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Positivity Theorems for Solid-Angle Polynomials

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Abstract. For a lattice polytope $\mathcal{P}$, define $A_{\mathcal{P}}(t)$ as the sum of the solid angles of all the integer points in the dilate $t\mathcal{P}$. Ehrhart and Macdonald proved that $A_{\mathcal{P}}(t)$ is a polynomial in the positive integer variable t. We study the numerator polynomial of the solid-angle series $\sum_{t\geq 0} A_{\mathcal{P}}(t) z^t$. In particular, we examine nonnegativity of its coefficients, monotonicity and unimodality questions, and study extremal behavior of the sum of solid angles at vertices of simplices. Some of our results extend to more general valuations.

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1. Introduction

Suppose $\mathcal{P} \subset \mathbf{R}^d$ is a d-dimensional polytope with integer vertices (a *lattice polytope*). Unless otherwise stated, we shall assume throughout that $\mathcal{P}$ is full-dimensional. Let $B(r, \mathbf{x})$ be the d-dimensional ball of radius r centered at the point $\mathbf{x} \in \mathbf{R}^d$. Then we define the *solid angle at* $\mathbf{x}$ *with respect to* $\mathcal{P}$ to be

$$\omega_{\mathcal{P}}(\mathbf{x}) := \lim_{r \to 0} \frac{\mathrm{vol}(B(r, \mathbf{x}) \cap \mathcal{P})}{\mathrm{vol}(B(r, \mathbf{x}))} .$$

The fraction above measures the proportion of a small sphere of radius r that intersects the polytope $\mathcal{P}$ and is hence constant for all sufficiently small $r > 0$, so that the limit always exists. We note that the notion of a solid angle $\omega_{\mathcal{P}}(\mathbf{x})$ is equivalent to the notion of the volume of a spherical polytope on the unit sphere, normalized by dividing by the volume of the boundary of the unit sphere.

We are interested in weighing every lattice point $x \in \mathbf{Z}^d$ by its corresponding solid angle $\omega_{t\mathcal{P}}(\mathbf{x})$, and summing these weights over the whole lattice. To this end, we consider the function $A_{\mathcal{P}} \colon \mathbf{Z}_{>0} \to \mathbf{R}$ defined by

$$A_{\mathcal{P}}(t) := \sum_{\mathbf{x} \in \mathbf{Z}^d} \omega_{t\mathcal{P}}(\mathbf{x}) ,$$

which we call the *solid-angle polynomial* of $\mathcal{P}$. Here $t\mathcal{P} = \{t\mathbf{x} \mid \mathbf{x} \in \mathcal{P}\}$ denotes the t^{th} dilate of $\mathcal{P}$. The fact that $A_{\mathcal{P}}(t)$ is indeed a polynomial follows from Ehrhart's celebrated theorem that the lattice-point enumerator

$$L_{\mathcal{P}}(t) := \# \left(t\mathcal{P} \cap \mathbf{Z}^d \right) ,$$

also initially defined only on the positive integers, is in fact a polynomial in $t \in \mathbf{Z}$. We will review a few basic facts about $A_{\mathcal{P}}(t)$ and $L_{\mathcal{P}}(t)$ in Section 2.

Solid-angle polynomials are not as widely known and studied as they deserve to be. This paper contains a few results that seem basic and yet have been unknown thus far.

Some of our results answer various open problems in [3, Chapter 11]. While Ehrhart polynomials can be computed using programs such as LattE [9], [13], normaliz [6], or polylib [5], there is currently no software available for computing solid-angle polynomials, so it is difficult to obtain data for making conjectures. Recent activity on solid angles can be found in [7] and [11].

In Section 3, we give some formulas related to calculations of solid angles. We also address the question of whether there are polytopes for which the polynomial $A_{\mathcal{P}}$ has negative coefficients. The equivalent question regarding Ehrhart polynomials $L_{\mathcal{P}}(t)$ has a positive answer, as exemplified by *Reeve's tetrahedron* $\mathcal{P}_h$ whose vertices are $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(1,1,h)$, with h a fixed positive integer. This tetrahedron was used by Reeve [20] to show that no linear analogue of Pick's theorem can hold in dimension 3. The Ehrhart polynomial of $\mathcal{P}_h$ is

$$L_{\mathcal{P}_h}(t) = \frac{h}{6} t^3 + t^2 + \left(2 - \frac{h}{6} \right) t + 1 . \tag{1}$$

Thus for $h > 12$ one obtains Ehrhart polynomials with negative coefficients. Reeve's tetrahedron allows us to construct solid-angle polynomials with negative coefficients as well:

Proposition 1. *The linear coefficient of $A_{\mathcal{P}_h}(t)$ is negative.*

In Section 4, we answer an open question (in the negative) raised in [3, Chapter 11], namely, whether the solid-angle vertex sum $\sum_{\mathbf{v}\text{ a vertex}} \omega_\Delta(\mathbf{v})$ is minimized when Δ is the regular d-simplex. We construct two infinite families of simplices in dimensions ≥ 3 that exhibit extreme asymptotic behavior (approaching 0 and 1/2, respectively) with respect to the sum of the solid angles at their vertices.

Although the existence of these constructions are special cases of theorems from [2], [18], and [19], the explicit nature of the examples we give here complement their existence proofs.

