A Positive Fraction Erdős-Szekeres Theorem and Its Applications

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Abstract

A famous theorem of Erdős and Szekeres states that any sequence of n distinct real numbers contains a monotone subsequence of length at least $\sqrt{n}$. Here, we prove a positive fraction version of this theorem. For $n > (k-1)^2$, any sequence A of n distinct real numbers contains a collection of subsets $A_1, \ldots, A_k \subset A$, appearing sequentially, all of size $s = \Omega(n/k^2)$, such that every subsequence $(a_1, \ldots, a_k)$, with $a_i \in A_i$, is increasing, or every such subsequence is decreasing. The subsequence $S = (A_1, \ldots, A_k)$ described above is called *block-monotone of depth k and block-size s*. Our theorem is asymptotically best possible and follows from a more general Ramsey-type result for monotone paths, which we find of independent interest. We also show that for any positive integer k, any finite sequence of distinct real numbers can be partitioned into $O(k^2 \log k)$ block-monotone subsequences of depth at least k, upon deleting at most $(k-1)^2$ entries. We apply our results to mutually avoiding planar point sets and biarc diagrams in graph drawing.

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1 Introduction

In 1935, Erdős and Szekeres [6] proved that any sequence of n distinct real numbers contains a monotone subsequence of length at least $\sqrt{n}$. This is a classical result in combinatorics and its generalizations and extensions have many important consequences in geometry, probability, and computer science. See Steele [13] for 7 different proofs along with several applications.

In this paper, we prove a positive fraction version of the Erdős-Szekeres theorem. We state this theorem using the following notion: A sequence $(a_1, a_2, \ldots, a_{ks})$ of ks distinct real numbers is said to be *block-increasing (block-decreasing)* with *depth k* and *block-size s* if every subsequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$, for $(j-1)s < i_j \leq js$, is increasing (decreasing). We call a sequence *block-monotone* if it's either block-increasing or block-decreasing.

▶ **Theorem 1.** *Let k and $n > (k-1)^2$ be positive integers. Then every sequence of n distinct real numbers contains a block-monotone subsequence of depth k and block-size $s = \Omega(n/k^2)$.*

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We prove Theorem 1 by establishing a more general Ramsey-type result for monotone paths, which we describe in detail in the next section. The theorem is also asymptotically best possible, see Remark 9.

By a repeated application of Theorem 1, we can decompose any sequence of n distinct real numbers into $O(k \log n)$ block-monotone subsequences of depth k upon deleting at most $(k-1)^2$ entries. Our next result shows that we can obtain such a partition, where the number of parts doesn't depend on n.

▶ **Theorem 2.** *For any positive integer k, every finite sequence of distinct real numbers can be partitioned into at most $O(k^2 \log k)$ block-monotone subsequences of depth at least k upon deleting at most $(k-1)^2$ entries.*

Our Theorem 2 is inspired by a similar problem of partitioning planar point sets into convex-positioned clusters, which is studied in [12]. A positive fraction Erdős-Szekeres-type result for convex polygons is given previously by Bárány and Valtr [3].

In the full version of this paper, we present a polynomial time algorithm that computes the block-monotone subsequence claimed by Theorem 1. Our proof of Theorem 2 is constructive hence implying a polynomial time algorithm for the claimed partition as well.

We give two applications of Theorems 1 and 2.

Mutually avoiding sets. Let A and B be finite point sets of $\mathbb{R}^2$ in *general position*, that is, no three points are collinear. We say that A and B are *mutually avoiding* if no line generated by a pair of points in A intersects the convex hull of B, and vice versa. Aronov et al. [1] used the Erdős-Szekeres Theorem to show that every n-element planar point set P in general position contains subsets $A, B \subset P$, each of size $\Omega(\sqrt{n})$, s.t. A and B are mutually avoiding. Valtr [14] showed that this bound is asymptotically best possible by slightly perturbing the points in an $\sqrt{n} \times \sqrt{n}$ grid. Following the same ideas of Aronov et al., we can use Theorem 1 to obtain the following.

▶ **Theorem 3.** *For every positive integer k there is a constant $\epsilon_k = \Omega(\frac{1}{k^2})$ s.t. every sufficiently large point set P in the plane in general position contains $2k$ disjoint subsets $A_1, \ldots, A_k, B_1, \ldots, B_k$, each of size at least $\epsilon_k|P|$, s.t. every pair of sets $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_k\}$, with $a_i \in A_i$ and $b_i \in B_i$, are mutually avoiding.*

This improves an earlier result of Mirzaei and the first author [9], who proved the theorem above with $\epsilon_k = \Omega(\frac{1}{k^4})$. The result above is asymptotically best possible for both k and $|P|$: Consider a $k \times k$ grid G and replace each point with a cluster of $|P|/k^2$ points placed very close to each other so that the resulting point set P is in general position. If we can find subsets A_i's and B_i's as in Theorem 3, but each of size $\epsilon'_k|P|$ with $\epsilon'_k = \omega(\frac{1}{k^2})$, then we can find mutually avoiding subsets in G of size $\omega(k)$, contradicting Valtr's [14].

Finally, let us remark that a recent result due to Pach, Rubin, and Tardos [11] shows that every n-element planar point set in general position determines at least $n/e^{O(\sqrt{\log n})}$ pairwise crossing segments. By using Theorem 3 instead of Lemma 3.3 from their paper, one can improve the constant hidden in the O-notation.

Monotone biarc diagrams. A *proper arc diagram* is a drawing of a graph in the plane, whose vertices are points placed on the x-axis, called the *spine*, and each edge is drawn as a half-circle. A classic result of Bernhard and Kainen [4] shows that a planar graph admits a *planar* proper arc diagram if and only if it's a subgraph of a planar Hamiltonian graph. A *monotone biarc diagram* is a drawing of a graph in the plane, whose vertices are placed on a spine, and each edge is drawn either as a half-circle or two half-circles centered on the spine, forming a continuous x-monotone biarc. See Figure 6 for an illustration. In [5], Di Giacomo et al. showed that every planar graph can be drawn as a *planar* monotone biarc diagram.

Using the Erdős-Szekeres Theorem, Bar-Yehuda and Fogel [2] showed that every graph $G = (V, E)$, with a given order on V, has a *double-paged book embedding* with at most $O(\sqrt{E})$ pages. That is, E can be partitioned into $O(\sqrt{|E|})$ parts, s.t. for each part E_i, (V, E_i) can be drawn as a planar monotone biarc diagram, and V appears on the spine with the given order. Our next result shows that we can significantly reduce the number of pages (parts), if we allow a small fraction of the pairs of edges to cross on each page.

▶ **Theorem 4.** *For any $\epsilon > 0$ and a graph $G = (V, E)$, where V is an ordered set, E can be partitioned into $O(\epsilon^{-2}\log(\epsilon^{-1})\log(|E|))$ subsets E_i s.t. each (V, E_i) can be drawn as a monotone biarc diagram having no more than $\epsilon|E_i|^2$ crossing edge-pairs, and V appears on the spine with the given order.*

This paper is organized as follows: In Section 2, we prove Theorem 1 in the setting of monotone paths in multicolored ordered graphs. Section 3 is devoted to the proof of Theorem 2. In Section 4, we present proofs for the applications claimed above. Section 5 lists some remarks.

