High Dimensional Expansion Implies Amplified Local Testability

Tali Kaufman ✉
Department of Computer Science, Bar-Ilan University, Ramat Gan, Israel

Izhar Oppenheim ✉
Department of Mathematics, Ben-Gurion University of the Negev, Be'er-Sheva, Israel

Abstract

In this work, we define a notion of local testability of codes that is strictly stronger than the basic one (studied e.g., by recent works on high rate LTCs), and we term it *amplified local testability*. Amplified local testability is a notion close to the result of optimal testing for Reed-Muller codes achieved by Bhattacharyya et al.

We present a scheme to get amplified locally testable codes from high dimensional expanders. We show that single orbit Affine invariant codes, and in particular Reed-Muller codes, can be described via our scheme, and hence are amplified locally testable. This gives the strongest currently known testability result of single orbit affine invariant codes, strengthening the celebrated result of Kaufman and Sudan.

2012 ACM Subject Classification Theory of computation → Expander graphs and randomness extractors; Theory of computation → Error-correcting codes

Keywords and phrases Locally testable codes, High dimensional expanders, Amplified testing

Digital Object Identifier 10.4230/LIPIcs.APPROX/RANDOM.2022.5

Category RANDOM

Related Version *Full Version*: https://arxiv.org/abs/2107.10488

Funding *Tali Kaufman*: This work was partially funded by ERC grant no. 336283 and BSF grant no. 2012256.
Izhar Oppenheim: This work was partially funded by ISF grant no. 293/18.

1 Introduction

High dimensional expansion implies amplified local testability

The aim of this work is to show that codes arising from high dimensional expanding set systems have a strong notion of local testability, which we call amplified locally testability (see exact definition below). Specifically, we define the notion of *High Dimensional Expanding System (*HDE-System*)* that is a two layer expanding set system that generalizes two layer set systems arising from high dimensional expanders. Using this new concept, we show that codes whose constraints form an HDE-System are amplified locally testable.

Testability of well studied codes via high dimensional expansion

We further show that most well studied locally testable codes such as Reed-Muller codes and more generally affine-invariant codes are, in fact, HDE-System codes. Hence, their (amplified) local testability could be re-inferred using our current work; and could be attributed to the high dimensional expansion phenomenon. Specifically, we give a high dimension expansion based proof to the amplified local testability of single orbit affine invariant codes, that strengthen the well known result of Kaufman and Sudan [10].

Amplified local testability

In our work, we study a strong notion of local testability for a family of locally testable codes and show that this strong testing property is holds for HDE-System codes.

In order to better explain the notion of amplified local testability, we recall the following formulation of locally testable codes:

▶ **Definition 1** (Locally testable code). *Given a linear code $C \subseteq \mathbb{F}_p^V$ defined by a set $\mathcal{E}_C$ of k-query tests (i.e., tests that query k-bits of the codeword), define* $\mathrm{rej} : \mathbb{F}_p^V \to [0,1]$ *where* $\mathrm{rej}(\underline{c})$ *is the fraction of k-query tests that $\underline{c}$ fails (by definition, $\underline{c} \in C$ if and only if* $\mathrm{rej}(\underline{c}) = 0$*). We refer to querying an equation in $\mathcal{E}_C$ as the* basic test *of the code.*

Let $\mathcal{C}$ be a sequence of codes such that there is $k = k(\mathcal{C})$ such that every $C \in \mathcal{C}$ is defined by a set $\mathcal{E}_C$ of k-query tests. We say that a family of linear codes $\mathcal{C}$ is locally testable *if there is a constant $r_{\mathcal{C}} > 0$ such that for every $C \in \mathcal{C}$ the following robustness property holds: For every $\underline{c} \in \mathbb{F}_p^V$,*

$$\mathrm{rej}(\underline{c}) \geq r_{\mathcal{C}} \min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\|,$$

where $\|\underline{c} - \underline{c}'\|$ is the fraction of the bits in which $\underline{c}$ and $\underline{c}'$ differ.

Note that in the above definition the number of bits queried in the basic test for a code $C \in \mathcal{C}$ is constant (independent of C) and thus one does not care if $r_{\mathcal{C}}$ depends on k. For instance in the recent celebrated work of Dinur at el. [4], the basic test queries k bits, and $r_{\mathcal{C}}$ is of the order of $\frac{1}{\sqrt{k}}$ (see [4, Theorem 4.5]).