In Section 5 we prove a nonnegativity result. We define the generating function of $A_{\mathcal{P}}(t)$ as

$$\mathrm{Solid}_{\mathcal{P}}(z) := \sum_{t\geq 0} A_{\mathcal{P}}(t)\, z^t.$$

The fact that $A_{\mathcal{P}}(t)$ is a degree d polynomial in t is equivalent to the fact that we can write $\mathrm{Solid}_{\mathcal{P}}(z)$ as a rational function in z with denominator $(1-z)^{d+1}$. When written this way, Macdonald [14] proved that the numerator of $\mathrm{Solid}_{\mathcal{P}}(z)$ is a palindromic polynomial or, equivalently, that

$$A_{\mathcal{P}}(-t) = (-1)^d A_{\mathcal{P}}(t)\,. \tag{2}$$

We will prove the following.

Theorem 2. *Given a lattice polytope $\mathcal{P} \subset \mathbf{R}^d$, write*

$$\mathrm{Solid}_{\mathcal{P}}(z) = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{(1-z)^{d+1}}\,.$$

Then $a_j > 0$ for $j = 1, 2, \ldots, d$ and $a_0 = 0$.

If we replace $A_{\mathcal{P}}(t)$ with $L_{\mathcal{P}}(t)$, there is a well-known result due to Stanley [24] that $a_j \geq 0$ for $j = 0, 1, \ldots, d$.

These theorems have a common generalization. Let M denote the set of measurable sets in $\mathbf{R}^d$. A *valuation* is a function $\nu\colon M \times \mathbf{R}^d \to \mathbf{R}$ that satisfies inclusion-exclusion:

$$\nu(K_1 \cup K_2, \mathbf{x}) = \nu(K_1, \mathbf{x}) + \nu(K_2, \mathbf{x}) - \nu(K_1 \cap K_2, \mathbf{x})\,.$$

For our purposes, we will replace M by the set of all polyhedral complexes. A valuation whose codomain is $\mathbf{R}_{\geq 0}$ is called *nonnegative.* A valuation ν is *translation-invariant* if $\nu(K + \mathbf{y}, \mathbf{x} + \mathbf{y}) = \nu(K, \mathbf{x})$ for all $K \in M$, and all $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$. See [12] for more about valuations.

It is clear that $\nu(K, \mathbf{x}) = \omega_K(\mathbf{x})$ is a (translation-invariant nonnegative) valuation, as is the function $\nu'(K, \mathbf{x}) = \#(K \cap \{\mathbf{x}\} \cap \mathbf{Z}^d)$, i.e., the function that reports whether or not $\mathbf{x}$ is a lattice point inside of K.

To state our generalization, let $N_{\mathcal{P}} \colon \mathbf{Z}_{>0} \to \mathbf{R}$ be defined through

$$N_{\mathcal{P}}(t) := \sum_{\mathbf{x} \in \mathbf{Z}^d} \nu(t\mathcal{P}, \mathbf{x}) \tag{3}$$

(in fact, we could replace $\mathbf{Z}^d$ here by an arbitrary lattice), and let $G_{\mathcal{P}}(z)$ be the generating function of $N_{\mathcal{P}}(t)$:

$$G_{\mathcal{P}}(z) := \sum_{t \ge 0} N_{\mathcal{P}}(t)\, z^t. \tag{4}$$

McMullen [17] proved that $N_{\mathcal{P}}(t)$ is a polynomial if $\mathcal{P}$ is a lattice polytope and, equivalently, that we can write $G_{\mathcal{P}}(z)$ as a rational function in z with denominator $(1-z)^{d+1}$. Recall that our polytopes are assumed to be full-dimensional.

Theorem 3. *Suppose ν is a translation-invariant nonnegative valuation and $\mathcal{P} \subset \mathbf{R}^d$ is a lattice polytope. Let $G_{\mathcal{P}}(z)$ be defined through* (3) *and* (4) *and written as*

$$G_{\mathcal{P}}(z) = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{(1-z)^{d+1}}\,.$$

Then $a_j \ge 0$ for $j = 0, 1, 2, \ldots, d$.

We also prove the following monotonicity theorem:

Theorem 4. *Let ν be a translation-invariant nonnegative valuation and $\mathcal{P} \subseteq \mathcal{Q} \subset \mathbf{R}^d$ two lattice polytopes. Writing*

$$G_{\mathcal{P}}(z) = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{(1-z)^{d+1}}, \quad G_{\mathcal{Q}}(z) = \frac{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0}{(1-z)^{d+1}},$$

one has $a_i \le b_i$ for $i = 0, 1, \ldots, d$.

Two immediate corollaries of this theorem are the appropriate monotonicity theorems for $L_{\mathcal{P}}(t)$ and $A_{\mathcal{P}}(t)$. The former is due to Stanley [24], whereas the latter seems to be new.

In Section 7, we discuss a phenomenon that can be observed with certain *rational polytopes*, i.e., polytopes whose vertices are in $\mathbf{Q}^d$. In this case, the functions $A_{\mathcal{P}}(t)$ and $L_{\mathcal{P}}(t)$ are examples of a *quasipolynomial*, that is, a function of the form $c_d(t)\, t^d + c_{d-1}(t)\, t^{d-1} + \cdots + c_0(t)$, where $c_0(t), \ldots, c_d(t)$ are periodic functions in t. The least common multiple of the denominators of the vertex coordinates of $\mathcal{P}$ is always a period of the coefficient functions of $A_{\mathcal{P}}(t)$ and $L_{\mathcal{P}}(t)$. The recent literature [10], [15] includes examples of rational polytopes whose Ehrhart quasipolynomials exhibit *period collapse*; that is, they are polynomials. We give a family of polytopes for which period collapse happens for the solid-angle quasipolynomials.