2 A positive fraction result for monotone paths

Several authors [7, 10, 8] observed that the Erdős-Szekeres theorem generalizes to the following graph-theoretic setting. Let G be a graph with vertex set $[n] = \{1, \ldots, n\}$. A *monotone path of length k* in G is a k-tuple $(v_1, \ldots, v_k)$ of vertices s.t. $v_i < v_j$ for all $i < j$ and all edges v_iv_{i+1}, for $i \in [k-1]$, are in G.

▶ **Theorem 5.** *Let χ be a q-coloring of the pairs of $[n]$. Then there must be a monochromatic monotone path of length at least $n^{1/q}$.*

Given subsets $A, B \subset [n]$, we write $A < B$ if every element in A is less than every element in B.

▶ **Definition 6.** *Let G be a graph with vertex set $[n]$ and let $V_1, \ldots, V_k \subset [n]$ and $p_1, \ldots, p_{k+1} \in [n]$. Then we say that $(p_1, V_1, p_2, V_2, p_3, \ldots, p_k, V_k, p_{k+1})$ is a block-monotone path of depth k and block-size s if*

1. *$|V_i| = s$ for all i,*
2. *we have $p_1 < V_1 < p_2 < V_2 < p_3 < \ldots < p_k < V_k < p_{k+1}$,*
3. *and every $(2k+1)$-tuple of the form*

$$(p_1, v_1, p_2, v_2, \ldots, p_k, v_k, p_{k+1}),$$

where $v_i \in V_i$, is a monotone path in G.

Our main result in this section is the following Ramsey-type theorem.

▶ **Theorem 7.** *There is an absolute constant $c > 0$ s.t. the following holds. Given integers $q \geq 2$, $k \geq 1$, and $n \geq (ck)^q$, let χ be a q-coloring of the pairs of $[n]$. Then χ produces a monochromatic block-monotone path of depth k and block-size $s \geq \frac{n}{(ck)^q}$.*

A careful calculation shows that we can take $c = 40$ in the theorem above. We will need the following lemma.

▶ **Lemma 8.** *Let $q \geq 2$ and $N > 3^q$. Then for any q-coloring of the pairs of $[N]$, there is a monochromatic block-monotone path of depth 1 and block-size $s \geq \frac{N}{q3^{3q}}$.*

Proof. Let χ be a q-coloring of the pairs of $[N]$, and set $r = 3^q$. By Theorem 5, every subset of size r of $[N]$ gives rise to a monochromatic monotone path of length 3. Hence, χ produces at least

$$\frac{\binom{N}{r}}{\binom{N-3}{r-3}} \geq \frac{6}{r^3}\binom{N}{3}$$

monochromatic monotone paths of length 3 in $[N]$. Hence, there are at least $\frac{6}{qr^3}\binom{N}{3}$ monochromatic monotone paths of length 3, all of which have the same color. By averaging, there are two vertices $p_1, p_2 \in [N]$, s.t. at least $\frac{N}{qr^3}$ of these monochromatic monotone paths of length 3 start at vertex p_1 and ends at vertex p_2. By setting V_1 to be the "middle" vertices of these paths, (p_1, V_1, p_2) is a monochromatic block-monotone path of depth 1 and block-size $s \geq \frac{N}{qr^3} = \frac{N}{q3^{3q}}$. ◀

Proof of Theorem 7. Let χ be a q-coloring of the pairs of $[n]$ and let c be a sufficiently large constant that will be determined later. Set $s = \lceil \frac{n}{(ck)^q} \rceil$. For the sake of contradiction, suppose χ does not produce a monochromatic block-monotone path of depth k and block-size s. For each element $v \in [n]$, we label v with $f(v) = (b_1, \ldots, b_q)$, where b_i denotes the depth of the longest block-monotone path with block-size s in color i, ending at v. By our assumption, we have $0 \leq b_i \leq k-1$, which implies that there are at most k^q distinct labels. By the pigeonhole principle, there is a subset $V \subset [n]$ of size at least n/k^q, s.t. the elements of V all have the same label.

By Lemma 8, there are vertices $p_1, p_2 \in V$, a subset $V' \subset V$, and a color α s.t. (p_1, V', p_2) is a monochromatic block-monotone path in color α, with block-size $t \geq \frac{|V|}{q3^{3q}}$. By setting c to be sufficiently large, we have

$$t \geq \frac{|V|}{q3^{3q}} \geq \frac{n}{k^q q3^{3q}} \geq \left\lceil \frac{n}{(ck)^q} \right\rceil = s.$$

However, this contradicts the fact that $f(p_1) = f(p_2)$, since the longest supported monotone path with block-size s in color α ending at vertex p_1 can be extended to a longer one ending at p_2. This completes the proof. ◀

Proof of Theorem 1. Let $A = (a_1, \ldots, a_n)$ be a sequence of distinct real numbers. Let χ be a red/blue coloring of the pairs of A s.t. for $i < j$, we have $\chi(a_i, a_j) = \text{red}$ if $a_i < a_j$ and $\chi(a_i, a_j) = \text{blue}$ if $a_i > a_j$. In other words, we color increasing pairs by red and decreasing pairs by blue.

If $n < (ck)^2$, notice that $n/(ck)^2 < 1$. By our assumption $n > (k-1)^2$, the classical Erdős-Szekeres theorem gives us a monotone subsequence in A of length at least k, which can be regarded as a block-monotone subsequence of depth at least k and block-size $s = 1 > n/(ck)^2$.

If $n \geq (ck)^2$, by Theorem 7, there is a monochromatic block-monotone path of depth k and block-size $s \geq n/(ck)^2$ in the complete graph on A, which can be regarded as a block-monotone subsequence of A with the claimed depth and block-size. ◀

▶ Remark 9. For each $k, q, s > 0$, the simple construction below shows Theorem 7 is tight up to the constant factor c^q. We first construct $K(k, q)$, for each k and q, a q-colored complete graph on $[k^q]$, whose longest monochromatic monotone path has length k: $K(k, 1)$ is just a monochromatic copy of the complete graph on $[k]$. To construct $K(k, q)$ from $K(k, q-1)$, take k copies of $K(k, q-1)$ with the same set of $q-1$ colors, place them in order and color the remaining edges by a new color. Now replace each point in $K(k, q)$ by a cluster of s points, where within each cluster one can arbitrarily color the edges. The resulting q-colored complete graph has no k subsets $V_1, V_2, \ldots, V_k \subset [n]$ each of size $s+1$ and edges between them monochromatic, otherwise $K(k, q)$ would have a monochromatic monotone path with length larger than k.

It's well-known that the sharpness of the classical Erdős-Szekeres theorem comes from sequences such as

$$S(k) = (k, k-1, \ldots, 1, 2k, 2k-1, \ldots, 2k+1, \ldots, k^2, k^2-1, k(k-1)+1).$$

We note that if we color the increasing pairs of $S(k)$ by red and the decreasing pairs of $S(k)$ by blue, we obtain the graph $K(k, 2)$. If we replace each entry $s_i \in S(k)$ by a cluster of s distinct real numbers very close to s_i, we obtain an example showing that Theorem 1 is asymptotically best possible.

