let $(U, \overline{U})$ be a pair of different points such that $\omega := [U, \overline{U}]$ is a real line, i.e., U and $\overline{U}$ can be real points or a pair of conjugate complex points.

Now we have to clarify, what are the projective orthogonal projections of a point X to the sides of Δ.

The involutive mapping $\iota : \omega \to \omega$ assigns a unique point X^ι to any point X. We say X is the ideal point of a pencil of lines and so X^ι is the ideal point of all lines ι-orthogonal to the lines of the pencil. In this way the ι-normals can be assigned to all points of k, i.e., a conic section through U and $\overline{U}$ tangent to Δ's sides. Note that the points $k \cap \omega = (U, \overline{U})$ (whether they are real or not) are self-adjoint and thus the lines through U and $\overline{U}$ are ι-self-orthogonal. Obviously,

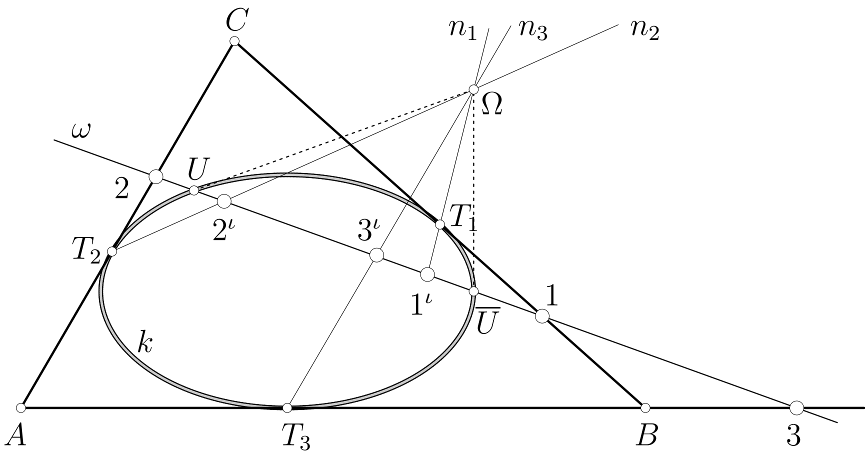

Figure 9. Projective normals of a conic k

the ι-normals of k are concurrent in ω's pole Ω with regard to k, i.e., Ω can be seen as a kind of center of k.

This allows the projective generalization of the construction of Darboux's cubic. We ask for all points X in Δ's plane with the property that for their ι-orthogonal-projections X_i to the sides of Δ the lines

$$[A, X_1], \quad [B, X_2], \quad [C, X_3]$$

are concurrent. The locus c of all points X with this property will from now on be called *generalized Darboux cubic*. An example is displayed in Figure 10.

We can summarize:

Theorem 4.1. *Given a triangle Δ and two points U and $\overline{U}$ in admissible position (cf. Theorem* 3.3*). U and $\overline{U}$ can be considered as the fixed points of an involutive projective mapping $\iota: \ \omega \to \omega$, with $\omega := [U, \overline{U}]$.*

There is a cubic curve c as locus of points X in Δ's plane with the property that for the cevians through the ι-orthogonal projections X_i of X to Δ's sides are concurrent.

Remark. Such cubics exist in pseudo-Euclidean and Euclidean planes but not in isotropic ones.

We note that another object of elementary Euclidean geometry appears in the more general setting. The midpoints of any side of Δ are the fourth harmonic points of the ideal lines of Δ's sides. Obviously we can define midpoints with help of any line ω. The generalization of pedals of Δ's altitudes is straight forward as well as the construction of midpoints on those parts of the altitudes which lie in between the vertices and the orthocenter of Δ. So we end up with nine points lying on one conic section f, which can be called the *Feuerbach conic*. This is illustrated in Figure 11.