2. Some background

We give a brief review of Ehrhart theory and the theory of solid angles without any proofs. The interested reader can find proofs and much more in [3]. See also [22] for proofs with a more valuation oriented mindset.

As mentioned in the introduction, for a given rational polytope $\mathcal{P} \subset \mathbf{R}^d$ (for this paragraph, we do not require that $\mathcal{P}$ is d-dimensional), the counting function $L_{\mathcal{P}}(t) := \#\left(t\mathcal{P} \cap \mathbf{Z}^d\right)$ is a quasipolynomial in the integer variable t. If $\mathcal{P}$ has integral vertices, then $L_{\mathcal{P}}(t)$ is a polynomial. Denote the interior of $\mathcal{P}$ as $\mathcal{P}^\circ$ (here we mean topological interior relative to the affine span of $\mathcal{P}$); the following *reciprocity law* holds:

$$L_{\mathcal{P}}(-t) = (-1)^{\dim \mathcal{P}} L_{\mathcal{P}^\circ}(t)\,. \tag{5}$$

We will need to know a few properties of Ehrhart polynomials. Namely, the degree of the polynomial is the dimension of the polytope, and the leading coefficient is its volume. (We always measure volume of a polytope relative to its affine span, normalized with respect to the lattice induced on this affine span.) The second leading coefficient is half of the sum of the volumes of the facets (the codimension-1 faces). In particular, these two coefficients are always positive.

Now we can explain why $A_{\mathcal{P}}(t)$ is also a quasipolynomial. For a face $\mathcal{F} \subseteq \mathcal{P}$, define the solid angle of $\mathcal{F}$ to be the solid angle of any point in $\mathcal{F}^\circ$, denoted by $\omega_{\mathcal{P}}(\mathcal{F})$. Thus

$$A_{\mathcal{P}}(t) = \sum_{\mathcal{F} \subseteq \mathcal{P}} \omega_{\mathcal{P}}(\mathcal{F}) L_{\mathcal{F}^\circ}(t)\,, \tag{6}$$

where the sum is over all faces $\mathcal{F}$ of $\mathcal{P}$ (see also [16] for more on the relationship between $A_{\mathcal{P}}(t)$ and $L_{\mathcal{P}}(t)$). So $A_{\mathcal{P}}(t)$ is indeed a quasipolynomial, since $L_{\mathcal{F}^\circ}(t)$ is a quasipolynomial for all faces $\mathcal{F}$. In particular, if $\mathcal{P}$ is a lattice polytope, then $A_{\mathcal{P}}(t)$ is a polynomial in t.

By (2), we have $A_{\mathcal{P}}(0) = 0$. An important relation which for rational polytopes is equivalent to this fact is the following.

Theorem 5. (Brianchon-Gram relation) *If $\mathcal{P}$ is a polytope, then*

$$\sum_{\mathcal{F} \subseteq \mathcal{P}} (-1)^{\dim \mathcal{F}} \omega_{\mathcal{P}}(\mathcal{F}) = 0\,,$$

where the sum is over all faces $\mathcal{F}$ of $\mathcal{P}$.

3. Formulas for the explicit computation of solid angles

In dimension 3, the following explicit formula can be used for calculating solid angles.

Proposition 6. *Given a simplicial cone $K \subset \mathbf{R}^3$ at the origin, generated by the linearly independent vectors v_1, v_2, v_3, the solid angle ω_K at the origin is given by:*

$$(4\pi)\omega_K = \cos^{-1}\left(\frac{(v_1 \times v_2)\cdot(v_1 \times v_3)}{\|v_1 \times v_2\|\|v_1 \times v_3\|}\right) + \cos^{-1}\left(\frac{(v_2 \times v_1)\cdot(v_2 \times v_3)}{\|v_2 \times v_1\|\|v_2 \times v_3\|}\right) + \cos^{-1}\left(\frac{(v_3 \times v_1)\cdot(v_3 \times v_2)}{\|v_3 \times v_1\|\|v_3 \times v_2\|}\right) - \pi,$$

where $\times$ denotes the cross product of 3-dimensional vectors, $\cdot$ denotes the dot product, and $\|\|$ is the usual Euclidean norm.

Proof. First note that computing ω_K is equivalent to taking a sphere of radius 1 at the origin and calculating the surface area of its intersection with K divided by the surface area of the sphere, which is 4π. The surface area of a spherical triangle is, as a consequence of Girard's theorem [8, §6.9], the sum of its spherical angles minus π. The spherical angle at v_i is precisely the dihedral angle θ_i between the two faces of K that intersect at v_i, and $\cos(\theta_i)$ is equal to the dot product of the normal vectors to the planes spanned by the faces, whence the formula. □

Proof of Proposition 1. Given a simplex Δ, let

$$S(\Delta) := \sum_{\mathbf{v} \text{ a vertex}} \omega_\Delta(\mathbf{v}).$$

Let $S = S(\Delta_h)$. By (2), the solid-angle polynomial of Δ_h is

$$A_{\Delta_h}(t) = \frac{h}{6}t^3 + \left(S - \frac{h}{6}\right)t.$$

Since $S < \frac{1}{2}$ by Proposition 11 below, we conclude that $S - \frac{h}{6} < 0$ if $h \geq 3$. A direct calculation using Proposition 6 shows that $S(\Delta_1) \approx 0.127 < \frac{1}{6}$ and $S(\Delta_2) \approx 0.171 < \frac{1}{3}$. □

To handle the situation in any dimension d we now describe a formula, discovered by Aomoto [1] in 1977, that allows us to use an infinite hypergeometric series to compute ω_K for a simplicial d-dimensional cone K. We follow Kenzi Sato's exposition [23], as it clarifies Aomoto's fundamental work a bit further.

