

# Limitations of Local Quantum Algorithms on Random MAX- $k$ -XOR and Beyond

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## Abstract

We introduce a notion of *generic local algorithm*, which strictly generalizes existing frameworks of local algorithms such as *factors of i.i.d.* by capturing local *quantum* algorithms such as the Quantum Approximate Optimization Algorithm (QAOA).

Motivated by a question of Farhi et al. [arXiv:1910.08187, 2019], we then show limitations of generic local algorithms including QAOA on random instances of constraint satisfaction problems (CSPs). Specifically, we show that any generic local algorithm whose assignment to a vertex depends only on a local neighborhood with  $o(n)$  other vertices (such as the QAOA at depth less than  $\varepsilon \log(n)$ ) cannot arbitrarily-well approximate boolean CSPs if the problem satisfies a geometric property from statistical physics called the coupled overlap-gap property (OGP) [Chen et al., Annals of Probability, 47(3), 2019]. We show that the random MAX- $k$ -XOR problem has this property when  $k \geq 4$  is even by extending the corresponding result for diluted  $k$ -spin glasses.

Our concentration lemmas confirm a conjecture of Brandao et al. [arXiv:1812.04170, 2018] asserting that the landscape independence of QAOA extends to logarithmic depth – in other words, for every fixed choice of QAOA angle parameters, the algorithm at logarithmic depth performs almost equally well on almost all instances. One of these lemmas is a strengthening of McDiarmid’s inequality, applicable when the random variables have a highly biased distribution, and may be of independent interest.

**2012 ACM Subject Classification** Theory of computation → Randomness, geometry and discrete structures; Mathematics of computing → Probabilistic algorithms; Mathematics of computing → Combinatorics; Theory of computation → Quantum complexity theory

**Keywords and phrases** Quantum Algorithms, Spin Glasses, Hardness of Approximation, Local Algorithms, Concentration Inequalities, Overlap Gap Property

**Digital Object Identifier** 10.4230/LIPIcs.ICALP.2022.41

**Category** Track A: Algorithms, Complexity and Games

**Related Version** *Full Version*: <https://arxiv.org/abs/2108.06049> [11]

**Funding** *Chi-Ning Chou*: Supported by NSF awards CCF 1565264 and CNS 1618026.

*Peter J. Love*: Supported by NSF STAQ award PHY-1818914 and DARPA ONISQ program award HR001120C0068.

*Juspreet Singh Sandhu*: Supported by NSF STAQ award PHY-1818914 and DARPA ONISQ program award HR001120C0068.

*Jonathan Shi*: Supported by European Research Council (ERC) award No. 834861.

**Acknowledgements** We thank Jonathan Wurtz for many insightful discussions about QAOA. We are grateful to Amartya Shankha Biswas for patiently explaining the *factors of i.i.d.* framework to us. We would also like to thank Antares Chen for many invigorating and profound discussions



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49th International Colloquium on Automata, Languages, and Programming (ICALP 2022).

Editors: Mikołaj Bojańczyk, Emanuela Merelli, and David P. Woodruff;

Article No. 41; pp. 41:1–41:20



Leibniz International Proceedings in Informatics

Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



which culminated as the open problem proposed in Problem 5.1. Lastly, we would like to thank Boaz Barak for providing detailed and helpful feedback on a prior version of this manuscript, and David Gamarnik for his explanations on the state of the art results in the research area.

## 1 Introduction

Recent developments [3, 26, 16] of noisy intermediate-scale quantum (NISQ) devices [40] have brought us to the door of near-term quantum computation. As experimentalists can now build programmable quantum simulators up to 256 qubits [16], this motivates an important theoretical question: what computational advantage can such a NISQ device provide?

One of the constraints of NISQ devices is the inability to create high-fidelity global entanglement. This motivates the study of the power of quantum algorithms that are *local*. A leading candidate in this regime of quantum algorithms is the *Quantum Approximate Optimization Algorithm (QAOA)* [19] at shallow depths. While there have been some recent results [29, 5, 33] that formally examine the QAOA algorithm at depth  $p = 1$  or  $2$ , very few results exist for super-constant depth QAOA [17, 18].

Given the imminent quest of demonstrating quantum computational advantage, it is important to clarify for what optimization problems can near-term quantum algorithms (such as *local* quantum algorithms) reliably be expected to demonstrate computational advantage.

We show that local *quantum* algorithms, a large natural class of NISQ algorithms, are obstructed by a geometric property of the solution space known as the *coupled Overlap-Gap Property* [10]. We conjecture that this property is satisfied by most CSPs (Conjecture 5.1). Specific problems known to have this property include the diluted  $k$ -spin glass Hamiltonian (equivalent to a max-cut problem on random  $k$ -hypergraphs) [10], independent set on random graphs [17], planted clique [25], and many other problems that so far seem to elude efficient algorithms and be algorithmically hard [22]. In this manuscript, we also demonstrate that the random Max- $k$ -XOR problem has this property (see subsection 2.4).

Our results lift the continuous coupled interpolation techniques of Chen et al. [10] to the generic quantum-inclusive setting, with stronger and more general concentration of measure statements about local quantum algorithms, extending the techniques of Farhi et al. [17].

Critical to our approach is a new definition of local algorithms we term *generic local algorithms* (See subsection 2.1). Previous work relating statistical-physics-derived OGP to local algorithms leveraged the *factors of i.i.d.* framework for local algorithms, which fails to contain local quantum algorithms, as we demonstrate in Proposition 2.3. Our definition of *generic local algorithms* subsumes local quantum and classical algorithms (see Proposition 2.3 and Proposition 4.3) but still satisfies strong concentration properties (see Theorems 5.3 and 5.4 in the full version), allowing techniques for local classical algorithms [10] to apply to the quantum case. Two of our core technical contributions involve showing that the random MAX- $k$ -XOR problem has a coupled Overlap Gap Property (see subsection 2.4) by extending the techniques of Chen et al. [10] and deriving a strengthened version of McDiarmid's inequality for highly-biased random variables using a martingale argument (see Lemma 2.7).

The rest of the paper is organized as follows: In subsection 1.1 we introduce the relevant spin glass literature, defining the notion of a diluted  $k$ -spin glass; in subsection 1.2 we introduce the relevant prior work; in subsection 1.3 we state our main theorems (informally); in subsection 1.4 we briefly explain the architecture of our proof and compare our techniques with those of Chen et al. [10] and Farhi et al. [17]; in subsection 1.5 we introduce the necessary mathematical preliminaries and notation, including the models of CSPs we work with and a rigorous definition of local classical algorithms; in section 2 we state our separation of different

families of local algorithms and our main concentration lemmas; in section 5 we conclude by summarizing our results and mentioning many natural open problems closely related to and/or motivated by our work. Complete proofs and technical details are delegated to the appendices.

## 1.1 Diluted $k$ -spin glasses, maximum cuts of sparse hypergraphs, and Max- $k$ -XOR

Spin glass theory is a central theoretical framework in statistical physics. The Sherrington-Kirkpatrick model (SK model) [43] is one of the most well studied mathematical models in the theory and consists of two variables: spins  $\{\sigma_i\}_{i \in [n]}$  and interactions  $\{J_{i,j}\}_{i,j \in [n]}$ . A spin  $\sigma_i$  takes values in  $\{\pm 1\}$  and the interaction  $J_{i,j}$  between two spins  $\sigma_i, \sigma_j$  is a real-valued variable that captures whether the physical system prefers the two spins to be the same ( $J_{i,j} > 0$ ) or different ( $J_{i,j} < 0$ ). The goal is to understand what spin configurations  $\sigma \in \{-1, 1\}^n$  maximize the following quantity (a.k.a. Hamiltonian):

$$H(\sigma) = \sum_{i,j} J_{i,j} \sigma_i \sigma_j.$$

The setting is easily generalized to higher order interactions, i.e.,  $J_{i_1, \dots, i_k}$  acting on  $k$  spins, and this is known as the  $k$ -spin model. See [35] for a comprehensive survey.

There is a natural correspondence between spin glass theory and combinatorial optimization problems. In a combinatorial optimization problem (e.g., MAX-CUT), a variable corresponds to a spin and a constraint corresponds to an interaction. Through this correspondence, the maximization of the above Hamiltonian  $H(\sigma)$  serves as a proxy for maximizing the number of satisfied constraints in the combinatorial optimization problem.

