Compositional Confluence Criteria

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Abstract

We show how confluence criteria based on decreasing diagrams are generalized to ones composable with other criteria. For demonstration of the method, the confluence criteria of orthogonality, rule labeling, and critical pair systems for term rewriting are recast into composable forms. In addition to them, we prove that Toyama's parallel closedness result based on parallel critical pairs subsumes his almost parallel closedness theorem.

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1 Introduction

Confluence is a property of rewriting that ensures uniqueness of computation results. In the last decades, various proof methods for confluence of term rewrite systems have been developed. They are roughly classified to three groups: (direct) confluence criteria based on critical pair analysis [18, 15, 30, 32, 10, 36, 23, 37, 40], decomposition methods based on modularity and commutation [31, 3, 27], and transformation methods based on simulation of rewriting [2, 17, 20, 27].

In this paper we present a confluence analysis based on *compositional* confluence criteria. Here a compositional criterion means a sufficient condition that, given a rewrite system $\mathcal{R}$ and its subsystem $\mathcal{C} \subseteq \mathcal{R}$, the confluence of $\mathcal{C}$ implies that of $\mathcal{R}$. Since such a subsystem can be analyzed by any other (compositional) confluence criterion, compositional criteria can be seen as a combination method for confluence analysis. Because the empty system is confluent, by taking the empty subsystem $\mathcal{C}$ compositional criteria can be used as ordinary (direct) confluence criteria.

In order to develop compositional confluence criteria we revisit van Oostrom's decreasing diagram technique [35, 37], which is known as a powerful confluence criterion for abstract rewrite systems. Most of existing confluence criteria for left-linear rewrite systems, including the ones listed above, can be proved by decreasingness of parallel steps or multi-steps. Recasting the decreasing diagram technique as a compositional criterion, we demonstrate how confluence criteria based on decreasing diagrams can be reformulated as compositional versions. We pick up the confluence criteria by orthogonality [26], rule labeling [40], and critical pair systems [11].

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In addition, we elucidate the hierarchy of Toyama's two parallel closedness theorems [30, 32] and rule labeling based on parallel critical pairs [40]. As a consequence, it turns out that rule labeling and its compositional version are generalizations of Huet's and Toyama's (almost) parallel closedness theorems.

The remaining part of the paper is organized as follows: In Section 2 we recall notions from rewriting. In Section 3 we show that Toyama's almost parallel closedness is subsumed by his earlier result based on parallel critical pairs. In Section 4, we introduce an abstract criterion for our approach, and in the subsequent three sections we derive compositional criteria from the confluence criteria of orthogonality (Section 5), rule labeling (Section 6), and the criterion by critical pair systems (Section 7). Section 8 reports experimental results. Discussing related work and potential future work in Section 9, we conclude the paper.

2 Preliminaries

Throughout the paper, we assume familiarity with abstract rewriting and term rewriting [4, 29]. We just recall some basic notions and notations for rewriting and confluence.

An (I-indexed) *abstract rewrite system* (ARS) $\mathcal{A}$ is a pair $(A, \{\to_\alpha\}_{\alpha \in I})$ consisting of a set A and the family of relations $\to_\alpha$ on A. Given a subset J of I, we write $x \to_J y$ if $x \to_\alpha y$ for some index $\alpha \in J$. The relation $\to_I$ is referred to as $\to_\mathcal{A}$. An ARS $\mathcal{A}$ is called *confluent* or *locally confluent* if ${}^*_\mathcal{A}\!\leftarrow \cdot \to^*_\mathcal{A} \subseteq \to^*_\mathcal{A} \cdot {}^*_\mathcal{A}\!\leftarrow$ or ${}_\mathcal{A}\!\leftarrow \cdot \to_\mathcal{A} \subseteq \to^*_\mathcal{A} \cdot {}^*_\mathcal{A}\!\leftarrow$ holds, respectively. We say that ARSs $\mathcal{A}$ and $\mathcal{B}$ *commute* if ${}^*_\mathcal{A}\!\leftarrow \cdot \to^*_\mathcal{B} \subseteq \to^*_\mathcal{B} \cdot {}^*_\mathcal{A}\!\leftarrow$ holds. A conversion of form $b \;{}_\mathcal{A}\!\leftarrow a \to_\mathcal{B} c$ is called a *local peak* (or simply a *peak*) between $\mathcal{A}$ and $\mathcal{B}$. An ARS $\mathcal{A}$ is *terminating* if there exists no infinite sequence $a_0 \to_\mathcal{A} a_1 \to_\mathcal{A} \cdots$. We define $\to_{\mathcal{A}/\mathcal{B}}$ as $\to^*_\mathcal{B} \cdot \to_\mathcal{A} \cdot \to^*_\mathcal{B}$. We say that $\mathcal{A}$ is *relatively terminating* with respect to $\mathcal{B}$, or simply $\mathcal{A}/\mathcal{B}$ is *terminating*, if $\to_{\mathcal{A}/\mathcal{B}}$ is terminating.

Positions are sequences of positive integers. The empty sequence ϵ is called the *root* position. We write $p \cdot q$ or simply pq for the concatenation of positions p and q. The prefix order $\leqslant$ on positions is defined as $p \leqslant q$ if $p \cdot p' = q$ for some p'. We say that positions p and q are *parallel* if $p \nleqslant q$ and $q \nleqslant p$. A set of positions is called *parallel* if all its elements are so.

Terms are built from a signature $\mathcal{F}$ and a countable set $\mathcal{V}$ of variables satisfying $\mathcal{F} \cap \mathcal{V} = \varnothing$. The set of all terms is denoted by $\mathcal{T}(\mathcal{F}, \mathcal{V})$. Let t be a term. The set of all variables in t is denoted by $\mathcal{V}\mathsf{ar}(t)$, and the set of all function positions and the set of variable positions in t by $\mathcal{P}\mathsf{os}_\mathcal{F}(t)$ and $\mathcal{P}\mathsf{os}_\mathcal{V}(t)$, respectively. The *subterm* of t at position p is denoted by $t|_p$. It is a *proper* subterm if $p \neq \epsilon$. By $t[u]_p$ we denote the term that results from replacing the subterm of t at p by term u. The size $|t|$ of t is the number of occurrences of functions symbols and variables in t. A term t is said to be *linear* if every variable in t occurs exactly once.

A *substitution* is a mapping $\sigma : \mathcal{V} \to \mathcal{T}(\mathcal{F}, \mathcal{V})$ whose domain $\mathcal{D}\mathsf{om}(\sigma)$ is finite. Here $\mathcal{D}\mathsf{om}(\sigma)$ stands for the set $\{x \in \mathcal{V} \mid \sigma(x) \neq x\}$. The term $t\sigma$ is defined as $\sigma(t)$ for $t \in \mathcal{V}$, and $f(t_1\sigma, \ldots, t_n\sigma)$ for $t = f(t_1, \ldots, t_n)$. A term u is called an *instance* of t if $u = t\sigma$ for some σ. A substitution is called a *renaming* if it is a bijection on variables.

A *term rewrite system* (TRS) over $\mathcal{F}$ is a set of rewrite rules. Here a pair (ℓ, r) of terms is a *rewrite rule* or simply a *rule* if $\ell \notin \mathcal{V}$ and $\mathcal{V}\mathsf{ar}(r) \subseteq \mathcal{V}\mathsf{ar}(\ell)$. We denote it by $\ell \to r$. The rewrite relation $\to_\mathcal{R}$ of a TRS $\mathcal{R}$ is defined on terms as follows: $s \to_\mathcal{R} t$ if $s|_p = \ell\sigma$ and $t = s[r\sigma]_p$ for some rule $\ell \to r \in \mathcal{R}$, position p, and substitution σ. We write $s \xrightarrow{p}_\mathcal{R} t$ if the rewrite position p is relevant. We call subsets of $\mathcal{R}$ *subsystems*. A TRS $\mathcal{R}$ is *left-linear* if ℓ is linear for all $\ell \to r \in \mathcal{R}$. Since any TRS $\mathcal{R}$ can be regarded as the ARS $(\mathcal{T}(\mathcal{F}, \mathcal{V}), \{\to_\mathcal{R}\})$, we use notions and notations of ARSs for TRSs. For instance, a TRS $\mathcal{R}$ is (locally) confluent if the ARS $(\mathcal{T}(\mathcal{F},\mathcal{V}),\{\to_\mathcal{R}\})$ is so. Similarly, two TRSs commute if their corresponding ARSs commute.

Local confluence of TRSs is characterized by notion of critical pair. We say that a rule $\ell_1 \to r_1$ is a *variant* of a rule $\ell_2 \to r_2$ if $\ell_1\rho = \ell_2$ and $r_1\rho = r_2$ for some renaming ρ.

▶ **Definition 1.** *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs. Suppose that the following conditions hold:*

- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ are variants of rules in $\mathcal{R}$ and in $\mathcal{S}$, respectively,*
- *$\ell_1 \to r_1$ and $\ell_2 \to r_2$ have no common variables,*
- *$p \in \mathcal{P}\mathsf{os}_\mathcal{F}(\ell_2)$,*
- *σ is a most general unifier of ℓ_1 and $\ell_2|_p$, and*
- *if $p = \epsilon$ then $\ell_1 \to r_1$ is not a variant of $\ell_2 \to r_2$.*

The local peak $(\ell_2\sigma)[r_1\sigma]_p \;{}_\mathcal{R}\!\overset{p}{\leftarrow}\; \ell_2\sigma \overset{\epsilon}{\to}_\mathcal{S} r_2\sigma$ is called a critical peak *between $\mathcal{R}$ and $\mathcal{S}$. When $t \;{}_\mathcal{R}\!\overset{p}{\leftarrow}\; s \overset{\epsilon}{\to}_\mathcal{S} u$ is a critical peak, the pair (t, u) is called a* critical pair*. To clarify the orientation of the pair, we denote it as the binary relation $t \;{}_\mathcal{R}\!\overset{p}{\leftarrow}\!\rtimes\!\overset{\epsilon}{\to}_\mathcal{S} u$, see [6]. Moreover, we write $t \;{}_\mathcal{R}\!\leftarrow\!\rtimes\!\overset{\epsilon}{\to}_\mathcal{S} u$ if $t \;{}_\mathcal{R}\!\overset{p}{\leftarrow}\!\rtimes\!\overset{\epsilon}{\to}_\mathcal{S} u$ for some position p.*

▶ **Theorem 2** ([15])**.** *A TRS $\mathcal{R}$ is locally confluent if and only if ${}_\mathcal{R}\!\leftarrow\!\rtimes\!\overset{\epsilon}{\to}_\mathcal{R} \subseteq \to_\mathcal{R}^* \cdot {}_\mathcal{R}^*\!\leftarrow$ holds.*

Combining it with Newman's Lemma [21], we obtain Knuth and Bendix' criterion [18].

▶ **Theorem 3** ([18])**.** *A terminating TRS $\mathcal{R}$ is confluent if and only if ${}_\mathcal{R}\!\leftarrow\!\rtimes\!\overset{\epsilon}{\to}_\mathcal{R} \subseteq \to_\mathcal{R}^* \cdot {}_\mathcal{R}^*\!\leftarrow$ holds.*

We define the parallel step relation, which plays a key role in analysis of local peaks.

▶ **Definition 4.** *Let $\mathcal{R}$ be a TRS and let P be a set of parallel positions. The* parallel step *$\overset{P}{\twoheadrightarrow}_\mathcal{R}$ is inductively defined on terms as follows:*

- *$x \overset{P}{\twoheadrightarrow}_\mathcal{R} x$ if x is a variable and $P = \varnothing$.*
- *$\ell\sigma \overset{P}{\twoheadrightarrow}_\mathcal{R} r\sigma$ if $\ell \to r$ is an $\mathcal{R}$-rule, σ is a substitution, and $P = \{\epsilon\}$.*
- *$f(s_1,\ldots,s_n) \overset{P}{\twoheadrightarrow}_\mathcal{R} f(t_1,\ldots,t_n)$ if f is an n-ary function symbol in $\mathcal{F}$, $s_i \overset{P_i}{\twoheadrightarrow}_\mathcal{R} t_i$ holds for all $1 \leqslant i \leqslant n$, and $P = \{i \cdot p \mid 1 \leqslant i \leqslant n \text{ and } p \in P_i\}$.*

We write $s \twoheadrightarrow_\mathcal{R} t$ if $s \overset{P}{\twoheadrightarrow}_\mathcal{R} t$ for some set P of positions.

