

Approximation Metatheorems for Classes with Bounded Expansion

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Abstract

We give a number of approximation metatheorems for monotone maximization problems expressible in the first-order logic, in substantially more general settings than previously known. We obtain

- a constant-factor approximation algorithm in any class of graphs with bounded expansion,
- a QPTAS in any class with strongly sublinear separators, and
- a PTAS in any fractionally treewidth-fragile class (which includes all common classes with strongly sublinear separators).

Moreover, our tools also give an exact subexponential-time algorithm in any class with strongly sublinear separators.

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1 Introduction

We are interested in approximation algorithms for problems such as the MAXIMUM INDEPENDENT SET and its variants (weighted, distance- d independent for a fixed parameter d , ...), MAXIMUM INDUCED MATCHING, MAXIMUM 3-COLORABLE INDUCED SUBGRAPH, and similar. There are many strong non-approximation results that preclude the existence of constant-factor approximation algorithms for these problems in general; for example, it is NP-hard to approximate the independence number [40] of an n -vertex graph up to the factor of $n^{1-\varepsilon}$ for every $\varepsilon > 0$. Hence, we need to consider more restricted settings.

There is a close connection between approximability and the existence of efficient algorithms parameterized by the solution size. Indeed, the existence of an EPTAS (arbitrarily precise polynomial-time approximation algorithm such that the degree of the polynomial bounding the complexity does not depend on the precision) directly implies fixed-parameter tractability, and a constant-factor approximation often forms a starting point for proving fixed-parameter tractability. A natural family of problems, namely those expressible in the first-order logic, is known to be fixed-parameter tractable in a subgraph-closed class of graphs if and (under standard complexity-theoretic assumptions) only if the class is nowhere-dense [26]. Moreover, in a slightly more restrictive setting of classes with bounded expansion, the parameterized algorithms have linear time complexity [17]. We refer the reader not familiar with the concept of bounded expansion to Section 1.1; here, let us just mention that examples of graph classes with this property are planar graphs and more generally all proper minor-closed classes, all graph classes with bounded maximum degree, and even more generally, graph classes closed under topological minors, as well as almost all Erdős-Rényi random graphs with bounded average degree.



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Motivated by this connection, we explore the approximability of maximization problems expressible in the first-order logic when restricted to classes with bounded expansion. As our first main result, we show that every monotone maximization problem expressible in the first-order logic admits a constant-factor approximation algorithm in every class of graphs of bounded expansion, even in the weighted setting. We need a few definitions to formulate the precise statement.

Let I be a finite index set and let S be a set of vertices of a graph G . An I -tuple of subsets of S is a system $A_I = \{A_i : i \in I\}$, where $A_i \subseteq S$ for each i . We say that the I -tuple covers a vertex $v \in V(G)$ if $v \in \bigcup_{i \in I} A_i$. A property of I -tuples of subsets in G is a set π of I -tuples of subsets of $V(G)$, listing the I -tuples that satisfy the property π . As an example, suppose $I = \{1, 2, 3\}$ and π consists exactly of the I -tuples $\{A_1, A_2, A_3\}$ such that A_1, A_2 , and A_3 are disjoint independent sets in G ; then an induced subgraph of G is 3-colorable if and only if its vertex set is covered by some I -tuple satisfying the property π . For I -tuples A_I and A'_I , we write $A'_I \subseteq A_I$ if A'_i is a subset of A_i for each $i \in I$. Similarly, we define $A_I \cup A'_I$, $A_I \cap X$ and $A_I \setminus X$ for a set $X \subseteq V(G)$ by applying the operation in each index separately. We say that the property π is *monotone* if for all I -tuples $A'_I \subseteq A_I$, if A_I satisfies the property π , then so does A'_I .

Our goal will be to maximize the weight of an I -tuple satisfying the given property. The weight of the I -tuple is the sum of the weights of the covered vertices, where the weight of vertex can depend on its membership in the elements of the I -tuple. More precisely, for a vertex $v \in V(G)$, let $\chi_{A_I}(v) \in 2^I$ be the set of indices $i \in I$ such that $v \in A_i$. A function $w : V(G) \times 2^I \rightarrow \mathbb{Z}$ is a *weight assignment* if $w(v, \emptyset) = 0$ for each $v \in V(G)$; with a few exceptions, we will only consider assignments of non-negative weights. Let us define $w(A_I) = \sum_{v \in V(G)} w(v, \chi_{A_I}(v))$, and let $\text{MAX}(\pi, w)$ be the maximum of $w(A_I)$ over the systems A_I satisfying the property π .

Let $X_I = \{X_i : i \in I\}$ be a system of unary predicate symbols (to be interpreted as subsets of vertices of the input graph). A *first-order I -formula* is a formula φ using quantification over vertices, the predicates X_i for $i \in I$, equality, and the standard logic conjunctions. A *first-order graph I -formula* can additionally use a binary symmetric *adjacency predicate* E . A formula is a *sentence* if it has no free variables. For a graph G , an I -tuple A_I of subsets of $V(G)$, and a first-order I -sentence φ , we write $G, A_I \models \varphi$ if the sentence φ holds when the variables in its quantifiers take values from $V(G)$, the adjacency predicate is interpreted as the adjacency in G , and for $i \in I$, X_i is interpreted as the set A_i . The *property π expressed by φ* consists of all I -tuples A_I such that $G, A_I \models \varphi$. For example, the property “ X_1 is a distance-2 independent set” is expressed by the first-order graph $\{1\}$ -sentence

$$(\forall x, y) (X_1(x) \wedge X_1(y) \wedge x \neq y) \Rightarrow (\neg E(x, y) \wedge \neg(\exists z) E(x, z) \wedge E(y, z)).$$

► **Theorem 1.** *Let I be a finite index set and let φ be a first-order graph I -sentence expressing a monotone property π . For any graph class \mathcal{G} with bounded expansion, there exists a constant $c \geq 1$ and a linear-time algorithm that, given*

- a graph $G \in \mathcal{G}$ and
 - a weight assignment $w : V(G) \times 2^I \rightarrow \mathbb{Z}_0^+$,
- returns an I -tuple A_I of subsets of $V(G)$ satisfying the property π such that

$$w(A_I) \geq \frac{1}{c} \cdot \text{MAX}(\pi, w).$$

Actually, the result applies to even more general class of properties, expressible by the fragment of monadic second-order logic where we allow quantification only over the subsets of the vertices in the solution. For a finite index set I that does not contain the integers

$1, \dots, n$, a *solution-restricted MSOL I -sentence with first-order graph core* ψ is a formula of form $(Q_1 X_1 \subseteq \bigcup_{i \in I} X_i) \dots (Q_n X_n \subseteq \bigcup_{i \in I} X_i) \psi$, where Q_1, \dots, Q_n are quantifiers, X_1, \dots, X_n are unary predicate symbols (interpreted as subsets of vertices of the input graph), and ψ is a first-order graph $I \cup \{1, \dots, n\}$ -sentence. For example, the property “ $G[X]$ is a union of cycles, and the distance in G between the distinct cycles is at least three” (or more natural properties such as “ $G[X]$ is planar” or “ $G[X]$ is acyclic” that however do not use the fact that in the first-order core, we are allowed to quantify also over the vertices not in X) can be expressed in this way.