A spin glass model additionally specifies a particular distribution on the interactions  $\{J_{i,j}\}$  for all  $i, j \in [n]$ . The quantity of interest is the asymptotic maximum value

$$H^* := \lim_{n \rightarrow \infty} \frac{1}{n} \max_{\sigma} H(\sigma),$$

(a.k.a. the ground state energy density), as well as spin configurations  $\sigma$  with  $H(\sigma) \approx H^*$ . There are many well-studied spin glass models in physics and various mathematical insights about these have been discovered over the years [12, 14, 42, 37]. For example, for the SK model [43], Parisi [38] proposed the infamous *Parisi Variational Principle* to capture the exact value of  $H^*$ . This was later rigorously proved by Talagrand [44] and again by Panchenko [36] in greater generality. These successes give hope to design local algorithms that simulate the physical system and output a final configuration as an approximation to the corresponding combinatorial optimization problem.

While traditional spin glass models consider the underlying non-trivial interactions as either lying on a certain physically-realistic graph (e.g., the non-zero  $J_{i,j}$  form a 2D-grid) or being a *mean field approximation* (for example, where every  $J_{i,j}$  is non-trivial), the applications in combinatorial optimization often require the underlying constraint graphs to be sparse and arbitrary. We use two methods of bridging the gap between the two settings:

- By studying the *diluted  $k$ -spin glass model* where one first samples a sparse hypergraph and then assigns non-trivial interactions on top of its hyperedges. Intuitively, approximating the  $H^*$  of the diluted  $k$ -spin glass corresponds to approximating the maximum cut over random sparse hypergraphs. This correspondence is made more precise in Definition 1.5.

- Using the techniques of *Gaussian interpolation* [27] and *Poisson interpolation* [21, 10] from statistical physics to relate the behavior of random dense spin glass models to random sparse CSPs. More specifically, we relate the random Max- $k$ -XOR problem to mean-field  $p$ -spin glasses (Section 8 in the full version) by modifying the Guerra-Tonnineli interpolation used in Chen et al. [10].

■ **Table 1** A dictionary between spin glass models and combinatorial optimization problems.

Spin glass models	Combinatorial optimization problems
Spins $\sigma \in \{-1, 1\}^n$	An assignment to boolean variables
Interactions $\{J_{i_1, \dots, i_k}\}_{i_1, \dots, i_k \in [n]}$	Constraints (i.e., hyperedges)
Hamiltonian $H(\sigma)$	Value of an assignment (i.e., $\text{val}_\Psi(\sigma)$ )
Ground state energy $H^*$	Optimal value (i.e., $\text{val}_\Psi$ )
Mean field model (e.g., SK model)	The underlying hypergraph <i>not</i> being a lattice
Diluted spin glass model	The underlying hypergraph being sparse

## 1.2 Prior Work

### 1.2.1 Constraint-satisfaction problems & hardness for classical algorithms

CSPs (described formally in Definition 1.4) are a natural class of combinatorial optimization problems that have been studied extensively in theoretical computer science [7, 32]. Many NP-Complete problems such as  $k$ -SAT,  $k$ -NAE-SAT, MAX-CUT and  $k$ -XOR, can be framed as CSPs. Consequently, unless  $\mathcal{P} = \mathcal{NP}$ , finding optimal solutions to these problems is infeasible. A natural question then is to understand how well can approximate answers to instances of these problems be constructed by efficient algorithms. Under the now widely believed *Unique Games Conjecture* [31], upper bounds on the approximability of CSPs are known [30, 41]. These bounds, however, are only worst-case and do not necessarily explicitly demonstrate a family of instances of a CSP that are hard to approximate. Additionally, they remain conditional on a positive resolution to the Unique Games Conjecture, which is still a difficult open problem in the field. In the average-case regime, the goal is to ask how well a *typical* instance of a CSP can be approximated, where the instance is chosen from a “natural” distribution over the set of instances. Perhaps surprisingly, great insight has been drawn about the algorithmic hardness (or lack thereof) about random instances of many CSPs based on work originating in the Statistical Physics community, particularly in Spin Glass Theory [34, 21, 39]. This was so because the problem of finding spin configurations of particles in many spin glass models that put a system in the ground state could naturally be interpreted as a CSP. Various iterative algorithms were proposed to study the problem of explicitly finding near-ground states of *typical* instances of various spin glass models [46, 9]. It was observed that these algorithms either consistently got better with the number of iterations, or hit a threshold which they could not exceed. To understand this, the work of Achlioptas et al. [1] studied the solution geometry of the  $k$ -SAT problem and found that most good solutions were in well separated clusters. Additionally, most variables in a good solution could only take a single value (i.e., they were “frozen”). This observation was stated as an intuitive reason for the failure of local algorithms on random instances of  $k$ -SAT. Gamarnik et al. [24] made this more formal and precise by showing that *no* classical local algorithm (described formally as *factors of i.i.d.*, see Definition 1.10) could approximate the

MAX-IND-SET problem arbitrarily well on sparse random graphs. Critical to their argument was the fact that *all* (not *most*) nearly-optimal solutions to the problem satisfied the *Overlap Gap Property* - they were in well separated clusters. In various works that followed up, many problems have been shown to have near-optimal solutions conform to this solution geometry and algorithmic hardness for various families of classical algorithms has been established [10, 23, 22].

## 1.2.2 The Overlap Gap Property & QAOA

A depth- $p$  QAOA algorithm, abbreviated as  $QAOA_p$ , applies a length- $p$  sequence of unitaries to each hyperedge and outputs the measurement result of the final state. The goal of QAOA is to approximate the optimal solutions on average-case (i.e., random) instances. For example, the work of Farhi et al. [19] showed that  $QAOA_1$  achieved 0.6924-approximation for MAX-CUT on triangle-free 3-regular graphs while such an approximation ratio was not then known to be achievable by local classical algorithms.

Shortly after  $QAOA_p$  (defined formally in Definition 2.2) was proposed as a way to solve hard optimization problems and possibly establish quantum computational advantage for MAX-CUT on 3-regular graphs [19], a local classical algorithm was designed that would outperform  $QAOA_p$  on these graphs at depth 1 [29]. Consequently, because of a flurry of follow up results, QAOA has been shown to be outmatched by local classical algorithms up to depth 2 [33, 5] for the MAX-CUT problem on  $d$ -regular graphs with large girth. In fact, under the widely-believed conjecture in the Spin-Glass Theory community that the SK model does not satisfy the *Overlap Gap Property* [4], an AMP algorithm was recently proposed that outputs arbitrarily good cuts for large (but constant) degree random regular graphs [2]. To analyze the performance of  $QAOA_p$  on a problem that possesses an OGP, Farhi et al. [17] was established that  $QAOA_p$  with depth  $p \leq \varepsilon \log(n)$  could not output independent sets of size better than .854 times the optimal for sparse random graphs. This work suggested that the OGP may broadly prove to be an obstacle for  $QAOA_p$  while it is local as much as it does for various classical algorithms. However, MAX-IND-SET is not a (maximum) CSP and, additionally, the prior work [17] does not give an analysis that generalizes to CSPs. Our work establishes this generalization (see Theorem 1.1) and also immediately positively resolves the “landscape independence” conjecture of  $QAOA_{\varepsilon \log(n)}$  (see Theorem 1.3) proposed by Brandao et al. [8]. This immediately suggests that quantum advantage is unlikely to be found up to this depth for CSPs with an OGP, and we conjecture that *almost all* CSPs will have an OGP (see subsection 5.1).

## 1.3 Our results

Our main result establishes that *any* local quantum algorithm, including  $QAOA_p$  with  $p \leq \varepsilon \log(n)$ , is obstructed from arbitrarily approximating *any* CSP that satisfies a “coupled” *Overlap Gap Property* (described in subsection 1.5.3). The precise family of CSPs we will work over will be notated as  $(k, d)$ -CSP( $f$ ), where  $k$  denotes the number of variables in a single clause,  $d$  denotes the average number of clauses any variable appears in, and  $f$  denotes the predicate applied to each clause. For a formal definition, refer to Definition 1.4. We state the informal version of our results, the formal versions of which may be found in Theorems 4.2 and 4.3 in the full version of this paper.