Note that $\twoheadrightarrow_\mathcal{R}$ is reflexive and the inclusions $\to_\mathcal{R} \subseteq \twoheadrightarrow_\mathcal{R} \subseteq \to_\mathcal{R}^*$ hold. As the latter entails $\to_\mathcal{R}^* = \twoheadrightarrow_\mathcal{R}^*$, we obtain the following useful characterizations.

▶ **Lemma 5.** *A TRS $\mathcal{R}$ is confluent if and only if $\twoheadrightarrow_\mathcal{R}$ is confluent. Similarly, TRSs $\mathcal{R}$ and $\mathcal{S}$ commute if and only if $\twoheadrightarrow_\mathcal{R}$ and $\twoheadrightarrow_\mathcal{S}$ commute.*

3 Parallel Closedness

Toyama made two variations of Huet's parallel closedness theorem [15] in 1981 [30] and in 1988 [32], but their relation has not been known. In this section we recall his and related results, and then show that Toyama's earlier result subsumes the later one. For brevity we omit the subscript $\mathcal{R}$ from $\to_\mathcal{R}$, $\twoheadrightarrow_\mathcal{R}$, and ${}_\mathcal{R}\!\leftarrow\!\rtimes\!\overset{\epsilon}{\to}_\mathcal{R}$ when it is clear from the contexts.

▶ **Definition 6** ([15])**.** *A TRS is* parallel closed *if $\leftarrow\!\rtimes\!\overset{\epsilon}{\to} \subseteq \twoheadrightarrow$ holds.*

▶ **Theorem 7** ([15])**.** *A left-linear TRS is confluent if it is parallel closed.*

In 1988, Toyama showed that the closing form for *overlay* critical pairs, originating from root overlaps, can be relaxed. We write $t \xleftarrow{>\epsilon}\!\rtimes\!\xrightarrow{\epsilon} u$ if $t \xleftarrow{p}\!\rtimes\!\xrightarrow{\epsilon} u$ holds for some $p > \epsilon$.

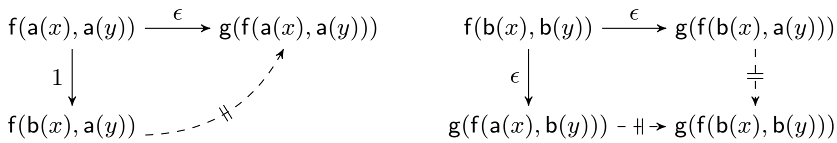
▶ **Definition 8** ([32]). *A TRS is* almost parallel closed *if* $\xleftarrow{\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{\|\!\!\!\!\to} \cdot {}^{*}\!\leftarrow$ *and* $\xleftarrow{>\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{\|\!\!\!\!\to}$ *hold.*

▶ **Theorem 9** ([32]). *A left-linear TRS is confluent if it is almost parallel closed.*

▶ **Example 10.** Consider the following left-linear and non-terminating TRS, which is a variant of the TRS in [10, Example 5.4].

$$\mathsf{a}(x) \to \mathsf{b}(x) \qquad \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \to \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$$
$$\mathsf{f}(\mathsf{b}(x),y) \to \mathsf{g}(\mathsf{f}(\mathsf{a}(x),y)) \qquad \mathsf{f}(x,\mathsf{b}(y)) \to \mathsf{g}(\mathsf{f}(x,\mathsf{a}(y)))$$

Out of the three critical pairs, two critical pairs including the next diagram (i) are closed by single parallel steps. The remaining pair (ii) joins by performing a single parallel step on each side:



(i) (ii)

Thus, the TRS is almost parallel closed. Hence, the TRS is confluent.

Inspired by almost parallel closedness, Gramlich [10] developed a confluence criterion based on *parallel critical pairs* in 1996. Let t be a term and let P be a set of parallel positions in t. We write $\mathcal{V}\mathsf{ar}(t,P)$ for the union of $\mathcal{V}\mathsf{ar}(t|_p)$ for all $p \in P$. By $t[u_p]_{p\in P}$ we denote the term that results from replacing in t the subterm at p by a term u_p for all $p \in P$.

▶ **Definition 11.** *Let $\mathcal{R}$ and $\mathcal{S}$ be TRSs, $\ell \to r$ a variant of an $\mathcal{S}$-rule, and $\{\ell_p \to r_p\}_{p\in P}$ a family of variants of $\mathcal{R}$-rules, where P is a set of positions. A local peak*

$$(\ell\sigma)[r_p\sigma]_{p\in P} \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\|}\; \ell\sigma \xrightarrow{\epsilon}_{\mathcal{S}} r\sigma$$

is called a parallel critical peak *between $\mathcal{R}$ and $\mathcal{S}$ if the following conditions hold:*

- *$P \subseteq \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell)$ is a non-empty set of parallel positions in ℓ,*
- *none of rules $\ell \to r$ and $\ell_p \to r_p$ for $p \in P$ shares a variable with other rules,*
- *σ is a most general unifier of $\{\ell_p \approx (\ell|_p)\}_{p\in P}$, and*
- *if $P = \{\epsilon\}$ then $\ell_\epsilon \to r_\epsilon$ is not a variant of $\ell \to r$.*

When $t \;{}_{\mathcal{R}}\!\overset{P}{\mathrel{\leftarrow\!\!\!\!\|}}\; s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is a parallel critical peak, the pair (t,u) is called a parallel critical pair*, and denoted by $t \;{}_{\mathcal{R}}\!\overset{P}{\mathrel{\leftarrow\!\!\!\!\|}}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$. In the case of $P \nsubseteq \{\epsilon\}$ the parallel critical pair is written as $t \;{}_{\mathcal{R}}\!\overset{>\epsilon}{\mathrel{\leftarrow\!\!\!\!\|}}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} u$. Whenever no confusion arises, we abbreviate ${}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\|}\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}}$ to $\mathrel{\leftarrow\!\!\!\!\|}\!\rtimes\!\xrightarrow{\epsilon}$.*

Consider a local peak $t \;{}_{\mathcal{R}}\!\overset{P}{\mathrel{\leftarrow\!\!\!\!\|}}\; s \xrightarrow{\epsilon}_{\mathcal{S}} u$ that employs a rule $\ell_p \to r_p$ at $p \in P$ in the left step and a rule $\ell \to r$ in the right step. We say that the peak is *orthogonal* if $P \cap \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell) = \varnothing$. A local peak $t \;{}_{\mathcal{R}}\!\xleftarrow{p}\; s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is *orthogonal* if $t \;{}_{\mathcal{R}}\!\overset{\{p\}}{\mathrel{\leftarrow\!\!\!\!\|}}\; s \xrightarrow{\epsilon}_{\mathcal{S}} u$ is.

▶ **Theorem 12** ([10]). *A left-linear TRS is confluent if the inclusions $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{\|\!\!\!\!\to} \cdot {}^{*}\!\leftarrow$ and $\overset{>\epsilon}{\mathrel{\leftarrow\!\!\!\!\|}}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \to^{*}$ hold.*

Unfortunately, this criterion by Gramlich does not subsume (almost) parallel closedness.

▶ **Example 13** (Continued from Example 10)**.** The TRS admits the parallel critical peak $\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)) \overset{\{1,2\}}{\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+}} \mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) \xrightarrow{\epsilon} \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$. However, $\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y)) \to^* \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)))$ does not hold.

As noted in the paper [10], Toyama [30] had already obtained in 1981 a closedness result that subsumes Theorem 12. His idea is to impose variable conditions on parallel steps $\mathrel{+\!\!\!+\!\!\!\!\!\to}$.

▶ **Theorem 14** ([30])**.** *A left-linear TRS is confluent if the following conditions hold:*

(a) *The inclusion* $\leftarrow\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{+\!\!\!+\!\!\!\!\!\to} \cdot \, {}^*\!\!\leftarrow$ *holds.*

(b) *For every parallel critical peak* $t \overset{P}{\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+}} s \xrightarrow{\epsilon} u$ *there exist a term* v *and a set* P' *of parallel positions such that* $t \to^* v \overset{P'}{\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+}} u$ *and* $\mathcal{V}\mathsf{ar}(v,P') \subseteq \mathcal{V}\mathsf{ar}(s,P)$.

▶ **Example 15** (Continued from Example 13)**.** The confluence of the TRS in Example 10 can be shown by Theorem 14. Since condition (a) of Theorem 14 follows from the almost parallel closedness, it is enough to verify condition (b). The following parallel critical peak, which Theorem 12 fails to handle, admits the following diagram:

$$\begin{array}{ccc}
\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)) & \xrightarrow{\epsilon} & \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y))) \\
\{1,2\}\ \big\downarrow\!\!\!\!\!+\!\!\!+ & & \vdots\!\!\!+\ \{1\cdot 2\} \\
\mathsf{g}(\mathsf{f}(\mathsf{b}(x),\mathsf{b}(y))) & \dashrightarrow & \mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{b}(y)))
\end{array}$$

Because $\mathcal{V}\mathsf{ar}(\mathsf{g}(\mathsf{f}(\mathsf{a}(x),\mathsf{b}(y))),\{1\cdot 2\}) = \{y\} \subseteq \{x,y\} = \mathcal{V}\mathsf{ar}(\mathsf{f}(\mathsf{a}(x),\mathsf{a}(y)),\{1,2\})$ holds, the parallel critical peak satisfies condition (b) in Theorem 14. Similarly, we can find suitable diagrams for the other parallel critical peaks. Hence, (b) holds for the TRS.

Now we show that Theorem 14 even subsumes Theorem 9. Revisiting the Parallel Moves Lemma [4, Lemma 6.4.4], we show that the variable condition of Theorem 14 is generalized to local peaks of form $\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+} \cdot \xrightarrow{\epsilon}$. We write $\sigma \mathrel{+\!\!\!+\!\!\!\!\!\to}_{\mathcal{R}} \tau$ if $x\sigma \mathrel{+\!\!\!+\!\!\!\!\!\to}_{\mathcal{R}} x\tau$ for all variables x.

▶ **Lemma 16.** *Let* $\mathcal{R}$ *be a TRS and* $\ell \to r$ *a left-linear rule. Consider a local peak* Γ *of the form* $t \;{}_{\mathcal{R}}\!\overset{P}{\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+}} s \xrightarrow{\epsilon}_{\{\ell\to r\}} u$.

(a) *If* Γ *is orthogonal,* $t \xrightarrow{\epsilon}_{\{\ell\to r\}} v \;{}_{\mathcal{R}}\!\overset{P'}{\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+}} u$ *and* $\mathcal{V}\mathsf{ar}(v,P') \subseteq \mathcal{V}\mathsf{ar}(s,P)$ *for some* v *and* P'.

(b) *Otherwise, there exist a parallel critical peak* $t_0 \;{}_{\mathcal{R}}\!\overset{P_0}{\mathrel{\leftarrow\!\!\!\!\!+\!\!\!+}} s_0 \xrightarrow{\epsilon}_{\{\ell\to r\}} u_0$ *and substitutions* σ *and* τ *such that* $s = s_0\sigma$, $t = t_0\tau$, $u = u_0\sigma$, $\sigma \mathrel{+\!\!\!+\!\!\!\!\!\to}_{\mathcal{R}} \tau$, $t_0\sigma \overset{P\setminus P_0}{\mathrel{+\!\!\!+\!\!\!\!\!\to}}_{\mathcal{R}} t_0\tau$*, and* $P_0 \subseteq P$.