However, a sequence of locally testable codes is usually a part of a larger family of codes where k does vary. The motivating example is (binary) Reed-Muller codes $\mathrm{RM}(d,n)$, where d is the degree of the polynomial and n is the number of variables. When fixing d and considering the sequence of codes where n tends to infinity, it is a classical result that this sequence is a locally testable code with the number of bits queried in the basic test is $k = 2^d$ (see [1]). If we consider the family of codes $\mathrm{RM}(d,n)$ where d also varies (and n is large enough with respect to d), we get a larger family of codes in which $k(C)$ is no longer constant. For this example of binary Reed-Muller codes, Bhattacharyya at el. [2] proved a striking result of optimal testing for binary Reed-Muller codes:

▶ **Theorem 2** ([2, Theorem 1]). *Let*

$$\mathcal{C} = \{\mathrm{RM}(d,n) : d \in \mathbb{N}, d \geq 2, n \in \mathbb{N} \text{ sufficiently large with respect to } d\}$$

be the family of all binary Reed-Muller codes. There is a constant $r_{\mathrm{RM}} > 0$ such that for every $C = \mathrm{RM}(d,n) \in \mathcal{C}$ with $k(C) = 2^d$, it holds for every $\underline{c} \in \mathbb{F}_2^{\mathbb{F}_2^n}$, that

$$\mathrm{rej}(\underline{c}) \geq k(C) r_{\mathrm{RM}} \min\{\min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\|, \frac{1}{k(C)}\}.$$

In other words, [2] show that the family of all binary Reed-Muller codes is locally testable even when changing the degree! This Theorem can be interpreted as follows: As noted above, for every $\mathrm{RM}(d,n) \in \mathcal{C}$, $k(\mathrm{RM}(d,n)) = 2^d$. Thus the above theorem states that for every $\underline{c} \in \mathbb{F}_2^{\mathbb{F}_2^n}$, if $\min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\| \leq \frac{1}{2^d}$, then

$$\mathrm{rej}(\underline{c}) \geq 2^d r_{\mathrm{RM}} \min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\|$$

and if $\min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\| \geq \frac{1}{2^d}$, then

$$\mathrm{rej}(\underline{c}) \geq r_{\mathrm{RM}} \geq r_{\mathrm{RM}} \min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\|.$$

Motivated by the above result, we define amplified local testability as a relaxation of optimal testability:

▶ **Definition 3** (Amplified locally testable codes). *Let $\mathcal{C}$ be a family of codes such that every $C \in \mathcal{C}$ is defined by a set $\mathcal{E}_C$ of $k(C)$-query (basic) tests. We say that a family of linear codes $\mathcal{C}$ is* amplified locally testable *if there are constants $t_{\mathcal{C}} \geq 1$ and $r_{\mathcal{C}} > 0$ such that for every $C \in \mathcal{C}$ the following robustness property holds: For every $\underline{c} \in \mathbb{F}_p^V$,*

$$\text{rej}(\underline{c}) \geq k(C) r_{\mathcal{C}} \min \left\{ \min_{\underline{c}' \in C} \| \underline{c} - \underline{c}' \|, \frac{1}{(k(C))^{t_{\mathcal{C}}}} \right\}.$$

▶ Remark 4 (Role of $t_{\mathcal{C}}$). The best we can hope for amplified local testing is $t_{\mathcal{C}} = 1$. If this happens, then the family has optimal local testability as in the result of [2]. Our methods below do not give optimal local testability, but only amplified local testability with $t_{\mathcal{C}} = 3$.

▶ Remark 5. A similar relaxation of optimal testability was studied for lifted codes by Haramaty, Ron-Zewi and Sudan in [6].

In this work we show that a code which can be described via HDE-System, not only we can infer local testability for it, but rather we can infer amplified local testability for it. As already noted above, this is not the case of the the analysis of the family of codes of [4]: In [4] the basic test samples k-bits and $r_{\mathcal{C}}$ (in the notation of Definition 1 above) behaves like $\frac{1}{\sqrt{k}}$ and thus decreases as k increases.

By applying our machinery to single orbit affine invariant codes, we get that these codes are amplified locally testable, which is the strongest notion of testability known for these codes, strengthening the well known work of Kaufman and Sudan [10].