► **Theorem 1.1** (Obstruction to  $QAOA_p$  given coupled OGP, informal). *Given a uniformly random instance  $\Psi$  of a  $(k, d)$ -CSP( $f$ ) that satisfies a coupled OGP, with high probability a depth- $p$   $QAOA_p$  circuit with*

$$p \leq \frac{\log(n)}{2 \log(d(k-1)/\ln(2))} - 1$$

*cannot output a solution that is better than a  $(1 - \varepsilon_0)$ -approximation for some  $\varepsilon_0 > 0$ .*

In particular, this immediately implies an obstruction for approximating maximum cuts of random sparse hypergraphs, as a coupled OGP is known to exist for that problem [10]. Furthermore, though the obstruction in Theorem 1.1 is stated for  $QAOA_p$ , it will apply to any generic local algorithm, and will, therefore, also apply to any local quantum algorithm. This is stated more precisely in Theorem 2.8.

► **Theorem 1.2** (Obstruction to generic local algorithms on random Max- $k$ -XOR, informal). *For every even  $k \geq 4$ , there exists  $d_0 \in \mathbb{N}$  and the following holds. There exists  $\varepsilon_0 > 0$  such that if  $QAOA_p$  outputs a solution  $\sigma \in \{-1, 1\}^n$  with  $H(\sigma)$  being  $(1 - \varepsilon_0)$ -close to the  $H^*$  of a random Max- $k$ -XOR instance of average degree  $d \geq d_0$  with probability at least 0.99, then  $p = \Omega(\log n)$ .*

This is stated formally in Corollary 4.4 in the full version, and answers a question of Farhi et al. [20], where the authors ask if  $QAOA_p$  would perform well on  $k$ -spin generalizations of the SK model, citing Max- $k$ -XOR in particular [10]. The above result is immediately implied by a proof of a coupled Overlap-Gap Property for the Max- $k$ -XOR problem, stated in Lemma 2.9 and proved in Section 8 in the full version.

To prove Theorem 1.1, two key lemmas about the concentration of output of local quantum algorithms need to be proved (Theorems 5.3 and 5.4 in the full version). A corollary to these two lemmas is that the quality of solution output by  $QAOA_p$  (with  $p$  as stated in Theorem 1.1) concentrates heavily around the expected value. More specifically, if we let  $\text{val}_\Psi(\sigma)$  denote the number of clauses of  $\Psi$  satisfied by an assignment  $\sigma \in \{\pm 1\}^n$  to the variables, then this value has small deviation on almost all instances.

► **Theorem 1.3** (Landscape-independence of  $QAOA_{\varepsilon \log(n)}$ , informal). *Given a random instance  $\Psi$  of a  $(k, d)$ -CSP( $f$ ) and a  $QAOA_p$  circuit with depth  $p$  as stated in Theorem 1.1, the solution  $\sigma$  output by  $QAOA_p$  with value  $\text{val}_\Psi(\sigma)$  concentrates as,*

$$\Pr [|\text{val}_\Psi(\sigma) - \mathbb{E}[\text{val}_\Psi(\sigma)]| \geq \delta n] \leq e^{O(n^\gamma)},$$

*for every  $\delta > 0$  and some  $\gamma > 0$ , and the probability taken over both the input distribution and internal randomness of the algorithm.*

The theorem above immediately confirms a conjecture by Brandao et al. [8] about the “landscape independence” of  $QAOA_p$  upto depth  $\varepsilon \log(n)$ . The “landscape independence” of  $QAOA_p$  is a term which asserts that the algorithm performs almost equally well on almost all instances.

## 1.4 Technical overview

### 1.4.1 Chen et al. [10] analysis

Chen et al. [10] establish a coupled overlap-gap property (OGP) for the maximum cut of random hypergraphs. The property says that for two “coupled” random instances and any nearly optimal solutions  $\sigma_1, \sigma_2 \in \{-1, 1\}^n$  of these, the solutions either have large or small

overlap on the assignment values to the variables, i.e., there exists an interval  $0 < a < b < 1$  such that  $\langle \sigma_1, \sigma_2 \rangle / n \notin [a, b]$ . The coupled OGP holds over an interpolation of a pair of hypergraphs  $\{(G_1(t), G_2(t))\}_{t \in [0,1]}$  with the following three properties: for every  $t \in [0, 1]$ , denote  $\sigma_1(t)$  and  $\sigma_2(t)$  as the outputs of a factors i.i.d. algorithm on inputs  $G_1(t)$  and  $G_2(t)$  respectively. (i) when  $t = 0$ ,  $(G_1(0), G_2(0))$  are independent random hypergraphs and  $\langle \sigma_1(0), \sigma_2(0) \rangle / n < a$  with high probability; (ii) when  $t = 1$ ,  $G_1(1) = G_2(1)$  are the same random hypergraph and  $\langle \sigma_1(1), \sigma_2(1) \rangle / n = 1$  with high probability; (iii) for each  $t \in [0, 1]$ , the correlation  $\langle \sigma_1(t), \sigma_2(t) \rangle / n$  between the two solutions is highly concentrated (with respect to the randomness of  $G_1(t), G_2(t)$  and the algorithm) to a value  $R(t)$ , and  $R(t)$  is a continuous function of  $t$ . By the intermediate value theorem, this contradicts the OGP if the solutions are nearly optimal and hence no such factors of i.i.d. algorithm can exist. Note that it is also important to assert that the hamming weight and the objective function values output by the algorithm also concentrate.

### 1.4.2 Our analysis

The key part of the proof in Chen et al. [10] that does not work for  $QAOA_p$  is item (iii) of step 2. Specifically, local quantum algorithms are not factors of i.i.d. algorithms and hence their concentration analysis on the correlation between solutions to coupled instances does not apply. Intuitively, this is because local quantum circuits can induce entanglement between qubits in a local neighborhood which cannot be explained by a local hidden variable theory [6]. We overcome this issue by first generalizing the notion of factors of i.i.d. algorithms to what we call *generic local algorithms* (Definition 2.1).

To establish concentration of overlap for quantum local algorithms, the challenge lies in how to capture the local correlations of  $G_1(t)$  and  $G_2(t)$ . We achieve this by defining a new notion of a random vector being *locally independent* (Definition 5.2 in the full version). This structure enables us to show concentration on a fixed instance over multiple runs of the generic local algorithm with respect to its internal randomness (Theorem 5.3 in the full version). Finally, to establish concentration between a pair of correlated instances  $(G_1(t)$  and  $G_2(t))$ , we strengthen McDiarmid's inequality for biased distributions (Lemma 2.7) and this allows the concentration analysis of the correlation function  $R(t)$  to pull through (Theorem 5.1 in the full version). We complete the analysis by showing that the hamming weight and objective function values output by a generic  $p$ -local algorithm also concentrate on any  $(k, d)$ -CSP( $f$ ), obtained as corollaries to the main concentration lemmas (Corollaries 5.10 and 5.13 in the full version).

### 1.4.3 Comparison with Chen et al. [10] and Farhi et al. [17]

We augment the techniques of Farhi et al. [17] to handle a coupled OGP over a continuous interpolation, as opposed to the coupled OGP in Farhi et al. [17] which is over a fixed discrete interpolation. The advantage of this is to enable the use of a broader family of coupled OGPs provable using statistical mechanics methods, whereas the coupled OGP of Farhi et al. [17] requires reasoning about explicit sequences of instances in a way that does not clearly generalize from their independent set analysis to the setting of CSPs.

Our statements additionally show stronger concentration than those of Chen et al. [10] which is necessary to demonstrate that polynomially many runs of the algorithm will (with high probability) not succeed. Lastly, we extend the coupled OGP from the setting of diluted  $k$ -spin glasses shown by Chen et al. [10] to the setting of random Max- $k$ -XOR.

## 1.5 Preliminaries & notation

### 1.5.1 Constraint-satisfaction problems & hypergraphs

Constraint satisfaction problems are a class of optimization problems where a set of constraints on the underlying variables need to be satisfied in tandem. We restrict our attention to the setting where the variables are boolean valued and the number of constraints are *sparse*. Furthermore, every constraint involves  $k$  variables, where  $k$  is a constant independent of the number of variables.

