Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 51 (2010), No. 2, 509-517.

Associated Polyhedra and Dual Linear Programs

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Abstract. The duality theorem of linear programming is set in a very general context, which is then mediated through the context of associated polyhedra; these latter are related by the representation theory of polyhedra. A feature of this approach is that it is made evident that two complementarity conditions are involved in the theorem.

1. Introduction

The duality theorem of linear programming is justly famous. Recall that a *linear program* optimizes (that is, maximizes or minimizes) a linear functional over a feasible region of a finite dimensional (real) coordinate vector space determined by finitely many linear equations and inequalities. With it is associated a dual linear program; if both programs are feasible (so that their feasible regions are non-empty), then the duality theorem says that the two programs have the same (finite) optimal value.

There are many expositions of the duality theorem. A very readable one of the conventional approach is that of Gale [1], which relies on the so-called Farkas Lemma; this is a special case of the separation theorem for convex sets, and was proved earlier than the general result. However, a novel (and much quicker) proof has been produced by Gritzmann and Klee [2, Section 5]. Their treatment assumes the background theory of polyhedra; we shall briefly survey those parts we need in Section 2, mainly in order to establish some notation and terminology. General references to polytope theory are [3], [8]; particularly relevant to our treatment

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are the introductory chapters of the latter, because these appeal to the basics of linear programming.

We shall present here yet another proof of the duality theorem or, rather, a more general version of it. This is related to that of [2]; actually, it was discovered by us considerably earlier (and thus independently). We were prompted to write up our proof by finding recently how to set everything in an even broader context, which at first sight seems to have little to do with linear programs; the ensuing main result of the paper is Theorem 3.1. A specialization of this in Section 4 (which was our original result) will involve what we call associated polyhedra, which are related by the representation theory of [4]. This will make it clear when we finally come to it that the duality theorem itself results from two complementarity conditions; the first corresponds to the usual equilibrium theorem, and the second to the complementary nature of the constraints of the programs.

2. Polyhedra

We shall assume that the reader is familiar with the theory of polyhedra (or polyhedral sets); as mentioned in the introduction, general background references are [3], [8]. However, we give a brief survey of those basic features of the theory which we need, in order to introduce the notation and conventions which we use.

We work throughout in finite dimensional vector spaces (denoted by $\mathbb{V}$, $\mathbb{X}$, $\mathbb{Y}$, and so on) over an ordered field $\mathbb{F}$. Only in the last Section 5 will we explicitly use coordinate spaces. Our treatment will require us to distinguish between a vector space $\mathbb{X}$ and its dual space $\mathbb{X}^* := \mathrm{Hom}(\mathbb{X}, \mathbb{F})$ of linear functionals on it; recall that (in the finite dimensional case) we may identify $\mathbb{X}^{**} := (\mathbb{X}^*)^*$ with $\mathbb{X}$. We write $\langle \cdot, \cdot \rangle$ for the pairing between $\mathbb{X}$ and $\mathbb{X}^*$.

A *half-space* in $\mathbb{X}$ is a set of the form

$$H^-(u, \beta) := \{x \in \mathbb{X} \mid \langle x, u \rangle \leqslant \beta\},$$

where $u \in \mathbb{X}^*$ (it is usual – but occasionally not necessary – to assume that $u \neq o$, the zero vector) and $\beta \in \mathbb{F}$. The boundary of $H^-(u, \beta)$ is the hyperplane

$$H(u, \beta) := \{x \in \mathbb{X} \mid \langle x, u \rangle = \beta\}.$$

A *polyhedron* is the intersection of finitely many half-spaces; we denote by $\mathcal{Q}(\mathbb{X})$ the family of non-empty polyhedra in $\mathbb{X}$.

The *support functional* $\eta(P, \cdot)$ of $P \in \mathcal{Q}(\mathbb{X})$ is defined for $u \in \mathbb{X}^*$ by

$$\eta(P, u) := \max\{\langle x, u \rangle \mid x \in P\};$$

we allow $\eta(P, u)$ to take the value ∞. Formally, if P is the empty polyhedron, then $\eta(P, \cdot) \equiv -\infty$. The *support hyperplane* of P *in direction* $u \in \mathbb{X}^*$, or with *outer normal* u, is $H(u, \eta(P, u))$ (provided that $\eta(P, u)$ is finite), and the corresponding *face* of P is then

$$F(P, u) := \{x \in P \mid \langle x, u \rangle = \eta(P, u)\}.$$

We write $F \leqslant P$ to mean that F is a face of P (in some direction u – allowing $u = o$, which will yield P itself). Recall that the intersection of any family of faces of P is either a face of P or is empty.