Proof. As (b) is a known result [40, Lemma 55], we only show (a). Suppose that Γ is orthogonal. Since $s \xrightarrow{\epsilon}_{\{\ell\to r\}} u$ holds, there exists a substitution σ with $s = \ell\sigma$ and $u = r\sigma$. As ℓ is linear and Γ is orthogonal, $t = \ell\tau$ and $\sigma \mathrel{+\!\!\!+\!\!\!\!\!\to} \tau$ for some τ. Take $v = r\tau$ and define P' as follows:

$$P' = \{p_1' \cdot p_2 \mid p_1 \cdot p_2 \in P,\ p_1' \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(r), \text{ and } \ell|_{p_1} = r|_{p_1'} \text{ for some } p_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\ell)\}$$

Clearly, $t \xrightarrow{\epsilon}_{\{\ell\to r\}} v$ holds. So it remains to show $u \overset{P'}{\mathrel{+\!\!\!+\!\!\!\!\!\to}}_{\mathcal{R}} v$ and $\mathcal{V}\mathsf{ar}(v,P') \subseteq \mathcal{V}\mathsf{ar}(s,P)$. Let p' be an arbitrary position in P'. There exist positions $p_1 \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(\ell)$, $p_1' \in \mathcal{P}\mathsf{os}_{\mathcal{V}}(r)$, and p_2 such that $p' = p_1' \cdot p_2$, $p_1 \cdot p_2 \in P$, and $\ell|_{p_1} = r|_{p_1'}$. Denoting $p_1 \cdot p_2$ by p, we have the identities:

$$u|_{p'} = (r\sigma)|_{p_1'\cdot p_2} = (r|_{p_1'}\sigma)|_{p_2} = (\ell|_{p_1}\sigma)|_{p_2} = (\ell\sigma)|_{p_1\cdot p_2} = s|_p$$
$$v|_{p'} = (r\tau)|_{p_1'\cdot p_2} = (r|_{p_1'}\tau)|_{p_2} = (\ell|_{p_1}\tau)|_{p_2} = (\ell\tau)|_{p_1\cdot p_2} = t|_p$$

From $s \xrightarrow{P}\!\!\!\!\!\!\!\!+\!\!\!\!\!\!+\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\to_{\mathcal{R}} t$ we obtain $s|_p \xrightarrow{\epsilon}_{\mathcal{R}} t|_p$ and thus $u|_{p'} \xrightarrow{\epsilon}_{\mathcal{R}} v|_{p'}$. Therefore, $u \xrightarrow{P'}_{\mathcal{R}} v$ is obtained. Moreover, we have $\mathcal{V}\mathsf{ar}(v|_{p'}) = \mathcal{V}\mathsf{ar}(t|_p) \subseteq \mathcal{V}\mathsf{ar}(s|_p) \subseteq \mathcal{V}\mathsf{ar}(s, P)$. As $\mathcal{V}\mathsf{ar}(v, P')$ is the union of $\mathcal{V}\mathsf{ar}(v|_{p'})$ for all $p' \in P'$, the desired inclusion $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$ follows. ◀

For almost parallel closed TRSs the above statement is extended to local peaks $\nleftarrow\!\!\!\!+\!\!\!\!\!\!+ \cdot \,+\!\!\!\!\!\!+\!\!\!\!\!\!\to$ of parallel steps. In its proof we measure parallel steps $s \xrightarrow{P} t$ in such a local peak by the *amount of contractums* $|t|_P$, namely the sum of $|(t|_p)|$ for all $p \in P$. Note that this measure attributes to [24, 19].

▶ **Lemma 17.** *Consider a left-linear almost parallel closed TRS. If $t \xleftarrow{P_1} s \xrightarrow{P_2} u$ then*

- *$t \to^* v_1 \xleftarrow{P_1'} u$ for some v_1 and P_1' with $\mathcal{V}\mathsf{ar}(v_1, P_1') \subseteq \mathcal{V}\mathsf{ar}(s, P_1)$, and*
- *$t \xrightarrow{P_2'} v_2 \;{}^*\!\!\leftarrow u$ for some v_2 and P_2' with $\mathcal{V}\mathsf{ar}(v_2, P_2') \subseteq \mathcal{V}\mathsf{ar}(s, P_2)$.*

Proof. Let $\Gamma\colon t \xleftarrow{P_1} s \xrightarrow{P_2} u$ be a local peak. We show the claim by well-founded induction on $(|t|_{P_1} + |u|_{P_2}, s)$ with respect to $\succ$. Here $(m, s) \succ (n, t)$ if either $m > n$, or $m = n$ and t is a proper subterm of s. Depending on the shape of Γ, we distinguish six cases.

1. If P_1 or P_2 is empty then the claim follows from the fact: $\mathcal{V}\mathsf{ar}(v, P) \subseteq \mathcal{V}\mathsf{ar}(w, P)$ if $w \xrightarrow{P} v$.
2. If P_1 or P_2 is $\{\epsilon\}$ and Γ is orthogonal then Lemma 16(a) applies.
3. If $P_1 = P_2 = \{\epsilon\}$ and Γ is not orthogonal then Γ is an instance of a critical peak. By almost parallel closedness $t \to^* v_1 \xleftarrow{Q_1} u$ and $t \xrightarrow{Q_2} v_2 \;{}^*\!\!\leftarrow u$ for some v_1, v_2, Q_1, and Q_2. For each $k \in \{1, 2\}$ we have $s \to^* v_k$, so $\mathcal{V}\mathsf{ar}(v_k) \subseteq \mathcal{V}\mathsf{ar}(s)$ follows. Thus, $\mathcal{V}\mathsf{ar}(v_k, Q_k) \subseteq \mathcal{V}\mathsf{ar}(v_k) \subseteq \mathcal{V}\mathsf{ar}(s) = \mathcal{V}\mathsf{ar}(s, \{\epsilon\})$. The claim holds.
4. If $P_1 \nsubseteq \{\epsilon\}$, $P_2 = \{\epsilon\}$, and Γ is not orthogonal then there is $p \in P_1$ such that $s' \xleftarrow{p} s \xrightarrow{\epsilon} u$ is an instance of a critical peak and $s' \xrightarrow{P_1 \setminus \{p\}} t$ follows by Lemma 16(b) where $P = \{p\}$. By the almost parallel closedness $s' \xrightarrow{P_2'} u$ for some P_2'. Since P_2' is a set of parallel positions in u, we have $|u|_{\{\epsilon\}} = |u| \geqslant |u|_{P_2'}$. As $|u|_{\{\epsilon\}} \geqslant |u|_{P_2'}$ and $|t|_{P_1} > |t|_{P_1 \setminus \{p\}}$ yield $|t|_{P_1} + |u|_{\{\epsilon\}} > |t|_{P_1 \setminus \{p\}} + |u|_{P_2'}$, we obtain the inequality:
   $$(|t|_{P_1} + |u|_{P_2}, s) \succ (|t|_{P_1 \setminus \{p\}} + |u|_{P_2'}, s')$$
   Thus, the claim follows by the induction hypothesis for $t \xleftarrow{P_1 \setminus \{p\}} s' \xrightarrow{P_2'} u$ and the inclusions $\mathcal{V}\mathsf{ar}(s', P_1 \setminus \{p\}) \subseteq \mathcal{V}\mathsf{ar}(s, P_1)$ and $\mathcal{V}\mathsf{ar}(s', P_2') \subseteq \mathcal{V}\mathsf{ar}(s, \{\epsilon\})$.
5. If $P_1 = \{\epsilon\}$, $P_2 \nsubseteq \{\epsilon\}$, and Γ is not orthogonal then the proof is analogous to the last case.
6. If $P_1 \nsubseteq \{\epsilon\}$ and $P_2 \nsubseteq \{\epsilon\}$ then we may assume $s = f(s_1, \ldots, s_n)$, $t = f(t_1, \ldots, t_n)$, $u = f(u_1, \ldots, u_n)$, and $t_i \xleftarrow{P_1^i} s_i \xrightarrow{P_2^i} u_i$ for all $1 \leqslant i \leqslant n$. Here P_k^i denotes the set $\{p \mid i \cdot p \in P_k\}$. For each $i \in \{1, \ldots, n\}$, we have $|t|_{P_1} \geqslant |t_i|_{P_1^i}$ and $|u|_{P_2} \geqslant |u_i|_{P_2^i}$, and therefore $|t|_{P_1} + |u|_{P_2} \geqslant |t_i|_{P_1^i} + |u_i|_{P_2^i}$. So we deduce the following inequality:
   $$(|t|_{P_1} + |u|_{P_2}, s) \succ (|t_i|_{P_1^i} + |u_i|_{P_2^i}, s_i)$$
   Consider an i-th peak $t_i \xleftarrow{P_1^i} s_i \xrightarrow{P_2^i} u_i$. By the induction hypothesis it admits valleys of the forms $t_i \to^* v_1^i \xleftarrow{Q_1^i} u_i$ and $t_i \xrightarrow{Q_2^i} v_2^i \;{}^*\!\!\leftarrow u_i$ such that $\mathcal{V}\mathsf{ar}(v_k^i, Q_k^i) \subseteq \mathcal{V}\mathsf{ar}(s_i, P_k^i)$ for both $k \in \{1, 2\}$. For each k, define $Q_k = \{i \cdot q \mid 1 \leqslant i \leqslant n \text{ and } q \in Q_k^i\}$ and $v_k = f(v_k^1, \ldots, v_k^n)$. Then we have $t \to^* v_1 \xleftarrow{Q_1} u$ and $t \xrightarrow{Q_2} v_2 \;{}^*\!\!\leftarrow u$. Moreover,
   $$\mathcal{V}\mathsf{ar}(v_k, Q_k) = \bigcup_{i=1}^{n} \mathcal{V}\mathsf{ar}(v_k^i, Q_k^i) \subseteq \bigcup_{i=1}^{n} \mathcal{V}\mathsf{ar}(s_i, P_k^i) = \mathcal{V}\mathsf{ar}(s, P_k)$$
   holds. Hence, the claim follows. ◀

▶ **Theorem 18.** *Every left-linear and almost parallel closed TRS satisfies conditions (a) and (b) of Theorem 14. In other words, Theorem 14 subsumes Theorem 9.*

Proof. Since (parallel) critical peaks are instances of $\leftarrow\!\!\!\!\!\!+\!\!\!\!+ \cdot +\!\!\!\!+\!\!\!\!\!\!\rightarrow$, Lemma 17 entails the claim. ◀

Note that Theorem 9 does not subsume Theorem 14 as witnessed by the TRS consisting of the four rules $\mathsf{f}(\mathsf{a}) \to \mathsf{c}$, $\mathsf{a} \to \mathsf{b}$, $\mathsf{f}(\mathsf{b}) \to \mathsf{b}$, and $\mathsf{c} \to \mathsf{b}$. In Section 6 we will see that Theorem 14 is subsumed by a variant of rule labeling.

4 Decreasing Diagrams with Commuting Subsystems

We make a variant of decreasing diagrams [35, 37]. First we recall the commutation version of the technique [37]. Let $\mathcal{A} = (A, \{\to_{1,\alpha}\}_{\alpha \in I})$ and $\mathcal{B} = (A, \{\to_{2,\beta}\}_{\beta \in J})$ be I-indexed and J-indexed ARSs on the same domain, respectively. Let $>$ be a well-founded order $>$ on $I \cup J$. By $\curlyvee\alpha$ we denote the set $\{\beta \in I \cup J \mid \alpha > \beta\}$, and by $\curlyvee\alpha\beta$ we denote $(\curlyvee\alpha) \cup (\curlyvee\beta)$. We say that a local peak $b \;{}_{1,\alpha}\!\!\leftarrow a \to_{2,\beta} c$ is *decreasing* if

$$b \xleftrightarrow[\curlyvee\alpha]{*} \cdot \xrightarrow[2,\beta]{=} \cdot \xleftrightarrow[\curlyvee\alpha\beta]{*} \cdot \xleftarrow[1,\alpha]{=} \cdot \xleftrightarrow[\curlyvee\beta]{*} c$$

holds. Here $\leftrightarrow_K$ stands for the union of ${}_{1,\gamma}\!\!\leftarrow$ and $\to_{2,\gamma}$ for all $\gamma \in K$. The ARSs $\mathcal{A}$ and $\mathcal{B}$ are *decreasing* if every local peak $b \;{}_{1,\alpha}\!\!\leftarrow a \to_{2,\beta} c$ with $(\alpha, \beta) \in I \times J$ is decreasing. In the case of $\mathcal{A} = \mathcal{B}$, we simply say that $\mathcal{A}$ is decreasing.

▶ **Theorem 19** ([37]). *If two ARSs are decreasing then they commute.*

We present the abstract principle of our compositional criteria. The idea of using the minimum index in the decreasing diagram technique is taken from [16, 9, 7].