3 Block-monotone sequence partition

This section is devoted to the proof of Theorem 2. We shall consider this problem geometrically by identifying each entry a_i of a given sequence $A = (a_i)_{i=1}^n$ as a planar point $(i, a_i) \in \mathbb{R}^2$. As we consider sequences of distinct real numbers, throughout this section, we assume that all point sets have the property that no two members share the same x-coordinate or the same y-coordinate.

Thus, we analogously define block-monotone point sets as follows: A set of ks planar points is said to be *block-increasing (block-decreasing)* with *depth* k and *block-size* s if it can be written as $\{(x_i, y_i)\}_{i=1}^{ks}$ s.t. $x_i < x_{i+1}$ for all i and every sequence $(y_{i_1}, y_{i_2}, \ldots, y_{i_k})$, for $(j-1)s < i_j \leq js$, is increasing (decreasing). We say that a point set is *block-monotone* if it's either block-increasing or block-decreasing. For each $j \in [k]$ we call the subset $\{(x_i, y_i)\}_{i=(j-1)s+1}^{js}$ the *j-th block* of this block-monotone point set.

Hence, Theorem 2 immediately follows from the following.

▶ **Theorem 10.** *For any positive integer k, every finite planar point set can be partitioned into at most $O(k^2 \log k)$ block-monotone point subsets of depth at least k and a remaining set of size at most $(k-1)^2$.*

Given a point set $P \subset \mathbb{R}^2$, let

$$U(P) := \{(x, y) \in \mathbb{R}^2; y > y',\ \forall (x', y') \in P\}, \quad \text{(up)}$$
$$D(P) := \{(x, y) \in \mathbb{R}^2; y < y',\ \forall (x', y') \in P\}, \quad \text{(down)}$$
$$L(P) := \{(x, y) \in \mathbb{R}^2; x < x',\ \forall (x', y') \in P\}, \quad \text{(left)}$$
$$R(P) := \{(x, y) \in \mathbb{R}^2; x > x',\ \forall (x', y') \in P\}. \quad \text{(right)}$$

Our proof of Theorem 10 relies on the following definitions. The constant c below (and throughout this section) is from Theorem 7. See Figure 1 for an illustration.

▶ **Definition 11.** *A point set $P \subset \mathbb{R}^2$ is said to be a (k, t)-*configuration *if P can be written as a disjoint union of subsets $P = Y_1 \cup Y_2 \cup \cdots \cup Y_{2t+1}$ s.t.*

- *$\forall i \in [t]$, Y_{2i} is a block-monotone point set of depth k and block-size at least $|Y_{2j+1}|/(3ck)^2$ for all $j \in \{0\} \cup [t]$;*
- *either $\cup_{j=i+1}^{2t+1} Y_j$ is located entirely in $R(Y_i) \cap U(Y_i)$ for all $i \in [2t]$, or $\cup_{j=i+1}^{2t+1} Y_j$ is located entirely in $R(Y_i) \cap D(Y_i)$ for all $i \in [2t]$.*

▶ **Definition 12.** *A point set $P \subset \mathbb{R}^2$ is said to be a (k,l,t)*-pattern *if P can be written as a disjoint union of subsets $P = S_1 \cup S_2 \cup \cdots \cup S_l \cup Y$ s.t.*
- *Y is a (k,t)-configuration;*
- *$\forall i \in [l]$, S_i is a block-monotone point set of depth k and block-size at least $|Y|/(3ck)^2$;*
- *$\forall i \in [l]$, the set $(\cup_{j=i+1}^{l} S_j) \cup Y$ is located entirely in one of the following regions: $U(S_i) \cap L(S_i)$, $U(S_i) \cap R(S_i)$, $D(S_i) \cap L(S_i)$ and $D(S_i) \cap R(S_i)$.*

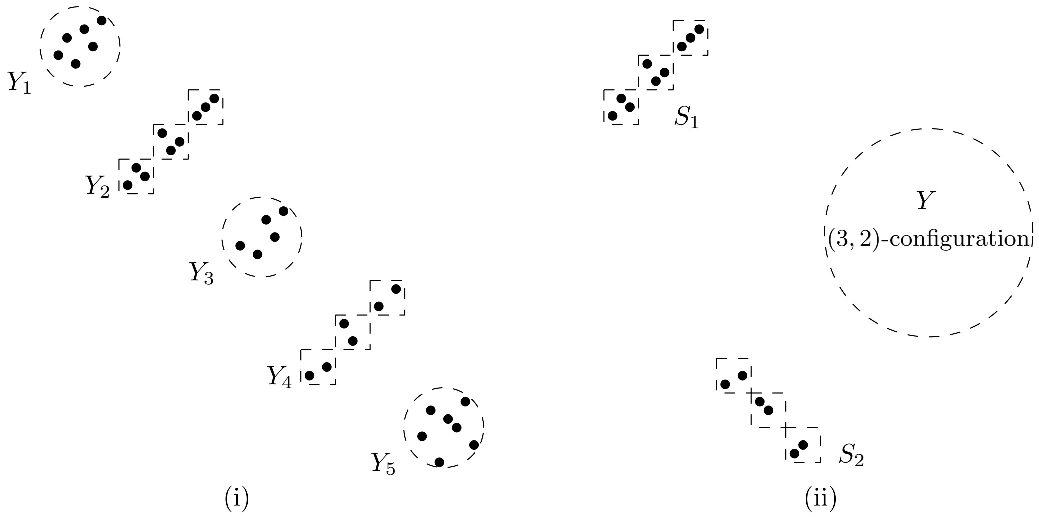

Figure 1 (i) A $(3,2)$-configuration. (ii) A $(3,2,2)$-pattern.

If a planar point set P is a $(k,4k,t)$-pattern or a (k,l,k)-pattern, the next two lemmas state that we can efficiently partition P into few block-monotone point sets of depth at least k and a small remaining set.

▶ **Lemma 13.** *If P is a $(k,4k,t)$-pattern, then P can be partitioned into $O(k \log k)$ block-monotone point sets of depth at least k and a remaining set of size $O(k^2)$.*

▶ **Lemma 14.** *If P is a (k,l,k)-pattern, then P can be partitioned into $O(k^2 \log k + l)$ block-monotone point sets of depth at least k and a remaining set of size $O(k^3)$.*

Starting with an arbitrary point set P, which can be regarded as a $(k,0,0)$-pattern, we will repeatedly apply the following lemma until P is partitioned into few block-monotone point sets, a set P' that is either a $(k,4k,t)$-pattern or a (k,l,k)-pattern, and a small remaining set.

▶ **Lemma 15.** *For $l < 4k$ and $t < k$, a (k,l,t)-pattern P can be partitioned into r block-monotone point sets with depth at least k, a point set P', and a remaining set E s.t.*
1. *$r = O(k)$, $|P'| \leq k(3k-1)^2$ and $E = \emptyset$; or*
2. *$r = O(k \log k)$, P' is a $(k,l,t+1)$-pattern and $|E| = O(k^2)$; or*
3. *$r = O(k \log k)$, P' is a $(k,l+t,0)$-pattern and $|E| = O(k^2)$.*

Moreover, when $t = 0$, we can always have this partition of P as in either case 1 or case 2.

Before we prove the lemmas above, let us use them to prove Theorem 10.

Proof of Theorem 10. Let P be the given point set. For $i \geq 0$, we inductively construct a partition $\mathcal{F}_i \cup \{P_i, E_i\}$ of P s.t.