Local testability of single orbit affine invariant codes via HDE-System

In the following we refer to single orbit affine invariant codes which were shown to be locally testable by the Kaufman-Sudan work [10]. These codes contain the well known Reed-Muller codes. We show that they are HDE-System codes, so their local testability is implied by our current work. Kaufman and Sudan have shown that single orbit affine invariant codes which are characterized by k-weight constraints that form a single orbit are locally testable. We will show that the Kaufman-Sudan requirement allows to show that single orbit affine invariant codes are modelled over HDE-System and thus are amplified locally testable.

▶ **Theorem 6** (Testability of single orbit affine invariant codes – informal, for formal, see the related full version of this paper). *Let $\mathcal{C}_{\text{affine-inv},p}$ be the family of all single orbit affine invariant codes $C \subseteq \mathbb{F}_p^{\mathbb{K}(C)^{n(C)}}$ with*

$$|\mathbb{K}(C)|^{n(C)} \geq 2^{11} p^2 (k(C))^4,$$

where $k(C)$ is the size of the support of the constraint defining C. Then the family of all these codes is amplified locally testable. Explicitly, for every $C \in \mathcal{C}_{\text{affine-inv},p}$ and every $\underline{c} \in \mathbb{F}_p^{\mathbb{K}(C)^{n(C)}}$ it holds that

$$\text{rej}(\underline{c}) \geq k(C) \frac{1}{2^{15} p^4} \min \left\{ \min_{\underline{c}' \in C} \| \underline{c} - \underline{c}' \|, \frac{1}{k(C)^3} \right\}.$$

We compare this result to the (non-amplified) local testing for affine invariant codes of Kaufman and Sudan [10, Theorem 2.9] who showed the following:

▶ **Theorem 7** ([10, Theorem 2.9]). *For every $C \in \mathcal{C}_{\textit{affine-inv},p}$ it holds that*

$$\mathrm{rej}(\underline{c}) \geq \frac{1}{2} \min \left\{ \min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\|, \frac{1}{k(C)^2} \right\}.$$

Our Theorem and [10, Theorem 2.9] both give a rejection of $\Omega(\frac{1}{k(C)^2})$ when $\min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\|$ is large. However, when $\min_{\underline{c}' \in C} \|\underline{c} - \underline{c}'\| << \frac{1}{k(C)^3}$, and $k(C)$ is large, our result gives a much better rejection rate.

Local testability via unique neighbor expansion

We show that λ-expanding HDE-System has some form of unique neighbor expansion property associated with it. We also show that if the HDE-system has a strong enough unique neighbor expansion property, then a linear code defined based on this system is amplified locally testable. We prove that this is the case for affine-invariant codes with the single orbit property. Thus, HDE-system provides a mechanism to get amplified local testability of codes.

2 Comparison to prior works

We already mentioned the celebrated work of Dinur at el. [4] that uses ideas from high dimensional expansion to construct locally testable codes with constant rate, distance and locality. As noted above, our work is in a different direction and achieves different goals (we do not achieve the result of [4], but do achieve amplified local testability).

Another work that seems superficially close to the methods of this paper is the recent work of Dikstein at el. [3] that also relies on ideas from high dimensional expansion to deduce local testability. The reader should note that there are major difference between the works:

- Our work has the benefit of deducing not only local testability, but rather amplified local testability which was not achieved in [3].
- As far as we know, the work of [3] does not apply to the family of affine invariant codes, but only to a sub-family of lifted codes. Thus, in terms of generality, our work seems to apply in a more general setting.
- The work of Dikstein at el. [3] relies on the idea that "global" local testability can be inferred from "local" local testability. I.e., in [3], the assumption is that the code contains many small (i.e., "local") locally testable codes and by expansion considerations, it follows that the global code is locally testable. This is also the point of view of [7, 5, 8] that considered what can be thought of as "co-cycle codes" and the global testability was derived assuming they are composed of small local codes that are locally testable (aka "the links" code). In contrast to [3] (and to [7, 5, 8]), the focus of this current work is to get local testability of codes *directly* from high dimensional expansion phenomenon. Deducing local testability of codes directly from high dimensional expansion (without relying on any local code that is locally testable) is new and is achieved here for the first time.