Finally we observe that the generalized Gergonne points of Δ determine their own Nagel points:

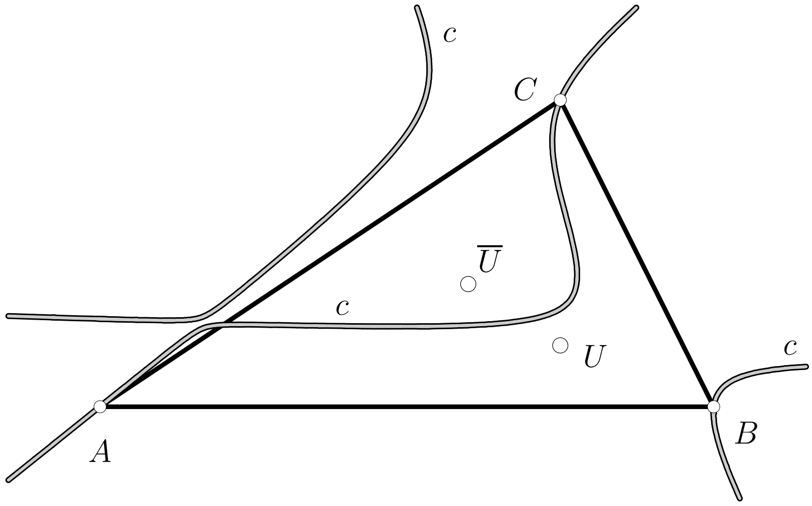

Figure 10. Projective version of Darboux's cubic d

Theorem 4.2. *The incidences given in Theorem* 2.4 *hold true in the generalized case.*

Proof. The isotomic transformation can be performed in the general setting. In order to find the isotomic conjugate of a point P (which is not contained in Δ's sides) we first project P from Δ's vertices to the opposite edges and obtain P_{AB}, P_{BC}, and P_{CA}. Then we reflect these points about the midpoints of AB, BC, CA, i.e., we apply the harmonic (involutive) projective mapping on each side of Δ that fixes the ideal point on the side (side's intersection with ω) and its harmonic conjugate. We obtain points P'_{AB}, P'_{BC}, and P'_{CA}. The cevians through the latter points are concurrent. □

In Figure 12 the generalized Nagel points as well as the generalized Gergonne points are displayed.

5. Trilinear coordinates

For sake of completeness we give the trilinear coordinates of the Gergonne points[1] G_1, G_2, and G_3 for triangles Δ in a Euclidean plane. The lengths of the sides of Δ shall be $|AB| = c$, $|BC| = a$, $|CA| = b$ and the coordinates are sorted in the same order. So we have

$$\begin{aligned} G_1 &= \left[\frac{ab}{a-b+c} : \frac{-bc}{a+b+c} : \frac{ac}{a+b-c} \right], \\ G_2 &= \left[\frac{ab}{-a+b+c} : \frac{bc}{a+b-c} : \frac{-ac}{a+b+c} \right], \\ G_3 &= \left[\frac{-ab}{a+b+c} : \frac{bc}{a-b+c} : \frac{ac}{a+b-c} \right]. \end{aligned}$$

[1]It is not necessary to give the coordinates of $G = X_7$. They can be found in [10, 11].