- P_i is a (k, l_i, t_i)-pattern,
- $|E_i| = O(ik^2)$,
- $\mathcal{F}_i$ is a disjoint family of block-monotone point sets of depth at least k, and $|\mathcal{F}_i| = O(ik \log k)$.

We start with $P_0 = P$, which is a $(k, 0, 0)$-pattern, and $\mathcal{F}_0 = E_0 = \emptyset$. Suppose we have constructed the i-th partition $\mathcal{F}_i \cup \{P_i, E_i\}$ of P. If $|P_i| \leq k(3k-1)^2$, or $l_i \geq 4k$, or $t_i \geq k$, we end this inductive construction process, otherwise, we construct the next partition $\mathcal{F}_{i+1} \cup \{P_{i+1}, E_{i+1}\}$ as follows.

According to Lemma 15, P_i can be partitioned into r block-monotone point sets with depth at least k, denoted as $\{P_{i,1}, \ldots, P_{i,r}\}$, a point set P', and a remaining set E, s.t. either one of the following cases happens.

Case 1. We have $r = O(k)$, $|P'| \leq k(3k-1)^2$, and $E = \emptyset$. In this case, we define $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \{P_{i,1}, \ldots, P_{i,r}\}$, $P_{i+1} = P'$, and $E_{i+1} = E_i \cup E$. Notice that we have $|\mathcal{F}_{i+1}| = |\mathcal{F}_i| + O(k) = O((i+1)k \log k)$ and $|E_{i+1}| = |E_i| + 0 = O((i+1)k^2)$.

Case 2. We have $r = O(k \log k)$, P' is a $(k, l_i, t_i + 1)$-pattern, and $|E| = O(k^2)$. In this case, we define $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \{P_{i,1}, \ldots, P_{i,r}\}$, $P_{i+1} = P'$, and $E_{i+1} = E_i \cup E$. This means $l_{i+1} = l_i$ and $t_{i+1} = t_i + 1$. Notice that we have $|\mathcal{F}_{i+1}| = |\mathcal{F}_i| + O(k \log k) = O((i+1)k \log k)$ and $|E_{i+1}| = |E_i| + O(k^2) = O((i+1)k^2)$.

Case 3. We have $r = O(k \log k)$, P' is a $(k, l_i + t_i, 0)$-pattern, and $|E| = O(k^2)$. In this case, we define $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \{P_{i,1}, \ldots, P_{i,r}\}$, $P_{i+1} = P'$, and $E_{i+1} = E_i \cup E$. This means $l_{i+1} = l_i + t_i$ and $t_{i+1} = 0$. Again, we have $|\mathcal{F}_{i+1}| = O((i+1)k \log k)$ and $|E_{i+1}| = O((i+1)k^2)$.

When $t_i = 0$, by Lemma 15, we can always partition P_i as in Case 1 or Case 2. So we always construct $\mathcal{F}_{i+1} \cup \{P_{i+1}, E_{i+1}\}$ according to Case 1 or Case 2 when $t_i = 0$.

Let $\mathcal{F}_w \cup \{P_w, E_w\}$ be the last partition of P constructed in this process. Here, P_w is a (k, l_w, t_w)-pattern. We must have either $|P_w| \leq k(3k-1)^2$, or $l_w \geq 4k$, or $t_w \geq k$. Since $t_{i+1} \leq t_i + 1$ and $l_{i+1} \leq l_i + t_i$ for all i, we have $t_w \leq k$ and $l_w \leq 5k$. Since we always construct the $(i+1)$-th partition according to Case 1 or Case 2 when $t_i = 0$, the sum $l_i + t_i$ always increases by at least 1 after 2 inductive process. So we have $w/2 \leq t_w + l_w \leq 6k$ and hence $w \leq 12k$.

Now we handle $\mathcal{F}_w \cup \{P_w, E_w\}$ based on how the construction process ends.

If the construction process ended with $|P_w| \leq k(3k-1)^2$, we define $E_{w+1} = E_w \cup P_w$ and $\mathcal{F}_{w+1} = \mathcal{F}_w$. Since $w \leq 12k$, we have $|\mathcal{F}_{w+1}| = O(k^2 \log(k))$ and $|E_{w+1}| = O(k^3)$.

If the construction process ended with $l_w \geq 4k$, by Definition 12, we can partition P_w into $l_w - 4k$ many block-monotone point sets of depth k, denoted as $\{P_{w,1}, \ldots, P_{w,l_w-4k}\}$, and a $(k, 4k, t_w)$-pattern P'_w. Then, by Lemma 13, P'_w can be partitioned into $r = O(k \log k)$ block-monotone point sets of depth at least k, denoted as $\{P'_{w,1}, \ldots, P'_{w,r}\}$, and a remaining set E of size $O(k^2)$. We define $E_{w+1} = E_w \cup E$ and

$$\mathcal{F}_{w+1} = \mathcal{F}_w \cup \{P_{w,1}, \ldots, P_{w,l_w-4k}, P'_{w,1}, \ldots, P'_{w,r}\}.$$

Using $w \leq 12k$ and other bounds we mentioned above, we can check $|\mathcal{F}_{w+1}| = O(k^2 \log(k))$ and $|E_{w+1}| = O(k^3)$.

If the construction process ended with $t_w \geq k$, we actually have $t_w = k$ and $l_w < 4k$. By Lemma 14, we can partition P_w into $r = O(k^2 \log(k) + l_w)$ block-monotone point sets of depth at least k, denoted as $\{P_{w,1}, \ldots, P_{w,r}\}$, and a remaining set E of size $O(k^3)$. We define $E_{w+1} = E_w \cup E$ and $\mathcal{F}_{w+1} = \mathcal{F}_w \cup \{P_{w,1}, \ldots, P_{w,r}\}$. Again, we can check $|\mathcal{F}_{w+1}| = O(k^2 \log(k))$ and $|E_{w+1}| = O(k^3)$.

Overall, we can always obtain a partition $\mathcal{F}_{w+1} \cup \{E_{w+1}\}$ of P with $|\mathcal{F}_{w+1}| = O(k^2 \log(k))$ and $|E_{w+1}| = O(k^3)$. Using the classical Erdős-Szekeres theorem, we can always find a monotone sequence of length at least k in E_{w+1} when $|E_{w+1}| > (k-1)^2$. By a repeated application of this fact, we can partition E_{w+1} into $O(k^2)$ block-monotone point sets of depth k and block-size 1, and a remaining set E of size at most $(k-1)^2$. We define $\mathcal{F}$ to be the union of $\mathcal{F}_{w+1}$ and these block-monotone sequences. The partition $\mathcal{F} \cup \{E\}$ of P has the desired properties and concludes the proof. ◀

We now give proofs for Lemmas 13, 14, and 15. We need the following facts.

▶ **Fact 16.** *For any positive integer k, every point set P can be partitioned into $O(k \log(k))$ block-monotone point sets of depth k and a remaining set P' with $|P'| \leq \max\{|P|/k, (k-1)^2\}$.*

This fact can be established by repeatedly using Theorem 1 to pull out block-monotone point sets and applying the elementary inequality $(1 - x^{-1})^{x \log(x)} \leq x^{-1}$ for any $x > 1$.