We begin with the hyperplane description of a spherical simplex in $\mathbf{R}^d$, defined by

$$\Delta = \{x \in \mathrm{S}^{d-1} \mid \langle n_i, x\rangle \geq 0,\ i = 0, \ldots, d-1\},$$

where the n_i are linearly independent integer vectors, normal to the facets of Δ (they are inward-pointing normal vectors). We define $\theta_{i,j}$ to be the dihedral angle between the two facets whose normal vectors are n_i and n_j. Thus, we have

$$\cos(\theta_{i,j}) = -\langle n_i, n_j\rangle.$$

The solid angle ω_Δ, i.e. the volume of the spherical simplex Δ, is determined by the $\binom{d}{2}$ dihedral angles $\theta_{0,1}, \theta_{0,2}, \ldots, \theta_{d-2,d-1}$. Here is Aomoto's hypergeometric series:

$$\omega_\Delta = C \sum_{\mathbf{m} \in \mathbf{Z}_{>0}^{d(d-1)/2}} \frac{\prod_{i<j}(-2b_{i,j})^{m_{i,j}}}{\prod_{i<j} m_{i,j}!} \prod_{k=0}^{d-1} \Gamma\left(\frac{1}{2}(m_{0,k} + \cdots + m_{k-1,k} + m_{k,k+1} + \cdots + m_{k,d-1}) + \frac{1}{2}\right),$$

where the sum is extended over all integer vectors of the form $\mathbf{m} = (m_{0,1}, m_{0,2}, \ldots, m_{d-2,d-1}) \in \mathbf{Z}_{>0}^{d(d-1)/2}$, where Γ denotes the Euler gamma function, where $C = \sqrt{\det B}/\pi^{d/2}$, and where the matrix $B := (b_{i,j})$ is a Gram-like matrix defined as follows. First, let

$$G = \Big(\langle n_i, n_j \rangle\Big),$$

a $d \times d$ Gram matrix. Next, let

$$G_k = G \text{ except with its } k\text{th row and } k\text{th column deleted},$$

so that G_k is a $(d-1) \times (d-1)$ matrix. Next, let

$$K_{i,j} = \delta(i,j) \frac{\det G_i}{\det G},$$

a diagonal matrix, by definition of the Kronecker delta function $\delta(i,j)$. Finally, let

$$B = (b_{i,j}) = K^{-1} G^{-1} K^{-1}.$$

In a recent paper, Ribando [21] rediscovered Aomoto's results, but gave different proofs, so that his paper has the redeeming feature of having a somewhat simplified proof of a simpler version of Aomoto's results.

4. Solid angles at the vertices of a polytope

As before, given any convex polytope $\mathcal{P}$, let $S(\mathcal{P})$ denote the sum of the solid angles at the vertices of $\mathcal{P}$. Our goal in this section is to study the extremal behavior of $S(\mathcal{P})$, and especially in the case that $\mathcal{P}$ is a simplex. In passing, we note that a conjecture of [3, Chapter 12] that the regular simplex minimizes $S(\Delta)$ is false, but that similar questions on minimizing or maximizing $S(\mathcal{P})$ are still quite interesting. In particular, [2] David Barnette has given an amusing and beautiful equivalence for the minimization of $S(\mathcal{P})$ in terms of the existence of a Hamiltonian circuit along the edge graph of $\mathcal{P}$. There are also two papers by Perles and Shephard ([18] and [19]) with very general results along these lines as well. The combinatorial type of a polytope shall refer to isomorphism type of its face lattice.

Theorem 7. (Barnette) *Any* 3*-dimensional polytope* $\mathcal{P}$ *has a Hamiltonian circuit along its edge-graph if and only if there are polytopes with the same combinatorial type as* $\mathcal{P}$*, with arbitrarily small vertex angle sums.*

Since it is obvious that a simplex has a Hamiltonian circuit along its edge graph, there are always simplices that have arbitrarily small vertex angle sums. It is also worth noting that [2] has a general upper bound for $S(\mathcal{P})$ in any dimension.

These results about $S(\mathcal{P})$ are mostly existential, so we complement them with some constructive examples below. We now compute some explicit examples of solid-angle polynomials that we shall need later.

Proposition 8. *Let* $\pi \in \mathfrak{S}_d$ *be a permutation on* d *elements, and let* $\mathbf{e}_1, \ldots, \mathbf{e}_d$ *be the standard basis vectors. The polytope* $\Delta_\pi = \operatorname{conv}\{\mathbf{e}_{\pi(1)}, \mathbf{e}_{\pi(1)} + \mathbf{e}_{\pi(2)}, \ldots, \mathbf{e}_{\pi(1)} + \cdots + \mathbf{e}_{\pi(d)}\}$ *has solid angle polynomial* $\frac{1}{d!}t^d$.

