

Computing Generalized Rank Invariant for 2-Parameter Persistence Modules via Zigzag Persistence and Its Applications

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Abstract

The notion of generalized rank invariant in the context of multiparameter persistence has become an important ingredient for defining interesting homological structures such as generalized persistence diagrams. Naturally, computing these rank invariants efficiently is a prelude to computing any of these derived structures efficiently. We show that the generalized rank over a finite interval I of a \mathbf{Z}^2 -indexed persistence module M is equal to the generalized rank of the zigzag module that is induced on a certain path in I tracing mostly its boundary. Hence, we can compute the generalized rank over I by computing the barcode of the zigzag module obtained by restricting the bifiltration inducing M to that path. If the bifiltration and I have at most t simplices and points respectively, this computation takes $O(t^\omega)$ time where $\omega \in [2, 2.373)$ is the exponent of matrix multiplication. Among others, we apply this result to obtain an improved algorithm for the following problem. Given a bifiltration inducing a module M , determine whether M is interval decomposable and, if so, compute all intervals supporting its summands.

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1 Introduction

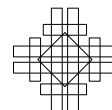
In Topological Data Analysis (TDA) one of the central tasks is that of decomposing persistence modules into direct sums of *indecomposables*. In the case of a persistence module M over the integers \mathbf{Z} , the indecomposables are interval modules, which implies that M is isomorphic to a direct sum of *interval* modules $\mathbb{I}([b_\alpha, d_\alpha])$, for integers $b_\alpha \leq d_\alpha$ and α in some index set A . This follows from a classification theorem for quiver representations established by Pierre Gabriel in the 1970s. The multiset of intervals $\{[b_\alpha, d_\alpha], \alpha \in A\}$ that appear in this decomposition constitutes the *persistence diagram*, or equivalently, the *barcode* of M – a central object in TDA [19, 21].



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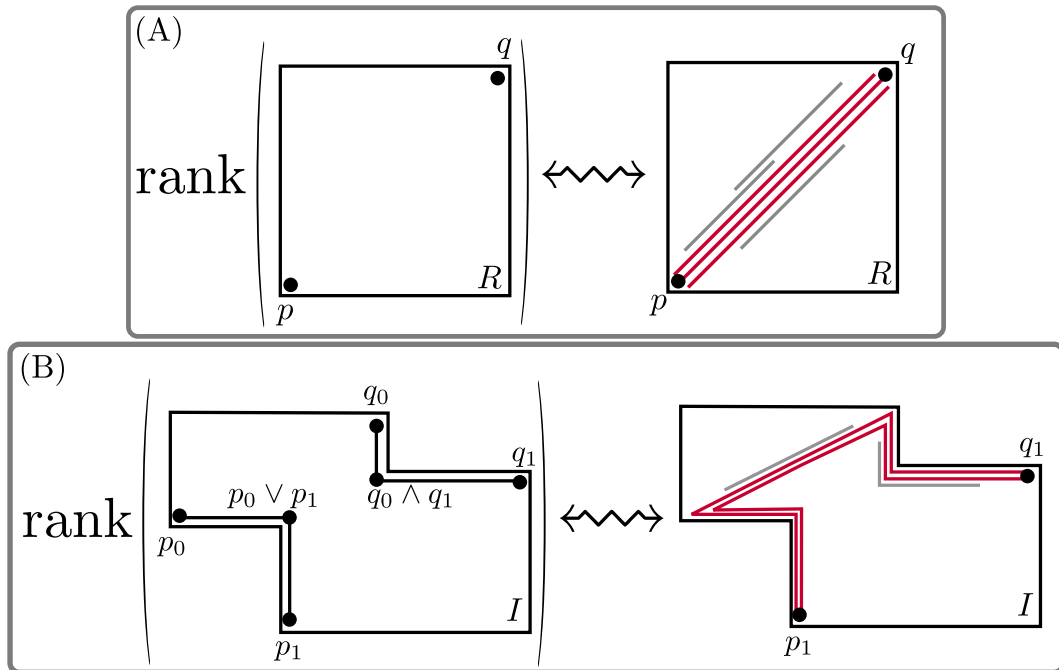


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There are many situations in which data naturally induce persistence modules over posets which are different from \mathbf{Z} [4, 12, 13, 14, 17, 22, 27, 29, 30, 33]. Unfortunately, the situation already becomes “wild” when the domain poset is \mathbf{Z}^2 . In that situation, one must contend with the fact that a direct analogue of the notion of persistence diagrams may not exist [14], namely it may not be possible to obtain a lossless up-to-isomorphism representation of the module as a direct sum of interval modules.