▶ **Fact 17.** *For any positive integers k and m, every block-monotone point set P with depth k and $|P| \geq m$ can be partitioned into a block-monotone point set of depth k, a subset of size exactly m, and a remaining set of size less than k.*

This fact can be established by taking out $\lceil m/k \rceil$ points from each block of P. Then we have taken out $k \cdot \lceil m/k \rceil = m + r$ points, where $0 \leq r < k$.

Proof of Lemma 13. Write the given $(k, 4k, t)$-pattern $P = S_1 \cup \cdots \cup S_{4k} \cup Y$ as in Definition 12. By definition, each block-monotone point set S_i is contained in one of the 4 regions: $U(Y) \cap L(Y)$, $U(Y) \cap R(Y)$, $D(Y) \cap L(Y)$ and $D(Y) \cap R(Y)$. By Pigeonhole principle, there are k indices $i_1, \ldots, i_k$ s.t. all S_{i_j}, for $j \in [k]$, are contained in one of the regions above. Without loss of generality, we assume $S_1, \ldots, S_k$ are all located entirely in $U(Y) \cap L(Y)$.

We have $S_{i_2} \subset D(S_{i_1}) \cap R(S_{i_1})$ for all $1 \leq i_1 < i_2 \leq k$. Indeed, since $Y \subset D(S_{i_1}) \cap R(S_{i_1})$, Definition 12 guarantees that $(\cup_{j=i_1+1}^{k} S_j) \cup Y$ to be contained in $D(S_{i_1}) \cap R(S_{i_1})$ and, in particular, S_{i_2} is contained in this region. See Figure 2 for an illustration.

Now apply Fact 16 to Y, we can partition Y into $\{A_1, \ldots, A_w, Y'\}$, where $w = O(k \log(k))$, s.t. each A_j is block-monotone of depth $9c^2k$, and either $|Y'| \leq |Y|/(9c^2k)$ or $|Y'| \leq (9c^2k - 1)^2$. If $|Y'| \leq (9c^2k - 1)^2$, we have partitioned P into $O(k \log(k))$ block-monotone point sets of depth at least k, which are $\{A_1, \ldots, A_w, S_1, \ldots, S_{4k}\}$, and a remaining set Y' of size $O(k^2)$, as wanted.

If $|Y'| \leq |Y|/(9c^2k)$, by Definition 12 we have $|Y'| \leq |S_i|$ for $i \in [k]$. We can apply Fact 17 with $m := |Y'|$ to S_i to obtain a partition $S_i = S'_i \cup B_i \cup E_i$ where S'_i is block-monotone of depth k, $|B_i| = |Y'|$ and $|E_i| \leq k$. We observe that $C = B_1 \cup B_2 \cup \cdots \cup B_k \cup Y'$ is block-monotone of depth $k+1$ by its construction. Then we have partitioned P into $O(k \log(k))$ many block-monotone point sets, which are $\{A_1, \ldots, A_w, S'_1, \ldots, S'_k, S_{k+1}, \ldots, S_{4k}, C\}$, and a remaining set $E := \cup_{i=1}^{k} E_i$ of size $O(k^2)$, as wanted. ◀

Proof of Lemma 14. Write the given (k, l, k)-pattern $P = S_1 \cup \cdots \cup S_l \cup Y$ as in Definition 12 and the (k, k)-configuration $Y = Y_1 \cup \cdots \cup Y_{2k+1}$ as in Definition 11. Since each S_i is block-monotone of depth k, it suffices to partition Y into $O(k^2 \log(k))$ many block-monotone point sets of depth at least k and a remaining set of size $O(k^3)$.

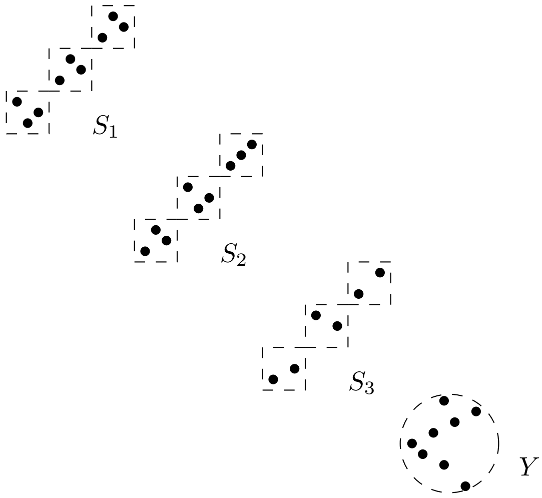

Figure 2 In proof of Lemma 13, $S_{i_2} \subset D(S_{i_1}) \cap R(S_{i_1})$ for $i_1 < i_2$.

For each $j \in \{0\} \cup [k]$, we apply Fact 16 to obtain a partition of Y_{2j+1} into $O(k\log(k))$ many block-monotone point sets of depth $9c^2k$ and a remaining set Y'_{2j+1} of size at most $|Y_{2j+1}|/(9c^2k)$ or at most $(9c^2k-1)^2$. We can apply Fact 16 again to partition Y'_{2j+1} into $O(k\log(k))$ many block-monotone point sets of depth $k+1$ and a remaining set Y''_{2j+1} with

$$|Y''_{2j+1}| \le \max\{|Y_{2j+1}|/(9c^2k(k+1)), (9c^2k-1)^2\}. \tag{1}$$

Denote the block-monotone point sets produced in this process as $\{A_{j,x}; x \in [w_j]\}$, where $w_j = O(k\log(k))$.

Next we denote $J_1 := \{j \in \{0\} \cup [k]; |Y''_{2j+1}| > (9c^2k-1)^2\}$ and $J_2 := (\{0\} \cup [k]) \setminus J_1$. For each $j \in J_1$ and $i \in [k]$, we must have

$$|Y''_{2j+1}| \le |Y_{2j+1}|/(9c^2k(k+1)) \le |Y_{2i}|/(k+1),$$

where the second inequality is by Definition 11. Hence $|Y_{2i}| \ge |\cup_{j\in J_1} Y''_{2j+1}|$. We can apply Fact 17 with $m := |\cup_{j\in J_1} Y''_{2j+1}|$ to Y_{2i} to obtain a partition $Y_{2i} = Y'_{2i} \cup B_i \cup E_i$ where Y'_{2i} is block-monotone of depth k, $|B_i| = m$, and $|E_i| \le k$. Since $|B_i| = |\cup_{j\in J_1} Y''_{2j+1}|$, we can take a further partition $B_i = \cup_{j\in J_1} B_{j,i}$ with $|B_{j,i}| = |Y''_{2j+1}|$ for each $j \in J_1$. Then we observe that $C_j = B_{j,1} \cup \cdots \cup B_{j,j} \cup Y''_{2j+1} \cup B_{j,j+1} \cup \cdots \cup B_{j,k}$ is block-monotone of depth $k+1$ for each $j \in J_1$ by its construction.