► **Definition 1.4** (Random  $(k, d)$ -CSP( $f$ )). *A (signed) random  $(k, d)$ -CSP( $f$ ) instance with a local constraint function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  is constructed as follows:*

1. Choose  $r \sim \text{Poisson}(dn/k)$ .
2. Sample  $r$  clauses of size  $k$  by choosing each clause  $C_i$  independently as a collection of  $k$  variables uniformly at random from  $\{x_1, \dots, x_n\}^k$ , and, in the case of a signed random CSP, random signs  $s_{i,1}, \dots, s_{i,k} \in \{\pm 1\}$ .

To each clause  $C_i$  there are  $k$  variables associated:  $\{x_{i_1}, \dots, x_{i_k}\}$ . A clause is satisfied if there is some assignment to every  $x_{i_j} \in \{-1, 1\}$ , such that,  $f(x_{i_1}, \dots, x_{i_k}) = 1$  (or  $f(s_{i,1}x_{i_1}, \dots, s_{i,k}x_{i_k}) = 1$  if signed). The value of an assignment  $\sigma \in \{-1, 1\}^n$  is defined as  $\text{val}_\Psi(\sigma) := \#\{C_i : f(\sigma_{i_1}, \dots, \sigma_{i_k}) = 1\}$  (or  $\#\{C_i : f(s_{i,1}\sigma_{i_1}, \dots, s_{i,k}\sigma_{i_k}) = 1\}$  if signed). The optimal value of  $\Psi$  is defined as  $\text{val}(\Psi) := \max_\sigma \text{val}_\Psi(\sigma)$ .

When unspecified, we will be referring to unsigned CSPs. In a diluted  $k$ -spin glass, the underlying particles can be thought of as vertices of a hypergraph with  $\frac{dn}{k}$  hyperedges. Each hyperedge is a tuple of  $k$ -vertices, and any interaction  $J_{i_1, \dots, i_k}$  over a tuple  $(i_1, \dots, i_k)$  is  $-1$  if  $(i_1, \dots, i_k)$  is a hyperedge of  $G$  and 0 otherwise.

► **Definition 1.5** (Random  $k$ -uniform hypergraph). *A random  $k$ -uniform hypergraph consists of choosing a number of edges  $|E| \sim \text{Poisson}(\frac{dn}{k})$  and then choosing hyperedges  $e_1, \dots, e_{|E|}$  independently and uniformly at random from the set  $\{1, \dots, n\}^k$  of all vertex  $k$ -tuples.*

The underlying hamiltonian of a diluted  $k$ -spin glass then is,

$$H_{n,k,d}^G(\sigma) = - \sum_{e \in E} \sigma_{e_1} \dots \sigma_{e_k}, \quad (1)$$

which on maximization corresponds to the MAX-CUT of a random  $k$ -uniform hypergraph. Observe that a  $(k, d)$ -CSP( $f$ ) can be encoded as a diluted  $k$ -spin glass by choosing the variables to be the set of vertices, the clauses to be the hyperedges, and the hamiltonian to be the same as in Equation 1 with the additional requirement that  $f$  acts on the variables in every hyperedge.

### 1.5.2 Vanishing Local Neighborhoods of Sparse Random Hypergraphs

We state a bound on sufficiently local neighborhoods of random sparse  $k$ -uniform hypergraphs. This bound is used in the obstruction result to precisely quantify the size of the neighborhood  $p(n)$  up to which a  $p(n)$ -generic local algorithm does not “see the whole hypergraph” around any vertex.

► **Lemma 1.6** (Vanishing local neighborhoods of random sparse  $k$ -uniform hypergraphs). *Let  $k \geq 2$  and  $d \geq 2$  and  $\tau \in (0, 1)$ . Then there exists  $a > 0$  and  $0 < A < 1$ , such that, for  $n$  large enough and  $p$  satisfying*

$$2p + 1 \leq \frac{(1 - \tau) \log n}{\log\left(\frac{d(k-1)}{\ln 2}\right)},$$

the following are true:

$$\Pr_{G \sim \mathcal{H}_{n,d,k}} \left[ \max_i B_G(v_i, 2p) \geq n^A \right] \leq e^{-n^a}, \quad \text{and} \quad \Pr_{G \sim \mathcal{H}_{n,d,k}} \left[ \max_i B_G(v_i, p) \geq n^{\frac{A}{2}} \right] \leq e^{-n^{\frac{a}{2}}}.$$

Intuitively, the above lemma says that the local neighborhood of each vertex is vanishingly small with high probability. To prove Lemma 1.6, we utilize a modified version of the proof of Farhi et al. [17, Neighborhood Size Theorem] to handle the case of sparse random hypergraphs and we defer the complete proof to Appendix B in the full version.

### 1.5.3 The Overlap Gap Property

We say that a  $(k, d)$ -CSP( $f$ ) (signed or unsigned) satisfies a *coupled overlap-gap property (OGP)* if, given two instances  $\Psi, \Psi'$  constructed so that they share a random  $t$ -fraction of clauses with the remaining  $(1 - t)$ -fraction chosen independently, any two “good” solutions  $\sigma$  of  $\Psi$  and  $\sigma'$  of  $\Psi'$  are either very similar or dissimilar.

► **Definition 1.7** (Coupled OGP, informal). *A signed or unsigned  $(k, d)$ -CSP( $f$ ) satisfies a coupled OGP if there exists  $\varepsilon_0 > 0$  and  $0 < a < b < 1$  such that the following hold for every  $t \in [0, 1]$ : Given two  $(k, d)$ -CSP( $f$ ) instances  $\Psi, \Psi'$  constructed so that they share a random  $t$ -fraction of their clauses and have the remaining  $(1 - t)$ -fraction of clauses chosen independently and uniformly at random, then for every  $0 < \varepsilon < \varepsilon_0$ , the overlap between any  $(1 - \varepsilon)$ -optimal solution  $\sigma$  of  $\Psi$  and  $\sigma'$  of  $\Psi'$  satisfies*

$$\frac{1}{n} \langle \sigma, \sigma' \rangle = \frac{1}{n} \left( \sum_{i=1}^n \sigma_i \sigma'_i \right) \notin [a, b]$$

with high probability.

A formal definition of the interpolation procedure described in Definition 1.7 and a complete and formal statement of a coupled OGP are provided in section 3.

### 1.5.4 Local classical algorithms

A local classical algorithm takes as input a hypergraph  $G$  and a label set  $S$ , runs a stochastic process  $\{X^G(t)\}_t$  that associates to each vertex  $v$  a label  $X_v^G(t) \in S$  at time  $t$ , and outputs an assignment  $\sigma_v$  to each vertex  $v$  according to its final label. While there is a huge design space for local classical algorithms, a *factors of i.i.d. algorithm* of radius  $p$  has the following restrictions:

1. The initial label for each vertex  $v$  is set to be one from an i.i.d. set of random variables  $X_v^G(0)$ .
2. For each vertex  $v$ , the assignment  $\sigma_v$  is a random variable that only depends on the labels from a  $p$ -neighborhood of  $v$  and is updated via a stochastic process.
3. The assignment function (also known as the *factor*) for each vertex is the same.
4. The label assignment of two vertices  $v, v'$  in hypergraphs  $G$  and  $G'$  are equivalent if the two vertices have *isomorphic*  $p$ -neighborhoods.

As we shall see, in subsection 2.1, we relax the first three conditions to include a larger class of local algorithms, which subsume local quantum algorithms. We now state the definition of the  $p$ -neighborhood of a vertex in a hypergraph, which generalizes the  $p$ -neighborhood of a graph by considering two vertices  $v$  and  $w$  to be adjacent if they belong to the same hyperedge  $e$ .

► **Definition 1.8** ( $p$ -neighborhood and hypergraphs with radius  $p$ ). *Let  $G$  be a hypergraph,  $v \in V(G)$ , and  $p \in \mathbb{N}$ . The  $p$ -neighborhood of  $v$  is defined as*

$$B_G(v, p) := \{w \in V(G) \mid w \text{ can be reached in } p \text{ steps from } v\}.$$

*Let  $G$  be a hypergraph,  $v \in V(G)$ , and  $p \in \mathbb{N}$ . We say  $(G, v)$  has radius  $p$  if  $B_p(G_v) = V(G)$ . Further, let  $k \in \mathbb{N}$ , we define*

$$\mathcal{G}_p := \{(G, v) \mid (G, v) \text{ has radius } p \text{ and } G \text{ is connected, finite, and } k\text{-uniform}\}$$

*be the collections of hypergraphs with radius at most  $p$ .*

To capture the fact that classical local algorithms assign the value of a vertex  $v$  by only looking at a stochastic process of the  $p$ -neighborhood, we define a factor  $f$  as a measurable function which gives a label  $\sigma_v \in \{\pm 1\}$  to every vertex  $v \in V(G)$ . We restrict to the case where the label set  $S = [0, 1]$ .