Much energy has been put into finding ways in which one can extract incomplete but still stable invariants from persistence modules $M : \mathbf{Z}^d \rightarrow \mathbf{vec}$ (which we will refer to as a \mathbf{Z}^d -module). Biasotti et al. [6] proposed considering the restriction of a \mathbf{Z}^d -module to lines with positive slope. This was further developed by Lesnick and Wright in the RIVET project [31] which facilitates the interactive visualization of \mathbf{Z}^2 -modules. Cai et al. [11] considered a certain elder-rule on the \mathbf{Z}^2 -modules which arise in multiparameter clustering. Other efforts have identified algebraic conditions which can guarantee that M can be decomposed into interval modules of varying degrees of complexity (e.g. rectangle modules etc) [7, 16].



■ **Figure 1 Generalized rank via zigzag persistence.** Let M be a \mathbf{Z}^2 -module. (A) Standard rank: Let $p \leq q$ in \mathbf{Z}^2 . The rank of the structure map from p to q coincides with the multiplicity of full bars (red) over the diagonal path, which is three. (B) Generalized rank: Let I be an interval of \mathbf{Z}^2 . Let us consider the zigzag poset $\partial I : p_1 \leq (p_0 \vee p_1) \geq p_0 \leq q_0 \geq (q_0 \wedge q_1) \leq q_1$. The generalized rank of M over I is equal to the multiplicity of full bars (red) in the zigzag module $M_{\partial I}$, which is two (Theorem 24) (note: by definition, the zigzag poset ∂I does not fully inherit the partial order on \mathbf{Z}^2 . For example, the partial order on ∂I does not contain the pair (p_1, q_1) whereas $p_1 \leq q_1$ in \mathbf{Z}^2).

A distinct thread has been proposed by Patel in [35] through the reinterpretation of the persistence diagram of a \mathbf{Z} -module as the Möbius inversion of its rank function. Patel’s work was then extended by Kim and Mémoli [26] to the setting of modules defined over any suitable locally finite poset. They generalized the rank invariant via the *limit-to-colimit map* over subposets and then conveniently expressed its Möbius inversion. In fact the limit-to-colimit map was suggested by Amit Patel to the authors of [26] who in [25] used it to define a notion

of rank invariant for zigzag modules. Chambers and Letscher [15] also considered a notion of persistent homology over directed acyclic graphs using the limit-to-colimit map. Asashiba et al. [2] study the case of modules defined on an $m \times n$ grid and propose a high-level algorithm for computing both their generalized rank function and their Möbius inversions with the goal of providing an approximation of a given module by interval decomposables. Asashiba et al. [1] tackle the interval decomposability of a given \mathbf{Z}^d -module via quiver representation theory.

One fundamental algorithmic problem is that of determining whether a given \mathbf{Z}^2 -module is interval decomposable, and if so, computing the intervals. There are some existing solutions to this problem in the literature. Suppose that the input \mathbf{Z}^2 -module is induced by a bifiltration comprising at most t simplices on a grid of cardinality $O(t)$. First, the decomposition algorithm by Dey and Xin [20] can produce all indecomposables from such a module in $O(t^{2\omega+1})$ time (see [24] for comments about its implementation) where $\omega \in [2, 2.373]$ is the exponent of matrix multiplication. Given these indecomposables, one could then test whether they are indeed interval modules. However, the algorithm requires that the input module be such that no two generators or relations in the module have the same grade. Then, Asashiba et al. [1] give an algorithm which requires enumerating an exponential number (in t) of intervals. Finally, the algorithm by Meataxe sidesteps both of the above issues, but incurs a worst-case cost of $O(t^{18})$ as explained in [20].

See also [5, 9, 10, 28, 32] for related recent work.

Contributions. One of our key results is the following. We prove that for an interval I in \mathbf{Z}^2 we can compute the generalized rank invariant $\text{rk}(M)(I)$ of a \mathbf{Z}^2 -module M through the computation of the zigzag persistence barcode of the *restriction* of M to the *boundary cap* of I , which is a certain zigzag path in I ; see Figure 1 for an illustration.