Finally, let $E := (\cup_{i=1}^k E_i) \cup (\cup_{j\in J_2} Y''_{2j+1})$, it easy to check that $E = O(k^3)$. So we have partitioned Y into $O(k^2\log(k))$ many block-monotone point sets, which are

$$\{A_{j,x}\}_{j\in\{0\}\cup[k], x\in[w_j]} \cup \{C_j\}_{j\in J_1} \cup \{Y'_{2i}\}_{i\in[k]},$$

and a remaining set E of size $O(k^3)$, as wanted. ◀

Proof of Lemma 15. Write the given (k,l,t)-pattern $P = S_1 \cup \cdots \cup S_l \cup Y$ as in Definition 12 and the (k,t)-configuration $Y = Y_1 \cup \cdots \cup Y_{2t+1}$ as in Definition 11. Without loss of generality, we assume $\cup_{j=i+1}^{2t+1} Y_j$ is located entirely in $R(Y_i) \cap U(Y_i)$ for all $i \in [2t]$. We also assume that Y_1 has the largest size among $\{Y_{2j+1}; j \in \{0\} \cup [t]\}$ because other scenarios can be proved similarly.

If $|Y_1| \leq (3k-1)^2$, we can partition P into $r = l + t = O(k)$ many block-monotone point sets of depth k, which are $\{S_1, \ldots, S_l, Y_2, Y_4, \ldots, Y_{2t}\}$, and a remaining set $P' := \cup_{j=0}^{t} Y_{2j+1}$ of size at most $k(3k-1)^2$, since $t < k$. So we conclude the lemma in case (1).

Now we assume $|Y_1| > (3k-1)^2$. Apply Theorem 1 to extract a block-monotone point set $S \subset Y_1$ of depth $3k$ and block-size at least $|Y_1|/(3ck)^2$ and name the i-th block of S as B_i for $i \in [3k]$. Our proof splits into two cases: S being block-increasing or S being block-decreasing.

Case 1. Suppose S is block-increasing, write $S_{l+i} := Y_{2(t+1-i)}$ for each $i \in [t]$ and set $P' = S_1 \cup \cdots \cup S_{l+t} \cup (Y_1 \setminus S)$. We can check that P' is a $(k, k+l, 0)$-pattern by Definition 12. Let $Z := \cup_{j=1}^{t} Y_{2j+1}$. By an argument similar to (1), we can apply Fact 16 three times to partition Z into $\{A_1, \ldots, A_w, Z'\}$, where $w = O(k \log(k))$, s.t. each A_i is block-monotone of depth at least k and $|Z'| \leq \max\{|Z|/(9c^2k^3), (9c^2k-1)^2\}$.

If $|Z'| \leq (9c^2k-1)^2$, let $E = Z'$. We have partitioned P into $O(k \log(k))$ block-monotone point sets of depth at least k, which are $\{A_1, \ldots, A_w, S\}$, a $(k, k+l, 0)$-pattern P', and a remaining set E of size $O(k^2)$. So we conclude the lemma in case (3).

If $|Z'| \leq |Z|/(9c^2k^3)$, notice that $|Z| \leq k|Y_1|$ since $t < k$, we have $|Z'| \leq |Y_1|/(3ck)^2 \leq |B_i|$, for each $i \in [3k]$. We can take a partition $B_i = B_i' \cup B_i''$ with $|B_i'| = |Z'|$. We observe that $C := B_1' \cup \cdots \cup B_{3k}' \cup Z'$ is block-increasing of depth $3k+1$ and $S' := B_1'' \cup \cdots \cup B_{3k}''$ is block-increasing of depth $3k$ by their construction. We have partitioned P into $O(k \log(k))$ block-monotone point sets of depth at least k, which are $\{A_1, \ldots, A_w, C, S'\}$, and a $(k, k+l, 0)$-pattern P'. So we conclude the lemma in case (3).

Case 2. Suppose S is block-decreasing, we choose two points in the following regions:

$$(x_1, y_1) \in R(B_k) \cap D(B_k) \cap L(B_{k+1}) \cap U(B_{k+1}),$$
$$(x_2, y_2) \in R(B_{2k}) \cap D(B_{2k}) \cap L(B_{2k+1}) \cap U(B_{2k+1}).$$

Also we require x_1 or x_2 isn't the x-coordinate of any element in P, and y_1 or y_2 isn't the y-coordinate of any element in P. We use the lines $x = x_i$ and $y = y_i$ for $i = 1, 2$ to divide the plane into a 3×3 grid and label the regions $R_i, i = 1, \ldots, 9$ as in Figure 3.

Let $C := B_{k+1} \cup \cdots \cup B_{2k}$ and notice that C is block-monotone of depth k and block-size at least $|Y_1|/(3ck)^2$. Define

$$Y' := (R_7 \cap Y_1) \cup C \cup (R_3 \cap Y_1) \cup Y_2 \cup Y_3 \cup \cdots \cup Y_{2t+1}.$$

We can check that Y' is a $(k, t+1)$-configuration and $P' := S_1 \cup \cdots \cup S_l \cup Y'$ is a $(k, l, t+1)$-pattern according to Definitions 11 and 12.

Next, we set $Z_1 := (Y_1 \setminus S) \cap (R_5 \cup R_6 \cup R_8 \cup R_9)$ and $Z_2 := (Y_1 \setminus S) \cap (R_1 \cup R_2 \cup R_4)$. By an argument similar to (1), we can apply Fact 16 twice to partition Z_j into $\{A_{j,1}, \ldots, A_{j,w_j}, Z_j'\}$, where $w_j = O(k \log(k))$, s.t. each $A_{j,x}$ is block-monotone of depth at least k and $|Z_j'| \leq \max\{|Z_j|/(3ck)^2, (9c^2k-1)^2\}$.

Writing $C_1 := B_1 \cup \ldots B_k$ and $C_2 = B_{2k+1} \cup \ldots B_{3k}$, then, for $j = 1, 2$, either $|Z_j'| = O(k^2)$ or $C_j \cup Z_j'$ can be partitioned into two block-decreasing point sets of depth at least k. Indeed, if $|Z_1'| > (9c^2k-1)^2$, we must have

$$|Z_1'| \leq |Z_1|/(3ck)^2 \leq |Y_1|/(3ck)^2 \leq |B_i|,$$

for each $i \in [k]$. Take a partition $B_i = B_i' \cup B_i''$ with $|B_i'| = |Z_1'|$, then we can observe $C_1 := B_1' \cup \cdots \cup B_k' \cup Z_1'$ is block-decreasing of depth $k+1$ and $C_1' = B_1'' \cup \cdots \cup B_k''$ is block-decreasing of depth k by their construction, as wanted. A similar argument applies to $C_2 \cup Z_2'$.

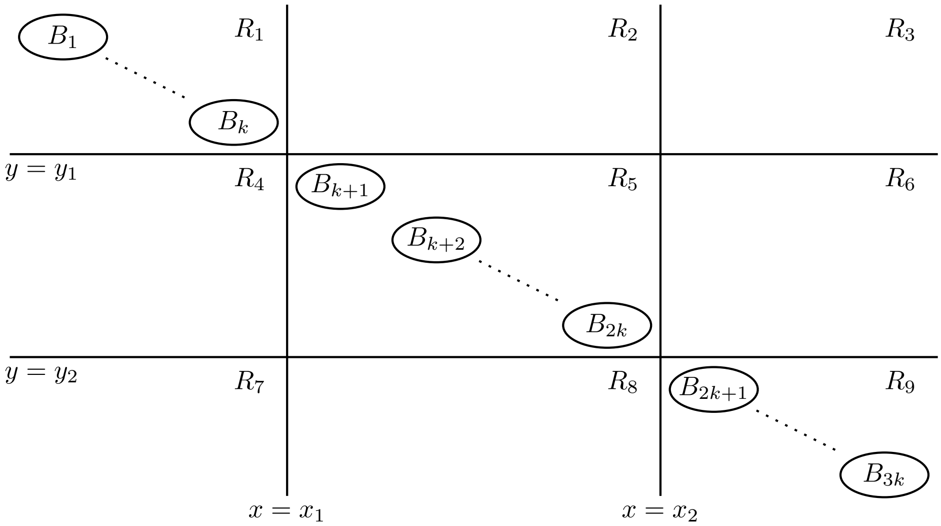

Figure 3 Division of the plane into 9 regions according to $(x_i, y_i), i = 1, 2$. Each ellipse represents a cluster of points as defined in the proof.