Each $S \subseteq P$ is contained in a unique minimal face of P (with respect to inclusion), which is called its *carrier*, and is denoted by $\operatorname{carr}(S, P)$; thus

$$\operatorname{carr}(S, P) = \bigcap \{F \leqslant P \mid S \subseteq F\}.$$

For a point-set $S = \{a\}$, we write $\operatorname{carr}(a, P)$ for its carrier; then $a \in \operatorname{relint} \operatorname{carr}(a, P)$, where relint denotes the relative interior (that is, the interior relative to the affine hull).

We call $K \in \mathcal{Q}(\mathbb{X})$ a *cone with apex* $a \in \mathbb{X}$ if $(1 - \lambda)a + \lambda x \in K$ for every $x \in K$ and $\lambda \geqslant 0$. We denote by $\mathcal{C}(\mathbb{X})$ the family of polyhedral cones with apex o. The *positive hull* $\operatorname{pos} A$ of a finite subset $A = \{a_1, \ldots, a_k\} \subseteq \mathbb{X}$ is the set of non-negative combinations of points of A, of the form $x = \sum_{i=1}^{k} \lambda_i a_i$ with $\lambda_j \geqslant 0$ for $j = 1, \ldots, k$; then $\operatorname{pos} A \in \mathcal{C}(\mathbb{X})$. More generally, the positive hull of $S \subseteq \mathbb{X}$ is

$$\operatorname{pos} S := \bigcup \{\operatorname{pos} A \mid A \subseteq S \text{ finite}\}.$$

If $a \in P \in \mathcal{Q}(\mathbb{X})$, then $A(a, P) := \operatorname{pos}(P - a)$ is the *angle cone* of P at a; if $F \leqslant P$, then $A(F, P) := A(a, P)$ is independent of the choice of $a \in \operatorname{relint} F$.

Another important example is the *recession cone* of P, which is

$$\operatorname{rec} P := \{x \in \mathbb{X} \mid a + \lambda x \subseteq P \text{ for each } a \in P \text{ and } \lambda \geqslant 0\}.$$

In the definition, we can actually take just a single $a \in P$. It is clear that $\operatorname{rec} P \subseteq A(F, P)$ for each $F \leqslant P$.

We next introduce polarity. If $K \in \mathcal{C}(\mathbb{X})$ is a cone, then its *polar* K^* is given by

$$K^* := \{u \in \mathbb{X}^* \mid \langle x, u \rangle \leqslant 0 \text{ for all } x \in K\}.$$

(We shall not need polarity in any more general context.) It is fairly easy to see that $K^* \in \mathcal{C}(\mathbb{X}^*)$, and that $K^{**} = K$. In fact, if $A = \{a_1, \ldots, a_k\}$ as before and $K = \operatorname{pos} A$, then

$$K^* = H^-(a_1, 0) \cap \cdots \cap H^-(a_k, 0).$$

Moreover, if $J \leqslant K$, then there is a corresponding face $\widehat{J} \leqslant K^*$ given by

$$\widehat{J} := \{u \in K^* \mid \langle x, u \rangle = 0 \text{ for all } x \in J\}.$$

Then $J \leftrightarrow \widehat{J}$ is an inclusion-reversing correspondence between the sets of faces of K and K^*.

The *normal cone* to $P \in \mathcal{Q}(\mathbb{X})$ at $a \in P$ is $N(a, P) := A(a, P)^*$; since this is independent of $a \in \operatorname{relint} F$ for $F \leqslant P$, we similarly write $N(F, P) := N(a, P)$. Thus $N(F, P)$ is the cone of outer normal vectors to support hyperplanes of P which contain F. We write $\mathcal{N}(P) := \{N(F, P) \mid F \leqslant P\}$ for the *normal fan* of P, and $\operatorname{nor} P := \bigcup \mathcal{N}(P)$ for the underlying set of all normals to faces of P. It is straightforward to see that

$$\operatorname{nor} P = (\operatorname{rec} P)^*.$$

In particular,

Proposition 2.1. *If $P \in \mathcal{Q}(\mathbb{X})$, then $\eta(P,u) < \infty$ if and only if $u \in \operatorname{nor} P$.*

Remark 2.2. Observe from the preceding discussion that, if $K \in \mathcal{C}(\mathbb{X})$ and $J \leqslant K$, then the corresponding face $\widehat{J}$ of K^* is

$$\widehat{J} = N(J,K) = N(a,K)$$

for any $a \in \operatorname{relint} J$.