▶ **Theorem 20.** *Let $\mathcal{A} = (A, \{\to_{1,\alpha}\}_{\alpha \in I})$ and $\mathcal{B} = (A, \{\to_{2,\beta}\}_{\beta \in I})$ be I-indexed ARSs equipped with a well-founded order $>$ on I. Suppose that $\bot$ is the minimum element in I and $\to_{1,\bot}$ and $\to_{2,\bot}$ commute. The ARSs $\mathcal{A}$ and $\mathcal{B}$ commute if every local peak ${}_{1,\alpha}\!\!\leftarrow \cdot \to_{2,\beta}$ with $(\alpha, \beta) \in I^2 \setminus \{(\bot, \bot)\}$ is decreasing.*

Proof. We define the two ARSs $\mathcal{A}' = (A, \{\Rightarrow_{1,\alpha}\}_{\alpha \in I})$ and $\mathcal{B}' = (A, \{\Rightarrow_{2,\alpha}\}_{\alpha \in I})$ as follows:

$$\Rightarrow_{i,\alpha} = \begin{cases} \to_{i,\alpha}^{*} & \text{if } \alpha = \bot \\ \to_{i,\alpha} & \text{otherwise} \end{cases}$$

Since $\to_{\mathcal{A}}^{*} = \Rightarrow_{\mathcal{A}}^{*}$ and $\to_{\mathcal{B}}^{*} = \Rightarrow_{\mathcal{B}}^{*}$, the commutation of $\mathcal{A}$ and $\mathcal{B}$ follows from that of $\mathcal{A}'$ and $\mathcal{B}'$. We show the latter by proving decreasingness of $\mathcal{A}'$ and $\mathcal{B}'$ with respect to the given well-founded order $>$. Let Γ be a local peak of form ${}_{1,\alpha}\!\!\Leftarrow \cdot \Rightarrow_{2,\beta}$. We distinguish four cases.

- If neither α nor β is $\bot$ then decreasingness of Γ follows from the assumption.
- If both α and β are $\bot$ then the commutation of $\to_{1,\bot}$ and $\to_{2,\bot}$ yields the inclusion:

  $$\xLeftarrow[1,\bot]{} \cdot \xRightarrow[2,\bot]{} \subseteq \xRightarrow[2,\bot]{} \cdot \xLeftarrow[1,\bot]{}$$

  Thus Γ is decreasing.
- If $\beta > \alpha = \bot$ then we have ${}_{1,\alpha}\!\!\leftarrow \cdot \to_{2,\beta} \subseteq \to_{2,\beta}^{=} \cdot \leftrightarrow_{\curlyvee\beta}^{*}$ Therefore, easy induction on n shows the inclusion ${}_{1,\alpha}^{\;n}\!\!\leftarrow \cdot \to_{2,\beta} \subseteq \to_{2,\beta}^{=} \cdot \leftrightarrow_{\curlyvee\beta}^{*}$ for all $n \in \mathbb{N}$. Thus,

  $$\xLeftarrow[1,\alpha]{} \cdot \xRightarrow[2,\beta]{} = \xleftarrow[1,\alpha]{*} \cdot \xrightarrow[2,\beta]{} \subseteq \xrightarrow[2,\beta]{=} \cdot \xleftrightarrow[\curlyvee\beta]{*} = \xRightarrow[2,\beta]{=} \cdot \xLeftrightarrow[\curlyvee\beta]{*}$$

  holds, where $\Leftrightarrow_J$ stands for ${}_{1,J}\!\!\Leftarrow \cup \Rightarrow_{2,J}$. Hence Γ is decreasing.
- The case that $\alpha > \beta = \bot$ is analogous to the last case. ◀

5 Orthogonality

As a first example of compositional confluence criteria and its derivation, we pick up Rosen's confluence criterion by orthogonality [26]. *Orthogonal* TRSs are left-linear TRSs having no critical pairs. Their confluence property can be shown by decreasingness of parallel steps. We briefly recall its proof. Left-linear TRSs are *mutually orthogonal* if ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} = \varnothing$ and ${}_{\mathcal{S}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}} = \varnothing$. Note that orthogonality of $\mathcal{R}$ and mutual orthogonality of $\mathcal{R}$ and $\mathcal{R}$ are equivalent.

▶ **Lemma 21** ([4, Theorem 9.3.11]). *For mutually orthogonal TRSs $\mathcal{R}$ and $\mathcal{S}$ the inclusion ${}_{\mathcal{R}}\!\nleftarrow\!\!\!\!\!\nleftarrow \cdot \nrightarrow\!\!\!\!\!\nrightarrow_{\mathcal{S}} \subseteq \nrightarrow\!\!\!\!\!\nrightarrow_{\mathcal{S}} \cdot {}_{\mathcal{R}}\!\nleftarrow\!\!\!\!\!\nleftarrow$ holds.*

▶ **Theorem 22** ([26]). *Every orthogonal TRS $\mathcal{R}$ is confluent.*

Proof. Let $\mathcal{A} = (\mathcal{T}(\mathcal{F},\mathcal{V}), \{\nrightarrow\!\!\!\!\!\nrightarrow_1\})$ be the ARS equipped with the empty order $>$ on $\{1\}$, where $\nrightarrow\!\!\!\!\!\nrightarrow_1 = \nrightarrow\!\!\!\!\!\nrightarrow_{\mathcal{R}}$. According to Lemma 5 and Theorem 19, it is enough to show that $\mathcal{A}$ is decreasing. Since Lemma 21 yields ${}_1\!\nleftarrow\!\!\!\!\!\nleftarrow \cdot \nrightarrow\!\!\!\!\!\nrightarrow_1 \subseteq \nrightarrow\!\!\!\!\!\nrightarrow_1 \cdot {}_1\!\nleftarrow\!\!\!\!\!\nleftarrow$, the decreasingness of $\mathcal{A}$ follows. ◀

The theorem can be recast as a compositional criterion that uses a confluent subsystem $\mathcal{C}$ of a given TRS $\mathcal{R}$. For this sake we switch the underlying criterion from Theorem 19 to Theorem 20, setting the relation of the minimum index $\bot$ to $\nrightarrow\!\!\!\!\!\nrightarrow_{\mathcal{C}}$.

▶ **Theorem 23.** *A left-linear TRS $\mathcal{R}$ is confluent if $\mathcal{R}$ and $\mathcal{R} \setminus \mathcal{C}$ are mutually orthogonal for some confluent TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$.*

Proof. Let $\mathcal{A} = (\mathcal{T}(\mathcal{F},\mathcal{V}), \{\nrightarrow\!\!\!\!\!\nrightarrow_0, \nrightarrow\!\!\!\!\!\nrightarrow_1\})$ be the ARS equipped with the well-founded order $1 > 0$, where $\nrightarrow\!\!\!\!\!\nrightarrow_0 = \nrightarrow\!\!\!\!\!\nrightarrow_{\mathcal{C}}$ and $\nrightarrow\!\!\!\!\!\nrightarrow_1 = \nrightarrow\!\!\!\!\!\nrightarrow_{\mathcal{R}\setminus\mathcal{C}}$. According to Lemma 5 and Theorem 19, it is enough to show that $\mathcal{A}$ is decreasing. Since $\mathcal{R}$ and $\mathcal{R} \setminus \mathcal{C}$ are mutually orthogonal, Lemma 21 yields ${}_k\!\nleftarrow\!\!\!\!\!\nleftarrow \cdot \nrightarrow\!\!\!\!\!\nrightarrow_m \subseteq \nrightarrow\!\!\!\!\!\nrightarrow_m \cdot {}_k\!\nleftarrow\!\!\!\!\!\nleftarrow$ for all $(k,m) \in \{0,1\}^2 \setminus \{(0,0)\}$, from which the decreasingness of $\mathcal{A}$ follows. Hence, Theorem 20 applies. ◀

We can derive a more general criterion by exploiting the flexible valley form of decreasing diagrams. We will adopt parallel critical pairs. It causes no loss of confluence proving power of Theorem 23 as ${}_{\mathcal{R}}\!\nleftarrow\!\!\!\!\!\nleftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} = \varnothing$ is equivalent to ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{S}} = \varnothing$.

▶ **Theorem 24.** *A left-linear TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}\!\nleftarrow\!\!\!\!\!\nleftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ holds for some confluent TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$.*

Proof. Recall the ARS used in Theorem 23. According to Lemma 5 and Theorem 20, it is sufficient to show that every local peak

$$\Gamma : t \underset{k}{\overset{P}{\nleftarrow\!\!\!\!\!\nleftarrow}} s \underset{m}{\overset{Q}{\nrightarrow\!\!\!\!\!\nrightarrow}} u$$

with $(k,m) \neq (0,0)$ is decreasing. To this end, we show $t \nrightarrow\!\!\!\!\!\nrightarrow_m \cdot \leftrightarrow\!\!\!\!\!\leftrightarrow^*_0 \cdot {}_k\!\nleftarrow\!\!\!\!\!\nleftarrow u$ by structural induction on s. Depending on the shape of Γ, we distinguish four cases.

1. If P or Q is empty then the claim is trivial.
2. If P or Q is $\{\epsilon\}$ and Γ is orthogonal then Lemma 16(a) yields the join form $t \nrightarrow\!\!\!\!\!\nrightarrow_m \cdot {}_k\!\nleftarrow\!\!\!\!\!\nleftarrow u$.
3. If $P \neq \varnothing$, $Q = \{\epsilon\}$, and Γ is not orthogonal then by Lemma 16(b) there exist a parallel critical peak $t_0 \;{}_k\!\nleftarrow\!\!\!\!\!\nleftarrow s_0 \xrightarrow{\epsilon}_m u_0$ and substitutions σ and τ such that $s = s_0\sigma$, $t = t_0\tau$, $u = u_0\sigma$, and $\sigma \nrightarrow\!\!\!\!\!\nrightarrow_k \tau$. The assumption $t_0 \leftrightarrow^*_{\mathcal{C}} u_0$ yields $t_0\tau \leftrightarrow\!\!\!\!\!\leftrightarrow^*_0 u_0\tau$. Therefore, $t = t_0\tau \leftrightarrow\!\!\!\!\!\leftrightarrow^*_0 u_0\tau \;{}_k\!\nleftarrow\!\!\!\!\!\nleftarrow u_0\sigma = u$ follows.

4. If $P = \{\epsilon\}$, $Q \neq \varnothing$, and Γ is not orthogonal then the proof is analogous to the last case.
5. If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then s, t, and u can be written as $f(s_1, \ldots, s_n)$, $f(t_1, \ldots, t_n)$, and $f(u_1, \ldots, u_n)$ respectively, and moreover, $t_i\ {}_k\!\twoheadleftarrow\!\!\!\!\!+\ s_i \twoheadrightarrow\!\!\!\!\!+_m u_i$ holds for all $1 \leqslant i \leqslant n$. For every i the induction hypothesis yields $t_i \twoheadrightarrow\!\!\!\!\!+_m v_i \leftrightarrow\!\!\!\!\!+^*_0 w_i\ {}_k\!\twoheadleftarrow\!\!\!\!\!+\ u_i$ for some v_i and w_i. Therefore, the desired conversion $t \twoheadrightarrow\!\!\!\!\!+_m v \leftrightarrow\!\!\!\!\!+^*_0 w\ {}_k\!\twoheadleftarrow\!\!\!\!\!+\ u$ holds for $v = f(v_1, \ldots, v_n)$ and $w = f(w_1, \ldots, w_n)$. ◀

From Takahashi's proposition [28] (see also [29, Proposition 9.3.5]) we can deduce that ${}_{\mathcal{R}}\!\twoheadleftarrow\!\!\!\!\!+ \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq =$ is equivalent to ${}_{\mathcal{R}}\!\leftarrow \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq =$. Thus, Theorem 24 subsumes Theorem 23. Note that when $\mathcal{C} = \varnothing$, Theorem 24 simulates the weak orthogonality criterion.