It is also beneficial to compare the results of this paper to previous results regarding single orbit affine invariant codes. In [10, Theorem 2.9], it was shown that single orbit affine invariant codes are locally testable. Using our new machinery, we improve on this result, showing the the family of all single orbit affine invariant codes has amplified local testability. As noted above, a stronger result was known for Reed-Muller codes (which is a sub-family of the family of affine invariant codes), but, prior to our work, no general treatment was available to the entire family of single orbit affine invariant codes.

3 High Dimensional Expanding System (HDE-System)

Our main definition towards defining high dimensional expander codes is called High-Dimensional-Expanding-System or HDE-System for short.

We start by defining a (s, k, K)-Two layer system:

▶ **Definition 8** ((s, k, K)-Two layer system)**.** *A two layer system X is a system $X = (V, E, T)$ of three sets:*

1. *A finite set V whose elements are called vertices.*
2. *A set $E \subseteq 2^V$ such that $|\tau| = k$ for every $\tau \in E$ and $\bigcup_{\tau \in E} \tau = V$.*
3. *A set $T \subseteq 2^E$ such that $|\sigma| = K$ for every $\sigma \in T$ and $\bigcup_{\sigma \in T} \sigma = E$.*
4. *By abuse of notation, we will denote $v \in \sigma$ for $v \in V, \sigma \in T$ if there is $\tau \in \sigma$ such that $v \in \tau$. Using this notation, for every $\sigma \in T$ and every $v \in \sigma$,*

$$2 \leq |\{\tau \in \sigma : v \in \tau\}| \leq s.$$

Roughly speaking, HDE-System is a two layer system with good expansion properties. In order to give the definition, we need to define several graphs associated with a two layer system. We note that all the graphs defined below will be actually considered as weighted graphs with a weight function induced by weights on T, but in the introduction we suppress this fact in order to keep things simple.

▶ **Definition 9** (The ground graph)**.** *For a two layer system $X = (V, E, T)$, the ground graph of X is the graph whose vertices are V and edges are $\{\{v, u\} : \exists \tau \in E, u, v \in \tau\}$.*

▶ **Definition 10** (Link of a vertex)**.** *For a two layer system $X = (V, E, T)$ and $v \in V$, the link of v is the graph whose vertex set is $E_v = \{\tau \in E : v \in \tau\}$ and whose edge set is*

$$T_v = \{\{\tau, \tau'\} : \tau \neq \tau' \text{ and } \exists \sigma \in T \text{ such that } \tau, \tau' \in \sigma\}.$$

▶ **Definition 11** (The non-intersecting graph)**.** *For a two layer system $X = (V, E, T)$, the non-intersecting graph of X is a graph whose vertex set is E and edge set is*

$$\{\{\tau, \tau'\} : \tau \cap \tau' = \emptyset \text{ and } \exists \sigma \in T, \text{ such that } \tau, \tau' \in \sigma\}.$$

This graph corresponds to the Non-Intersecting Walk*, i.e., to the walk from a between elements of E that* do NOT intersect *(as subsets of V) via a T element that contains both of them.*

An HDE-System is a two layer system X in which all these graphs are expanding. More precisely, for $0 \leq \lambda < 1$, we call a (weighted) graph G a λ-expander if it is connected and either the second largest eigenvalue of the is $\leq \lambda$ or (which is less restrictive) its (generalized) Cheeger constant is $\geq 1 - \lambda$ (see related full version of this paper for exact definition).

▶ **Definition 12** (High Dimensional Expanding System (HDE-System) – informal, for formal see the related full version of this paper)**.** *For $0 \leq \lambda < 1$, a (weighted) two layer system $X = (V, E, T)$ is called λ-expanding-*HDE-System *if the ground graph and the links of all the vertices are λ-expanders and the non-intersecting graph is either totally disconnected (i.e., it has no edges) or a λ-expander.*

High Dimensional expanders imply HDE-System

Part of our motivation for the Definition of HDE-systems is to mimic the definition of high dimensional expanders based on simplicial complexes (called λ-local spectral expander – see [9, Definitions 2,3]). The simplest example is when Y is a 2-dimensional simplicial complex. In this case, we define a two layer system $X = (V, E, T)$ as follows: V is the vertex set of Y, E is the edge set of Y and T is the sets of triples of edges that form a triangle in Y. We note that in this case the parameters of X are $s = 2, k = 2, K = 3$. Note that the ground graph is the 1-skeleton of Y, the link of each vertex in X is the link in the simplicial complex and the non-intersecting graph is totally disconnected (since every two edges that are in the same triangle share a vertex). Thus, by definition if Y is a λ-local spectral expander, then the 1-skeleton of Y and all the links of Y are λ-expanders and it follows that X is λ-expanding.