► **Definition 1.9** (Factor of radius  $p$ ). *Let  $p \in \mathbb{N}$ . We define the collection of all  $[0, 1]$ -labelled hypergraphs of radius at most  $p$  as*

$$\Lambda_p := \left\{ (G, v, X) \mid (G, v) \in \mathcal{G}_p \text{ and } X \in [0, 1]^{V(G)} \right\}$$

*We say  $(G_1, v_1, X_1), (G_2, v_2, X_2) \in \Lambda_r$  are isomorphic if there exists a hypergraph isomorphism  $\phi : V(G_1) \rightarrow V(G_2)$  such that (i)  $\phi(v_1) = v_2$  and (ii)  $X_1 = X_2 \circ \phi$ .*

*Finally, we say  $f : \Lambda_p \rightarrow \{-1, 1\}$  is a factor of radius  $p$  function if*

1.  $f$  is measurable.
2.  $f(G_1, v_1, X_1) = f(G_2, v_2, X_2)$  for every isomorphic  $(G_1, v_1, X_1), (G_2, v_2, X_2) \in \Lambda_r$ .

Intuitively, the output distribution of a factors of i.i.d. algorithm with radius  $p$  on a vertex  $v$  is determined by the  $p$ -neighborhood of  $v$ .

► **Definition 1.10** (Factors of i.i.d., [10, Section 2]). *Let  $k, p \in \mathbb{N}$ . A factors of i.i.d. algorithm  $A$  with radius  $p$  is associated with a factor of radius  $p$  function  $f$  with the following property. On input a  $k$ -uniform hyper graph  $G$ , the algorithm  $A$  samples a random labeling  $X = \{X(v)\}_{v \in V(G)}$  where  $X(v)$ 's are i.i.d. uniform random variables on  $[0, 1]$ . The output of  $A$  is  $\sigma \in \{-1, 1\}^{V(G)}$  where  $\sigma_v := f(B_p(G, v), v, \{X(w)\}_{w \in B_p(G, v)})$  for each  $v \in V(G)$ .*

Common local algorithms such as Glauber dynamics and Belief Propagation are examples of factors of i.i.d. algorithms.

## 2 Main Technical Theorems

The main results include showing that  $QAOA_p$  is a generic  $p$ -local algorithm, a concept which subsumes *factors of i.i.d.* local algorithms and local quantum algorithms, and then establishing that this more general class of algorithms also have strong concentration properties. The concentration allows the original interpolation argument of Chen et al. [10] stated in subsection 1.4.1 to continue to pose an obstruction to generic  $p$ -local algorithms from outputting arbitrarily good solutions on almost all instances of a  $(k, d)$ -CSP( $f$ ) that has a coupled OGP.

## 2.1 Generic local algorithms

Generic local algorithms, including local quantum algorithms, can be defined from an information-based perspective. Specifically, given a parameter  $p$  that characterizes the locality of the information an algorithm has about the input, we can assert that a *generic*  $p$ -local algorithm makes independent decisions on appropriately far-away vertices and that all decisions for a vertex are sufficiently local.

► **Definition 2.1** (Generic local algorithms, informal). *Let  $p \in \mathbb{N}$  and let  $S$  be a finite label set. We say an algorithm  $A$  (which takes a hypergraph  $G$  as an input) is generic  $p$ -local if the following hold:*

- **(Local distribution determination).** *For every set of vertices  $L \subset V$ , the joint marginal distribution of the labels  $(A(G)_v)_{v \in L}$  depends only on the union of the  $p$ -neighborhoods of  $v \in L$  in  $G$ .*
- **(Local independence).**  *$A(G)_v$  is statistically independent of the joint distribution of  $\{A(G)_{v'}\}$  for every  $v'$  that is farther than a distance of  $2p$  from  $v$ .*

The definition makes no assumptions about how the randomness of the algorithm is instantiated, what sort of function is used to decide the label locally, or whether it is the same for every vertex. A formal definition is provided in section 4, along with a procedure to partially sample from the run of the algorithm on correlated instances. We now briefly introduce the  $QAOA_p$  circuit.

► **Definition 2.2** ( $QAOA_p$  algorithm, [19]). *The  $QAOA$  circuit parametrized by angle vectors  $\hat{\gamma} = (\gamma_1, \dots, \gamma_p)$  and  $\hat{\beta} = (\beta_1, \dots, \beta_p)$  looks as follows,*

$$U_p(\hat{\beta}, \hat{\gamma}) = \prod_{j=1}^p e^{-i\beta_j \sum_{k=1}^n X_k} e^{-i\gamma_j H_C(G)}.$$

In the circuit above,  $X_i$  is the Pauli-X matrix acting on the  $i$ -th qubit (with identity action everywhere else) and  $H_C(G)$  is the hamiltonian that encodes the problem instance. For us,  $H_C(G) = H_{n,k,d}^G$  where every  $k$ -spin interaction is replaced by a Pauli-Z interaction  $Z_{e_1} \otimes \dots \otimes Z_{e_k}$ . Typically, the initial state on which the circuit is applied is a *symmetric product state*, most notably  $|0\rangle^{\otimes n}$  or  $|+\rangle^{\otimes n}$ . The expected value that  $QAOA$  outputs after applying the circuit on some initial state  $|\psi_0\rangle$  is,

$$\langle \psi_0 | U^\dagger H_C(G) U | \psi_0 \rangle.$$

We will notate by  $QAOA(\hat{\beta}, \hat{\gamma})$  a  $QAOA$  circuit of depth  $2p$  with angle parameters  $\hat{\beta}$  and  $\hat{\gamma}$ . In our regime, we will work with *any* collection of *fixed* angles  $(\hat{\beta}, \hat{\gamma})$ .

As stated below, generic  $p$ -local algorithms strictly generalize the *factors of i.i.d.* framework as well as  $QAOA_p$ .

► **Proposition 2.3** (Generic local strictly generalizes factors of i.i.d.). *There exists a generic  $p$ -local algorithm as defined in Definition 2.1 that is not a  $p$ -local factors of i.i.d. algorithm as defined in Definition 1.10.*

► **Proposition 2.4** ( $QAOA_p$  is generic  $p$ -local). *For every  $p > 0$  and fixed angle vectors  $\hat{\beta}$  and  $\hat{\gamma}$ ,  $QAOA_p(\hat{\beta}, \hat{\gamma})$  is generic  $p$ -local under Definition 2.1.*

The proofs for the two propositions above are provided in section 4.

## 2.2 Concentration of outputs of local quantum algorithms

Having shown that generic  $p$ -local algorithms capture classical & quantum local algorithms, we now first show that the outputs of these algorithms concentrate even when run on multiple fixed correlated instances. Additionally, the outputs concentrate over the input distribution from which the instances are drawn. The proof for the lemmas in this subsection are delegated to Section 5 in the full version.

### 2.2.1 Concentration over internal randomness of the algorithm

We first show that the output of generic  $p$ -local algorithms on sufficiently “local” functions of the underlying spins concentrates heavily, even when run on multiple correlated instances. We state an informal version of the lemma. For a formal and detailed version, please refer to Theorem 5.3 in the full version. “Spins” here refer to the assignment of a variable.