▶ **Example 25.** By successive application of Theorem 24 we show the confluence of the left-linear TRS $\mathcal{R}$ (COPS [13] number 62), taken from [25]:

$$\begin{array}{rrcl rrcl rrcl}
1: & x - 0 & \to & x & 7: & \mathsf{gcd}(x, 0) & \to & x & 13: & \mathsf{if}(\mathsf{true}, x, y) & \to & x \\
2: & 0 - x & \to & 0 & 8: & \mathsf{gcd}(0, x) & \to & x & 14: & \mathsf{if}(\mathsf{false}, x, y) & \to & y \\
3: & \mathsf{s}(x) - \mathsf{s}(y) & \to & x - y & 9: & \mathsf{gcd}(x, y) & \to & \mathsf{gcd}(y, \mathsf{mod}(x, y)) & & & & \\
4: & x < 0 & \to & \mathsf{false} & 10: & \mathsf{mod}(x, 0) & \to & x & & & & \\
5: & 0 < \mathsf{s}(y) & \to & \mathsf{true} & 11: & \mathsf{mod}(0, y) & \to & 0 & & & & \\
6: & \mathsf{s}(x) < \mathsf{s}(y) & \to & x < y & 12: & \mathsf{mod}(x, \mathsf{s}(y)) & \to & \mathsf{if}(x < \mathsf{s}(y), x, \mathsf{mod}(x - \mathsf{s}(y), \mathsf{s}(y))) & & & &
\end{array}$$

Let $\mathcal{C} = \{5, 7, 8, 10, 11, 13\}$. The six non-trivial parallel critical pairs of $\mathcal{R}$ are

$$(x, \mathsf{gcd}(0, \mathsf{mod}(x, 0))) \qquad (y, \mathsf{gcd}(y, \mathsf{mod}(0, y))) \qquad (0, \mathsf{if}(0 < \mathsf{s}(y), 0, \mathsf{mod}(0 - \mathsf{s}(y), \mathsf{s}(y))))$$

and their symmetric versions. All of them are joinable by $\mathcal{C}$. So it remains to show that $\mathcal{C}$ is confluent. Because $\mathcal{C}$ only admits trivial parallel critical pairs, ${}_{\mathcal{C}}\!\twoheadleftarrow\!\!\!\!\!+ \rtimes \xrightarrow{\epsilon}_{\mathcal{C}} \subseteq \leftrightarrow^*_{\varnothing}$ holds. Therefore, the confluence of $\mathcal{C}$ is concluded if we show the confluence of the empty system. The latter claim is trivial. This completes the proof.

Theorem 24 is a generalization of Toyama's yet another theorem:

▶ **Corollary 26** ([33]). *A left-linear TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}\!\twoheadleftarrow\!\!\!\!\!+ \rtimes \xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ holds for some terminating and confluent TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$.*

6 Rule Labeling

In this section we recast the *rule labeling* criterion [37, 40, 7] in a compositional form. Rule labeling is a direct application of decreasing diagrams to confluence proofs for TRSs. It labels rewrite steps by their employed rewrite rules and compares indexes of them. Among others, we focus on the variant of rule labeling based on parallel critical pairs, introduced by Zankl et al. [40].

▶ **Definition 27.** *Let $\mathcal{R}$ be a TRS. A* labeling function *for $\mathcal{R}$ is a function from $\mathcal{R}$ to $\mathbb{N}$. Given a labeling function ϕ and a number $k \in \mathbb{N}$, we define the TRS $\mathcal{R}_{\phi,k}$ as follows:*

$$\mathcal{R}_{\phi,k} = \{\ell \to r \in \mathcal{R} \mid \phi(\ell \to r) \leqslant k\}$$

The relations $\to_{\mathcal{R}_{\phi,k}}$ and $\twoheadrightarrow\!\!\!\!\!+_{\mathcal{R}_{\phi,k}}$ are abbreviated to $\to_{\phi,k}$ and $\twoheadrightarrow\!\!\!\!\!+_{\phi,k}$. Let ϕ and ψ be labeling functions for $\mathcal{R}$. We say that a local peak $t \xleftarrow[\phi,k]{P}\!\!\!\!\!+\ s \xrightarrow[\psi,m]{\epsilon} u$ is (ψ, ϕ)-decreasing if

$$t \xleftrightarrow[\curlyvee k]{*} \cdot \xrightarrow[\psi,m]{}\!\!\!\!\!\!+ \cdot \xleftrightarrow[\curlyvee km]{*} v \xleftarrow[\phi,k]{P'}\!\!\!\!\!+ \cdot \xleftrightarrow[\curlyvee m]{*} u$$

and $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$ for some set P' of parallel positions and term v. Here $\leftrightarrow_K$ stands for the union of ${}_{\phi,k}\!\leftarrow$ and $\to_{\psi,k}$ for all $k \in K$.

The following theorem is a commutation-based version of the rule labeling method [40, Theorem 56].

▶ **Theorem 28.** *Let $\mathcal{R}$ be a left-linear TRS, and ϕ and ψ its labeling functions. The TRS $\mathcal{R}$ is confluent if the following conditions hold for all $k, m \in \mathbb{N}$.*

- *Every parallel critical peak of form $t \underset{\phi,k}{\leftarrow\!\!\shortparallel\!\!-} s \overset{\epsilon}{\underset{\psi,m}{\longrightarrow}} u$ is (ψ,ϕ)-decreasing.*
- *Every parallel critical peak of form $t \underset{\psi,m}{\leftarrow\!\!\shortparallel\!\!-} s \overset{\epsilon}{\underset{\phi,k}{\longrightarrow}} u$ is (ϕ,ψ)-decreasing.*

With a small example we illustrate the usage of rule labeling.

▶ **Example 29.** Consider the left-linear TRS $\mathcal{R}$:

$$(x + y) + z \to x + (y + z) \qquad\qquad x + (y + z) \to (x + y) + z$$

We define the labeling functions ϕ and ψ as follows: $\phi(\ell \to r) = 0$ and $\psi(\ell \to r) = 1$ for all $\ell \to r \in \mathcal{R}$. Because $\mathcal{R}$ is reversible, all parallel critical peaks can be closed by $\to_{\phi,0}$-steps, like the following diagram:

$$\begin{array}{ccccc}
s = ((x+y)+z)+w & & \xrightarrow[\psi,1]{\epsilon} & & (x+y)+(z+w) \\
\{1\} \big\downarrow\!\!\shortparallel\ \phi,0 & & & & \varnothing\ \big\downarrow\!\!\shortparallel\ \phi,0 \\
(x+(y+z))+w & \underset{\phi,0}{\dashrightarrow} & ((x+y)+z)+w & \underset{\phi,0}{\dashleftarrow} & (x+y)+(z+w) = v
\end{array}$$

As $\mathcal{V}\mathsf{ar}(v, \varnothing) = \varnothing \subseteq \{x, y, z\} = \mathcal{V}\mathsf{ar}(s, \{1\})$, this parallel critical peak is (ψ, ϕ)-decreasing. In a similar way the other peaks can also be verified. Hence, the TRS $\mathcal{R}$ is confluent.

We make the rule labeling compositional. The following lemma is used for composing parallel steps.

▶ **Lemma 30** ([40, Lemma 51(b)]). *If $s \overset{P}{\twoheadrightarrow\!\!\!\shortparallel}_{\mathcal{R}} t$, $\sigma \twoheadrightarrow\!\!\!\shortparallel_{\mathcal{R}} \tau$, and $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(s, P)$ then $s\sigma \twoheadrightarrow\!\!\!\shortparallel_{\mathcal{R}} t\tau$.*

The next theorem is a compositional version of the rule labeling criterion. Note that by taking $\mathcal{C} := \mathcal{R}_{\phi,0} = \mathcal{R}_{\psi,0}$ it can be used as a compositional confluence criterion parameterized by $\mathcal{C}$.

▶ **Theorem 31.** *Let $\mathcal{R}$ be a left-linear TRS, and ϕ and ψ its labeling functions. Suppose that $\mathcal{R}_{\phi,0}$ and $\mathcal{R}_{\psi,0}$ commute. The TRS $\mathcal{R}$ is confluent if the following conditions hold for all $(k, m) \in \mathbb{N}^2 \setminus \{(0, 0)\}$.*

- *Every parallel critical peak of form $t \underset{\phi,k}{\leftarrow\!\!\shortparallel\!\!-} s \overset{\epsilon}{\underset{\psi,m}{\longrightarrow}} u$ is (ψ,ϕ)-decreasing.*
- *Every parallel critical peak of form $t \underset{\psi,m}{\leftarrow\!\!\shortparallel\!\!-} s \overset{\epsilon}{\underset{\phi,k}{\longrightarrow}} u$ is (ϕ,ψ)-decreasing.*

Proof. Consider the ARSs $(\mathcal{T}(\mathcal{F}, \mathcal{V}), \{\twoheadrightarrow\!\!\!\shortparallel_{\phi,k}\}_{k\in\mathbb{N}})$ and $(\mathcal{T}(\mathcal{F}, \mathcal{V}), \{\twoheadrightarrow\!\!\!\shortparallel_{\psi,m}\}_{m\in\mathbb{N}})$. According to Lemma 5 and Theorem 20, it is sufficient to show that every local peak

$$\Gamma \colon t \overset{P}{\underset{\phi,k}{\leftarrow\!\!\shortparallel\!\!-}} s \overset{Q}{\underset{\psi,m}{-\!\!\shortparallel\!\!\rightarrow}} u$$

with $(k, m) \neq (0, 0)$ is decreasing. To this end, we perform structural induction on s. Depending on the shape of Γ, we distinguish five cases.

1. If P or Q is empty then the claim is trivial.
2. If P or Q is $\{\epsilon\}$ and Γ is orthogonal then Lemma 16(a) yields the join form $t \xrightarrow[\psi,m]{\Vert} \cdot \xleftarrow[\phi,k]{\Vert} u$.
3. If $P \neq \varnothing$, $Q = \{\epsilon\}$, and Γ is not orthogonal then by Lemma 16(b) there exist a parallel critical peak $t_0 \xleftarrow[\phi,k]{P_1} s_0 \xrightarrow[\psi,m]{\epsilon} u_0$ and substitutions σ and τ such that $t = t_0\tau$, $u = u_0\sigma$, $\sigma \xrightarrow{\Vert\!|} \tau$, $t_0\sigma \xrightarrow{P \setminus P_1}_{\mathcal{R}} t_0\tau$, and $P_1 \subseteq P$. Since $\xleftrightarrow{\Vert\!|}{}^*_J = \leftrightarrow^*_J$ holds in general, the assumption yields

   $$t_0 \xleftrightarrow[\curlyvee k]{\Vert\,*} \cdot \xrightarrow[\psi,m]{\Vert} \cdot \xleftrightarrow[\curlyvee km]{\Vert\,*} v_0 \xleftarrow[\phi,k]{P_1'} w_0 \xleftrightarrow[\curlyvee m]{\Vert\,*} u_0$$

   and $\mathcal{V}\mathsf{ar}(v_0, P_1') \subseteq \mathcal{V}\mathsf{ar}(s_0, P_1)$ for some v_0, w_0, and P_1'. Since the rewrite steps are closed under substitutions, the following relations are obtained:

   $$t_0\tau \xleftrightarrow[\curlyvee k]{\Vert\,*} \cdot \xrightarrow[\psi,m]{\Vert} \cdot \xleftrightarrow[\curlyvee km]{\Vert\,*} v_0\tau \qquad\qquad w_0\sigma \xleftrightarrow[\curlyvee m]{\Vert\,*} u_0\sigma$$

   Since $t_0\sigma|_p = t_0\tau|_p$ holds for all $p \in P_1$, the identity $x\sigma = x\tau$ holds for all $x \in \mathcal{V}\mathsf{ar}(s_0, P_1)$. Therefore, $x\sigma = x\tau$ holds for all $x \in \mathcal{V}\mathsf{ar}(v_0, P_1')$. Because $v_0 \xrightarrow[\phi,k]{P_1'} v_0$, $\sigma \xrightarrow{\Vert\!|} \tau$, and $x\sigma = x\tau$ for all $x \in \mathcal{V}\mathsf{ar}(v_0, P_1')$ hold, Lemma 30 yields $w_0\sigma \xrightarrow[\phi,k]{\Vert} v_0\tau$. Hence, the decreasingness of Γ is witnessed by the following sequence:

   $$t = t_0\tau \xleftrightarrow[\curlyvee k]{\Vert\,*} \cdot \xrightarrow[\psi,m]{\Vert} \cdot \xleftrightarrow[\curlyvee km]{\Vert\,*} v_0\tau \xleftarrow[\phi,k]{\Vert} w_0\sigma \xleftrightarrow[\curlyvee m]{\Vert\,*} u_0\sigma = u$$