We really need only one result in this section which may not be completely familiar to the reader; it is Theorem 2.4, whose proof is taken from [6]. We recall that, if $\Phi\colon \mathbb{X} \to \mathbb{V}$ is a linear mapping, then there is a *dual* linear mapping $\Phi^*\colon \mathbb{V}^* \to \mathbb{X}^*$, given by

$$\langle x, \Phi^* v\rangle = \langle \Phi x, v\rangle$$

for all $x \in \mathbb{X}$ and $v \in \mathbb{V}^*$. As a preliminary, we have

Lemma 2.3. *Let $\Phi\colon \mathbb{X} \to \mathbb{V}$ be a linear mapping, and let $K \in \mathcal{C}(\mathbb{V})$. Then*

$$\Phi^{-1}K = (\Phi^* K^*)^*. \tag{2.1}$$

Here, of course, $\Phi^{-1}K = \{x \in \mathbb{X} \mid \Phi x \in K\}$ is the *inverse image* of K under Φ.

Proof. Indeed, we have

$$\begin{aligned}
x \in (\Phi^* K^*)^* &\iff \langle x, u\rangle \leqslant 0 && \text{for all } u \in \Phi^* K^* \\
&\iff \langle x, \Phi^* v\rangle \leqslant 0 && \text{for all } v \in K^* \\
&\iff \langle \Phi x, v\rangle \leqslant 0 && \text{for all } v \in K^* \\
&\iff \Phi x \in K^{**} = K \\
&\iff x \in \Phi^{-1}K,
\end{aligned}$$

as claimed. □

Note that (2.1) can also be expressed in the form

$$(\Phi^{-1}K)^* = \Phi^* K^*.$$

In our application of this lemma, let $K \in \mathcal{Q}(\mathbb{V})$ (now a general polyhedron), and suppose that $P = \Phi^{-1}K$ is the inverse image of K under Φ. If $F \leqslant P$, then there is a unique $J \leqslant K$ such that $\Phi(\operatorname{relint} F) = \operatorname{relint}(\Phi F) \subseteq \operatorname{relint} J$; thus $J = \operatorname{carr}(\Phi F, K) = \operatorname{carr}(\Phi a, K)$ for any $a \in \operatorname{relint} F$. Moreover, since it is clear that $A(F,P) = \Phi^{-1}A(J,K)$, Lemma 2.3 then tells us that

$$N(F,P) = \Phi^* N(J,K). \tag{2.2}$$

Further, if we write

$$\mathcal{N}(K;\Phi) := \{N(J,K) \in \mathcal{N}(K) \mid J = \operatorname{carr}(\Phi F, K) \text{ for some } F \leqslant P\}, \tag{2.3}$$

we then have

Theorem 2.4. *With the preceding notation,*

$$\mathcal{N}(\Phi^{-1}K) = \Phi^* \mathcal{N}(K;\Phi).$$

By the latter expression, we just mean the set of images under Φ^* of normal cones in $\mathcal{N}(K;\Phi)$.

Proof. This follows from Lemma 2.3 or, rather, its consequence (2.2), taken over all $F \leqslant P$. □

Corollary 2.5. *With the notation of Theorem* 2.4*,*

$$\operatorname{nor}(\Phi^{-1}K) \subseteq \Phi^*(\operatorname{nor} K).$$

3. General duality

Before we specialize to a result which is close to the usual duality theorem for linear programs, we establish a much more general one. We now work in the following setting. We have a short exact sequence