► **Lemma 2.5** (Concentration of local functions of spins, informal). *Let  $z, m \in \mathbb{N}$  and  $A < 1$ . Let  $\sigma$  be the output of a generic  $p$ -local algorithm on a fixed hypergraph  $G$ . Let  $v_{j,i}$  denote the vertex corresponding to the  $i$ -th vertex (or variable) in the  $j$ -th hyperedge (or clause) with  $j \in [m]$  and  $i \in [z]$ . Let  $r = (r_1, \dots, r_m)$  be a random vector of a correlated hypergraph  $G'$  with a certain joint independence structure (Definition 5.2 in the full version). Now, consider a sum  $X = \sum_{j \in [m]} h(\sigma_{v_{j,1}}, \dots, \sigma_{v_{j,z}}) r_j$ , where  $|h| \leq 1$ . Suppose that each vertex  $v$  occurs at most  $C$  times among the different  $v_{j,i}$ . Then, provided the  $p$ -neighborhood of every vertex in  $G \cup G'$  has at most  $n^A$  vertices:*

$$\Pr_{\sigma, r} [|X - \mathbb{E}_{\sigma, r} [X]| \geq \delta n] \leq e^{-\Omega(\delta^2 m / (Czn^A))}.$$

The lemma above states that even when choosing two correlated instances  $G$  and  $G'$  that share, say  $t$ -fraction of their clauses, provided every vertex sees not too many vertices in the local neighborhood of the combined input  $G \cup G'$ , the value of sufficiently local functions  $h$  acting on the output of the algorithm concentrate. To get concentration of overlap between two solutions  $\sigma$  of  $G$  and  $\sigma'$  of  $G'$ , we choose  $h(\sigma_{v_{j,1}}, \dots, \sigma_{v_{j,z}}) = \sigma_{v_j}$ ,  $m = n$  and  $r = \sigma'$ . To get concentration for  $\text{val}_G(\sigma)$ , we choose  $h = f$  for the underlying  $(k, d)$ -CSP( $f$ ) and  $m = \frac{dn}{k}$ . When working to obtain concentration on a single instance, it suffices to set  $G' = G$  and  $r$  to be the all 1s vector.

### 2.2.2 Concentration over instances

The prior lemma merely asserts that the output is concentrated for *fixed* correlated instances. However, this still doesn't imply that the output will be concentrated *across* random input instances. To do so, we need concentration over the distribution from which the inputs are sampled. This is accomplished by Theorem 5.4 in the full version, an informal version of which is stated below.

► **Lemma 2.6** (Concentration of differences of coupled hypergraphs, informal). *Let  $f$  be a function of two hypergraphs over  $n$  vertices such that*

$$|f(G_1, G_2) - f(G'_1, G'_2)| \leq r(n),$$

*for some  $r$  whenever  $(G_1, G_2)$  differs from  $(G'_1, G'_2)$  by the addition and/or removal of a single hyperedge  $e \in [n]^k$  from one or both hypergraphs. Furthermore, assume that the largest  $p$ -neighborhood of  $G_1, G_2, G'_1, G'_2$  has no more than  $n^{\frac{A}{2}}$  vertices for some  $A < 1$ . Then  $\exists a \in (0, 1)$ , such that*

$$\Pr_{\substack{G_1, G_2 \sim \\ \mathcal{H}_{n, k, d, t}}} \left[ \left| f(G_1, G_2) - \mathbb{E}_{G_1, G_2} f(G_1, G_2) \right| \geq \delta n r(n) \right] \leq 2 \exp\left(\frac{-3k\delta^2 n}{12(2-t)d+2k\delta}\right) + 2 \exp(-n^{a/2}).$$

The notation  $G_1, G_2 \sim \mathcal{H}_{n,k,d,t}$  indicates that  $G_1$  and  $G_2$  are  $t$ -correlated *random* hypergraphs. This means that  $t$ -fraction of their hyperedges are chosen i.i.d. from  $[n]^k$  and these are common. The remaining  $(1-t)$  fraction of edges are chosen independently for each. For a formal definition of the coupled distribution, refer to Definition 3.3. The above lemma relies on two critical facts:

1. For  $p \leq \varepsilon \log(n)$  (for an appropriately chosen  $\varepsilon$ ), the  $p$ -neighborhood of a *random*  $k$ -uniform hypergraph has no more than  $n^{\frac{A}{2}}$  vertices (Lemma 1.6).
2. A strengthening of *McDiarmid's Inequality* (Lemma 2.7) for biased distributions. This strengthening is proved via a martingale argument.

### 2.2.3 Strengthened McDiarmid's Inequality

We end by stating the strengthened version of McDiarmid's Inequality (Lemma 2.7) for biased distributions, which is a generic concentration statement that may be of independent interest. The inequality adds a Bernstein-like variance term to the tail-bound that controls the likelihood of deviation, where the variance term depends on a precise quantification of how unlikely certain very rare events are.

► **Lemma 2.7** (McDiarmid's inequality for biased distributions). *Suppose that  $X_1, \dots, X_n$  are sampled i.i.d. from a distribution  $D$  over a finite set  $\mathcal{X}$ , such that  $D$  assigns probability  $1-p$  to a particular outcome  $\chi_0 \in \mathcal{X}$ . Let  $f : \mathcal{X}^n \rightarrow \mathbb{R}$  satisfy a bounded-differences inequality, so that*

$$|f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq c$$

for all  $x_1, \dots, x_n, x'_i \in \mathcal{X}$ . Then

$$\Pr[|f(X_1, \dots, X_n) - \mathbb{E} f(X_1, \dots, X_n)| \geq \varepsilon] \leq 2 \exp\left(\frac{-\varepsilon^2}{2np(2-p)c^2 + 2c\varepsilon/3}\right).$$

The above inequality is robust in that if there are highly unlikely or likely (think  $p = o_n(1)$ ) events that cause large deviations, then these deviations can be absorbed into the Bernstein-like variance term in the tail bound term. This means that one can get strong concentration bounds even in the presence of a vanishingly small set of events that cause large deviations. This property ends up being critical when showing concentration of outputs of generic local algorithms over the input distribution.

## 2.3 Obstruction of generic local algorithms by the Overlap Gap Property

We state our main theorem (informally) that obstructs all generic  $p$ -local algorithms on random instances  $\Psi$  of a  $(k, d)$ -CSP( $f$ ) that satisfies the coupled Overlap-Gap Property.

► **Theorem 2.8** (Obstruction to Generic  $p$ -Local Algorithms given coupled OGP, informal). *Let  $f$  be a boolean predicate defined on  $k$  variables and  $p(n)$  be such that it satisfies the requirements of Lemma 2.5 and Lemma 2.6. Furthermore, assume that  $(k, d)$ -CSP( $f$ ) satisfies the coupled Overlap Gap Property. Then, given an instance  $\Psi \sim (k, d)$ -CSP( $f$ ) and a generic  $p(n)$ -local algorithm  $\mathcal{A}$  that outputs a solution  $\sigma$  on input  $\Psi$ , with high probability, the solution will be no more than  $(1 - \varepsilon_0)$ -optimal for some  $\varepsilon_0 > 0$ .*

The theorem above is stated in full precision and formality in Theorem 4.3 in the full version and proved formally in Section 6 of the full version. The theorem effectively obstructs any algorithm that makes assignments for variables by looking at  $o(n)$  sized

local neighborhoods *irrespective* of how these decisions are made and what concrete model of randomness is used, provided the problem exhibits a coupled OGP. For instance, the obstruction would hold even for quantum algorithms that assign labels based on the local neighborhoods of qubits *and* start and end in a state that is moderately locally entangled - By this, we mean a state that exhibits entanglement only between any qubit  $i$  and  $o(n)$  qubits in its  $p(n)$ -sized local neighborhood.

## 2.4 Overlap Gap Property for MAX- $k$ -XOR

In Section 8 in the full version, we establish that the random Max- $k$ -XOR problem obeys a coupled OGP, so that the obstruction implied by Theorem 2.8 holds for this problem.

► **Lemma 2.9** (Coupled OGP for random Max- $k$ -XOR). *The random Max- $k$ -XOR problem satisfies Definition 1.7.*

This is proven by means of a Guerra-Tonelli interpolation [28, 10], where a dense mean-field  $k$ -spin glass model is continuously transformed to the random Max- $k$ -XOR model. Treated as a physical system, the thermodynamic equilibrium at each point in the interpolation is calculated at a low enough temperature so that the equilibrium states are approximately optimal. The change in equilibrium energy over the course of the interpolation is then shown to be small.

A coupled OGP is shown in this way by interpolating a pair of systems constrained so that the overlap between the two states of the systems is bounded.