Proof. The set $\{\Delta_\pi \mid \pi \in \mathfrak{S}_d\}$ is a triangulation of the unit cube $[0,1]^d$, and the Δ_π are all congruent to one another, i.e., any such simplex can be obtained from any other through a series of rotations, reflections, and translations. Note that $[0,t]^d$ is tiled by t^d copies of $[0,1]^d$, so $A_{[0,1]^d}(t) = t^d$ since $A_{[0,1]^d}(1) = 1$. Hence $A_{\Delta_\pi}(t) = \frac{1}{d!}t^d$. □

Now let $\Delta = \operatorname{conv}\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}$, which is a regular tetrahedron. The solid angle at an edge is the dihedral angle $\frac{1}{2\pi}\cos^{-1}(\frac{1}{3})$, and they are all the same by symmetry, so Brianchon-Gram gives

$$0 = -1 + 4 \cdot \frac{1}{2} - \frac{3}{\pi}\cos^{-1}\left(\frac{1}{3}\right) + 4\omega\,,$$

where ω is the solid angle at a vertex. Thus $S(\Delta) \approx 0.175$. But by Proposition 8, the solid-angle polynomial of $\mathcal{Q} = \operatorname{conv}\{(0,0,0),(0,0,1),(0,1,1),(1,1,1)\}$ is $\frac{1}{6}t^3$, so $S(\mathcal{Q}) = \frac{1}{6}$, which is less than 0.175. Also, the sum of the solid angles at the vertices of the standard simplex $\operatorname{conv}\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$ is approximately 0.206, so $S(\Delta)$ is neither a maximum nor a minimum. Using Proposition 9, one observes the same behavior in higher dimensions.

Despite our negative answer for the original conjecture, we can rephrase it as follows: In fixed dimension, which simplices minimize/maximize $S(\Delta)$? We note that a similar question of angle sums is addressed in [7], but here we are concerned with integral polytopes.

Proposition 9. *For* $d > 2$*, let* $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ *be the standard basis vectors of* $\mathbf{R}^d$*, and define* $\Delta(h_1, \ldots, h_{d-1}) = \operatorname{conv}\{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{d-1}, (h_1, h_2, \ldots, h_{d-1}, 1)\}$*. Then*

(a) $S(\Delta(h_1, \ldots, h_{d-1}))$ *is arbitrarily close to* $\frac{1}{2}$ *for sufficiently large negative values of all of the* h_i*'s, and*

(b) $S(\Delta(h, h, 1, 1, \ldots, 1)) \to 0$ *as* $h \to +\infty$.

Proof. We first prove (a). To show that $S(\Delta(h_1, \dots, h_{d-1})) \to \frac{1}{2}$ as $h_i \to -\infty$, it is enough to show that the solid angle at the origin approaches $\frac{1}{2}$ by Proposition 11 below. Fix $\varepsilon > 0$ sufficiently small. (It is enough to choose ε such that any point x within ε of the origin satisfies $|x_1| + \cdots + |x_d| < 1$.) To show that the solid angle at the origin approaches $\frac{1}{2}$, it is enough to show that any point with positive x_d coordinate that is within ε of the origin is contained in $\Delta(h_1, \dots, h_{d-1})$ for sufficiently negative values of h_i. But this is clear: given such a point $\mathbf{x} = (x_1, \dots, x_d)$, assume for notational simplicity that its first k coordinates and x_d are the only positive entries. Then

$$\mathbf{x} = (1 - x_1 - \cdots - x_k - x_d)\mathbf{0} + x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k + x_d(h_1, \dots, h_{d-1}, 1)$$

with $h_1 = \cdots = h_k = 0$ and $h_i = x_i/x_d$ for $i = k+1, \dots, d-1$. In the case that $\mathbf{x}$ has different positive entries other than the first k coordinates, it is trivial to modify the above argument.

For (b), we need to show that the solid angle at each vertex approaches 0 as h tends to $+\infty$. We do so by induction on dimension, starting with $d = 3$ where we can use Proposition 6. Then the solid angles at $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, (h, h, 1)$, respectively, are as follows:

$$\begin{aligned}
&\frac{1}{4\pi}\left(2\cos^{-1}\left(\frac{h}{\sqrt{h^2+1}}\right) + \cos^{-1}\left(\frac{-h^2}{h^2+1}\right) - \pi\right), \\
&\frac{1}{4\pi}\left(\cos^{-1}\left(\frac{h}{\sqrt{h^2+1}}\right) + \cos^{-1}\left(\frac{-2h+1}{\sqrt{4h^2-4h+3}}\right)\right. \\
&\qquad\qquad\qquad \left. + \cos^{-1}\left(\frac{2h^2-h+1}{\sqrt{(h^2+1)(4h^2-4h+3)}}\right) - \pi\right), \\
&\frac{1}{4\pi}\left(\cos^{-1}\left(\frac{h}{\sqrt{h^2+1}}\right) + \cos^{-1}\left(\frac{-2h+1}{\sqrt{4h^2-4h+3}}\right)\right. \\
&\qquad\qquad\qquad \left. + \cos^{-1}\left(\frac{2h^2-h+1}{\sqrt{(h^2+1)(4h^2-4h+3)}}\right) - \pi\right), \\
&\frac{1}{4\pi}\left(\cos^{-1}\left(\frac{-h^2}{\sqrt{h^2+1}}\right) + 2\cos^{-1}\left(\frac{2h^2-h+1}{\sqrt{(h^2+1)(4h^2-4h+3)}}\right) - \pi\right),
\end{aligned}$$

and these all tend to 0 as $h \to \infty$.

Now let Δ_d be the d-dimensional simplex $\Delta(h, h, 1, \dots, 1)$. Then $\Delta_{d+1} \subset \Delta_d \times [0, 1]$, and the vertices of Δ_{d+1} are a subset of the vertices of $\Delta_d \times [0, 1]$. Hence $S(\Delta_{d+1}) \le S(\Delta_d \times [0, 1])$, and we finish by using Lemma 10 below. $\square$

Lemma 10. *Given a polyhedral cone K with vertex $\mathbf{x}$, one has*

$$\omega_{K\times[0,1]}((\mathbf{x}, 0)) = \omega_{K\times[0,1]}((\mathbf{x}, 1)) \le c\,\omega_K(\mathbf{x})$$

where c is a constant that only depends on $d = \dim K$.