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O}$$

of finite dimensional spaces over $\mathbb{F}$, where $\mathbb{O}$ denotes the zero space. Recall that this means that $\operatorname{im}\Phi = \ker\Psi$, where im and ker denote the image space and kernel, respectively. There is then the corresponding dual short exact sequence

$$\mathbb{O} \longleftarrow \mathbb{X}^* \xleftarrow{\Phi^*} \mathbb{V}^* \xleftarrow{\Psi^*} \mathbb{Y}^* \longleftarrow \mathbb{O}.$$

Further, let $K \in \mathcal{C}(\mathbb{V})$, so that $K^* \in \mathcal{C}(\mathbb{V}^*)$ is its dual cone. We take $b \in \mathbb{V}$ and define $P := \Phi^{-1}(K - b)$; similarly, let $c \in \mathbb{V}^*$ and define $Q := (\Psi^*)^{-1}(K^* - c)$. Finally, define $p := \Psi b$ and $q := \Phi^* c$. Our main result is a general duality theorem.

Theorem 3.1. *With the foregoing notation, suppose that* $P \neq \emptyset$*.*

(a) *If* $Q \neq \emptyset$ *also, then*

$$\eta(P, q) + \eta(Q, p) = -\langle b, c\rangle.$$

(b) *If* $Q = \emptyset$*, then* $\langle \cdot, q\rangle$ *is unbounded above on* P*.*

Note that, if $b \in K$ and $c \in K^*$, then $-\langle b, c\rangle \geqslant 0$.

Proof. For (a), first let $x \in P$ and $y \in Q$. Thus $\Phi x + b \in K$ and $\Psi^* y + c \in K^*$, and hence

$$\begin{aligned} 0 &\geqslant \langle \Phi x + b, \Psi^* y + c\rangle \\ &= \langle \Psi\Phi x, y\rangle + \langle x, \Phi^* c\rangle + \langle \Psi b, y\rangle + \langle b, c\rangle \\ &= \langle x, q\rangle + \langle y, p\rangle + \langle b, c\rangle \end{aligned}$$

since $\Psi\Phi = O$, so that $\langle x, q\rangle + \langle y, p\rangle \leqslant -\langle b, c\rangle$. Further, we have equality if and only if $\langle \Phi x + b, \Psi^* y + c\rangle = 0$.

Therefore, suppose that x is optimal, so that we have $\langle x, q\rangle = \eta(P, q)$. Let $J := \operatorname{carr}(\Phi x + b, K)$. If $F(P, q) = F$, then

$$\Phi^* c = q \in \operatorname{relint} N(F, P) = \operatorname{relint}(\Phi^* N(J, K)) = \Phi^*(\operatorname{relint} N(J, K)) = \Phi^*(\operatorname{relint} \widehat{J}).$$

Hence, $\Psi^* y + c \in \operatorname{relint} \widehat{J}$ for some $y \in \mathbb{Y}^*$, and this implies that $y \in (\Psi^*)^{-1}(K^* - c) = Q$. Indeed, if $G = \operatorname{carr}(y, Q)$, then $G = (\Psi^*)^{-1}(\widehat{J} - c)$.

Now we have

$$\Phi x + b \in J, \qquad \Psi^* y + c \in \widehat{J},$$

and consequently we have equality above, namely, $\langle x, q\rangle + \langle y, p\rangle = -\langle b, c\rangle$.

For (b), we now have

$$Q = \emptyset \implies \operatorname{im} \Psi^* \cap (K^* - c) = \emptyset \implies (\operatorname{im} \Psi^* + c) \cap K^* = \emptyset.$$

Applying Φ^* then shows that $q = \Phi^* c \notin \Phi^* K^* \supseteq \operatorname{nor} P$ by Corollary 2.5; in turn, Proposition 2.1 implies that $\langle \cdot, q\rangle$ is unbounded above on P. □