We have partitioned $P \setminus (C_1 \cup Z_1' \cup C_2 \cup Z_2')$ into $O(k \log(k))$ block-monotone sequence of depth at least k, which are $\{A_{j,x}; j = 1, 2, x \in [w_j]\}$, and a $(k, l, t + 1)$-pattern P'. Combined with the claim in previous paragraph, we conclude the lemma in case (2).

Finally, when we are in the special case $t = 0$ and S is block-increasing, we can still use the arguments for the case when S is block-decreasing and conclude the lemma in case (2). The condition $t = 0$ can be used to verify Y' is a $(k, t + 1)$-configuration, which is generally not true when $t > 0$ and S is block-increasing. ◀

4 Applications

4.1 Mutually avoiding sets

We devote this subsection to the proof of Theorem 3. The proof is essentially the same as in [1], but we include it here for completeness. Given a non-vertical line L in the plane, we denote L^+ to be the closed upper-half plane defined by L, and L^- to be the closed lower-half plane defined by L. We need the following result, which is Lemma 1 in [1].

▶ **Lemma 18.** *Let $P, Q \subset \mathbb{R}^2$ be two n-element point sets with P and Q separated by a non-vertical line L and $P \cup Q$ in general position. Then for any positive integer $m \leq n$, there is another non-vertical line H s.t. $|H^+ \cap P| = |H^+ \cap Q| = m$ or $|H^- \cap P| = |H^- \cap Q| = m$.*

Proof of Theorem 3. Let k be as given and $n > 24k^2$. Let P be an n-element point set in the plane in general position. We start by taking a non-vertical line L to partition the plane s.t. each half-plane contains $\lfloor \frac{n}{2} \rfloor$ points from P. Then by Lemma 18, we obtain a non-vertical line H with, say, $H^+ \cap (L^+ \cap P) = H^+ \cap (L^- \cap P) = \lfloor \frac{n}{6} \rfloor$. Next, we find a third line N, by first setting $N = H$, and then sweeping N towards the direction of H^-, keeping it parallel with H, until $H^- \cap N^+ \cap L^+$ or $H^- \cap N^+ \cap L^-$ contains $\lfloor \frac{n}{6} \rfloor$ points from P. Without loss of generality, let us assume $Q := P \cap (H^- \cap N^+ \cap L^+)$ first reaches $\lfloor \frac{n}{6} \rfloor$ points, and the region $H^- \cap N^+ \cap L^-$ has less than $\lfloor \frac{n}{6} \rfloor$ points from P. Hence, both $Q_l := P \cap (H^+ \cap L^-)$ and $Q_r := P \cap (N^- \cap L^-)$ have at least $\lfloor \frac{n}{6} \rfloor$ points. See Figure 4 for an illustration.

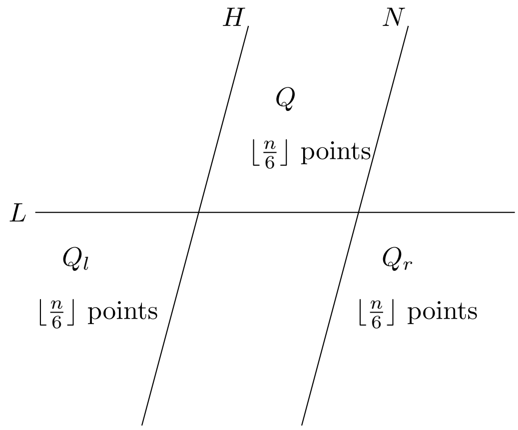

Figure 4 The division of plane into regions according to L, H, N.

We can apply an affine transformation so that L and H are perpendicular, and N is on the right side of H. Think of L as the x-axis, H as the y-axis, and N as a vertical line with a positive x-coordinate. After ordering the elements in Q according to their x-coordinates, we apply Theorem 1 to Q to obtain disjoint subsets $Q_1, \dots, Q_{2k+1} \subset Q$ s.t. $(Q_1, \dots, Q_{2k+1})$ is block-monotone of depth $2k+1$ and block-size $\Omega(n/k^2)$, where each entry represents its y-coordinate. Without loss of generality, we can assume it is block-decreasing, otherwise we can work with Q_r rather than Q_l in the following arguments.

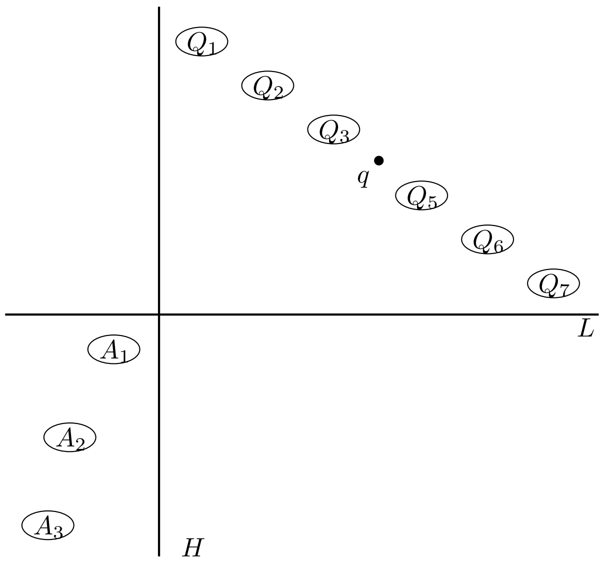

Figure 5 An example when A_i's are increasing. Each ellipse represents a cluster of points as defined in the proof.

Now fix a point $q \in Q_{k+1}$. We express the points in Q_l in polar coordinates (ρ, θ) with q being the origin. We can assume no two points in Q_l are at the same distance to q, otherwise a slight perturbation may be applied. By ordering the points in Q_l with respect to θ, in counter-clockwise order, we apply Theorem 1 to Q_l to obtain disjoint subsets $A_1, \dots, A_k \subset Q_l$ s.t. $(A_1, \dots, A_k)$ is block-monotone of depth k and block-size $\Omega(n/k^2)$, where each entry represents its distance to q. If it's block-decreasing, take $B_i = Q_i$ for $i \in [k]$, and if it's block-increasing, take $B_i = Q_{k+1+i}$. It is easy to check that the sets $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ have the claimed properties. See Figure 5 for an illustration. ◀

4.2 Monotone biarc diagrams

We devote this subsection to the proof of Theorem 4. Our proof is constructive, hence implying an recursive algorithm for the claimed outcome.