These are our main results assuming that the input is a bifiltration with $O(t)$ simplices:

1. We reduce the problem of computing the generalized rank invariant of a \mathbf{Z}^2 -module to computing zigzag persistence (Theorem 24).
2. We provide an algorithm INTERVAL (page 13) to compute the barcode of any finite interval decomposable \mathbf{Z}^2 -module in time $O(t^{\omega+2})$ (Proposition 38).
3. We provide an algorithm ISINTERVALDECOMP (page 15) to decide the interval decomposability of a finite \mathbf{Z}^2 -module in time $O(t^{3\omega+2})$ (Proposition 39).

2 Preliminaries

In §2.1, we review the notion of interval decomposability of persistence modules. In §2.2, we review the notions of generalized rank invariant and generalized persistence diagram. In §2.3, we discuss how to compute the limit and the colimit of a given functor $P \rightarrow \mathbf{vec}$.

2.1 Persistence Modules and their decompositions

We fix a certain field \mathbb{F} and every vector space in this paper is over \mathbb{F} . Let \mathbf{vec} denote the category of *finite dimensional* vector spaces and linear maps over \mathbb{F} .

Let P be a poset. We regard P as the category that has points of P as objects. Also, for any $p, q \in P$, there exists a unique morphism $p \rightarrow q$ if and only if $p \leq q$. For a positive integer d , let \mathbf{Z}^d be given the partial order defined by $(a_1, a_2, \dots, a_d) \leq (b_1, b_2, \dots, b_d)$ if and only if $a_i \leq b_i$ for $i = 1, 2, \dots, d$.

A (P -indexed) **persistence module** is any functor $M : P \rightarrow \mathbf{vec}$ (which we will simply refer to as a P -module). In other words, to each $p \in P$, a vector space M_p is associated, and to each pair $p \leq q$ in P , a linear map $\varphi_M(p, q) : M_p \rightarrow M_q$ is associated. Importantly, whenever $p \leq q \leq r$ in P , it must be that $\varphi_M(p, r) = \varphi_M(q, r) \circ \varphi_M(p, q)$.

We say that a pair of $p, q \in P$ is **comparable** if either $p \leq q$ or $q \leq p$.

► **Definition 1** ([8]). An **interval** I of P is a subset $I \subseteq P$ such that:

- (i) I is nonempty.
- (ii) If $p, q \in I$ and $p \leq r \leq q$, then $r \in I$.
- (iii) I is **connected**, i.e. for any $p, q \in I$, there is a sequence $p = p_0, p_1, \dots, p_\ell = q$ of elements of I with p_i and p_{i+1} comparable for $0 \leq i \leq \ell - 1$.

By $\mathbf{Int}(P)$ we denote the set of all finite intervals of P . When P is finite and connected, $P \in \mathbf{Int}(P)$ will be referred to as **the full interval**.

For an interval I of P , the **interval module** $\mathbb{I}_I : P \rightarrow \mathbf{vec}$ is defined as

$$\mathbb{I}_I(p) = \begin{cases} \mathbb{F} & \text{if } p \in I, \\ 0 & \text{otherwise,} \end{cases} \quad \varphi_{\mathbb{I}_I}(p, q) = \begin{cases} \text{id}_{\mathbb{F}} & \text{if } p, q \in I, p \leq q, \\ 0 & \text{otherwise.} \end{cases}$$

Direct sums and quotients of P -modules are defined pointwisely at each index $p \in P$.

► **Definition 2.** Let M be any P -module. A **submodule** N of M is defined by subspaces $N_p \subseteq M_p$ such that $\varphi_M(p, q)(N_p) \subseteq N_q$ for all $p, q \in P$ with $p \leq q$. These conditions guarantee that N itself is a P -module, with the structure maps given by the restrictions $\varphi_M(p, q)|_{N_p}$. In this case we write $N \leq M$.

A submodule N is a **summand** of M if there exists a submodule N' which is complementary to N , i.e. $M_p = N_p \oplus N'_p$ for all p . In that case, we say that M is a direct sum of N, N' and write $M \cong N \oplus N'$. Note that this direct sum is an internal direct sum.

► **Definition 3.** A P -module M is called **interval decomposable** if M is isomorphic to a direct sum of interval modules, i.e. there exists an indexing set \mathcal{J} such that $M \cong \bigoplus_{j \in \mathcal{J}} \mathbb{I}_{I_j}$ (external direct sum). In this case, the multiset $\{I_j : j \in \mathcal{J}\}$ is called the **barcode** of M , which will be denoted by $\text{barc}(M)$.