4. Associated polyhedra

We now consider a special case of what we did in the previous section. Let $U = (u_1, \ldots, u_n)$ be a fixed (ordered) set of normal vectors in $\mathbb{X}^*$ which spans that space linearly (this is for convenience), and denote by $\mathcal{Q}(\mathbb{X}; U)$ the family of polyhedra of the form

$$P(U; b) := \{x \in \mathbb{X} \mid \langle x, u_j\rangle \leqslant \beta_j \text{ for } j = 1, \ldots, n\}, \tag{4.1}$$

where $b = (\beta_1, \ldots, \beta_n)^{\mathsf{T}} \in \mathbb{F}^n$. (Here, $^{\mathsf{T}}$ denotes the transpose, because we find it more convenient to write vectors in text as rows.) In fact, we immediately introduce the following convention. Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{F}$ with ordered basis $E = (e_1, \ldots, e_n)$, identify b with $\sum_{i=1}^n \beta_i e_i$, define $\Phi\colon \mathbb{X} \to \mathbb{V}$ by $\Phi x := -\sum_{i=1}^n \langle x, u_i\rangle e_i$, and define $K \in \mathcal{C}(\mathbb{V})$ by $K := \operatorname{pos} E$. Then we see at once that $P := P(U; b) = \Phi^{-1}(K - b)$.

The dual cone $\widehat{K} \in \mathcal{C}(\mathbb{V}^*)$ is then $K^* = \operatorname{pos} E^*$, with $E^* = (e_1^*, \ldots, e_n^*)$ the *negative* of the basis of $\mathbb{V}^*$ dual to E, so that

$$\langle e_j, e_k^*\rangle = \begin{cases} -1, & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

Thus $u_j = \Phi^* e_j^*$ for $j = 1, \ldots, n$. If, as in Section 3, we complete to a short exact sequence

$$\mathbb{O} \longrightarrow \mathbb{X} \xrightarrow{\Phi} \mathbb{V} \xrightarrow{\Psi} \mathbb{Y} \longrightarrow \mathbb{O},$$

and define $\overline{U} = (\overline{u}_1, \ldots, \overline{u}_n)$ by $\overline{u}_j := \Psi e_j$ for $j = 1, \ldots, n$, then Ψ^* is similarly given by $\Psi^* y = -\sum_{i=1}^n \langle y, \overline{u}_i\rangle e_i^*$.

The relationship just described means that $\overline{U}$ is a *linear transform* of U and that $p = \Psi b$ *represents* P in $\mathbb{Y}$ in the sense of [4]. (See also the survey article [5], and [7] for a more abstract definition of linear transforms. Since we are explicitly using polar cones, we have followed here the convention of [6] in defining representations.)

In a similar way, we have a family $\mathcal{Q}(\mathbb{Y}^*; \overline{U})$ of polyhedra in $\mathbb{Y}^*$ of the form

$$P(\overline{U}; c) = \{y \in \mathbb{Y}^* \mid \langle y, \overline{u}_j \rangle \leqslant \gamma_j \text{ for } j = 1, \ldots, n\}$$

with $c = (\gamma_1, \ldots, \gamma_n)^{\mathsf{T}} \in \mathbb{F}^n$ identified with $\sum_{i=1}^n \gamma_i e_i^* \in \mathbb{V}^*$. Then $Q := P(\overline{U}; c) = (\Psi^*)^{-1}(K^* - c)$, and $q := \Phi^* c$ represents Q in $\mathbb{X}^*$. We shall call P and Q *associated* polyhedra.

Theorem 3.1 can now be sharpened. At this point, it is appropriate to introduce a little more terminology. We define $\mathsf{N} := \{1, \ldots, n\}$ and, for each $\mathsf{J} \subseteq \mathsf{N}$, write $U(\mathsf{J}) := \{u_j \in U \mid j \in \mathsf{J}\}$. Moreover, if $V := U(\mathsf{J})$, we define $\widetilde{V} := \overline{U}(\mathsf{N} \setminus \mathsf{J}) \subseteq \overline{U}$. For $j \in \mathsf{N}$, we let

$$F_j := \{x \in P \mid \langle x, u \rangle = \beta_j\}.$$

We think of F_j as being a facet of P (that is, a face of codimension 1), but it need not be, and could even be empty. We call $V = U(\mathsf{J})$ the *facial set* of $F \leqslant P$ when $F \leqslant F_j$ if and only if $j \in \mathsf{J}$, and denote it by $\operatorname{fac}(F, P)$; then $\widetilde{V}$ is the corresponding *cofacial set* in $\overline{U}$, denoted by $\operatorname{cofac}(F, P)$. (Note that these terms strictly depend on U as well, but we do not want to overburden the notation.)