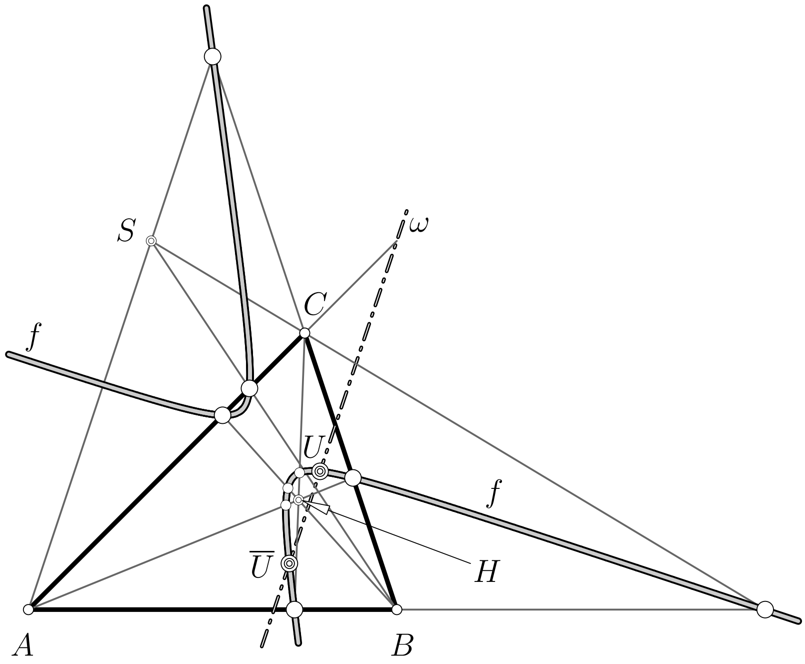

Figure 11. Feuerbach's nine point conic f and the projective versions of the barycenter S and orthocenter H

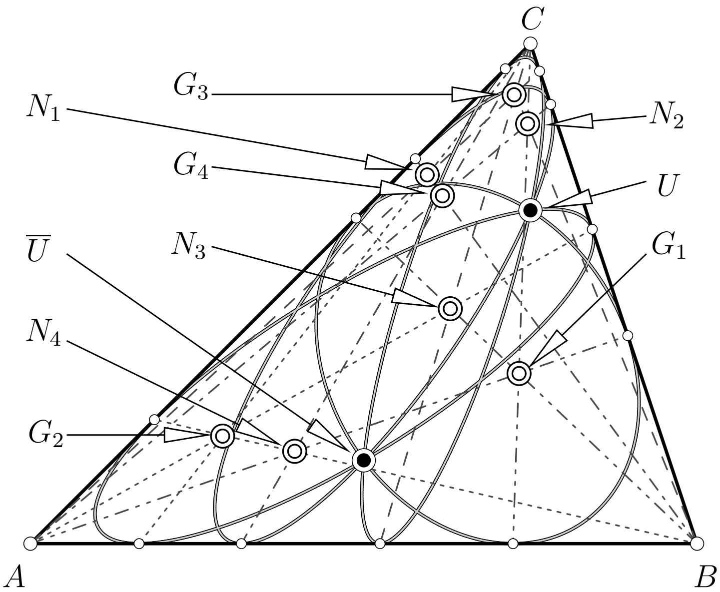

Figure 12. The generalized Gergonne and Nagel points

According to Theorem 2.4 the Nagel points N_i are the isotomic conjugates of the Gergonne points G_i for $i \in \{1, 2, 3\}$. So we compute the trilinear coordinates of the N_i from those of G_i. For a point X with trilinear coordinates $(\xi_0 : \xi_1 : \xi_2)$ the trilinear coordinates of its isotomic conjugate X' are $(c^{-2}\xi_0^{-1} : a^{-2}\xi_1^{-1} : b^{-2}\xi_2^{-1})$, cf. [11]. Therefore the trilinear coordinates of the Nagel points[2] N_1, N_2, and N_3 are

$$\begin{aligned} N_1 &= \left[\ \frac{a-b+c}{-c} : \frac{a+b+c}{a} : \frac{a+b-c}{-b}\ \right], \\ N_2 &= \left[\ \frac{-a+b+c}{-c} : \frac{a+b-c}{-a} : \frac{a+b+c}{b}\ \right], \\ N_3 &= \left[\ \frac{a+b+c}{c} : \frac{a-b+c}{-a} : \frac{a+b-c}{-b}\ \right]. \end{aligned}$$

Acknowledgements. I have to express my sincere thanks to Prof. Hellmuth Stachel for comments on an earlier version of this paper.

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Received September 16, 2009