► **Theorem 2.** *Let I be a finite index set and let φ be a solution-restricted MSOL I -sentence with first-order graph core expressing a monotone property π . For any class \mathcal{G} with bounded expansion, there exists a constant $c \geq 1$ and a linear-time algorithm that, given*

■ a graph $G \in \mathcal{G}$ and

■ a weight assignment $w : V(G) \times 2^I \rightarrow \mathbb{Z}_0^+$,

returns an I -tuple A_I of subsets of $V(G)$ satisfying the property π such that

$$w(A_I) \geq \frac{1}{c} \cdot \text{MAX}(\pi, w).$$

We are also interested in the graph classes for which every monotone maximization problem expressible in the first-order logic admits a polynomial-time approximation scheme (PTAS), i.e., an arbitrarily precise polynomial-time approximation algorithm. Note that it is hard to approximate the maximum independent set size within the factor of 0.995 in graphs of maximum degree at most three [4], and thus we do not aim to obtain PTAS in all classes with bounded expansion. The class of graphs of maximum degree three has exponential expansion, motivating us to consider the classes with polynomial expansion. Dvořák and Norin [18] proved these are exactly the graph classes with strongly sublinear separators¹, and approximation questions have been intensively studied for various graph classes with this property (such as planar graphs or more generally for proper minor-closed classes); see Section 1.2 for an overview. While we were not able to obtain a PTAS for all classes with strongly sublinear separators, we were at least able to obtain a quasi-polynomial time approximation scheme (but only for properties expressed by first-order graph sentences, rather than solution-restricted MSOL sentences with first-order graph core).

► **Theorem 3.** *Let I be a finite index set and let φ be a first-order graph I -sentence expressing a monotone property π . For any class \mathcal{G} with strongly sublinear separators, there exists a polynomial p and an algorithm that, given*

■ a graph $G \in \mathcal{G}$,

■ a weight assignment $w : V(G) \times 2^I \rightarrow \mathbb{Z}_0^+$, and

■ a positive integer o ,

returns in time $\exp(p(o \cdot \log |V(G)|))$ an I -tuple A_I of subsets of $V(G)$ satisfying the property π such that

$$w(A_I) \geq \left(1 - \frac{1}{o}\right) \cdot \text{MAX}(\pi, w).$$

Interestingly, the ideas used to prove Theorem 3 also lead to exact subexponential-time algorithms for classes with strongly sublinear separators¹.

¹ For an n -vertex graph G , a set $X \subseteq V(G)$ is a *balanced separator* if each component of $G - X$ has at most $2n/3$ vertices. Let $s(G)$ denote the minimum size of a balanced separator in G , and for a class \mathcal{G} of graphs, let $s_{\mathcal{G}} : \mathbb{Z}^+ \rightarrow \mathbb{Z}_0^+$ be defined by $s_{\mathcal{G}}(n) = \max\{s(H) : H \subseteq G \in \mathcal{G}, |V(H)| \leq n\}$. The class \mathcal{G} has *strongly sublinear separators* if $s_{\mathcal{G}}(n) = O(n^{1-\beta})$ for some $\beta > 0$.

► **Theorem 4.** *Let I be a finite index set and let φ be a first-order graph I -sentence expressing a property π . Let \mathcal{G} be a class of graphs such that $s_{\mathcal{G}}(n) = O(n^{1-\beta})$ for some positive $\beta < 1$. There exists an algorithm that, given*

- a graph $G \in \mathcal{G}$ and
 - a weight assignment $w : V(G) \times 2^I \rightarrow \mathbb{Z}$ (of not necessarily non-negative weights),
- returns in time $\exp(O(|V(G)|^{1-\beta} \log^{1/2} |V(G)|))$ an I -tuple A_I of subsets of $V(G)$ satisfying the property π such that $w(A_I) = \text{MAX}(\pi, w)$.

We can obtain PTASes under a slightly stronger assumption on the considered class of graphs, efficient *fractional treewidth-fragility*. For a positive integer s and a positive real number $\delta \leq 1$, a multiset \mathcal{Z} of subsets of vertices of a graph G is an (s, δ) -generic cover of G if for every set $S \subseteq V(G)$ of size at most s , we have $S \subseteq Z$ for at least $\delta|\mathcal{Z}|$ sets $Z \in \mathcal{Z}$. The *treewidth* of the cover is the maximum of $\text{tw}(G[Z])$ over all $Z \in \mathcal{Z}$. We say that a class of graphs \mathcal{G} is *fractionally treewidth-fragile* if for some function $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$, the following claim holds: for every $G \in \mathcal{G}$ and any positive integers s and o , there exists an $(s, 1 - 1/o)$ -generic cover of G of treewidth at most $f(os)$. The class is *efficiently fractionally treewidth-fragile* if such a cover can be found in time polynomial in $|V(G)|$, and in particular, \mathcal{Z} has polynomial size².

► **Theorem 5.** *Let I be a finite index set and let φ be a solution-restricted MSOL I -sentence with first-order graph core expressing a monotone property π . For any class \mathcal{G} that is efficiently fractionally treewidth-fragile, there exists a function f , a polynomial p , and an algorithm that, given*

- a graph $G \in \mathcal{G}$,
 - a weight assignment $w : V(G) \rightarrow \mathbb{Z}_0^+$, and
 - a positive integer o ,
- returns in time $f(o)p(|V(G)|)$ an I -tuple A_I of subsets of $V(G)$ satisfying the property π such that

$$w(A_I) \geq \left(1 - \frac{1}{o}\right) \cdot \text{MAX}(\pi, w).$$

Efficiently fractionally treewidth-fragile classes include many of the known graph classes with strongly sublinear separators, in particular

- all hereditary classes with sublinear separators and bounded maximum degree [13],
- all proper minor-closed classes, as an easy consequence of the result of DeVos et al. [10] (or [16] without using the Robertson-Seymour structure theorem), and
- many geometric graph classes, such as the intersection graphs of convex sets with bounded aspect ratio in a fixed Euclidean space that have bounded clique number (as can be seen using the idea of [22]).

Indeed, it is possible (and I have conjectured) that all classes with sublinear separators are fractionally treewidth-fragile.

² Note this is a somewhat non-standard formulation of fractional treewidth-fragility. In the usual definition [13, 14], one requires the existence of a system of sets whose deletion results in a graph of treewidth at most $f(o)$ and such that each vertex belongs to at most $1/o$ fraction of the sets, i.e., the complements of the sets of the system form a $(1, 1 - 1/o)$ -generic cover of treewidth at most $f(o)$. To match this with our definition, it suffices to observe that a $(1, 1 - \frac{1}{os})$ -generic cover is also $(s, 1 - 1/o)$ -generic.