Expanding HDE's have unique neighbor expansion for small sets that are also locally small

Our main motivation for the definition of HDE-System is the ability to deduce unique neighbor expansion theorem from them, for "small" sets that are also "locally small". This unique neighbor expansion theorem that we state below will play a major role in proving local testability based on HDE-System.

In order to state this Theorem, we will need the following definition:

▶ **Definition 13** (δ-Locally-small set – informal, for formal see the related full version of this paper)**.** *Let $X = (V, E, T)$ be a two layer system and let $A \subseteq E$ be a non-empty set. For a vertex $v \in V$, define $A_v = \{\tau \in A : v \in \tau\}$. For a constant $0 \leq \delta < 1$, a vertex v is called δ-small if the size of A_v in the link of v (when accounting for the weight function on the link) is smaller than δ fraction of the size of E_v. Vertices that are not δ-small are called δ-large. A set $A \subseteq E$ is called* δ-locally small*, if the fraction of its mass that is distributed on vertices that are δ-large is negligible with respect to the total mass of A.*

Following we define a notion of unique neighbor expansion that applies for small sets that are also δ-locally small.

▶ **Definition 14** (Unique neighbor expansion property – informal, for formal see the related full version of this paper)**.** *We say that $A \subset E$ has a unique neighbor expansion into T if there exists $\sigma \in T$ that contains exactly one k-set from A. Let $X = (V, E, T)$ be a two layer system and let $A \subseteq E$ be a non-empty set. For constants $\varepsilon_0 > 0$, $\delta > 0$, we say that X has (δ, ε_0)-*unique neighbor expansion property *if for every non-empty set $A \subseteq E$ and every $\varepsilon < \varepsilon_0$ if A ε-small (i.e., its mass is at most a ε-fraction of the total mass of E) and δ-locally small, then A has unique neighbor expansion into T.*

▶ **Theorem 15** (**Main Theorem 1:** Unique neighbor expansion property for HDE-System – informal, for formal see the related full version of this paper)**.** *Given a λ-expanding* HDE-System *X, with λ sufficiently small, there are $\delta > 0$ and $\varepsilon_0 > 0$ such that X has the (δ, ε_0)-unique neighbor expansion property. Moreover, if $s = 2$, then $\delta \to 1$ as $\lambda \to 0$.*

On the ability to get unique neighbor expansion from HDE-systems

The idea behind the proof of Main Theorem 1 is to use the expansion of the links in order to derive unique neighbor expansion. The links are very good expanders so a set that is locally small has the property that its local views in the links expand a lot. Each link induces by its local view many "potential unique neighbors". However, it could be that the local views of the links will interfere and the "potential unique neighbors" by the "links opinion" will turn out to be non unique neighbors. Since the system is expanding the total interference between links is small and thus the overall unique neighbor property is implied.

4 HDE-System Codes

Given a two layer system $X = (V, E, T)$ as above, we want to use it as a "foundation" and for constructing a code. Such a construction is not unique and cannot be done for every X. However, for a code that "could be constructed via X", its testability could be inferred from the expansion properties of X.

Before describing this construction, we need to establish some terminology and notation: Let $C \subseteq \mathbb{F}_p^V$ be a linear code (where V is a finite set) with a check matrix H.

- We denote by $\mathcal{E} = \mathcal{E}(H)$ the rows H and we refer to $\mathcal{E}$ as the constraints of the code (or k-constraints if they all have a support of size k – see below). Thus, $\mathcal{E}$ are $1 \times n$ vectors and for $\underline{c} \in \mathbb{F}_p^V$, $\underline{c} \in C$ if and only if for every $\underline{e} \in \mathcal{E}$, $\underline{e} \cdot \underline{c} = 0$ (recall that $\underline{e}, \underline{c}$ are indexed by the elements in V, thus $\underline{e} \cdot \underline{c} = \sum_v \underline{e}(v)\underline{c}(v)$).
- For $\underline{e} \in \mathcal{E}$, we define the support of $\underline{e}$ as