Various features of representation theory follow directly from this geometric description:

- each translate $P + t$ of P (with $t \in \mathbb{X}$) is also represented by p;
- $P \neq \emptyset$ if and only if $p \in \operatorname{pos} \overline{U}$;
- $V \subseteq U$ is the facial set of some face $F \leqslant P$ if and only if $p \in \operatorname{relint} \operatorname{pos} \widetilde{V}$, and then $N(F, P) = \operatorname{pos} V$.

Let us now see what Theorem 3.1 looks like in this restricted situation. Part (b) remains the same, but we can be more specific about part (a). If $\langle \cdot, q \rangle$ is maximized on $F \leqslant P$ and $\langle \cdot, p \rangle$ is maximized on $G \leqslant Q$, then $\operatorname{carr}(\Phi F + b, K) = J \leqslant K$ and $\operatorname{carr}(\Psi^* G + c, K^*) = \widehat{J} \leqslant K^*$ for some J. In accord with the convention employed above, if $J = \operatorname{pos} E(\mathsf{N} \setminus \mathsf{J})$, with $\mathsf{J} \subseteq \mathsf{N}$, then $\widehat{J} = \operatorname{pos} E^*(\mathsf{J})$, and $q \in \operatorname{relint} J$, $p \in \operatorname{relint} \widehat{J}$. If it were not already obvious, this says that $V = U(\mathsf{J}) = \operatorname{fac}(F, P) = \operatorname{cofac}(G, Q)$ and $\widetilde{V} = \overline{U}(\mathsf{N} \setminus \mathsf{J}) = \operatorname{cofac}(F, P) = \operatorname{fac}(G, Q)$.

Recall that $x \in \mathbb{X}$ is called *feasible* if $x \in P$. Moreover, if $z = \sum_{i=1}^n \zeta_i e_i \in \mathbb{V}$, then its *support* is $\operatorname{supp} z := \{j \in \mathsf{N} \mid \zeta_j \neq 0\}$; two vectors $z \in \mathbb{V}$ and $z^* \in \mathbb{V}^*$ are then *complementary* if $\operatorname{supp} z \cap \operatorname{supp} z^* = \emptyset$. Theorem 3.1 then implies the *equilibrium* or *complementarity conditions.*

Theorem 4.1. *The feasible vectors $x \in P$ and $y \in Q$ are optimal for $\langle \cdot, q \rangle$ and $\langle \cdot, p \rangle$, respectively, if and only if the vectors $\Phi x + b \in \mathbb{V}$ and $\Psi^* y + c \in \mathbb{V}^*$ are complementary.*

5. Linear programs

We finally come to linear programs themselves. We shall take these in standard form in the usage of [1] (the reader should be aware that different writers use different terms).

Our context now is of coordinate spaces over the ordered field $\mathbb{F}$, whose elements are column vectors. Thus the inner product of $x, c \in \mathbb{F}^n$ is written

$$\langle c, x \rangle = c^{\mathsf{T}} x = x^{\mathsf{T}} c,$$

with $^{\mathsf{T}}$ as before denoting transpose, and the multiplication being ordinary matrix multiplication. The *standard maximal linear program* is then

$$\max c^{\mathsf{T}} x \qquad \text{subject to } x \geqq o \text{ and } Ax \leqq b. \tag{5.1}$$

Here, $\leqq$ means the corresponding inequality $\leqslant$ between each coordinate, $c, x \in \mathbb{F}^n$ as before, $b \in \mathbb{F}^m$, and A is an $m \times n$ matrix over $\mathbb{F}$. This is often also called the *primal* program. The corresponding *dual* program, which is a *standard minimal linear program*, is

$$\min b^{\mathsf{T}} y \qquad \text{subject to } y \geqq o \text{ and } A^{\mathsf{T}} y \geqq c; \tag{5.2}$$

thus $y \in \mathbb{F}^m$. Recall that the linear program (5.1) is *feasible* if it has a feasible vector, that is, an $x \in \mathbb{F}^n$ which satisfies the *constraints* $x \geqq o$ and $Ax \leqq b$; the set of feasible vectors is called the *feasible region*, and corresponds to the polyhedron P of Section 4. Similarly, the feasible region of (5.2) is Q. Moreover, feasible vectors x or y which achieve the maximum or minimum, respectively, are called *optimal*, and the corresponding $c^{\mathsf{T}} x$ or $b^{\mathsf{T}} y$ are their *optimal values*.