We start by making the simple observation that our main results hold for sequences of (not necessarily distinct) real numbers, if the term *block-monotone* now refers to being block-nondecreasing or block-nonincreasing. More precisely, a sequence $(a_1, a_2, \ldots, a_{ks})$ of real numbers is said to be *block-nondecreasing* (*block-nonincreasing*) with *depth* k and *block-size* s if every subsequence $(a_{i_1}, a_{i_2}, \ldots, a_{i_k})$, for $(j-1)s < i_j \leq js$, is nondecreasing (nonincreasing).

▶ **Theorem 19.** *For any positive integer k, every finite sequence of real numbers can be partitioned into at most $C_k = O(k^2 \log k)$ block-monotone subsequences of depth at least k upon deleting at most $(k-1)^2$ entries.*

To see our main results imply the above variation, it suffices to slightly perturb the possibly equal entries of a given sequence until all entries are distinct. Algorithms for our main results can also be applied after such a perturbation.

We need the following lemma in [2] for Theorem 4.

▶ **Lemma 20.** *For any graph $G = (V, E)$ with $V = [n]$, there exists $b \in [n]$ s.t. both the induced subgraphs of G on $\{1, 2, \ldots, b\}$ and $\{b+1, b+2, \ldots, n\}$ have no more than $|E|/2$ edges.*

Proof. For $U \subset [n]$, let G_U denote the induced subgraph of G on U. Let b be the largest among $[n]$ s.t. $E(G_{[b]}) \leq \frac{|E|}{2}$, so $E(G_{[b+1]}) > \frac{|E|}{2}$. Notice that $E(G_{[b+1]})$ and $E(G_{[n]\setminus[b]})$ are two disjoint subsets of E, so $E(G_{[n]\setminus[b]}) \leq |E| - E(G_{[b+1]}) < \frac{|E|}{2}$, as wanted. ◀

Proof of Theorem 4. We prove by induction on $|E|$. The base case when $|E| = 1$ is trivial. For the inductive step, by the given order on V, we can identify V with $[n]$. We find such a b according to Lemma 20. Consider the set E' of edges between $[b]$ and $[n] \setminus [b]$. By writing each edge $e \in E'$ as (x, y), where $x \in [b]$ and $y \in [n] \setminus [b]$, we order the elements in E' lexicographically: for $(x, y), (x', y') \in E$, we have $(x, y) < (x, y)$ when $x < x'$ or when $x = x'$ and $y < y'$.

Given the order on E' described above, consider the sequence of right-endpoints in E'. We apply Theorem 19 with parameter $k = \lceil \epsilon^{-1} \rceil$ to this sequence, to decompose it into C_k many block-monotone sequences of depth k, upon deleting at most $(k-1)^2$ entries. For each block-monotone subsequence of depth k, we draw the corresponding edges on a single page as follows. If the subsequence is block-nonincreasing of depth k and block-size s, we draw the corresponding edges as semicircles above the spine. Then, two edges cross only if they come from the same block. Since there are a total of $\binom{ks}{2}$ pairs of edges, and only $k\binom{s}{2}$ such pairs from the same block, the fraction of pairs of edges that cross in such a drawing is at most $1/k$. See Figure 6(i). Similarly, if the subsequence is block-nondecreasing of depth k and block-size s, we draw the corresponding edges as monotone biarcs, consisting of two semicircles with the first (left) one above the spine, and the second (right) one below the spine. Furthermore, we draw the monotone biarc s.t. it crosses the spine at $b+1-\ell/n-r/(2n^2)$, where ℓ and r are the left and right endpoints of the edge respectively. See Figure 6(ii). By the same argument above, the fraction of pairs of edges that cross in such a drawing is at most $1/k$.

Hence, E' can be decomposed into $C_k+(k-1)^2$ many monotone biarc diagrams, s.t. each monotone biarc diagram has at most $1/k$-fraction of pairs of edges that are crossing.

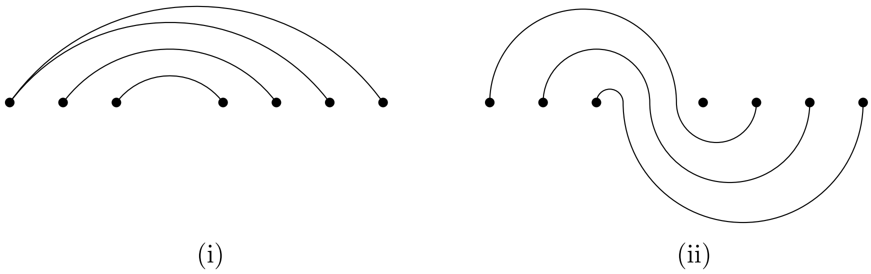

Figure 6 (i) A proper arc diagram. (ii) A monotone biarc diagram.

For edges within $[b]$, Lemma 20 and the inductive hypothesis tell us that they can be decomposed into $(C_k+(k-1)^2)(\log|E|-1)$ monotone biarc diagrams, s.t. the fraction of pairs of edges that are crossing in each diagram is at most $1/k$. The same argument applies to the edges within $[n]\setminus[b]$. However, notice that two such monotone biarc diagrams, one in $[b]$ and another in $[n]\setminus[b]$, can be drawn on the same page without introducing more crossings. Hence, we can decompose $E\backslash E'$ into at most $(C_k+(k-1)^2)(\log|E|-1)$ such monotone biarc diagrams, giving us a total of $(C_k+(k-1)^2)\log|E|$ monotone biarc diagrams. ◀

5 Final remarks

1. We call a sequence $(a_1,a_2,\ldots,a_n)$ of n distinct real numbers *ϵ-increasing* (*ϵ-decreasing*) if the number of decreasing (increasing) pairs (a_i,a_j), where $i<j$, is less than ϵn^2. And we call a sequence *ϵ-monotone* if it's either ϵ-increasing or ϵ-decreasing. Clearly, a block-monotone sequence of depth k is an ϵ-monotone sequence with $\epsilon=k^{-1}$. Hence, Theorem 1 implies the following.

▶ **Corollary 21.** *For all $n>0$ and $\epsilon>0$, every sequence of n distinct real numbers contains an ϵ-monotone subsequence of length at least $\Omega(\epsilon n)$.*

This corollary is also asymptotically best possible. To see this, for $n>(k-1)^2$ and a sequence $A=(a_i)_{i=1}^n$ of distinct real numbers, we can apply Corollary 21 with $\epsilon=(64k)^{-1}$ to A and obtain an ϵ-monotone subsequence $S\subset A$ and then apply Lemma 2.1 in [11] to S to obtain a block-monotone subsequence of depth k and block-size $\Omega(n/k^2)$. So Corollary 21 implies Theorem 1.

2. Let $f(k)$ be the smallest number N s.t. every finite sequence of distinct real numbers can be partitioned into at most N block-monotone subsequences of depth at least k upon deleting $(k-1)^2$ entries. Our Theorem 2 is equivalent to saying $f(k)=O(k^2\log(k))$. The $K(k,2)$-type construction in Remark 9 implies $f(k)\geq k$. What is the asymptotic order of $f(k)$?

3. We suspect our algorithm for Theorem 1 presented in the full version of this paper can be improved. How fast can we compute a block-monotone subsequence as large as claimed in Theorem 1? Can we do it within time almost linear in n for all k?

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