Proof. The first equality follows by symmetry. We can write

$$\omega_K(\mathbf{x}) = \frac{\mathrm{vol}(B(1,\mathbf{x}) \cap K)}{\mathrm{vol}(B(1,\mathbf{x}))}$$

since K is a polyhedral cone. Now let $C = B(1,\mathbf{x}) \times [0,1]$. The upper hemisphere of the $(d+1)$-dimensional ball of radius 1 centered at $(\mathbf{x},0)$ is contained in C. Hence we have

$$\mathrm{vol}(B(1,\mathbf{x}) \cap K) = \mathrm{vol}(C \cap (K \times [0,1])) \geq \mathrm{vol}(B(1,(\mathbf{x},0)) \cap (K \times [0,1])),$$

and setting

$$c = \frac{\mathrm{vol}(B(1,\mathbf{x}))}{\mathrm{vol}(B(1,(\mathbf{x},0)))}$$

proves the inequality. □

Notice that 0 is an obvious lower bound for $S(\Delta)$. It turns out that $\frac{1}{2}$ is the upper bound, as was already shown in [2]. We provide another proof here, with a pretty argument due to Herbert Edelsbrunner and Igor Rivin (personal communication).

Proposition 11. *Let Δ be a d-simplex. For $d = 2$, $S(\Delta) = \frac{1}{2}$, and for $d > 2$, $S(\Delta) < \frac{1}{2}$.*

Proof. We may assume that one of the vertices is the origin. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d$ be the other vertices of Δ. Let K be the cone generated by $\mathbf{v}_1, \ldots, \mathbf{v}_d$. The fundamental parallelepiped defined by

$$\Pi = \{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d \mid 0 \leq \lambda_i < 1\}$$

tiles K, and thus the sum of the solid angles of the vertices of its closure is 1. Define

$$\begin{aligned}
\Delta_1 &= \{\lambda_1 \mathbf{v}_1 + \cdots + \lambda_d \mathbf{v}_d \mid 0 \leq \lambda_i,\ \textstyle\sum \lambda_i \leq 1\}, \\
\Delta_2 &= \{(1-\lambda_1)\mathbf{v}_1 + \cdots + (1-\lambda_d)\mathbf{v}_d \mid 0 \leq \lambda_i,\ \textstyle\sum \lambda_i \leq 1\},
\end{aligned}$$

which are subsets of Π. Then $\Delta_1 \cap \Delta_2 = \varnothing$ and $\Delta_1 \cup \Delta_2 \neq \Pi$ if $d > 2$. So Π contains two congruent disjoint copies of Δ, and hence $2S(\Delta) < 1$ when $d > 2$. □

5. Numerator polynomial of solid-angle series

Proof of Theorem 2. Since solid angles are additive, it suffices to prove the statement for lattice simplices. If $\Delta = \mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}\} \subset \mathbf{R}^d$ is a lattice d-simplex, we form the *cone over* Δ:

$$\begin{aligned}
\mathrm{cone}(\Delta) := \{&\lambda_1(\mathbf{v}_1,1) + \lambda_2(\mathbf{v}_2,1) + \cdots + \lambda_{d+1}(\mathbf{v}_{d+1},1) \mid \lambda_1, \lambda_2, \ldots, \lambda_{d+1} \geq 0\} \\
&\subset \mathbf{R}^{d+1}.
\end{aligned} \tag{7}$$

We now consider codimension-1 solid angles in $\mathbf{R}^{d+1}$, by setting $f_{\mathrm{cone}(\Delta)}(\mathbf{x})$ of a point $\mathbf{x} \in \mathrm{cone}(\Delta)$ to be the solid angle of $\mathbf{x}$ relative to the hyperplane through $\mathbf{x}$ with normal vector $(0,0,\ldots,0,1)$. To be more precise, let $H = \{\mathbf{x} \in \mathbf{R}^{d+1} \mid x_{d+1} = 0\}$, then

$$f_{\mathrm{cone}(\Delta)}(\mathbf{x}) = \omega_{\mathrm{cone}(\Delta)\cap(x_{d+1}+H)}(\mathbf{x}),$$

where we are treating $x_{d+1} + H$ as an isomorphic copy of $\mathbf{R}^d$. Now we need a generating function that lists all function values of f of the lattice points in a polyhedron $\mathcal{P}' \subset \mathbf{R}^{d+1}$:

$$g_{\mathcal{P}'}(\mathbf{z}) := \sum_{\mathbf{m}\in\mathbf{Z}^{d+1}} f_{\mathcal{P}'}(\mathbf{m})\, \mathbf{z}^{\mathbf{m}}.$$

Here we are using the multivariate notation $\mathbf{z}^{\mathbf{m}} = z_1^{m_1} z_2^{m_2} \cdots z_{d+1}^{m_{d+1}}$. As a function of $\mathcal{P}'$, $g_{\mathcal{P}'}(\mathbf{z})$ is totally additive as long as the involved polyhedra have no facets parallel to H. We have now set the stage to use the machinery of [3, Chapter 3], which in fact goes back to Ehrhart's original ideas. The cone over Δ can be tiled with translates of the parallelepiped

$$\Pi := \{\lambda_1(\mathbf{v}_1, 1) + \lambda_2(\mathbf{v}_2, 1) + \cdots + \lambda_{d+1}(\mathbf{v}_{d+1}, 1) \mid 0 \le \lambda_1, \lambda_2, \ldots, \lambda_{d+1} < 1\}, \quad (8)$$

by nonnegative integer combinations $\sum n_i \mathbf{v}_i$. The total additivity of $g_{\mathcal{P}'}(\mathbf{z})$ implies that

$$g_{\mathrm{cone}(\Delta)}(\mathbf{z}) = \frac{g_\Pi(\mathbf{z})}{\left(1 - \mathbf{z}^{(\mathbf{v}_1,1)}\right)\left(1 - \mathbf{z}^{(\mathbf{v}_2,1)}\right)\cdots\left(1 - \mathbf{z}^{(\mathbf{v}_{d+1},1)}\right)}.$$