Let us finish the introduction by giving two natural open questions. Our results only apply to maximization problems. More precisely, the technique we use can be applied to minimization problems (with *monotone* meaning the supersets of valid solutions are also valid solutions) as well, but the resulting algorithms have error bounded by a fraction of the total weight of all vertices, rather than a fraction of the optimal solution weight.

► **Problem 6.** *Do monotone minimization problems expressible in the first order logic admit constant factor approximation in all classes with bounded expansion? And PTASes in all efficiently fractionally treewidth-fragile graph classes?*

As a simplest example, we do not know whether there exists a PTAS for weighted vertex cover in fractionally treewidth-fragile graph classes.

Secondly, many results for classes of graphs with bounded expansion extend to nowhere-dense graph classes, up to replacement of some constants by terms of order $n^{o(1)}$. Our approach does not apply to this setting, since it is based on a quantifier elimination result specific to graph classes with bounded expansion.

► **Problem 7.** *Do monotone maximization problems expressible in the first order logic admit an $O(n^{o(1)})$ -factor approximation for n -vertex graphs from nowhere-dense classes?*

Let us remark that if a hereditary class \mathcal{G} of graphs is not nowhere-dense, then for some fixed integer r , it contains r -subdivisions of all graphs. Consequently, using the non-approximability results for the independence number [40], we conclude that for every $\varepsilon > 0$, MAXIMUM-WEIGHT DISTANCE- r INDEPENDENT SET cannot be approximated in polynomial time for n -vertex graphs from \mathcal{G} up to the factor of $O(n^{1/2-\varepsilon})$, unless $P = NP$.

1.1 Bounded expansion

The theory of bounded expansion and nowhere-density was developed chiefly by Nešetřil and Ossona de Mendez in a series of papers [32, 33, 34] to capture the notion of graph sparsity with respect to the expressive power of the first-order logic.

For a non-negative integer r , an r -shallow minor of a graph G is any graph obtained from a subgraph of G by contracting pairwise vertex-disjoint subgraphs of radius at most r . A class of graphs \mathcal{G} has *expansion bounded by a function* $f : \mathbb{Z}_0^+ \rightarrow \mathbb{Z}_0^+$ if for every $r \geq 0$, every r -shallow minor of a graph belonging to \mathcal{G} has average degree at most $f(r)$. We say that a class has *bounded expansion* if it has expansion bounded by some function, and *polynomial expansion* if it has expansion bounded by a polynomial.

Many natural graph classes have bounded expansion, thus making it possible to treat them uniformly within this framework. For example, Dvořák and Norin [18] proved that a class of graphs has polynomial expansion if and only if it has strongly sublinear separators. This includes

- planar graphs [29], and more generally all proper minor-closed classes [2];
- graphs drawn in the plane (or on a fixed surface) with a bounded number of crossings on each edge [36]; and
- many geometric graph classes, such as the intersection graphs of convex sets with bounded aspect ratio in a fixed Euclidean space that have bounded clique number, or nearest-neighbor graphs of point sets in a fixed Euclidean space [31].

Classes with bounded (but superpolynomial) expansion include

- Graph classes with bounded maximum degree, and more generally all graph classes closed under topological minors [32];
- graphs with bounded stack or queue number [36]; and,
- almost all Erdős-Rényi random graphs with linear number of edges [36].

For a more in-depth introduction to the topic, the reader is referred to the book of Nešetřil and Ossona de Mendez [35]. The classes with bounded expansion have found many applications in the design of parameterized algorithms [17, 11, 1, 28, 21]. Their applications in the context of approximation algorithms are discussed in the following section.

1.2 Related work

Distance versions of both minimum dominating set and the maximum independent set are known to admit constant-factor approximation algorithms in classes with bounded expansion [12, 15]. A constant-factor approximation algorithm for weighted and distance version of the minimum dominating set also follows from [6] combined with the bounds on the neighborhood complexity in classes with bounded expansion [38].

There are many known techniques to obtain approximation schemes for specific classes of graphs with strongly sublinear separators (polynomial expansion). We illustrate the power of these techniques on variants of the maximum INDEPENDENT SET problem: The DISTANCE- r INDEPENDENT SET problem (parameterized by a fixed positive integer r), where we require the distance between distinct vertices of the chosen set to be greater than r , and the WEIGHTED version of the problem, where the input contains an assignment of weights to vertices and we maximize the sum of weights of vertices in the set rather than the size of the set; see Table 1 for a summary.

- Let us start with the techniques that apply to all classes with strongly sublinear separators. Lipton and Tarjan [30] observed that for each $\varepsilon > 0$, one can split the input graph G by iteratively deleting sublinear separators into components of size $\text{poly}(1/\varepsilon)$, where the resulting set R of removed vertices has size at most $\varepsilon|V(G)|$. One can then solve the problem in each component separately by brute force and obtain an approximation with the additive error $\varepsilon|V(G)|$. In addition to only giving an additive approximation bound, this technique is limited to the problems for which a global solution can be obtained from the partial solutions in $G - R$; e.g., it does not apply to the DISTANCE-2 INDEPENDENT SET problem. It also does not apply in the WEIGHTED setting. Har-Peled and Quanrud [27] proved that in any hereditary class with sublinear separators, a simple local search approach (incrementally improving an initial solution by changes of bounded size) gives PTAS for a number of natural optimization problems, including the r -INDEPENDENT SET problem for any fixed $r \geq 1$ (this is not explicitly stated in their paper, but it is easy to work out the argument). On the other hand, it is not clear which problems are amenable to this approach, and it fails even for some very simple problems (e.g., finding the maximum monochromatic set in an edgeless graphs with vertices colored red and blue). The technique does not apply in the weighted setting.
- The property of *fractional treewidth-fragility*, which as noted above is satisfied by many classes with strongly sublinear separators, was developed as a way to extend Baker's technique (discussed below) to more general graph classes. While a direct application (solving the problem separately in each subgraph induced by the cover) fails for the distance versions of the problems, we have overcome this restriction in a joint work with Lahiri [20]. The approach presented in this paper can be seen as a substantial generalization in terms of the algorithmic problems to which it applies (Dvořák and Lahiri [20] only consider problems expressible in terms of distances between the solution vertices). The approach works for weighted problems. However, as a major restriction, it generally only applies to maximization problems.
- In [19], I made a rather technical attempt to improve upon the fractional treewidth-fragility, by introducing a more powerful notion of *thin systems of overlays*. All hereditary classes with sublinear separators and bounded maximum degree have this property, and

■ **Table 1** PTAS design techniques in hereditary classes with sublinear separators.

Technique	Applies to	INDEP. SET	r -INDEP. S.	WEIGHTED I.S.	monot. FO
Iterated separators	all classes with sublinear separators	✓			
Local search	all classes with sublinear separators	✓	✓		
Fractional treewidth-fragility	bounded max. degree, proper minor closed, ...; maybe all?	✓	✓	✓	✓ (this work)
Thin systems of overlays	bounded max. degree, proper minor closed, ...	✓	✓	✓	
Baker's technique	proper minor closed, some geometric settings	✓	✓	✓	✓

so do all proper minor-closed classes. Thin systems of overlays make it possible to design PTAS for the r -INDEPENDENT SET problem for any fixed $r \geq 1$, as well as for other problems defined in terms of distances between the vertices in the solution (including minimization problems such as the distance version of the minimum dominating set). However, the notion is not suitable for the problems where more complex relationships need to be considered.