## 3 Coupled Interpolation & Overlap-Gap Properties

We now state the OGP formally as it holds for the diluted  $k$ -spin glass model in both uncoupled and coupled form. To do so, we begin by introducing the notion of an overlap between two spin-configurations  $\sigma_1$  and  $\sigma_2$ , which is equivalent to the number of spins that are the same in both configurations subtracted by the number of different spins, normalized by the number of particles in the system. Formally,

► **Definition 3.1** (Overlap between spin configuration vectors). *Given any two vectors  $\sigma_1, \sigma_2 \in \{-1, 1\}^n$ , the overlap between them is defined as,*

$$R(\sigma_1, \sigma_2) = \frac{1}{n} \langle \sigma_1, \sigma_2 \rangle = \frac{1}{n} \sum_{i \in [n]} (\sigma_1)_i (\sigma_2)_i.$$

We first state the OGP for diluted  $k$ -spin glasses about the overlap gaps in a *single instance*.

► **Theorem 3.2** (OGP for Diluted  $k$ -Spin Glasses, [10, Theorem 2]). *For every even  $k \geq 4$ , there exists an interval  $0 < a < b < 1$  and parameters  $d_0 > 0$ ,  $0 < \eta_0 < P(k)$  and  $n_0 > 1$ , such that, for  $d \geq d_0$ ,  $n \geq n_0$  and  $L = L(\eta_0, d)$ , with probability at least  $1 - Le^{-n/L}$  over the random hypergraph  $G \sim \mathcal{H}_{n,d,k}$ , whenever two spins  $\sigma_1, \sigma_2$  satisfy*

$$\frac{H^G(\sigma_i)}{n} \geq M(k, d) \left( 1 - \frac{\eta_0}{P(k)} \right),$$

*then also,  $|R(\sigma_1, \sigma_2)| \notin (a, b)$ .*

A more general version of the OGP excludes, with high probability, a certain range of overlaps between any two solutions of two different instances jointly drawn from a coupled random process. We first introduce this process, and then state the *coupled* version of the OGP as proven in Chen et al [10].

► **Definition 3.3** (Coupled Interpolation, [10, Section 3.2]). *The coupled interpolation  $\mathcal{H}_{d,k,n,t}$  generates a coupled pair of hypergraphs  $(G_1, G_2) \sim \mathcal{H}_{d,k,n,t}$  as follows:*

1. *First, a random number is sampled from  $\text{Poisson}(tdn/k)$ , and that number of random  $k$ -hyperedges are uniformly drawn from the set  $[n]^k$  and put into a set  $E$ .*
2. *Then, two more random numbers are independently sampled from  $\text{Poisson}((1-t)dn/k)$ , and those numbers of random  $k$ -hyperedges are independently drawn from  $[n]^k$  to form the sets  $E_1$  and  $E_2$  respectively.*
3. *Lastly, the two hypergraphs are constructed as  $G_1 = (V, E \cup E_1)$  and  $G_2 = (V, E \cup E_2)$ .*

► **Theorem 3.4** (OGP for Coupled Diluted  $k$ -Spin Glasses, [10, Theorem 5]). *For every even  $k \geq 4$ , there exists an interval  $0 < a < b < 1$  and parameters  $d_0 > 0$ ,  $0 < \eta_0 < P(k)$  and  $n_0 > 1$ , such that, for any  $t \in [0, 1]$ ,  $d \geq d_0$ ,  $n \geq n_0$  and constant  $L = L(\eta_0, d)$ , with probability at least  $1 - Le^{-n/L}$  over the hypergraph pair  $(G_1, G_2) \sim \mathcal{H}_{n,d,k,t}$ , whenever two spins  $\sigma_1, \sigma_2$  satisfy*

$$\frac{H^{G_i}(\sigma_i)}{n} \geq M(k, d) \left(1 - \frac{\eta_0}{P(k)}\right),$$

*then their overlap satisfies  $|R(\sigma_1, \sigma_2)| \notin [a, b]$ .*

We also provide a corresponding coupled OGP for random Max- $k$ -XOR in Theorem 8.12 in the full version.

## 4 Locality and Shared Randomness

### 4.1 Generic $p$ -local algorithms

We introduce a concept of “local random algorithm” which will allow for different runs of the same local algorithm to “share their randomness”, even when run on mostly-different instances. Later we will demonstrate that QAOA is a local algorithm under this definition.

► **Definition 4.1** (Generic local algorithms). *We consider randomized algorithms on hypergraphs whose output  $A(G) \in S^V$  assigns a label from some set  $S$  to each vertex in  $V$ . Such an algorithm is generic  $p$ -local if the following hold.*

- **(Local distribution determination).** *For every set of vertices  $L \subset V$ , the joint marginal distribution of its labels  $(A(G)_v)_{v \in L}$  is identical to the joint marginal distribution of  $(A(G')_v)_{v \in L}$  whenever  $\bigcup_{v \in L} B_G(v, p) \cong_L \bigcup_{v \in L} B_{G'}(v, p)$ , and,*
- **(Local independence).**  *$A(G)_v$  is statistically independent of the joint distribution of  $A(G)_{v'}$  over all  $v' \notin B_G(v, 2p)$ .*

Consequently, it will be possible to sample  $A(G)_v$  without even knowing what the hypergraph looks like beyond a distance of  $p$  away from  $v$ .

This definition is more general than the factors of i.i.d. concept used in probability theory [24, 10]. Our definition, for instance, encompasses local quantum circuits whereas factors of i.i.d. algorithms satisfy Bell’s inequalities and do not capture quantum mechanics.

► **Proposition 4.2** (Generic local strictly generalizes factors of i.i.d. (Restatement of Proposition 2.3)). *There exists a generic 1-local algorithm as defined in section 4 that is not a 1-local factors of i.i.d. algorithm as defined in Definition 1.10.*

A proof of this proposition is provided in Appendix A in the full version, and consists of setting up a Bell’s inequality experiment within the framework of a generic 1-local algorithm.

## 4.2 Locality properties of QAOA for hypergraphs

We show that any QAOA circuit of depth  $p$  with some *fixed* angle parameters  $(\hat{\beta}, \hat{\gamma})$  is a  $p$ -local algorithm. This allows us to describe a process to sample outputs of this circuit when it is run on two different input hypergraphs.

► **Proposition 4.3.** *For every  $p > 0$ , angle vectors  $\hat{\beta}$  and  $\hat{\gamma}$ ,  $\text{QAOA}_p(\hat{\beta}, \hat{\gamma})$  is generic  $p$ -local under Definition 4.1.*

**Proof.** To see this, consider the structure of QAOA: we start with a product state  $|\psi_0\rangle$  where each qubit corresponds to a vertex in the hypergraph, apply the unitary transformation  $U = U_p(\hat{\beta}, \hat{\gamma})$  to the state, and then measure each vertex  $v$  in the computational basis with the Pauli-Z operator  $\sigma_z(v)$ . Equally valid and equivalent is the Heisenberg picture interpretation of this process, where we keep the product state  $|\psi_0\rangle$  fixed but transform the measurements according to the reversed unitary transformation  $U^\dagger$ , so that we end up taking the measurements  $U^\dagger \sigma_z(v) U$  on the fixed initial state.

Because the  $\sigma_z(v)$  operators all commute with each other, their unitarily transformed versions  $U^\dagger \sigma_z(v) U$  also mutually commute, and the measurements can be taken in any order without any change in results. Let  $M(v) = U^\dagger \sigma_z(v) U$  and  $M(L) = \{U^\dagger \sigma_z(u) U \mid u \in L\}$ .

To show that QAOA satisfies the first property of generic  $p$ -local algorithms, we need to show that the marginal distribution of its assignments to any set  $L' \subseteq V$  of vertices depends only on the union of the  $p$ -distance neighborhoods of  $L'$ . To show this, since we are allowed to take the measurements in any order, take the measurements in  $M(L')$  before any other measurement. Then since the action of the unitary  $U = U_p(\hat{\beta}, \hat{\gamma})$  on qubits in  $L'$  does not depend on any feature of the hypergraph outside of a radius of  $p$  around  $L'$ , the operators  $M(L')$  are fully determined by the  $p$ -local neighborhoods of  $L'$ , and since we take them before every other measurement, the qubits are simply in their initial states when we make these measurements, thus the distribution of outputs is fully determined.