Setting all but the last variable equal to 1 gives the generating function of the solid-angle polynomial of Δ:

$$\mathrm{Solid}_\Delta(z) = g_{\mathrm{cone}(\Delta)}(1,1,\ldots,1,z) = \frac{g_\Pi(1,1,\ldots,1,z)}{(1-z)^{d+1}}.$$

The coefficient of z^k in the polynomial $g_\Pi(1,1,\ldots,1,z)$ records the solid-angle sum of the points in $\Pi \cap \{\mathbf{x} \in \mathbf{Z}^{d+1} \mid x_{d+1} = k\}$, which is positive for $1 \le k \le d$ and 0 for $k = 0$. $\square$

It is tempting to conjecture that the coefficients of the numerator polynomial form a unimodal sequence because of the palindromy, but this turns out not to be the case. For example, let Δ be a lattice 3-simplex whose only integer points are its vertices, and let S be the sum of the solid angles at the vertices of Δ. Then

$$A_\Delta(t) = \frac{1}{6}t^3 + \left(S - \frac{1}{6}\right)t,$$

so

$$\mathrm{Solid}_\Delta(z) = \frac{Sz^3 + (1-2S)z^2 + Sz}{(1-z)^4}.$$

If the numerator polynomial is unimodal, then $1 - 2S \ge S$, which implies $S \le \frac{1}{3}$. In Section 4, we gave a class of simplices for all dimensions whose only integer points are its vertices and whose solid-angle sum S converges to $\frac{1}{2}$. A similar computation in dimension 4 shows that $S > \frac{1}{4}$ means that Δ does not have a unimodal numerator polynomial.

It would be interesting to find nice conditions for when the numerator polynomial is unimodal, however. In dimension 3, if $\mathrm{vol}(\mathcal{P})t^3 + ct$ is the solid-angle polynomial of a polytope $\mathcal{P}$, then the numerator polynomial is

$$(\mathrm{vol}(\mathcal{P}) + c)z^3 + (4\,\mathrm{vol}(\mathcal{P}) - 2c)z^2 + (\mathrm{vol}(\mathcal{P}) + c)z,$$

so is unimodal if and only if $c \le \mathrm{vol}(\mathcal{P})$. Proposition 1 gives an infinite family of 3-polytopes whose solid-angle polynomial has $c < 0$, which at least gives some examples.

6. Nonnegativity results for valuation generating functions

Proof of Theorem 3. We shall be able to use the same philosophy as in the proof of Theorem 2, but there are some intricate refinements necessary. Suppose $\mathcal{P} = \mathrm{conv}\,\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subset \mathbf{R}^d$ and define $\mathrm{cone}(\mathcal{P}) \subset \mathbf{R}^{d+1}$ as in (7). We can triangulate $\mathrm{cone}(\mathcal{P})$ into simplicial rational cones $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_m$, each of whose $d+1$ generators are among the generators $(\mathbf{v}_i, 1)$ of $\mathrm{cone}(\mathcal{P})$. Now we use a trick from [4]: there exists a vector $\mathbf{v} \in \mathbf{R}^{d+1}$ such that

$$\mathrm{cone}(\mathcal{P}) \cap \mathbf{Z}^{d+1} \ = \ (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbf{Z}^{d+1}$$

and no facet of any cone $\mathbf{v} + \mathcal{K}_j$ contains any integral points. Thus every integral point in $\mathbf{v} + \mathrm{cone}(\mathcal{P})$ belongs to exactly one simplicial cone $\mathbf{v} + \mathcal{K}_j$, and we have

$$\mathrm{cone}(\mathcal{P}) \cap \mathbf{Z}^{d+1} \ = \ (\mathbf{v} + \mathrm{cone}(\mathcal{P})) \cap \mathbf{Z}^{d+1} \ = \ \coprod_{j=1}^{m} \left((\mathbf{v} + \mathcal{K}_j) \cap \mathbf{Z}^{d+1} \right), \qquad (9)$$

where $\coprod$ denotes disjoint union. We now extend the valuation ν naturally by setting $\nu(\mathrm{cone}(\mathcal{P}), \mathbf{x}) = \nu(\mathrm{cone}(\mathcal{P}) \cap H, \mathbf{x})$, where H is a hyperplane through $\mathbf{x}$ with normal vector $(0, 0, \ldots, 0, 1)$. Let

$$\sigma_K(\mathbf{z}) := \sum_{\mathbf{m} \in K \cap \mathbf{Z}^{d+1}} \nu(K, \mathbf{m})\, \mathbf{z}^{\mathbf{m}}.$$

Then (9) gives rise to the following identity of generating functions:

$$\sigma_{\mathrm{cone}(\mathcal{P})}\left(\mathbf{z}\right) \ = \ \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}\left(\mathbf{z}\right),$$

which, in turn, allows us to compute

$$G_{\mathcal{P}}(z) = \sum_{t \ge 0} N_{\mathcal{P}}(t)\, z^t = \sigma_{\mathrm{cone}(\mathcal{P})}\left(1, 1, \ldots, 1, z\right) = \sum_{j=1}^{m} \sigma_{\mathbf{v}+\mathcal{K}_j}\left(1, 1, \ldots, 1, z\right).$$