- Baker [3] designed a very powerful technique for planar graphs based on finding a partition of the graph into layers (where the edges are allowed only within the layers and between consecutive layers) such that the union of a bounded number of these layers induces a subgraph of bounded treewidth. This is a substantially more restrictive condition than fractional treewidth-fragility (not even all proper minor-closed classes have this property). In a trade-off, the range of problems for which it applies is much wider. In particular, it can deal with all (maximization or minimization) problems expressible in monotone first-order logic [8], which includes all the discussed variants of the INDEPENDENT SET problem. A modified version of this technique (where the layering step is iterated and combined with removal of a bounded number of vertices) also can be used in less restricted settings [16], including for example all proper minor-closed classes (but not all classes with sublinear separators).
- The arguments based on bidimensionality [9] are rather powerful, but limited in scope to (subclasses of) the proper minor-closed classes. With regards to the PTAS design, they essentially build on the Baker's technique framework.

1.3 Proof outline

The proofs of all our results are based on three ingredients:

(1) A strong locality result for first-order properties in graphs from classes with bounded expansion, proved using a modification of the quantifier elimination procedure of [17]. To state the result, we need a few more definitions. A *simple signature* σ is a set of unary predicate and function symbols. For a finite index set I and a system X_I of unary predicate symbols disjoint from σ , a *first-order graph (I, σ) -formula* is a formula φ using all the ingredients from the definition of a first-order graph I -formula and additionally the predicates and unary functions from σ . For a graph G , a unary function $f : V(G) \rightarrow V(G)$ is *guarded by G* if for each $v \in V(G)$, either $f(v) = v$ or v is adjacent to $f(v)$ in G . A *G -interpretation \mathcal{I} of σ* assigns to each unary predicate symbol P a subset $P_{\mathcal{I}}$ of vertices of G and to each

unary function symbol f a unary function $f_{\mathcal{I}}$ guarded by G . For a positive integer s and a graph G , a function $h : V(G) \rightarrow 2^{V(G)}$ is an s -shroud if for each $v \in V(G)$, $|h(v)| \leq s$ and $v \in h(v)$. The h -center of a set $Y \subseteq V(G)$ is the set $\{v \in Y : h(v) \subseteq Y\}$.

► **Theorem 8.** *Let I be a finite index set, let φ be a first-order graph I -sentence, and let \mathcal{G} be a class of graphs with bounded expansion. There exists a constant s , a simple signature σ disjoint from all symbols appearing in φ , and a first-order graph (I, σ) -sentence φ' such that the following claim holds.*

Given a graph $G \in \mathcal{G}$, we can in time $O(|V(G)|)$ find an s -shroud h with the following property: For any $Y \subseteq V(G)$, we can in linear time find a $G[Y]$ -interpretation \mathcal{I}_Y of σ for which every I -tuple A_I of subsets of the h -center of Y satisfies

$$G, A_I \models \varphi \text{ if and only if } G[Y], \mathcal{I}_Y, A_I \models \varphi'.$$

That is, for the I -tuples of subsets of the h -center of Y , we can evaluate whether they have the property π (in the whole graph G) just by looking at the induced subgraph $G[Y]$ enhanced by \mathcal{I}_Y . Note that Theorem 8 straightforwardly extends to solution-restricted MSOL I -sentences $(Q_1 X_1 \subseteq \bigcup_{i \in I} X_i) \dots (Q_n X_n \subseteq \bigcup_{i \in I} X_i) \psi$ with first-order graph core, since if X_I is interpreted as an I -tuple A_I of subsets of the h -center of Y , then X_1, \dots, X_n also correspond to subsets of the h -center of Y .

► **Corollary 9.** *Let I be a finite index set, let φ be a solution-restricted MSOL I -sentence with first-order graph core, and let \mathcal{G} be a class of graphs with bounded expansion. There exists a constant s , a simple signature σ disjoint from all symbols appearing in φ , and a solution-restricted MSOL (I, σ) -sentence φ' with first-order graph core such that the following claim holds.*

Given a graph $G \in \mathcal{G}$, we can in linear time find an s -shroud h with the following property: For any $Y \subseteq V(G)$, we can in linear time find a $G[Y]$ -interpretation \mathcal{I}_Y of σ for which every I -tuple A_I of subsets of the h -center of Y satisfies

$$G, A_I \models \varphi \text{ if and only if } G[Y], \mathcal{I}_Y, A_I \models \varphi'.$$

(2) The existence of sufficiently generic covers. For efficiently fractionally treewidth-fragile classes, we have them by definition. For classes with bounded expansion, we use covers obtained from *low-treedepth colorings*. A *rooted forest* F is an acyclic graph with a specified *root* vertex in each component. The *depth* of F is the number of vertices on the longest path from a root to a leaf. If the path in F from a root to a vertex v contains a vertex u , we say that u is an *ancestor* of v and v is a *descendant* of u . The *closure* of F is the graph with the vertex set $V(F)$ where each vertex is adjacent exactly to its ancestors and descendants in F . The *treedepth* of a graph H is the minimum d such that H is a subgraph of the closure of a rooted forest of depth d . A graph of treedepth d is known to have treewidth (in fact, even pathwidth) smaller than d [35]. For a positive integer s , a *treedepth- s coloring* of a graph G is a coloring such that the union of every s color classes induces a subgraph of treewidth at most s . Nešetřil and Ossona de Mendez [33] proved the following claim.

► **Theorem 10** (Nešetřil and Ossona de Mendez [33]). *For every class \mathcal{G} of graphs with bounded expansion and every positive integer s , there exists an integer a and a linear-time algorithm that given a graph $G \in \mathcal{G}$ returns a treedepth- s coloring of G using at most a colors.*

By considering the cover consisting of all $c = \binom{a}{s}$ unions of s -tuples of color classes, we obtain the following claim.

► **Corollary 11.** *For every class \mathcal{G} of graphs with bounded expansion and every positive integer s , there exists a positive integer c and a linear-time algorithm that given a graph $G \in \mathcal{G}$ returns an $(s, 1/c)$ -generic cover of size c and treedepth at most s .*

For classes with strongly sublinear separators, in [14] I proved they are “almost” fractionally treewidth-fragile, in the following sense (again, with a somewhat different notation, see the footnote at the definition of fractional treewidth-fragility).