$$\text{supp}(\underline{e}) = \{v \in V : \underline{e}(v) \neq 0\}.$$

- A *linear dependency* of $\mathcal{E}$ is a function $\text{ld} : \mathcal{E} \to \mathbb{F}_p$ such that for every $\underline{c} \in \mathbb{F}_p^V$, $\sum_{\underline{e} \in \mathcal{E}} \text{ld}(\underline{e})(\underline{e} \cdot \underline{c}) = 0$. In other words, if we think of the row vector $\underline{\text{ld}} = (\text{ld}(\underline{e}))_{\underline{e} \in \mathcal{E}}$, then $\underline{\text{ld}}H = \underline{0}$. As above, the support of ld is the set

$$\text{supp}(\text{ld}) = \{\underline{e} \in \mathcal{E} : \text{ld}(\underline{e}) \neq 0\}.$$

▶ **Example 16.** Consider $C \subseteq \mathbb{F}_2^V$, $V = \{v_1, v_2\}$ given by the parity check matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

If $\underline{e}_i$ denotes the i-th row of H, then $\text{ld} : \{\underline{e}_1, \underline{e}_2, \underline{e}_3\} \to \mathbb{F}_2$ defined by

$$\text{ld}(\underline{e}_i) = 1, \forall i = 1, 2, 3,$$

is a linear dependency. Indeed,

$$\underline{\text{ld}} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix},$$

and one can verify that $\underline{\text{ld}}H = \underline{0}$.

▶ **Definition 17** (Code modelled over a two layer system)**.** *Let $X = (V, E, T)$ be a two layer system. A code C is said to be modelled over X if the following holds:*

- *There is a prime power p such that $C \subseteq \mathbb{F}_p^V$.*
- *There is a check matrix H and $\mathcal{E} = \mathcal{E}(H)$ such that*

$$E = \{\text{supp}(\underline{e}) : \underline{e} \in \mathcal{E}\},$$

and such that for every $\underline{e}_1, \underline{e}_2 \in \mathcal{E}$, if $\underline{e}_1 \neq \underline{e}_2$, then $\text{supp}(\underline{e}_1) \neq \text{supp}(\underline{e}_2)$. In other words, there is a bijection $\Phi : \mathcal{E} \to E$ given by $\Phi(\underline{e}) = \text{supp}(\underline{e})$. Note that under this assumption, the size of the support of all the constraints is k (the constant of the system X) and we refer to the elements of $\mathcal{E}$ as the k-constraints of the code, when there is no chance for ambiguity.

- *There is a set $\mathcal{T}$ of linear dependencies such that*

$$T = \{\{\text{supp}(\underline{e}) : \underline{e} \in \text{supp}(\text{ld})\} : \text{ld} \in \mathcal{T}\}.$$

▶ **Example 18.** Let $X = (V, E, T)$ the following two layer system: $V = \{v_1, v_2, v_3\}$, $E = \{\tau_{i,j} = \{v_i, v_j\} : 1 \leq i < j \leq 3\}$ and $T = \{\sigma = \{\tau_{i,j} : 1 \leq i < j \leq 3\}\}$. Then for every prime power p, we can define a code $C \subseteq \mathbb{F}_p^V$ modelled over X as follows: define the check matrix of the code to be

$$H = \begin{pmatrix} 1 & p-1 & 0 \\ 0 & 1 & p-1 \\ p-1 & 0 & 1 \end{pmatrix}.$$

One can see that for this matrix the support of the i-the row is $\{v_i, v_{i+1 \mod 3}\} \in E$ and that no two rows have the same support. Further define a linear dependency $\text{ld} : \mathcal{E} \to \mathbb{F}_p$ to be the constant function 1, thus one can verify that the support of ld is $\sigma \in T$ and that this is indeed a linear dependency.