The Azumaya-Krull-Remak-Schmidt theorem guarantees that $\text{barc}(M)$ is well-defined [3]. Consider a **zigzag poset** of n points, $\bullet_1 \leftrightarrow \bullet_2 \leftrightarrow \dots \leftrightarrow \bullet_{n-1} \leftrightarrow \bullet_n$ where \leftrightarrow stands for either \leq or \geq . A functor from a zigzag poset to \mathbf{vec} is called a **zigzag module** [12]. Any zigzag module is interval decomposable [23] and thus admits a barcode.

The following proposition directly follows from the Azumaya-Krull-Remak-Schmidt theorem and will be useful in §4.

► **Proposition 4.** Let $M : P \rightarrow \mathbf{vec}$ be interval decomposable and let $N \leq M$ is a summand of M (Definition 2). Then, M/N is interval decomposable (proof in the full version).

2.2 Generalized rank invariant and generalized persistence diagrams

Let P be a finite connected poset and consider any P -module M . Then M admits a limit $\varprojlim M = (L, (\pi_p : L \rightarrow M_p)_{p \in P})$ and a colimit $\varinjlim M = (C, (i_p : M_p \rightarrow C)_{p \in P})$; see the full version for definitions. This implies that, for every $p \leq q$ in P , $\varphi_M(p \leq q) \circ \pi_p = \pi_q$ and $i_q \circ \varphi_M(p \leq q) = i_p$, which in turn imply $i_p \circ \pi_p = i_q \circ \pi_q : L \rightarrow C$ for any $p, q \in P$.

► **Definition 5** ([26]). The **canonical limit-to-colimit map** $\psi_M : \varprojlim M \rightarrow \varinjlim M$ is the linear map $i_p \circ \pi_p$ for any $p \in P$. The **generalized rank** of M is $\text{rank}(M) := \text{rank}(\psi_M)$.

The rank of M counts the multiplicity of the fully supported interval modules in a direct sum decomposition of M .

► **Theorem 6** ([15, Lemma 3.1]). The rank of M is equal to the number of indecomposable summands of M which are isomorphic to the interval module \mathbb{I}_P .

► **Definition 7.** The **(Int)-generalized rank invariant** of M is the map $\text{rk}_{\mathbb{I}}(M) : \mathbf{Int}(P) \rightarrow \mathbf{Z}_+$ defined as $I \mapsto \text{rank}(M|_I)$, where $M|_I$ is the restriction of M to I .

► **Definition 8.** The **(Int)-generalized persistence diagram** of M is the unique¹ function $\text{dgm}_{\mathbb{I}}(M) : \mathbf{Int}(P) \rightarrow \mathbf{Z}$ that satisfies, for any $I \in \mathbf{Int}(P)$,

$$\text{rk}_{\mathbb{I}}(M)(I) = \sum_{\substack{J \supseteq I \\ J \in \mathbf{Int}(P)}} \text{dgm}_{\mathbb{I}}(M)(J).$$

The following is a slight variation of [26, Theorem 3.14] and [2, Theorem 5.10].

► **Theorem 9.** If a given $M : P \rightarrow \mathbf{vec}$ is interval decomposable, then for all $I \in \mathbf{Int}(P)$, $\text{dgm}_{\mathbb{I}}(M)(I)$ is equal to the multiplicity of I in $\text{barc}(M)$ (proof in the full version).

We consider P to be a 2d-grid and focus on the setting of \mathbf{Z}^2 -modules.

► **Definition 10.** For any $I \in \mathbf{Int}(\mathbf{Z}^2)$, we define $\text{nb}_{\mathbb{I}}(I) := \{p \in \mathbf{Z}^2 \setminus I : I \cup \{p\} \in \mathbf{Int}(\mathbf{Z}^2)\}$.

Note that $\text{nb}_{\mathbb{I}}(I)$ is nonempty [2, Proposition 3.2]. When $A \subseteq \text{nb}_{\mathbb{I}}(I)$ contains more than one point, $A \cup I$ is not necessarily an interval of \mathbf{Z}^2 . However, there always exists a unique smallest interval that contains $A \cup I$ which is denoted by $\overline{A \cup I}$.

► **Remark 11** ([2, Theorem 5.3]). If in Definition 8 we assume that $P \in \mathbf{Int}(\mathbf{Z}^2)$ then we have that for every $I \in \mathbf{Int}(P)$,²

$$\text{dgm}_{\mathbb{I}}(M)(I) = \text{rk}_{\mathbb{I}}(M)(I) + \sum_{\substack{A \subseteq \text{nb}_{\mathbb{I}}(I) \cap P \\ A \neq \emptyset}} (-1)^{|A|} \text{rk}_{\mathbb{I}}(M)\left(\overline{A \cup I}\right). \tag{1}$$

2.3 Canonical constructions of limits and colimits

Let M be any P -module.