► **Theorem 12** (Dvořák [14]). *For every class \mathcal{G} with strongly sublinear separators, there exists a polynomial $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ and a polynomial-time algorithm that, for every $G \in \mathcal{G}$ and positive integers s and o , returns an $(s, 1 - 1/o)$ -generic cover of G of treewidth at most $f(os \log |V(G)|)$. Moreover, the algorithm also returns the corresponding tree decomposition for each element of the cover.*

(3) The means to solve the problem on graphs of bounded treewidth. For Theorems 2 and 5, we use the well-known result of Courcelle [7], in the following optimization version (note that we do not need to be given a tree decomposition, as for graphs of bounded treewidth, we can find an optimal tree decomposition in linear time [5]).

► **Theorem 13.** *Let I be a finite index set and let φ be a MSOL graph formula with free variables X_I , expressing a property π . For any positive integer t , there exists a linear-time algorithm that, given*

- a graph G of treewidth at most t ,
- a set $X \subseteq V(G)$, and
- a weight assignment $w : V(G) \times 2^I \rightarrow \mathbb{Z}$,

returns an I -tuple A_I of subsets of X satisfying the property π such that $w(A_I)$ is maximum among all such I -tuples, or decides no such I -tuple of subsets of X exists.

This is not sufficient for the proof of Theorem 3, where we work with covers of polylogarithmic treewidth, and thus we need a better control over the dependence of the time complexity on the treewidth. We use the following result proved using the locality property underlying Theorem 8; we believe this result to be of independent interest.

► **Theorem 14.** *Let I be a finite index set and let φ be a first-order graph I -sentence expressing a property π . For any class \mathcal{G} with bounded expansion, there exists a constant $c > 0$ and an algorithm that, given*

- a graph $G \in \mathcal{G}$,
- a tree decomposition τ of G with at most $|V(G)|$ nodes,
- a set $X \subseteq V(G)$, and
- a weight assignment $w : V(G) \times 2^I \rightarrow \mathbb{Z}_0$,

returns in time $O(\exp(ct)|V(G)|)$, where t is the width of the decomposition τ , an I -tuple A_I of subsets of X satisfying the property π such that $w(A_I)$ is maximum among all such I -tuples, or decides no such I -tuple of subsets of X exists.

Let us remark that the reason we are not able to generalize Theorem 3 to solution-restricted MSOL sentences with first-order graph core is that we cannot prove the analogue of Theorem 14 in that setting.

We are now ready to prove our results.

Proof of Theorems 2, 3 and 5. Note that \mathcal{G} has bounded expansion:

- In the situation of Theorem 2, this is an assumption.
- In the situation of Theorem 3, this is the case since classes with strongly sublinear separators have polynomial expansion [18].

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- In the situation of Theorem 5, this is the case since every fractionally treewidth-fragile class has bounded expansion [13].

Let s , σ , and φ' be obtained by applying Theorem 8 or Corollary 9 to φ and \mathcal{G} . Now, for the input graph G and the weight assignment w (and the precision o in the case of Theorems 3 and 5), we apply the following algorithm:

- Let h be the s -shroud obtained using Theorem 8 or Corollary 9.
- Let \mathcal{Z} be an (s, δ) -generic cover of treewidth at most t , where
 - in the situation of Theorem 2, \mathcal{Z} is obtained using Corollary 11, $\delta = 1/c$, and $t = s$;
 - in the situation of Theorem 3, \mathcal{Z} is obtained using Theorem 12 for the given o and s , $\delta = 1 - 1/o$, and $t = \text{poly}(os \log |V(G)|)$; and
 - in the situation of Theorem 5, \mathcal{Z} is obtained using the definition of efficient fractional treewidth-fragility for the given o and s , $\delta = 1 - 1/o$, and $t = f(os)$.
- For each $Y \in \mathcal{Z}$:
 - Let \mathcal{I}_Y be the $G[Y]$ -interpretation of σ obtained using Theorem 8 or Corollary 9.
 - Let Y' be the h -center of Y .
 - Using Theorem 13 or 14 in the bounded-treewidth graph $G[Y]$, find an I -tuple A_I^Y of subsets of Y' satisfying $G[Y], \mathcal{I}_Y, A_I^Y \models \varphi'$ such that $w(A_I^Y)$ is maximum possible.
- Return the I -tuple A_I^Y such that $w(A_I^Y)$ is maximum over all $Y \in \mathcal{Z}$.

Note that Theorem 8 or Corollary 9 implies that $G, A_I^Y \models \varphi$, and thus it suffices to bound the approximation ratio of the algorithm.

Let A_I be an I -tuple of subsets of vertices of G satisfying the property π such that $w(A_I)$ is maximum. Choose $Y \in \mathcal{Z}$ uniformly at random. Since h is an s -shroud and \mathcal{Z} is an (s, δ) -cover, for each $v \in V(G)$, the probability that $h(v) \subseteq Y$ is at least δ . Hence, letting $A_I' = A_I \cap Y'$ (where Y' is the h -center of Y), the expected value of $w(A_I')$ is at least $\delta \cdot w(A_I)$. Since π is monotone, we have $G, A_I' \models \varphi$, and by Theorem 8, $G, \mathcal{I}_Y, A_I' \models \varphi'$. This implies $w(A_I^Y) \geq w(A_I')$, and thus the expected value of $w(A_I^Y)$ is also at least $\delta \cdot w(A_I)$. Since we return the maximum over all elements of \mathcal{Z} , this implies the weight of the returned set is at least $\delta \cdot w(A_I)$. ◀

Proof of Theorem 4. Note that since \mathcal{G} has strongly sublinear separators, it has bounded (in fact, polynomial) expansion [18].

Using the algorithm of [23], we can for any n -vertex subgraph of G in polynomial time find a balanced separator of size $O(n^{1-\beta} \log^{1/2} n)$. Using this algorithm, let us construct a tree decomposition τ of G as follows:

- Find a balanced separator S in G .
- Recursively find a tree decomposition τ_C of each component C of $G - S$.
- Add a new root vertex adjacent to the root of τ_C for each component C of $G - S$, and add S to all bags (including the bag of the new root vertex).

This tree decomposition has width $t = O(|V(G)|^{1-\beta} \log^{1/2} |V(G)|)$. We then apply the algorithm from Theorem 14 for this tree decomposition and $X = V(G)$. ◀

Theorem 8 is a consequence of a quantifier-elimination result whose proof is inspired by the approach of Dvořák, Král', and Thomas [17] (but likely could also be proved using the alternative approaches to quantifier elimination in bounded expansion classes, such as [24, 25, 37, 39]); we state the result and derive Theorem 8 in Section 2. Theorem 14 follows by a standard dynamic programming approach, with Theorem 8 used to bound the number of states; we give the proof in Section 3.

2 The quantifier elimination result and its applications

In [13], it has been shown that any first-order formula on a graph from a class with bounded expansion can be transformed into an equivalent quantifier-free first-order formula on a graph from a (different) class with bounded expansion, by introducing unary functions guarded by the resulting graph and new unary predicates, both computable in linear time. Essentially the same procedure can be applied to first-order formulas describing set properties; however, for our application, we need to be more explicit in terms of how the newly introduced functions and predicates are defined and in particular, how they depend on the set system whose weight we are maximizing.