So it suffices to show that the rational generating functions $\sigma_{\mathbf{v}+\mathcal{K}_j}(1,1,\ldots,1,z)$ for the simplicial cones $\mathbf{v}+\mathcal{K}_j$ have nonnegative numerators (and denominators of the form $(1-z)^{d+1}$). For the simplicial cones, we can employ the same argument as in (8) (and this is where we need the valuation to be translation-invariant). As in the solid-angle case, the coefficient of z^k in the numerator of $\sigma_{\mathbf{v}+\mathcal{K}_j}(1,1,\ldots,1,z)$ records the sum of the valuation at the points in a parallelepiped at height $x_{d+1} = k$, which is nonnegative. □

Proof of Theorem 4. The result follows from the last sentence from the previous proof and the fact that

$$\nu(\mathcal{Q},\mathbf{v}) \le \nu(\mathcal{P},\mathbf{v})$$

for all points $\mathbf{v}$, which follows from the additivity

$$\nu(\mathcal{Q},\mathbf{v}) = \nu(\mathcal{P},\mathbf{v}) + \nu(\mathcal{Q}\setminus\mathcal{P},\mathbf{v})$$

because $\nu(\varnothing,\mathbf{v}) = 0$ for all valuations. □

7. Some remarks on period collapse

Suppose $\mathcal{P}$ is a rational polytope. Then $A_{\mathcal{P}}(t)$ is a quasipolynomial, whose period divides the least common multiple of the denominators of the vertex coordinates of $\mathcal{P}$. For a generic polytope, this number is the period of $A_{\mathcal{P}}(t)$. However, in analogy with Ehrhart quasipolynomials, we expect that there are special classes of polytopes that exhibit period collapse, i.e., $A_{\mathcal{P}}(t)$ is a polynomial. Here is one such family:

Proposition 12. *The polytope $\mathcal{P} = [0,\frac{1}{2}] \times [0,1]^{d-1}$ has solid-angle polynomial $\frac{1}{2}t^d$.*

Proof. As seen in the proof of Proposition 8, the solid-angle polynomial of the unit cube is t^d. In fact, $\mathcal{P}$ and $\mathcal{P}' = [\frac{1}{2},1] \times [0,1]^{d-1}$ are congruent to one another, so have the same solid-angle enumerator. Since their union is the unit cube, we conclude the desired result by additivity. □

We can completely classify period collapse in dimension 1. Let $\mathcal{P} = [a,b]$. There are two cases to consider. In the first case, $a \in \mathbf{Z}$ and $b \in \mathbf{Q}\setminus\mathbf{Z}$. Let $f_0,\ldots,f_{p-1}$ be the constituent polynomials of $A_{\mathcal{P}}(t)$; that is, $A_{\mathcal{P}}(t) = f_j(t)$ if $t \equiv j \bmod p$. Then $f_0 = (b-a)t$, so in order for $A_{\mathcal{P}}(t)$ to be a polynomial, we need that $A_{\mathcal{P}}(1) = b-a$. We know that $A_{\mathcal{P}}(1) = \lfloor b \rfloor - a + \frac{1}{2}$, so we conclude $b = \lfloor b \rfloor + \frac{1}{2}$, and thus $\mathcal{P}$ must have denominator 2. If both a and b are nonintegral, then $A_{\mathcal{P}}(1) = \lfloor b \rfloor - \lceil a \rceil + 1$, so for period collapse to occur, this must be equal to $b-a$. This means that $\mathcal{P}$ is of the form $[\frac{p}{q}, \frac{p}{q}+n]$ where $n \in \mathbf{Z}$. For $t \nmid q$, there is a bijection between points of $t\mathcal{P}$ and the points of $(0,tn]$, so $A_{\mathcal{P}}(t) = nt$ is given by adding $\frac{p}{q}$ to $(0,tn]$. We summarize these results.

Proposition 13. *Let $\mathcal{P} = [a, b]$ be a 1-dimensional rational polytope. Then $A_{\mathcal{P}}(t)$ is a polynomial if and only if one of the following holds:*

(1) $a \in \mathbf{Z}, 2b \in \mathbf{Z}$ *or* $2a \in \mathbf{Z}, b \in \mathbf{Z}$

(2) $b - a \in \mathbf{Z}$.

In higher dimensions we can at least say the following. For $0 \le j \le d = \dim \mathcal{P}$, define the *$j$-index* of $\mathcal{P}$ to be the minimal positive integer p_j such that the affine span of each j-dimensional face of $\mathcal{P}$ contains an integer point. Then the p_j give bounds on the periods of coefficient functions of $A_{\mathcal{P}}(t)$.

Proposition 14. *Given a rational d-polytope $\mathcal{P}$ with j-index p_j, write the solid-angle polynomial as $A_{\mathcal{P}}(t) = c_d(t)t^d + c_{d-2}(t)t^{d-2} + \cdots + c_1(t)t + c_0(t)$ (where $c_1(t) = 0$ if d is even, and $c_0(t) = 0$ if d is odd). Then the minimum period of $c_j(t)$ divides p_j for all j.*

Proof. The j-index of a face of $\mathcal{P}$ divides p_j, so this follows from the identity (6) and the fact that the corresponding theorem for Ehrhart polynomials holds (see [17, Theorem 6]). □

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