► **Notation 12.** Let $p, q \in P$ and let $v_p \in M_p$ and $v_q \in M_q$. We write $v_p \sim v_q$ if p and q are comparable, and either v_p is mapped to v_q via $\varphi_M(p, q)$ or v_q is mapped to v_p via $\varphi_M(q, p)$.

The following proposition gives a standard way of constructing a limit and a colimit of a P -module M . Since it is well-known, we do not prove it (see for example [26, Section E]).

► **Proposition 13.**

(i) The limit of M is (isomorphic to) the pair $(W, (\pi_p)_{p \in P})$ where:

$$W := \left\{ (v_p)_{p \in P} \in \bigoplus_{p \in P} M_p : \forall p \leq q \text{ in } P, v_p \sim v_q \right\} \tag{2}$$

and for each $p \in P$, the map $\pi_p : W \rightarrow M_p$ is the canonical projection. An element of W is called a **section** of M .

(ii) The colimit of M is (isomorphic to) the pair $(U, (i_p)_{p \in P})$ described as follows: For $p \in P$, let the map $j_p : M_p \hookrightarrow \bigoplus_{p \in P} M_p$ be the canonical injection. U is the quotient $(\bigoplus_{p \in P} M_p) / T$, where T is the subspace of $\bigoplus_{p \in P} M_p$ which is generated by the vectors of the form $j_p(v_p) - j_q(v_q)$, $v_p \sim v_q$, the map $i_p : M_p \rightarrow U$ is the composition $\rho \circ j_p$, where ρ is the quotient map $\bigoplus_{p \in P} M_p \rightarrow U$.

¹ The existence and uniqueness is guaranteed by properties of the Möbius inversion formula [36, 37].

² In [2], only the case $P = \{1, \dots, m\} \times \{1, \dots, n\} \subset \mathbf{Z}^2$ was considered. However, it is not difficult to check that Eq. (1) is still valid for any finite interval P in \mathbf{Z}^2 and any subinterval $I \subseteq P$.

► **Setup 1.** In the rest of the paper, limits and colimits of a P -module M will all be constructed as in Proposition 13. Hence, assuming that P is connected, the canonical limit-to-colimit map $\varprojlim M \rightarrow \varinjlim M$ is $\psi_M := i_p \circ \pi_p$ for any $p \in P$.

3 Computing generalized rank via boundary zigzags

In §3.1 we introduce the notions of lower and upper fences of a poset. In §3.2, we introduce the *boundary cap* ∂I of a finite interval I of \mathbf{Z}^2 , which is a path, a certain sequence of points in I . In §3.3, we show that the rank of any functor $M : I \rightarrow \mathbf{vec}$ can be obtained by computing the barcode of the zigzag module over the path ∂I .

3.1 Lower and upper fences of a poset

Let P be any connected poset. Given any $p \in P$, by p^\downarrow , we denote the set of all elements of P that are less than or equal to p . Dually p^\uparrow is defined as the set of all elements of P that are greater than or equal to p .

► **Definition 14.** A subposet $L \subset P$ (resp. $U \subset P$) is called a lower (resp. upper) fence of P if L is connected, and for any $q \in P$, the intersection $L \cap q^\downarrow$ (resp. $U \cap q^\uparrow$) is nonempty and connected.

► **Proposition 15.** Let L and U be a lower and an upper fences of P respectively. Given any P -module M , we have $\varprojlim M \cong \varprojlim M|_L$ and $\varinjlim M \cong \varinjlim M|_U$ (proof in the full version).³

The canonical isomorphism $\varprojlim M \cong \varprojlim M|_L$ in Proposition 15 is given by the canonical section extension $e : \varprojlim M|_L \rightarrow \varprojlim M$. Namely,