To this end, in addition to unary functions and predicates, we introduce *counters*. Semantically, a counter γ assigns a non-negative integer $\gamma(v)$ to each vertex v . For each counter symbol γ , our formulas can use expressions of form $\gamma(x) \geq m$, where x is a term and m is a positive integer, with the natural interpretation. The symbols for counters are linearly ordered, and we say that a formula is γ -dominated if all counter symbols appearing in the formula are strictly smaller than γ . For a variable x , we say that a formula θ is x -local if θ is quantifier-free, does not use unary functions, and x is the only variable appearing in θ .

The counters are used to keep track of the numbers of vertices that satisfy a prescribed property. Formally, the values of each counter γ are determined by an associated *trigger* (f, θ) , where f is a unary function symbol and θ is an x -local γ -dominated first-order formula. For each vertex v , the value $\gamma(v)$ is equal to the number of vertices $u \in V(G) \setminus \{v\}$ such that $f(u) = v$ and $\theta(u)$ holds.

A *global formula* is a formula that can additionally use elementary formulas of form $\#\theta \geq m$, where θ is an x -local formula and m is a positive integer; this elementary formula is true if there are at least m vertices $v \in V(G)$ such that $\theta(v)$ holds.

Let I be a finite index set. A *counter I -signature* σ is a set of unary predicate and function symbols and linearly ordered counter symbols, together with the triggers associated with these counter symbols, where the triggers are allowed to refer to the unary predicates from X_I and σ . For a graph G , a *G -interpretation* of σ is a G -interpretation of the simple signature consisting of the unary predicate and function symbols from σ . We can now state the quantifier elimination result.

► **Theorem 15.** *Let I be a finite index set, let φ be a first-order graph I -sentence, and let \mathcal{G} be a class of graphs with bounded expansion. There exists a counter I -signature σ and a global quantifier-free first-order (I, σ) -sentence φ' such that the following claim holds. Given a graph $G \in \mathcal{G}$, we can in linear time compute a G -interpretation \mathcal{I} of σ such that*

$$G, A_I \models \varphi \text{ if and only if } G, \mathcal{I}, A_I \models \varphi'$$

for every I -tuple A_I of subsets of $V(G)$.

As we mentioned before, Theorem 8 can be proved by a straightforward modification of known arguments [17, 24, 25, 37, 39] and we omit the proof in this extended abstract; it can be found in the full version of the paper at <https://arxiv.org/abs/2103.08698>.

Let us remark that since φ' is a quantifier-free sentence, it is a Boolean combination of formulas of form $\#\theta \geq m$, where θ is an x -local formula, and in particular φ' does not refer to any function symbols.

Importantly, the interpretation \mathcal{I} of the predicate and function symbols from σ is independent on A_I ; the choice of A_I only affects the values of counters. Let $\ell(\sigma)$ denote the number of counter symbols in σ , and for $i = 1, \dots, \ell(\sigma)$, let $f_{i, \mathcal{I}}$ be the interpretation

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of the unary function from the trigger of the i -th counter symbol in σ , in their fixed linear ordering. For $v \in V(G)$, let us define $h_{0,\mathcal{I}}(v) = \{v\}$ and for $i = 1, \dots, \ell(\sigma)$, let $h_{i,\mathcal{I}}(v) = h_{i-1,\mathcal{I}}(v) \cup \{f_{i,\mathcal{I}}(u) : u \in h_{i-1,\mathcal{I}}(v)\}$. We let $h_{\mathcal{I}} = h_{\ell(\sigma),\mathcal{I}}$; note that $h_{\mathcal{I}}$ is a $2^{\ell(\sigma)}$ -shroud.

For a counter γ of a counter I -signature σ , a G -interpretation \mathcal{I} , and an I -tuple A_I of subsets of $V(G)$, let $\gamma(\mathcal{I}, A_I, v)$ denote the value of the counter γ when the symbols of σ are interpreted according to \mathcal{I} and the predicates X_I are interpreted as A_I . The applications of Theorem 15 use the simple fact that the membership of a vertex v in the sets of an I -tuple A_I only affects the values of the counters in $h_{\mathcal{I}}(v)$.

► **Lemma 16.** *Let I be a finite index set, let σ be a counter I -signature, let G be a graph and let \mathcal{I} be a G -interpretation of σ . For any $X \subseteq V(G)$ and I -tuples A_I and A'_I of subsets of vertices of G , if $A_I \setminus X = A'_I \setminus X$, then $\gamma(\mathcal{I}, A_I, v) = \gamma(\mathcal{I}, A'_I, v)$ for every $v \in V(G) \setminus h_{\mathcal{I}}(X)$.*

Proof. For $i = 1, \dots, \ell(\sigma)$, let (θ_i, f_i) be the trigger of the i -th counter symbol γ_i in σ , in their fixed linear ordering, and let $f_{i,\mathcal{I}}$ be the interpretation of f_i in \mathcal{I} . By induction on i , we show that $\gamma_i(\mathcal{I}, A_I, v) = \gamma_i(\mathcal{I}, A'_I, v)$ for each $v \in V(G) \setminus h_{i,\mathcal{I}}(X)$. Recall that $\gamma_i(\mathcal{I}, A_I, v)$ is the number of vertices $u \in V(G) \setminus \{v\}$ such that $f_{i,\mathcal{I}}(u) = v$ and $G, \mathcal{I}, A_I \models \theta_i(u)$. Note that $u \in V(G) \setminus h_{i-1,\mathcal{I}}(X) \subseteq V(G) \setminus h_{j,\mathcal{I}}(X) \subseteq V(G) \setminus X$ for each $j \leq i-1$, and thus $\gamma_j(\mathcal{I}, A_I, u) = \gamma_j(\mathcal{I}, A'_I, v)$ for each such j by the induction hypothesis. Since θ_i is x -local, the value of $\theta_i(u)$ only depends on these counters, \mathcal{I} , and the interpretation of X_I on $u \notin X$, and thus $G, \mathcal{I}, A_I \models \theta_i(u)$ if and only if $G, \mathcal{I}, A'_I \models \theta_i(u)$. Consequently, $\gamma_i(\mathcal{I}, A_I, v) = \gamma_i(\mathcal{I}, A'_I, v)$, as required. ◀

This is useful in combination with the fact that vertices with fixed values of the counters can be deleted. Let I be a finite index set, let σ be a counter I -signature, and let φ be a global quantifier-free first-order (I, σ) -sentence. For a graph G and $Y \subseteq V(G)$, a (G, Y, σ, φ) -census is a function n that, letting M be the largest integer appearing in the formula φ and the triggers of σ , assigns an element of $\{0, \dots, M\}$ to

- each x -local formula θ appearing in an elementary formula $\#\theta \geq m$ in φ , and
- each pair (γ, v) , where γ is a counter symbol of σ and $v \in Y$ is a vertex with at least one neighbor in $V(G) \setminus Y$.