   Note that the construction is depicted in Figure 1.
4. If $P = \{\epsilon\}$, $Q \neq \varnothing$, and Γ is not orthogonal then the proof is analogous to the last case.
5. If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then s, t, and u can be written as $f(s_1, \ldots, s_n)$, $f(t_1, \ldots, t_n)$, and $f(u_1, \ldots, u_n)$ respectively, and moreover, $t_i \xleftarrow[\phi,k]{\Vert} s_i \xrightarrow[\psi,m]{\Vert} u_i$ holds for all $1 \leqslant i \leqslant n$. By the induction hypotheses we have $t_i \xleftrightarrow[\curlyvee k]{\Vert\,*} \cdot \xrightarrow[\psi,m]{\Vert} \cdot \xleftrightarrow[\curlyvee km]{\Vert\,*} \cdot \xleftarrow[\phi,k]{\Vert} \cdot \xleftrightarrow[\curlyvee m]{\Vert\,*} u_i$ for all $1 \leqslant i \leqslant n$. Therefore, we obtain the desired relations:

   $$t = f(t_1, \ldots, t_n) \xleftrightarrow[\curlyvee k]{\Vert\,*} \cdot \xrightarrow[\psi,m]{\Vert} \cdot \xleftrightarrow[\curlyvee km]{\Vert\,*} \cdot \xleftarrow[\phi,k]{\Vert} \cdot \xleftrightarrow[\curlyvee m]{\Vert\,*} f(u_1, \ldots, u_n) = u$$

   Hence Γ is decreasing. ◀

The original version of rule labeling (Theorem 28) is a special case of Theorem 31: Suppose that labeling functions ϕ and ψ for a left-linear TRS $\mathcal{R}$ satisfy the conditions of Theorem 28. By taking the labeling functions ϕ' and ψ' with

$$\phi'(\ell \to r) = \phi(\ell \to r) + 1 \qquad\qquad \psi'(\ell \to r) = \psi(\ell \to r) + 1$$

Theorem 31 applies for ϕ', ψ', and the empty TRS $\mathcal{C}$.

The next example shows the combination of our rule labeling variant (Theorem 31) with Knuth–Bendix' criterion (Theorem 3).

▶ **Example 32.** Consider the left-linear TRS $\mathcal{R}$:

$$1\colon\ \mathsf{0} + x \to x \qquad 2\colon\ (x + y) + z \to x + (y + z) \qquad 3\colon\ x + (y + z) \to (x + y) + z$$

Let $\mathcal{C} = \{1, 2\}$. We define the labeling functions ϕ and ψ as follows:

$$\phi(\ell \to r) = \psi(\ell \to r) = \begin{cases} 0 & \text{if } \ell \to r \in \mathcal{C} \\ 1 & \text{otherwise} \end{cases}$$

For instance, the parallel critical pairs involving rule 3 admit the following diagrams:

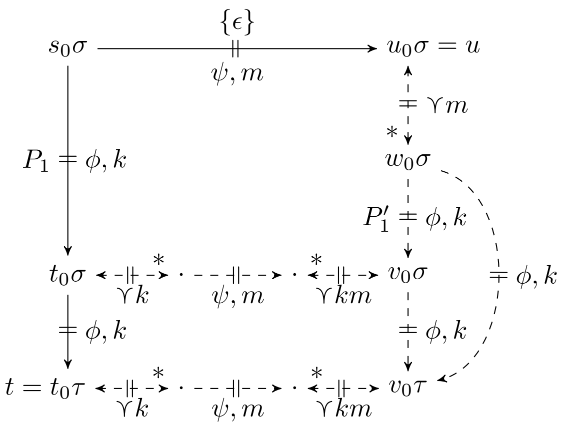
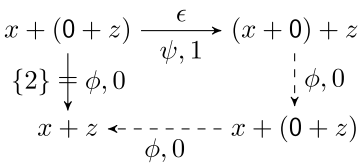
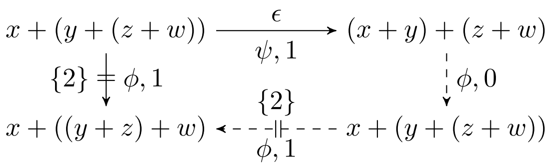


Figure 1 Proof of Theorem 31.





They fit for the conditions of Theorem 31. The other parallel critical pairs also admit suitable diagrams. Therefore, it remains to show that $\mathcal{C}$ is confluent. Since $\mathcal{C}$ is terminating and all its critical pairs are joinable, confluence of $\mathcal{C}$ follows by Knuth and Bendix' criterion (Theorem 3). Thus, $\mathcal{R}_{\phi,0}$ and $\mathcal{R}_{\psi,0}$ commute because $\mathcal{R}_{\phi,0} = \mathcal{R}_{\psi,0} = \mathcal{C}$. Hence, by Theorem 31 we conclude that $\mathcal{R}$ is confluent.

While a proof for Theorem 24 is given in Section 3, here we present an alternative proof based on Theorem 31.

Proof of Theorem 24. Define the labeling functions ϕ and ψ as in Example 32. Then Theorem 31 applies. ◀

We conclude the section by stating that rule labeling based on parallel critical pairs (Theorem 28) subsumes parallel closedness based on parallel critical pairs (Theorem 14): Suppose that conditions (a,b) of Theorem 14 hold. We define ϕ and ψ as the constant rule labeling functions $\phi(\ell \to r) = 1$ and $\psi(\ell \to r) = 0$. By using structural induction as well as Lemmata 16 and 30 we can prove the implication

$$t \xleftarrow[\phi,1]{P_1}\!\!\!\!\!\!\!\!\!\nparallel\;\; s \xrightarrow[\psi,0]{\;\nparallel\;} u \implies t \xrightarrow[\psi,0]{*} v \xleftarrow[\phi,1]{P_1'}\!\!\!\!\!\!\!\!\!\nparallel\;\; u \text{ and } \mathcal{V}\mathsf{ar}(v, P_1') \subseteq \mathcal{V}\mathsf{ar}(s, P_1) \text{ for some } P_1'$$

Thus, the conditions of Theorem 28 follow. As a consequence, our compositional version (Theorem 31) is also a generalization of parallel closedness.

7 Critical Pair Systems

The last example of compositional criteria is a variant of the confluence criterion by critical pair systems [11]. It is known that the original criterion is a generalization of the orthogonal criterion (Theorem 22) and Knuth and Bendix' criterion (Theorem 3) for left-linear TRSs.

▶ **Definition 33.** *The* critical pair system $\mathsf{CPS}(\mathcal{R})$ *of a TRS* $\mathcal{R}$ *is defined as the TRS:*

$$\{s \to t, s \to u \mid t \;{}_{\mathcal{R}}\!\leftarrow s \xrightarrow{\epsilon}_{\mathcal{R}} u \textit{ is a critical peak}\}$$

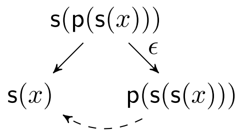
▶ **Theorem 34** ([11])**.** *A left-linear and locally confluent TRS* $\mathcal{R}$ *is confluent if* $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ *is terminating (i.e.,* $\mathsf{CPS}(\mathcal{R})$ *is relatively terminating with respect to* $\mathcal{R}$*).*

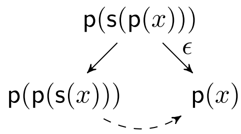
The theorem is shown by using the decreasing diagram technique (Theorem 19), see [11].

▶ **Example 35.** Consider the left-linear and non-terminating TRS $\mathcal{R}$:

$$\mathsf{s}(\mathsf{p}(x)) \to \mathsf{p}(\mathsf{s}(x)) \qquad \mathsf{p}(\mathsf{s}(x)) \to x \qquad \infty \to \mathsf{s}(\infty)$$

The TRS $\mathcal{R}$ admits two critical pairs and they are joinable:





The critical pair system $\mathsf{CPS}(\mathcal{R})$ consists of the four rules:

$$\begin{aligned} &\mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \to \mathsf{s}(x) && \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \to \mathsf{p}(\mathsf{p}(\mathsf{s}(x))) \\ &\mathsf{s}(\mathsf{p}(\mathsf{s}(x))) \to \mathsf{p}(\mathsf{s}(\mathsf{s}(x))) && \mathsf{p}(\mathsf{s}(\mathsf{p}(x))) \to \mathsf{p}(x) \end{aligned}$$

Termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ can be shown by, e.g., the termination tool NaTT [38]. Hence the confluence of $\mathcal{R}$ follows by Theorem 34.

We argue about the parallel critical pair version of $\mathsf{CPS}(\mathcal{R})$:

$$\mathsf{PCPS}(\mathcal{R}) = \{s \to t, s \to u \mid t \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\!\!+}\; s \xrightarrow{\epsilon}_{\mathcal{R}} u \text{ is a parallel critical peak}\}$$

Interestingly, replacing $\mathsf{CPS}(\mathcal{R})$ by $\mathsf{PCPS}(\mathcal{R})$ in Theorem 34 results in the same criterion (see [40]). Since $\to_{\mathsf{CPS}(\mathcal{R})} \subseteq \to_{\mathsf{PCPS}(\mathcal{R})} \subseteq \to_{\mathsf{CPS}(\mathcal{R})} \cdot \mathrel{+\!\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}}$ holds, $\to_{\mathsf{CPS}(\mathcal{R})/\mathcal{R}} = \to_{\mathsf{PCPS}(\mathcal{R})/\mathcal{R}}$ follows. So the termination of $\mathsf{PCPS}(\mathcal{R})/\mathcal{R}$ is equivalent to that of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$. However, a compositional form of Theorem 34 may benefit from the use of parallel critical pairs, as seen in Section 5.

▶ **Definition 36.** *Let* $\mathcal{R}$ *and* $\mathcal{C}$ *be TRSs. The* parallel critical pair system $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ *of* $\mathcal{R}$ *modulo* $\mathcal{C}$ *is defined as the TRS:*

$$\{s \to t, s \to u \mid t \;{}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\!\!+}\; s \xrightarrow{\epsilon}_{\mathcal{R}} u \textit{ is a parallel critical peak but not } t \leftrightarrow^*_{\mathcal{C}} u\}$$

Note that $\mathsf{PCPS}(\mathcal{R},\varnothing) \subseteq \mathsf{PCPS}(\mathcal{R})$ holds in general, and $\mathsf{PCPS}(\mathcal{R},\varnothing) \subsetneq \mathsf{PCPS}(\mathcal{R})$ when $\mathcal{R}$ admits a trivial critical pair. The next lemma relates $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ to closing forms of parallel critical peaks.

▶ **Lemma 37.** *Let* $\mathcal{R}$ *be a left-linear TRS and* $\mathcal{R}_1$*,* $\mathcal{R}_2$*, and* $\mathcal{C}$ *subsets of* $\mathcal{R}$*, and let* $\mathcal{P} = \mathsf{PCPS}(\mathcal{R},\mathcal{C})$*. Suppose that* ${}_{\mathcal{R}}\!\mathrel{\leftarrow\!\!\!\!\!\!+}\rtimes\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow$ *holds. If* $t \;{}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\!\!\!+}\; s \mathrel{+\!\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}_2} u$ *then*

(i) $t \mathrel{+\!\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}_2} \cdot \leftrightarrow^*_{\mathcal{C}} \cdot {}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\!\!\!+}\; u$*, or*

(ii) $t \;{}_{\mathcal{R}_1}\!\mathrel{\leftarrow\!\!\!\!\!\!+}\; t' \;{}_{\mathcal{P}}\!\leftarrow s \to_{\mathcal{P}} u' \mathrel{+\!\!\!\!\!\!\twoheadrightarrow}_{\mathcal{R}_2} u$ *and* $t' \to^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}\!\leftarrow u'$ *for some* t' *and* u'*.*

Proof. Let $\Gamma\colon t \;{}_{\mathcal{R}_1}\!\overset{P}{\mathrel{\leftarrow\!\!\!\!\!\!+}}\; s \overset{Q}{\mathrel{+\!\!\!\!\!\!\twoheadrightarrow}}_{\mathcal{R}_2} u$ be a local peak. We use structural induction on s. Depending on the form of Γ, we distinguish five cases.