Our motivation for considering codes modelled over two layer system is the following

▶ **Theorem 19** (**Main Theorem 2:** Codes modelled over two layer systems with unique neighbor property are amplified locally testable – informal, for formal see the related full version of this paper)**.** *For every p prime, $t' \in \mathbb{N}, t' > 0$, $\mu > 0$ and $\delta > \frac{p-1}{p}$, let $\mathcal{C}(\delta, p, t', \mu)$ be the family of p-ary codes (i.e., codes of the form $C \subseteq \mathbb{F}_p^{V(C)}$) modelled over two layer systems such that*

$$\mathcal{C}(\delta, p, t', \mu) = \left\{ C : \exists \varepsilon_0(C) > 0 \text{ such that } C \text{ has the } (\delta, \varepsilon_0(C))\text{-unique neighbor property and } \varepsilon_0 \geq \frac{\mu}{k(C)^{t'}} \right\}.$$

Then the family $\mathcal{C}(\delta, p, t', \mu)$ is amplified locally testable with $t_{\mathcal{C}} = t' + 1$.

On the ability to get amplified local testability from unique neighbor expansion

We will explain how to get amplified local testability from unique neighbor expansion for $p = 2$ (this is to avoid carrying p as a constant) and $\delta = \frac{3}{4}$.

We assume we are in a situation that we have a code that is modelled over an HDE-system. Thus, we know that each k-constraint of the code is participating in a linear dependency. This means that on every dependency, if there is one violated constraint that touches it, there must be another one that touches it.

We are given a vector $\underline{c}$ that falsifies $|A|$ constraints from the code and we want to show that such a vector is close to the code. We can try to correct it by flipping variables such that this flipping reduces the number of violated constraints by a fixed proportion: Assume that each bit is a member of N equations and we change the value of the bit if the number of falsified equations containing it is more than $\frac{3}{4}N$. In this case, flipping the bit corrects many equations (since all the equations that were false are now true and vice-versa): i.e., flipping the bit corrects at least $\frac{1}{2}N$ equations. Let us compute what is the maximal number of steps for such a correcting procedure to stop: we assumed that there were the corrupted code word had $|A|$ falsified equations, i.e., $\text{rej}(\underline{c}) = \frac{|A|}{|E|}$. Thus, the number of bits flipping in the correction procedure is at most $\frac{|A|}{\frac{1}{2}N} = \frac{2|A|}{N}$. and in the end of this procedure, each vertex is $\frac{3}{4}$-locally small.

Assume the HDE-System on which the code is modelled has (δ, ε_0)-unique neighbor expansion and that $\frac{|A|}{|E|} < \varepsilon_0$. The unique neighbor expansion implies that there are linear dependencies that "sees" only one violating constraint. However, as we said, this is not possible. So when we arrived at a situation where no more flipping is possible, we, in fact, arrived at a codeword that is close to our initial vector. Explicitly, the fraction of flipped bits is less or equal to $\frac{\frac{2|A|}{N}}{|V|}$.

Note that if $|V|$ denotes the number of bits, then the number of equations is $|E|= \frac{|V|N}{k}$ (each equation contains k-bits and each bit is a member of N equations). It follows that the number of flipped bits is less or equal to

$$\frac{\frac{2|A|}{N}}{|V|} = \frac{\frac{2|A|}{N}}{|E|}\frac{|E|}{|V|} = \frac{2|E|}{N|V|}\frac{|A|}{|E|} = \frac{2}{k}\operatorname{rej}(\underline{c})$$

and thus

$$\operatorname{rej}(\underline{c}) \geq k\frac{1}{2}\min_{\underline{c}'\in C}\|\underline{c} - \underline{c}'\|$$

as needed.

▶ **Definition 20** (HDE-System Code)**.** *We call a code C as above a* HDE-System*-code if it is modelled over a λ-expanding* HDE-System*.*

Codes that give rise to HDE-System with $s = 2$ are amplified locally testable

By Main Theorem 2, the family $\mathcal{C}_{\delta,p}$ of codes $C \subseteq \mathbb{F}_p^V$ modelled a two layer systems with a $(\delta, \varepsilon_0(C))$-unique neighbor property are locally testable given that $\delta > \frac{p-1}{p}$. We have furthered showed (see Main Theorem 1 above) that given any $\delta < 1$, there is λ sufficiently small such that every λ-expanding HDE-System with $s = 2$ has the (δ, ε_0)-unique neighbor property (where ε_0 depends on the parameters of the HDE-System). Thus, overall we get that the family of all codes $C \subseteq \mathbb{F}_p^V$ modelled over λ-expanding HDE-System with $s = 2$ (and λ sufficiently small) is amplified locally testable.