Given a G -interpretation \mathcal{I} of σ and an I -tuple A_I of subsets of vertices of G , we say that n is a $(G, Y, \mathcal{I}, \sigma, \varphi)$ -shadow of A_I if

- for each x -local formula θ appearing in an elementary formula $\#\theta \geq m$ in φ , $n(\theta)$ is the minimum of M and the number of vertices $u \in V(G) \setminus Y$ such that $G, \mathcal{I}, A_I \models \theta(u)$; and,
- for each counter γ with trigger (θ, f) and each vertex $v \in Y$ with at least one neighbor in $V(G) \setminus Y$, $n(\gamma, v)$ is the minimum of M and the number of vertices $u \in V(G) \setminus Y$ such that $f_{\mathcal{I}}(u) = v$ and $G, \mathcal{I}, A_I \models \theta(u)$.

► **Lemma 17.** *Let I be a finite index set, let σ be a counter I -signature, and let φ be a global quantifier-free first-order (I, σ) -sentence. There exists a signature σ' obtained from σ by adding unary predicate symbols and changing the triggers on the counter symbols, and a global quantifier-free first-order (I, σ') -sentence φ' such that the following claim holds. For any graph G , a G -interpretation \mathcal{I} of σ , a set $Y \subseteq V(G)$ and a (G, Y, σ, φ) -census n , we can in linear time find a $G[Y]$ -interpretation \mathcal{I}' of σ' such that*

$$G, \mathcal{I}, A_I \models \varphi \text{ if and only if } G[Y], \mathcal{I}', A_I \models \varphi'$$

for every I -tuple A_I of subsets of Y with $(G, Y, \mathcal{I}, \sigma, \varphi)$ -shadow n .

Proof. Let M be the largest integer appearing in the formula φ or the triggers of σ . For each counter symbol γ and each positive integer $m \leq M$, we add to σ' a unary predicate $P_{\gamma,m}$, interpreted in \mathcal{I}' as the set of vertices $v \in Y$ such that $n(\gamma, v) \geq m$. Each function symbol g is interpreted in \mathcal{I}' by setting $g_{\mathcal{I}'}(v) = g_{\mathcal{I}}(v)$ if $g_{\mathcal{I}}(v) \in Y$ and $g_{\mathcal{I}}(v) = v$ otherwise. The predicate symbols are interpreted in \mathcal{I}' as the restrictions of their interpretations in \mathcal{I} to Y .

For a counter symbol γ with trigger (θ, f) , we set the trigger of γ in σ' to be (θ', f) , where θ' is obtained from θ by replacing each formula of form $\gamma'(x) \geq m$ by the formula

$$(\gamma'(x) \geq m) \vee (P_{\gamma',1}(x) \wedge \gamma'(x) \geq m-1) \vee \dots \vee P_{\gamma',m}(x).$$

This ensures that for each $v \in Y$ and each I -tuple A_I of subsets of Y with $(G, Y, \mathcal{I}, \sigma, \varphi)$ -shadow n , we have

$$\min(M, \gamma(\mathcal{I}, A_I, v)) = \min(M, \gamma(\mathcal{I}', A_I, v) + n(\gamma, v)),$$

where $n(\gamma, v) = 0$ if all neighbors of v belong to Y . For each subformula of φ of form $\#\theta \geq m$, let θ' be obtained by performing the same replacements. The formula φ' is obtained from φ by replacing each such subformula by the formula $\#\theta' \geq \max(0, m - n(\theta))$. The fact that $G, \mathcal{I}, A_I \models \varphi$ if and only if $G[Y], \mathcal{I}', A_I \models \varphi'$ holds for every I -tuple A_I of subsets of Y with $(G, Y, \mathcal{I}, \sigma, \varphi)$ -shadow n is clear from the construction. \blacktriangleleft

We also need the fact that counters can be eliminated at the expense of re-introducing quantifiers.

► Lemma 18. *Let I be a finite index set, let σ be a counter I -signature, and let σ' be the simple signature consisting of the predicate and function symbols from σ . For any counter symbol $\gamma \in \sigma$ and a positive integer m , there exists a first-order (I, σ') -formula $\psi_{\gamma,m}$ with one free variable x such that the following claim holds: Let G be a graph and let \mathcal{I} be a G -interpretation of σ . For any I -tuple A_I of subsets of vertices of G , we have $\gamma(\mathcal{I}, A_I, v) \geq m$ if and only if $G, \mathcal{I}, A_I \models \psi_{\gamma,m}(v)$.*

Proof. We prove the claim by induction along the linear ordering of the counter symbols in σ , and thus we can assume that the claim holds for all counter symbols appearing in the trigger (θ, f) of γ . Let θ' be the (I, σ') -formula obtained from θ by replacing each formula of form $\gamma'(x) \geq m'$ by the formula $\psi_{\gamma',m'}(x)$. We let $\psi_{\gamma,m}$ be the formula

$$(\exists x_1) \dots (\exists x_m) \left(\bigwedge_{i < j} x_i \neq x_j \right) \wedge \bigwedge_{i=1}^m (x_i \neq x \wedge f(x_i) = x \wedge \theta'(x_i)). \quad \blacktriangleleft$$

We now straightforwardly compose the results.

Proof of Theorem 8. Let σ_1 be the counter I -signature and φ_1 the global quantifier-free first-order (I, σ_1) -sentence obtained using Theorem 15. Let $s = 2^{\ell(\sigma_1)}$. Let σ_2 and φ_2 be the counter I -signature and the global quantifier-free first-order (I, σ_2) -sentence obtained using Lemma 17 for σ_1 and φ_1 . We let σ be the simple signature consisting of the predicate and function symbols from σ_2 , and for each counter symbol γ of σ_2 and each positive integer m , we let $\psi_{\gamma,m}$ be the formula constructed in Lemma 18. We let φ' be the formula obtained from φ_2 by replacing every subformula of form $\gamma(x) \geq m$ by the formula $\psi_{\gamma,m}(x)$, and every subformula of form $\#\theta \geq m$ by the formula

$$(\exists x_1) \dots (\exists x_m) \left(\bigwedge_{i < j} x_i \neq x_j \right) \wedge \bigwedge_{i=1}^m \theta(x_i).$$

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Now, given a graph G , we first use the algorithm from Theorem 15 to compute a G -interpretation \mathcal{I}_1 of σ_1 such that

$$G, A_I \models \varphi \text{ if and only if } G, \mathcal{I}_1, A_I \models \varphi_1$$

for every I -tuple A_I of subsets of $V(G)$. We let h be the s -shroud $h_{\mathcal{I}_1}$ and let X be the h -center of Y . Let n be the $(G, Y, \mathcal{I}_1, \sigma_1, \varphi_1)$ -shadow of the I -tuple of empty sets. Note that by Lemma 16, every I -tuple A_I of subsets of X has $(G, Y, \mathcal{I}_1, \sigma_1, \varphi_1)$ -shadow n . Using the algorithm from Lemma 17, we compute a $G[Y]$ -interpretation \mathcal{I}_2 of σ_2 such that

$$G, \mathcal{I}_1, A_I \models \varphi_1 \text{ if and only if } G[Y], \mathcal{I}_2, A_I \models \varphi_2$$

for every I -tuple A_I of subsets of X . By construction, φ' is a first-order (I, σ) -sentence such that

$$G[Y], \mathcal{I}_2, A_I \models \varphi_2 \text{ if and only if } G[Y], \mathcal{I}_2, A_I \models \varphi'$$

for every I -tuple A_I of subsets of Y . Therefore,

$$G, A_I \models \varphi \text{ if and only if } G[Y], \mathcal{I}_2, A_I \models \varphi'$$

for every I -tuple A_I of subsets of the h -center X of Y , and we can set $\mathcal{I}_Y = \mathcal{I}_2$. ◀