We then have the classical duality theorem of linear programming.

Theorem 5.1. *Let* (5.1) *and* (5.2) *be dual standard linear programs.*

(a) *If both programs are feasible, then their optimal values are the same.*
(b) *If* (5.1) *is feasible but* (5.2) *is not, then* $c^{\mathsf{T}} x$ *is unbounded above on the feasible region.*

Proof. All we need do is translate Theorem 3.1 into the language of this section, bearing in mind the special case discussed in Section 4. First, we must put (5.2) into primal form; to do this, we consider the equivalent program

$$\max(-b)^{\mathsf{T}} y \qquad \text{subject to } y \geqq o \text{ and } (-A^{\mathsf{T}}) y \leqq -c.$$

Let $b = (\beta_1, \ldots, \beta_m)^{\mathsf{T}}$ and $c = (\gamma_1, \ldots, \gamma_n)^{\mathsf{T}}$. For the spaces, we have

$$\mathbb{X} \mapsto \mathbb{F}^n, \quad \mathbb{V} \mapsto \mathbb{F}^{n+m}, \quad \mathbb{Y} \mapsto \mathbb{F}^m.$$

For the mappings, we have

$$\varPhi \mapsto \begin{bmatrix} -I_n \\ -A \end{bmatrix}, \qquad \varPsi \mapsto [A \;\; -I_m];$$

note that $\Phi^* \mapsto [-I_n \; -A^\mathsf{T}]$, and so on. Finally,

$$b \mapsto (0^n, \beta_1, \dots, \beta_m)^\mathsf{T}, \quad c \mapsto (-\gamma_1, \dots, -\gamma_n, 0^m)^\mathsf{T},$$

where 0^k stands for a string $0, \dots, 0$ of length k, and hence

$$p \mapsto (-\beta_1, \dots, -\beta_m)^\mathsf{T}, \quad q \mapsto (\gamma_1, \dots, \gamma_n)^\mathsf{T}.$$

Then part (a) follows from

$$\max q^\mathsf{T} x + \max p^\mathsf{T} y = (0^n, \beta_1, \dots, \beta_m)(-\gamma_1, \dots, -\gamma_n, 0^m)^\mathsf{T} = 0,$$

while part (b) is as before. □

Remark 5.2. An important thing to observe is that a second complementarity condition is involved here, namely, that the constraint vectors $(0^n, \beta_1, \dots, \beta_m)^\mathsf{T}$ of (5.1) and $(-\gamma_1, \dots, -\gamma_n, 0^m)^\mathsf{T}$ of (5.2) are themselves complementary.

References

[1] Gale, D.: *The Theory of Linear Economic Models*. McGraw-Hill, New York-Toronto-London 1960. cf. Verlag für ausländische Literatur, Moskau 1963. Zbl 0114.12203

[2] Gritzmann, P.; Klee, V. L.: *Mathematical programming and geometric convexity*. In: P. M. Gruber and J. M. Wills (eds.), Handbook of Convex Geometry. 627–674. North-Holland, Amsterdam 1993. Zbl 0806.90098

[3] Grünbaum, B.: *Convex Polytopes*. Prepared by V. Kaibel, V. L. Klee and G. M. Ziegler, 2nd edition. Graduate Texts in Mathematics **221**, Springer, New York 2003. Zbl 1024.52001

[4] McMullen, P.: *Representations of polytopes and polyhedral sets*. Geom. Dedicata **2** (1973), 83–99. Zbl 0273.52006

[5] McMullen, P.: *Transforms, diagrams and representations*. In: J. Tölke and J. M. Wills (eds.), Contributions to Geometry, Proc. Symp., Siegen 1978, 92–130. Birkhäuser, Basel 1979. Zbl 0445.52006

[6] McMullen, P.: *Fibre tilings*. Mathematika **50** (2003), 1–33. Zbl 1086.52006

[7] Oda, T.; Park, H. S.: *Linear Gale transforms and Gelfand-Kapranov-Zelevinskij decompositions*. Tôhoku Math. J., II. Ser. **43** (1991), 375–399. Zbl 0782.52006

[8] Ziegler, G. M.: *Lectures on Polytopes*. Graduate Texts in Mathematics **152**, Springer, Berlin 1995. Zbl 0823.52002

Received July 30, 2009