▶ **Corollary 21** (Codes modelled over expanding-HDE-System with $s = 2$ are amplified locally testable – informal, for formal see the related full version of this paper)**.** *The family of all codes $C \subseteq \mathbb{F}_p^V$ of $k(C)$-constraints modelled over expanding HDE systems with $s = 2$ is amplified locally testable. Moreover, under some mild assumptions (passing to a large sub-family) $t_{\mathcal{C}} = 3$ where $t_{\mathcal{C}}$ is as in Definition 3.*

Above, we stated Theorem 6 regarding amplified local testability of single orbit affine invariant codes. This Theorem is deduced from the above Corollary, because we show that single orbit affine invariant codes are modelled over expanding HDE systems with $s = 2$.

Local testability when $s > 2$

The main focus of this work is proving amplified local testability for codes modelled over HDE-systems with $s = 2$ as all our examples satisfy the $s = 2$ assumption (Reed-Muller codes and single orbit affine invariant codes satisfy $s = 2$). We further have a more general treatment for codes modelled over HDE-system with general $s \geq 3$ under some extra-assumptions (although we currently do not have examples for such codes). Roughly speaking, for the case of $s \geq 3$ we need the extra assumption that the code is composed of local small codes that are locally testable. The difficulty in the case where $s \geq 3$ is that the bit flipping argument we described above can only correct the code to be $\frac{p-1}{p}$-locally small, while "The unique neighbor Theorem" says that we can deduce the (δ, ε_0)-unique neighbor property from expansion given that $\delta < \frac{1}{s-1}$. Thus, in the case where $s \geq 3$, we may not be able correct a corrupted codeword by bit flipping to a setting in which we can apply our unique neighbor argument. This difficulty is dealt by adding the assumption of "local" local testability that grantees that correcting by bit flipping converges to a word that is δ-locally small (and thus we can use our previous machinery). This new method requires some additional definitions and we refer the reader to the related full version of this paper for further details.

Distance of HDE codes

An additional result is that for codes modelled over HDE-systems, the distance of the code can be bounded in terms of the expansion of the HDE system.

References

1. Noga Alon, Tali Kaufman, Michael Krivelevich, Simon Litsyn, and Dana Ron. Testing Reed-Muller codes. *IEEE Trans. Inform. Theory*, 51(11):4032–4039, 2005. doi:10.1109/TIT.2005.856958.
2. Arnab Bhattacharyya, Swastik Kopparty, Grant Schoenebeck, Madhu Sudan, and David Zuckerman. Optimal testing of Reed-Muller codes. In *2010 IEEE 51st Annual Symposium on Foundations of Computer Science—FOCS 2010*, pages 488–497. IEEE Computer Soc., Los Alamitos, CA, 2010.
3. Yotam Dikstein, Irit Dinur, Prahladh Harsha, and Noga Ron-Zewi. Locally testable codes via high-dimensional expanders, 2020. arXiv:2005.01045.
4. Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. Locally testable codes with constant rate, distance, and locality, 2021. arXiv:2111.04808.
5. Shai Evra and Tali Kaufman. Bounded degree cosystolic expanders of every dimension. In *STOC'16—Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*, pages 36–48. ACM, New York, 2016. doi:10.1145/2897518.2897543.
6. Elad Haramaty, Noga Ron-Zewi, and Madhu Sudan. Absolutely sound testing of lifted codes. *Theory Comput.*, 11:299–338, 2015. doi:10.4086/toc.2015.v011a012.
7. Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Ramanujan complexes and bounded degree topological expanders. In *55th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2014*, pages 484–493. IEEE Computer Soc., Los Alamitos, CA, 2014. doi:10.1109/FOCS.2014.58.
8. Tali Kaufman and David Mass. Cosystolic expanders over any abelian group. *Electronic Colloquium on Computational Complexity (ECCC)*, 25:134, 2018. URL: https://eccc.weizmann.ac.il/report/2018/134.
9. Tali Kaufman and Izhar Oppenheim. High order random walks: beyond spectral gap. In *Approximation, randomization, and combinatorial optimization. Algorithms and techniques*, volume 116 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 47, 17. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.
10. Tali Kaufman and Madhu Sudan. Algebraic property testing: the role of invariance. In *STOC'08*, pages 403–412. ACM, New York, 2008. doi:10.1145/1374376.1374434.