1. If P or Q is the empty then (i) holds trivially.
2. If P or Q is $\{\epsilon\}$ and Γ is orthogonal then (i) follows by Lemma 16(a).
3. If $P \neq \varnothing$, $Q = \{\epsilon\}$, and Γ is not orthogonal then we distinguish two cases.
   - If there exist P_0, t_0, u_0, and σ such that "$P_0 \subseteq P$, $t \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; t_0\sigma \;{}_{\mathcal{R}_1}\!\overset{P_0}{\twoheadleftarrow}\; s \xrightarrow{\epsilon}_{\mathcal{R}_2} u_0\sigma = u$, and $t_0 \;{}_{\mathcal{R}}\!\twoheadleftarrow\rtimes\xrightarrow{\epsilon}_{\mathcal{R}}\; u_0$" but not $t_0 \leftrightarrow_{\mathcal{C}}^* u_0$. Take $t' = t_0\sigma$ and $u' = u_0\sigma$. Then $t_0\tau \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; t_0\sigma \;{}_{\mathcal{P}}\!\leftarrow s \rightarrow_{\mathcal{P}} u_0\sigma = u$ holds and by the assumption $t' \rightarrow_{\mathcal{R}}^* \cdot {}_{\mathcal{R}}^*\!\leftarrow u'$ also holds. Hence (ii) follows.
   - Otherwise, whenever P_0, t_0, u_0, and σ satisfy the conditions quoted in the last item, $t_0 \leftrightarrow_{\mathcal{C}}^* u_0$ holds. Because Γ is not orthogonal, by Lemma 16(b) there exist P_0, t_0, u_0, σ, and τ such that $P_0 \subseteq P$, $t = t_0\tau \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; t_0\sigma \;{}_{\mathcal{R}_1}\!\overset{P_0}{\twoheadleftarrow}\; s \xrightarrow{\epsilon}_{\mathcal{R}_2} u_0\sigma = u$, $\sigma \twoheadrightarrow_{\mathcal{R}_1} \tau$. Thus $t_0 \leftrightarrow_{\mathcal{C}}^* u_0$ follows. Therefore, $t = t_0\tau \leftrightarrow_{\mathcal{C}}^* u_0\tau \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; u_0\sigma = u$, and hence (i) holds.
4. If $P = \{\epsilon\}$, $Q \nsubseteq \{\epsilon\}$, and Γ is not orthogonal then the proof is analogous to the last case.
5. If $P \nsubseteq \{\epsilon\}$ and $Q \nsubseteq \{\epsilon\}$ then s, t, and u can be written as $f(s_1, \ldots, s_n)$, $f(t_1, \ldots, t_n)$, and $f(u_1, \ldots, u_n)$ respectively, and $\Gamma_i \colon t_i \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; s_i \twoheadrightarrow_{\mathcal{R}_2} u_i$ holds for all $1 \leqslant i \leqslant n$. For every peak Γ_i the induction hypothesis yields (i) or (ii). If (i) holds for all Γ_i then (i) is concluded for Γ. Otherwise, some Γ_i satisfies (ii). By taking $t' = f(s_1, \ldots, t_i, \ldots, s_n)$ and $u' = f(s_1, \ldots, u_i, \ldots, s_n)$ we have $t \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; t' \;{}_{\mathcal{P}}\!\leftarrow s \rightarrow_{\mathcal{P}} u' \twoheadrightarrow_{\mathcal{P}} u$. From $t_i \rightarrow_{\mathcal{R}}^* \cdot {}_{\mathcal{R}}^*\!\leftarrow u_i$ we obtain $t' \rightarrow_{\mathcal{R}}^* \cdot {}_{\mathcal{R}}^*\!\leftarrow u'$. Hence Γ satisfies (ii). ◀

The next theorem is a compositional confluence criterion based on parallel critical pair systems.

▶ **Theorem 38.** *Let $\mathcal{R}$ be a left-linear TRS and $\mathcal{C}$ a confluent TRS with $\mathcal{C} \subseteq \mathcal{R}$. The TRS $\mathcal{R}$ is confluent if ${}_{\mathcal{R}}\!\twoheadleftarrow\rtimes\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \rightarrow_{\mathcal{R}}^* \cdot {}_{\mathcal{R}}^*\!\leftarrow$ and $\mathcal{P}/\mathcal{R}$ is terminating, where $\mathcal{P} = \mathsf{PCPS}(\mathcal{R}, \mathcal{C})$.*

Proof. Let $\bot$ be a fresh symbol and let $I = \mathcal{T}(\mathcal{F}, \mathcal{V}) \cup \{\bot\}$. We define the relation $>$ on I as follows: $\alpha > \beta$ if $\alpha \neq \bot = \beta$ or $\alpha \rightarrow_{\mathcal{P}/\mathcal{R}}^+ \beta$. Since $\mathcal{P}/\mathcal{R}$ is terminating, $>$ is a well-founded order. Let $\mathcal{A} = (\mathcal{T}(\mathcal{F}, \mathcal{V}), \{\twoheadrightarrow_\alpha\}_{\alpha \in I})$ be the ARS, where $\twoheadrightarrow_\alpha$ is defined as follows: $s \twoheadrightarrow_\alpha t$ if either $\alpha = \bot$ and $s \twoheadrightarrow_{\mathcal{C}} t$, or $\alpha \neq \bot$ and $\alpha \rightarrow_{\mathcal{R}}^* s \twoheadrightarrow_{\mathcal{R} \setminus \mathcal{C}} t$. Since the commutation of $\mathcal{C}$ and $\mathcal{C}$ follows from confluence of $\mathcal{C}$, Lemma 5 yields the commutation of $\rightarrow_\bot$ and $\rightarrow_\bot$. According to Lemma 5 and Theorem 20, it is sufficient to show that every local peak

$$\Gamma \colon t \underset{\alpha}{\twoheadleftarrow} s \underset{\beta}{\twoheadrightarrow} u$$

with $(\alpha, \beta) \in I^2 \setminus \{(\bot, \bot)\}$ is decreasing. By the definition of $\mathcal{A}$ we have $s \twoheadrightarrow_{\mathcal{R}_1} t$ and $s \twoheadrightarrow_{\mathcal{R}_2} u$ for some TRSs $\mathcal{R}_1, \mathcal{R}_2 \in \{\mathcal{R} \setminus \mathcal{C}, \mathcal{C}\}$. Using Lemma 37, we distinguish two cases.

1. Suppose that Lemma 37(i) holds for Γ. Then $t \twoheadrightarrow_{\mathcal{R}_2} t' \leftrightarrow_{\mathcal{C}}^* u' \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; u$ holds for some t' and u'. If $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{C}$ then $t \twoheadrightarrow_\beta t'$ follows from $\beta \rightarrow_{\mathcal{R}}^* s \rightarrow_{\mathcal{R}}^* t \twoheadrightarrow_{\mathcal{R} \setminus \mathcal{C}} t'$. Otherwise, $\mathcal{R}_2 = \mathcal{C}$ yields $t \twoheadrightarrow_\bot t'$. In either case $t \twoheadrightarrow_{\{\beta, \bot\}} t'$ is obtained. Similarly, $u \twoheadrightarrow_{\{\alpha, \bot\}} u'$ is obtained. Moreover, $t' \twoheadleftarrow\!\!\twoheadrightarrow_\bot^* u'$ follows from $t' \leftrightarrow_{\mathcal{C}}^* u'$. Since $(\alpha, \beta) \neq (\bot, \bot)$ yields $\bot \in \curlyvee\alpha\beta$ and the reflexivity of $\twoheadrightarrow_\bot$ yields $\twoheadrightarrow_{\{\delta, \bot\}} \subseteq \twoheadrightarrow_\delta^= \cdot \twoheadrightarrow_\bot$ for any δ, we obtain the desirable conversion $t \overset{=}{\underset{\beta}{\twoheadrightarrow}} t' \overset{*}{\underset{\curlyvee\alpha\beta}{\twoheadleftarrow\!\!\twoheadrightarrow}} u' \overset{=}{\underset{\alpha}{\twoheadleftarrow}} u$. Hence, Γ is decreasing.
2. Suppose that Lemma 37(ii) holds for Γ. We have $t \;{}_{\mathcal{R}_1}\!\twoheadleftarrow\; t' \;{}_{\mathcal{P}}\!\leftarrow s \rightarrow_{\mathcal{P}} u' \twoheadrightarrow_{\mathcal{R}_2} u$ and $t' \rightarrow_{\mathcal{R}}^* v \;{}_{\mathcal{R}}^*\!\leftarrow u'$ for some t', u', and v. As $(\alpha, \beta) \neq (\bot, \bot)$, we have $\alpha \rightarrow_{\mathcal{R}}^* s \rightarrow_{\mathcal{P}} t'$ or $\beta \rightarrow_{\mathcal{R}}^* s \rightarrow_{\mathcal{P}} t'$, from which $\alpha > t'$ or $\beta > t'$ follows. Thus, $t' \in \curlyvee\alpha\beta$. If $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{C}$ then $t' \twoheadrightarrow_{t'} t$. Otherwise, $\mathcal{R}_2 = \mathcal{C}$ yields $t' \twoheadrightarrow_\bot t$. So in either case $t \twoheadleftarrow_{\curlyvee\alpha\beta} t'$ holds. Consider terms w and w' with $t' \rightarrow_{\mathcal{R}}^* w \rightarrow_{\mathcal{R}} w' \rightarrow_{\mathcal{R}}^* v$. We have $w \twoheadrightarrow_{t'} w'$ or $w \twoheadrightarrow_\bot w'$. So $w \twoheadrightarrow_{\curlyvee\alpha\beta} w'$ follows by $\{t', \bot\} \subseteq \curlyvee\alpha\beta$. Thus, $t \twoheadleftarrow_{\curlyvee\alpha\beta} t' \twoheadleftarrow\!\!\twoheadrightarrow_{\curlyvee\alpha\beta}^* v$. In a similar way $u \twoheadleftarrow_{\curlyvee\alpha\beta} u' \twoheadleftarrow\!\!\twoheadrightarrow_{\curlyvee\alpha\beta}^* v$ is obtained. Therefore $t \underset{\curlyvee\alpha\beta}{\twoheadleftarrow} t' \overset{*}{\underset{\curlyvee\alpha\beta}{\twoheadrightarrow}} v \overset{*}{\underset{\curlyvee\alpha\beta}{\twoheadleftarrow}} u' \underset{\curlyvee\alpha\beta}{\twoheadrightarrow} u$, and hence Γ is decreasing. ◀

We claim that Theorem 34 is subsumed by Theorem 38. Suppose that $\mathcal{C}$ is the empty TRS. Trivially $\mathcal{C}$ is confluent. Because $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ is a subset of $\mathsf{PCPS}(\mathcal{R})$, termination of $\mathsf{PCPS}(\mathcal{R},\mathcal{C})/\mathcal{R}$ follows from that of $\mathsf{PCPS}(\mathcal{R})/\mathcal{R}$, which is equivalent to termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$. Finally, ${}_{\mathcal{R}}{\leftarrow}\!\!\rtimes\!\!\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \to^*_{\mathcal{R}} \cdot {}^*_{\mathcal{R}}{\leftarrow}$ is a necessary condition of confluence. Thus, whenever Theorem 34 applies, Theorem 38 applies.

Theorem 38 also subsumes Theorem 24 too. Suppose that $\mathcal{C}$ is a confluent subsystem of $\mathcal{R}$. If ${}_{\mathcal{R}}{\leftarrow}\!\!\rtimes\!\!\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$ then $\mathsf{PCPS}(\mathcal{R},\mathcal{C}) = \varnothing$, which leads to the termination of $\mathsf{PCPS}(\mathcal{R},\mathcal{C})/\mathcal{R}$. Hence, Theorem 38 applies. Note that if $\mathcal{C} = \mathcal{R}$ then $\mathsf{PCPS}(\mathcal{R},\mathcal{C}) = \varnothing$.