The same type of reasoning shows that the assignment to each  $v \in V$  is statistically independent of the assignments to any set of vertices outside of a  $2p$ -distance neighborhood of  $v$ . Take  $L'' \subset V \setminus B(v, 2p)$ . Then  $M(v)$  acts on a radius- $p$  ball around  $v$ , and each measurement in  $M(L'')$  acts on a radius- $p$  ball around a vertex in  $L''$ , and by taking  $\{M(v)\} \cup M(L'')$  before any of the measurements in  $M(\{B(v, 2p) \setminus \{v\}\})$ , we ensure that the qubits being measured by  $M(v)$  are disjoint from and unentangled with those measured by anything in  $M(L'')$ . Hence the measurement  $M(v)$  is independent of all measurements in  $M(L'')$ . We conclude that  $\text{QAOA}_p$  is a generic  $p$ -local algorithm. ◀

## 4.3 Shared randomness between runs of a generic local algorithm

We describe a process to sample the outputs of a generic local algorithm when run twice on two different hypergraphs, so that the two runs of the algorithm can share randomness when the hypergraphs have some hyperedges in common. This is not meant as a constructive algorithm, but a statistical process with no guarantee of feasible implementation.

The idea is to start with two  $t$ -coupled hypergraphs, which for large enough  $n$ , are likely to have some set of vertices  $L^+$  whose  $p$ -neighborhoods are identical between the two hypergraphs. Since these vertices have identical  $p$ -neighborhoods, a generic  $p$ -local algorithm behaves identically on the vertices in  $L^+$ . We pick a random  $t^+$  fraction of the elements of  $L^+$ , and assign the same labels to those vertices in the two coupled instances. Then the remaining labels on each hypergraph are assigned by generic  $p$ -local algorithms, conditioned on the output being consistent with the already assigned labels.

The formal definition may be found in the full version of this paper, where it is combined with the coupled OGP to demonstrate a contradiction if a generic local algorithm achieves a good approximation.

## 5 Conclusion & Future Work

The full version of this work conclusively establishes the coupled OGP as an obstruction to all local quantum algorithms on *any*  $(k, d)$ -CSP( $f$ ). In doing this, the work hints at and leaves open many interesting questions for future work in areas that are at the intersection of Quantum inapproximability, Statistical Physics, Random Graph Theory, Combinatorial Optimization and Average-Case Complexity.

### 5.1 Which CSPs have an OGP?

While various sparse CSPs such as  $k$ -SAT, unsigned  $\max$ - $k$ -XOR and  $k$ -NAE-SAT have been shown to exhibit clustering in their solution spaces at different clause-to-variable ratios [1, 15, 10], it is not known whether this property is pervasive to most CSPs or something that happens to a select few. Therefore, in order to understand the complexity landscape of CSPs on typical instances better, the following open question is interesting to investigate:

► **Conjecture 5.1** (Random Predicate CSPs and coupled OGP). *Given a function  $f$  chosen uniformly at random from the set of functions  $\mathcal{B}_k = \{g \mid g : \{\pm 1\}^k \rightarrow \{0, 1\}\}$ ,  $(k, d)$ -CSP( $f$ ) has a coupled-OGP for sufficiently large  $k$  and  $d$  with high probability (over the choice of  $f$  and instance  $\Psi \sim (k, d)$ -CSP( $f$ )).*

Notice that the conjecture above is specifically interested in the solution geometry of a CSP in the *unsatisfiable* regime (large  $d$ ). A positive resolution to the above conjecture will make the obstructions stated in Theorem 2.8 hold for *almost all* CSPs.

Another question of interest is which properties about a predicate  $f$  can be identified which would conclusively imply that a random instance  $\Psi$  of a  $(k, d)$ -CSP( $f$ ) will have an OGP.

► **Problem 5.1** (Properties of coupled-OGP predicates). *Can we enumerate a set of necessary and sufficient conditions on  $f$  to be such that  $(k, d)$ -CSP( $f$ ) satisfies a coupled-OGP for sufficiently large  $k$  and  $d$ ?*

### 5.2 Beyond log-depth obstructions for $QAOA_p$ ?

Work on obstructing  $QAOA_p$  using an OGP heavily relies on the locality of the algorithm at shallow depths. It is interesting to investigate whether this obstruction can be extended beyond the  $\varepsilon \log(n)$ -depth regime to make this a non-local obstruction. Recent work [23, 45] suggests that the OGP may actually result in stronger obstructions than just local ones, and it would be interesting to see if these techniques can be generalized to the setting of  $QAOA_p$  to yield obstructions that are non-local.

► **Problem 5.2** (Polylogarithmic obstructions to  $QAOA_p$  in the OGP regime). *Given a  $QAOA_p$  circuit with depth  $p \leq \varepsilon (\log(n))^c$  for some  $c > 1$ , does there exist  $\varepsilon_0 > 0$ , such that  $QAOA_p$  is obstructed on a  $(k, d)$ -CSP( $f$ ) with a coupled OGP from outputting solutions that are better than  $(1 - \varepsilon_0)$  approximations to the optimal?*

### 5.3 A Quantum OGP and lifting “classical” obstructions

The idea of the OGP obstructing families of algorithms that are *stable* under small perturbations to the input [23] motivates the idea of a quantized version of the OGP, to apply to *quantum* CSPs. To define such a property over quantum states, however, there would need to be a metric that is very similar to the classical hamming distance over  $\mathbb{F}_2$  and has the property that it is invariant over permutations of the canonical basis, while still quantifying entanglement in a desired way. One such possible metric is a *quantum* version of the Wasserstein distance of first order that was proposed by De-Palma et al. [13]. In particular, given a natural generalization of Definition 1.7 to a quantized setting using a quantum version of the Wasserstein distance of first order, it is interesting to investigate if a larger family of quantum circuits up to some depth  $p(n)$  can be obstructed by a family of  $d$ -local hamiltonians  $\{H_n\}_{n \geq n_0}$  that possess a qOGP (quantized Overlap-Gap Property). A result of this type could imply a way to generically “lift” classical obstructions for *stable* classical algorithms to a corresponding family of quantum algorithms.

### 5.4 Message-Passing algorithm for MAX-CUT of *all* $d$ -regular graphs?

Finding an efficient classical algorithm that can output cuts that are arbitrary approximations of the optimal ones for  $d$ -regular graphs is a long-standing open problem in Random Graph Theory and Theoretical Computer Science. Recently, this problem was nearly completely solved by Alaoui et al. [2] as they constructed a Message-Passing algorithm for random regular graphs of very large degree under the widely believed no-OGP assumption about the SK model. However, the problem does not provide a complete solution as it needs the degree  $d$  to be larger than  $O(\frac{1}{\epsilon})$  in order to output a  $(1 - \epsilon)$ -optimal cut. A natural question is whether, under a no-OGP assumption, the result can be extended to output  $(1 - \epsilon)$ -optimal cuts for  $d$ -regular graphs for *any*  $d \geq 3$ .

► **Conjecture 5.2** (AMP algorithm for Random  $d$ -Regular Graphs). *There exists a  $\text{poly}(n, \frac{1}{\epsilon})$  time algorithm  $A$  that outputs a  $(1 - \epsilon)$ -approximate cut of a random  $d$ -regular graph  $G$  with high probability under a “no-OGP” assumption for any  $d \geq 3$ .*

Note that the approach of Alaoui et al. [2] critically relies on the Guerra-Tonnineli interpolation between the  $\mathcal{G}_{n,d}$  model and the SK-model which will *only* work for  $d \geq O(\frac{1}{\epsilon})$ . Consequently, a solution that works for *all*  $d \geq 3$  will require a fundamentally different approach. A natural question that is motivated by the above conjecture is to then investigate if there is any range of degree for which the MAX-CUT problem over  $d$ -regular graphs possesses an OGP. Given the belief that the SK model does not exhibit an OGP, this would only be an interesting question in the relatively low-degree regime.

► **Problem 5.3** (Random  $d$ -Regular Graphs don't have an OGP). *Does the MAX-CUT problem on random  $d$ -regular graphs have an OGP for some  $d \geq 3$ ? If so, for what  $\{d_0, d_1\} \subset \mathbb{N}$  does the problem exhibit an OGP?*

The study of  $QAOA_p$  was initiated on MAX-CUT for  $d$ -regular graphs: positive answers to the conjectures above would resolve the question of quantum advantage on the problem.

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