3 Deciding first-order properties in time exponential in treewidth

In this section, we give the algorithm from the statement of Theorem 14. Let I be a finite index set, let σ be a counter I -signature, let φ be a global quantifier-free first-order (I, σ) -sentence, let G be a graph, and let \mathcal{I} be a G -interpretation of σ . Suppose $G = L \cup R$ for some subgraphs L and R of G . In this context, for I -tuples A_I and A'_I of subsets of vertices of L , we write $A_I \equiv_{(L,R)} A'_I$ if for every I -tuple B_I of subsets of $V(R) \setminus V(L)$, we have

$$G, \mathcal{I}, A_I \cup B_I \models \varphi \text{ if and only if } G, \mathcal{I}, A'_I \cup B_I \models \varphi.$$

The algorithm is based on the following key fact.

► **Lemma 19.** *Let I be a finite index set, let σ be a counter I -signature, let φ be a global quantifier-free first-order (I, σ) -sentence, let $G = L \cup R$ be a graph and let \mathcal{I} be a G -interpretation of σ . Let $h_{\mathcal{I}}$ be the corresponding $2^{\ell(\sigma)}$ -shroud and let $S = h_{\mathcal{I}}(V(L \cap R)) \cap V(L)$. If A_I and A'_I are I -tuples of subsets of $V(L)$ such that*

- $A_I \cap S = A'_I \cap S$ and
- A_I and A'_I have the same $(G, V(L) \setminus S, \mathcal{I}, \sigma, \varphi)$ -shadow n ,

then $A_I \equiv_{(L,R)} A'_I$.

Proof. By the definition of $h_{\mathcal{I}}$, we have $S = h_{\mathcal{I}}(V(R)) \cap V(L)$. Consider any I -tuple B_I of subsets of $V(R) \setminus V(L)$. By Lemma 16 applied with $X = V(R) \setminus V(L)$, we have $\gamma(\mathcal{I}, A_I \cup B_I, v) = \gamma(\mathcal{I}, A_I, v)$ and $\gamma(\mathcal{I}, A'_I \cup B_I, v) = \gamma(\mathcal{I}, A'_I, v)$ for each $v \in V(L) \setminus S$. Consequently, the $(G, V(L) \setminus S, \mathcal{I}, \sigma, \varphi)$ -shadow of both $A_I \cup B_I$ and $A'_I \cup B_I$ is equal to n . Let σ' , φ' , and \mathcal{I}' be obtained using Lemma 17 for the census n and $Y = V(R) \cup S$. By the assumptions, we have $(A_I \cup B_I) \cap (V(R) \cup S) = (A'_I \cup B_I) \cap (V(R) \cup S)$, and thus

$$\begin{aligned} G, \mathcal{I}, A_I \cup B_I \models \varphi & \text{ if and only if} \\ G[V(R) \cup S], \mathcal{I}', (A_I \cup B_I) \cap (V(R) \cup S) \models \varphi' & \text{ if and only if} \\ G[V(R) \cup S], \mathcal{I}', (A'_I \cup B_I) \cap (V(R) \cup S) \models \varphi' & \text{ if and only if} \\ G, \mathcal{I}, A'_I \cup B_I \models \varphi, & \end{aligned}$$

as required. ◀

► **Corollary 20.** *Let I be a finite index set, let σ be a counter I -signature, let φ be a global quantifier-free first-order (I, σ) -sentence, let $G = L \cup R$ be a graph and let \mathcal{I} be a G -interpretation of σ . Then $\equiv_{(L,R)}$ has $\exp(O(|V(L \cap R)|))$ equivalence classes.*

Proof. Let M be the largest integer appearing in the formula φ and the triggers of σ and let a be the number of x -local formulas appearing in φ . Let

$$c = (M + 1)^{\ell(\sigma)2^{\ell(\sigma)}} 2^{|I|2^{\ell(\sigma)}}.$$

Let $h_{\mathcal{I}}$ be the $2^{\ell(\sigma)}$ -shroud corresponding to \mathcal{I} and let $S = h_{\mathcal{I}}(V(L \cap R)) \cap V(L)$.

By Lemma 19, each equivalence class of $\equiv_{(L,R)}$ is determined by

- the $(G, V(L) \setminus S, \sigma, \varphi)$ -census n : since only the vertices of S can have neighbors in $V(L) \setminus S$, the number of such censuses is at most $(M + 1)^{a + \ell(\sigma)|S|} \leq (M + 1)^{a + \ell(\sigma)2^{\ell(\sigma)} \cdot |V(L \cap R)|}$, and
- the restriction of the I -tuple to S : there are $2^{|I| \cdot |S|} \leq 2^{|I|2^{\ell(\sigma)} \cdot |V(L \cap R)|}$ options.

Hence, the number of equivalence classes of $\equiv_{(L,R)}$ is at most $(M + 1)^a c^{|V(L \cap R)|}$. ◀

Proof of Theorem 14. Let T be the tree of the tree decomposition τ , rooted arbitrarily. For each node $x \in V(T)$, let T_x denote the subtree of T induced by x and its descendants. Let $L_x = G[\bigcup_{y \in V(T_x)} \tau(y)]$ and $R_x = G[\bigcup_{y \in (V(T) \setminus V(T_x)) \cup \{x\}} \tau(y)]$, so that $L_x \cup R_x = G$ and $|V(L_x \cap R_x)| = |\tau(x)| \leq t + 1$.

Let σ, φ' and \mathcal{I} be obtained using Theorem 15. We use the standard dynamic programming approach, computing for each $x \in V(T)$ a table assigning to each equivalence class C of $\equiv_{(L_x, R_x)}$ an I -tuple $A_{I,C} \in C$ of subsets of $X \cap V(L_x)$ such that $w(A_{I,C})$ is maximized. Since $\equiv_{(L_x, R_x)}$ has $\exp(O(t))$ equivalence classes by Corollary 20, this can be done in total time $\exp(O(t))|V(G)|$ for all nodes of τ .

Let r be the root of T , and note that $L_r = G$ and $R_r = G[\tau(r)]$. We go over the equivalence classes C of $\equiv_{(L_r, R_r)}$ corresponding to I -tuples satisfying the property expressed by φ' (and thus also by φ), and return the I -tuple $A_{I,C}$ maximizing $w(A_{I,C})$. ◀

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