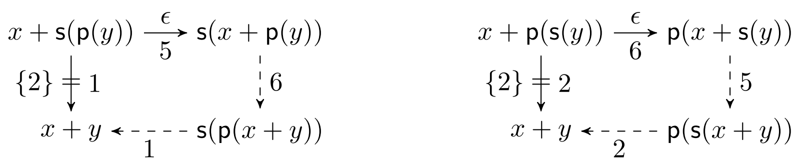
▶ **Example 39.** Consider the left-linear TRS $\mathcal{R}$:

$$
\begin{array}{lll}
1\colon\ \mathsf{s}(\mathsf{p}(x)) \to x & 3\colon\ x + 0 \to x & 5\colon\ x + \mathsf{s}(y) \to \mathsf{s}(x+y) \\
2\colon\ \mathsf{p}(\mathsf{s}(x)) \to x & 4\colon\ 0 + x \to x + 0 & 6\colon\ x + \mathsf{p}(y) \to \mathsf{p}(x+y)
\end{array}
$$

We show the confluence of $\mathcal{R}$ by the combination of Theorem 38 and orthogonality. Let $\mathcal{C} = \{3\}$. The TRS $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ consists of the eight rules:

$$
\begin{array}{ll}
\mathsf{0} + \mathsf{s}(x) \to \mathsf{s}(\mathsf{0} + x) & x + \mathsf{s}(\mathsf{p}(y)) \to \mathsf{s}(x + \mathsf{p}(y)) \\
\mathsf{0} + \mathsf{s}(x) \to \mathsf{s}(x) + \mathsf{0} & x + \mathsf{s}(\mathsf{p}(y)) \to x + y \\
\mathsf{0} + \mathsf{p}(x) \to \mathsf{p}(\mathsf{0} + x) & x + \mathsf{p}(\mathsf{s}(y)) \to \mathsf{p}(x + \mathsf{s}(y)) \\
\mathsf{0} + \mathsf{p}(x) \to \mathsf{p}(x) + \mathsf{0} & x + \mathsf{p}(\mathsf{s}(y)) \to x + y
\end{array}
$$

Termination of $\mathsf{PCPS}(\mathcal{R},\mathcal{C})/\mathcal{R}$ can be shown by, e.g., the termination tool NaTT [38]. Since $\mathcal{C}$ is orthogonal and $\mathcal{R}$ is locally confluent, Theorem 38 applies. Note that the confluence of $\mathcal{R}$ can neither be shown by Theorem 28 nor Theorem 34. The former fails due to the lack of suitable labeling functions for the following diagrams:



The latter fails due to non-termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$. The culprit is the rule $\mathsf{0} + \mathsf{0} \to \mathsf{0} + \mathsf{0}$ in $\mathsf{CPS}(\mathcal{R})$, originating from the critical peak $\mathsf{0} \leftarrow \mathsf{0} + \mathsf{0} \to \mathsf{0} + \mathsf{0}$. In contrast, the rule does not belong to $\mathsf{PCPS}(\mathcal{R},\mathcal{C})$ because the conversion $\mathsf{0} \leftrightarrow^*_{\mathcal{C}} \mathsf{0} + \mathsf{0}$ holds.

8 Experiments

In order to evaluate the presented approach we implemented the main three compositional confluence criteria (Theorems 24, 31, and 38) and their original versions (Theorems 22, 28, and 34) in our prototype confluence tool Hakusan.[1] The problem set used in experiments consists of 448 left-linear TRSs taken from the confluence problems database COPS [13]. Out of 448 TRSs, at least 179 are known to be non-confluent. The tests were run on a PC with Intel Core i7-1065G7 CPU (1.30 GHz) and 16 GB memory of RAM using timeouts of 120 seconds. Table 1 summarizes the results. The columns in the table stand for the following confluence criteria:

[1] The tool and the experimental data are available from: `https://www.jaist.ac.jp/project/saigawa/`

Table 1 Experimental results on 448 left-linear TRSs.

	O	**R**	**C**	**OO**	**RC**	**CR**	ACP	CoLL-Saigawa	CSI
# of proved TRSs	20	132	58	85	149	140	195	168	209
timeouts	0	20	8	13	82	32	47	169	3

- **O**: Orthogonality (Theorem 22).
- **R**: Rule labeling (Theorem 28).
- **C**: The criterion by critical pair systems (Theorem 34).
- **OO**: Successive application of Theorem 24, as illustrated in Example 25.
- **RC**: Theorem 31, where confluence of a subsystem $\mathcal{C}$ is shown by Theorem 38 with the empty subsystem.
- **CR**: Theorem 38, where confluence of a subsystem $\mathcal{C}$ is shown by Theorem 31 with the empty subsystem.

For the sake of comparison the results of the confluence tools ACP version 0.62 [3], CoLL-Saigawa version 1.6 [27], and CSI version 1.25 [39] are also included in the table.

We briefly explain how these criteria are automated in our tool. Suitable subsystems for the compositional criteria are searched by enumeration. Relative termination, required by Theorems 34 and 38, is checked by employing the termination tool NaTT version 1.9 [38]. Joinability of each (parallel) critical pairs (t, u) is tested by the relation:

$$t \xrightarrow{\leqslant 5} \cdot \xleftarrow{\leqslant 5} u$$

For rule labeling, the decreasingness of each parallel critical peak $t \;{}_{\phi,k}\overset{P}{\mathrel{\leftarrow\!\!\!\!\!\!+\!\!\!+}}\; s \xrightarrow{\epsilon}_{\psi,m} u$ is checked by existence of a conversion of the form

$$t \xrightarrow[\curlyvee k]{}{}^{i_1} \cdot \xrightarrow[\psi,m]{+\!+}{}^{i_2} \cdot \xrightarrow[\curlyvee km]{}{}^{i_3} \cdot {}^{j_3}\xleftarrow[\curlyvee km]{} v \;{}^{j_2}\overset{P'}{\underset{\phi,k}{\xleftarrow{+\!+}}} \cdot {}^{j_1}\xleftarrow[\curlyvee m]{} u$$

such that $i_1, i_3, j_1, j_3 \in \mathbb{N}$, $i_2, j_2 \in \{0, 1\}$, $i_1 + i_2 + i_3 \leqslant 5$, $j_1 + j_2 + j_3 \leqslant 5$, and the inclusion $\mathcal{V}\mathsf{ar}(v, P') \subseteq \mathcal{V}\mathsf{ar}(s, P)$ holds. This is encoded into linear arithmetic constraints [11], and they are solved by the SMT solver Z3 version 4.8.11 [5].

As theoretically expected, in the experiments **O**, **R**, and **C** are subsumed by their compositional versions **OO**, **RC**, and **CR**, respectively. Moreover, **OO** is subsumed by **R**, **RC**, and **CR**. Due to timeouts, **CR** misses three systems of which **R** can prove confluence. While the union of **R** and **C** amounts to 142, the union of **RC** and **CR** amounts to 150. Differences between **RC** and **CR** are summarized as follows:

- Three systems are proved by **RC** but not by **CR** nor **R**.[2] One of them is the next TRS (COPS number `994`). **RC** uses the subsystem $\{2, 4, 6\}$ whose confluence is shown by **C**.

$$\begin{array}{lll} 1\colon\ \mathsf{a}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{c}(x)) & 3\colon\ \mathsf{c}(\mathsf{b}(x)) \to \mathsf{a}(\mathsf{b}(x)) & 5\colon\ \mathsf{c}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{c}(x)) \\ 2\colon\ \mathsf{a}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{b}(x)) & 4\colon\ \mathsf{b}(\mathsf{c}(x)) \to \mathsf{a}(\mathsf{c}(x)) & 6\colon\ \mathsf{c}(\mathsf{c}(x)) \to \mathsf{c}(\mathsf{b}(x)) \end{array}$$

- The only TRS where **CR** is advantageous to **RC** is COPS number `132`:

$$\begin{array}{ll} 1\colon\ -(x+y) \to (-x) + (-y) & 3\colon\ -(-x) \to x \\ 2\colon\ (x+y)+z \to x+(y+z) & 4\colon\ x+y \to y+x \end{array}$$

  Its confluence is shown by the composition of Theorem 38 and Theorem 28, the latter of which proves the subsystem $\{1, 2, 4\}$ confluent.

[2] The three systems are COPS numbers `994`, `1001`, and `1029`. The aforementioned confluence tools also fail to prove confluence of these systems.

9 Conclusion

We studied how compositional confluence criteria can be derived from confluence criteria based on the decreasing diagrams technique, and showed that Toyama's almost parallel closedness theorem is subsumed by his earlier theorem based on parallel critical pairs. We conclude the paper by mentioning related work and future work.

Simultaneous critical pairs. van Oostrom [36] showed the almost development closedness theorem: A left-linear TRS is confluent if the inclusions

$$\xleftarrow{\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \xrightarrow{*} \cdot \mathrel{\leftarrow\!\!\circ\!\!-} \qquad\qquad \xleftarrow{>\epsilon}\!\rtimes\!\xrightarrow{\epsilon} \subseteq \mathrel{-\!\!\circ\!\!\rightarrow}$$

hold, where $\mathrel{-\!\!\circ\!\!\rightarrow}$ stands for the multi-step [29, Section 4.7.2]. Okui [23] showed the simultaneous closedness theorem: A left-linear TRS is confluent if the inclusion

$$\mathrel{\leftarrow\!\!\circ\!\!-}\!\rtimes\!\rightarrow \subseteq \xrightarrow{*} \cdot \mathrel{\leftarrow\!\!\circ\!\!-}$$

holds, where $\mathrel{\leftarrow\!\!\circ\!\!-}\!\rtimes\!\rightarrow$ stands for the set of simultaneous critical pairs [23]. As this inclusion characterizes the inclusion $\mathrel{\leftarrow\!\!\circ\!\!-} \cdot \rightarrow \subseteq \rightarrow^* \cdot \mathrel{\leftarrow\!\!\circ\!\!-}$, simultaneous closedness subsumes almost development closedness. The main result in Section 3 is considered as a counterpart of this relationship in the setting of parallel critical pairs.

Critical-pair-closing systems. A TRS $\mathcal{C}$ is called *critical-pair-closing* for a TRS $\mathcal{R}$ if

$${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\xrightarrow{\epsilon}_{\mathcal{R}} \subseteq \leftrightarrow^*_{\mathcal{C}}$$

holds. It is known that a left-linear TRS $\mathcal{R}$ is confluent if $\mathcal{C}_{\mathsf{d}}/\mathcal{R}$ is terminating for some confluent critical-pair-closing TRS $\mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{R}$, see [14]. Here $\mathcal{C}_{\mathsf{d}}$ denotes the set of all duplicating rules in $\mathcal{C}$. Theorem 24 imposes closedness by $\mathcal{C}$ on all *parallel* critical pairs in return to removal of the relative termination condition. Investigating whether the latter subsumes the former is our future work.

Rule labeling. Dowek et al. [7, Theorem 38] extended rule labeling based on parallel critical pairs [40] to take higher-order rewrite systems. If we restrict their method to a first-order setting, it corresponds to the case that a complete TRS is employed for $\mathcal{C}$ in Theorem 31, and thus, it can be seen as a generalization of Corollary 26 by Toyama [33].

Critical pair systems. The second author and Middeldorp [12] generalized Theorem 34 by replacing $\mathsf{CPS}(\mathcal{R})$ by the following subset:

$$\mathsf{CPS}'(\mathcal{R}) = \{s \to t, s \to u \mid t \;{}_{\mathcal{R}}\!\leftarrow s \xrightarrow{\epsilon}_{\mathcal{R}} u \text{ is a critical peak but not } t \mathrel{-\!\!\circ\!\!\rightarrow}_{\mathcal{R}} u\}$$

This variant subsumes van Oostrom's development closedness theorem [36]. We anticipate that in a similar way our compositional variant (Theorem 38) is extended to subsume the parallel closedness theorem based on parallel critical pairs (Theorem 14).

Modularity and Automation. Compositional criteria are conceived as a criterion that confluence of a subsystem implies confluence of the original system. In their automation searching suitable subsystems is a serious bottleneck. If a criterion for the converse direction is established, the bottleneck is resolved as a confluence problem reduces to that of a subsystem. Modularity-based decomposition methods [31, 34, 1, 22, 8] are capable of this type of reduction. Integrating modularity results in compositional criteria is our another future work.

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