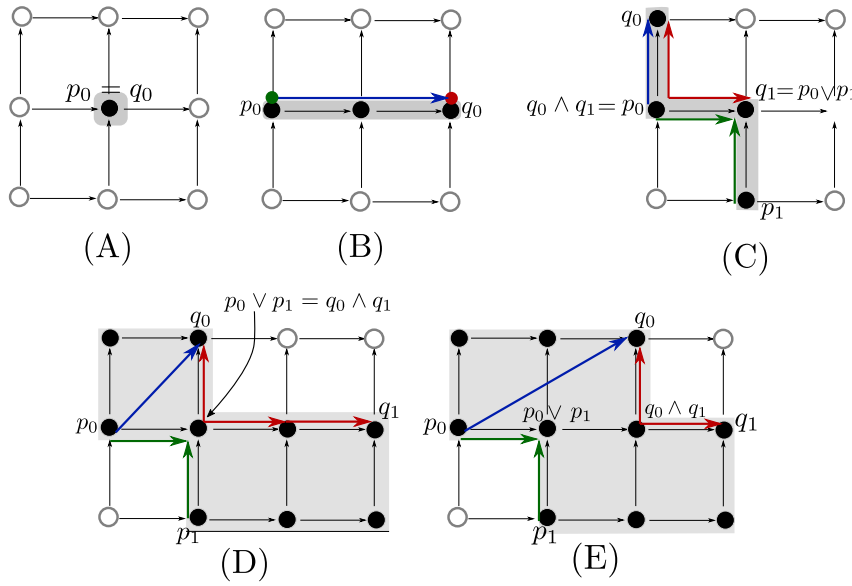
$$e : (\mathbf{v}_p)_{p \in L} \mapsto (\mathbf{w}_q)_{q \in P}, \quad (3)$$

where for any $q \in P$, the vector \mathbf{w}_q is defined as $\varphi_M(p, q)(\mathbf{v}_p)$ for any $p \in L \cap q^\downarrow$; the connectedness of $L \cap q^\downarrow$ guarantees that \mathbf{w}_q is well-defined. Also, if $q \in L$, then $\mathbf{w}_q = \mathbf{v}_q$. The inverse $r := e^{-1}$ is the canonical section restriction. The other isomorphism $\varinjlim M \cong \varinjlim M|_U$ in Proposition 15 is given by the map $i : \varinjlim M|_U \rightarrow \varinjlim M$ defined by $[v_p] \mapsto [v_p]$ for any $p \in U$ and any $v_p \in M_p$; the fact that this map i is well-defined will become clear from Proposition 22. Let us define $\xi : \varprojlim M|_L \rightarrow \varinjlim M|_U$ by $i^{-1} \circ \psi_M \circ e$. By construction, the following diagram commutes

$$\begin{array}{ccc} \varprojlim M|_L & \xrightarrow{\xi} & \varinjlim M|_U \\ \cong \downarrow e & & \cong \downarrow i \\ \varprojlim M & \xrightarrow{\psi_M} & \varinjlim M, \end{array} \quad (4)$$

where ψ_M is the canonical limit-to-colimit map of M . Hence we have the fact $\text{rank}(\psi_M) = \text{rank}(\xi)$, which is useful for proving Theorem 24.

³ This proposition appeared in an earlier version of [26] (see Proposition D.14 in the second arXiv version).



■ **Figure 2** Five different intervals I of \mathbf{Z}^2 . Relations in $\min_{\mathbf{ZZ}}(I)$ and $\max_{\mathbf{ZZ}}(I)$ are indicated by green and red arrows, respectively. The inequality $p_0 \leq q_0$ is indicated by blue arrows unless $p_0 = q_0$. Notice that ∂I , as defined in equation (7), has cardinality 2, 2, 6, 6, 6 in that order ((A),(B),(C),(D),(E)).

3.2 Boundary cap of an interval in \mathbf{Z}^2

Let $I \in \text{Int}(\mathbf{Z}^2)$, i.e. I is a finite interval of \mathbf{Z}^2 (Definition 1). By $\min(I)$ and $\max(I)$, we denote the collections of minimal and maximal elements of I , respectively. In other words,

$$\min(I) := \{p \in I : \text{there is no } q \in I \text{ s.t. } q < p\},$$

$$\max(I) := \{p \in I : \text{there is no } q \in I \text{ s.t. } p < q\}.$$

Note that $\min(I)$ and $\max(I)$ are nonempty and that $\min(I)$ and $\max(I)$ respectively form an *antichain* in I , i.e. any two different points in $\min(I)$ (or in $\max(I)$) are *not* comparable.

► Remark 16.

- (i) The least upper bound and the greatest lower bound of $p, q \in \mathbf{Z}^2$ are denoted by $p \vee q$ and $p \wedge q$ respectively. Let $p = (p_x, p_y)$ and $q = (q_x, q_y)$ in \mathbf{Z}^2 . Then,

$$p \vee q = (\max\{p_x, q_x\}, \max\{p_y, q_y\}), \quad p \wedge q = (\min\{p_x, q_x\}, \min\{p_y, q_y\}).$$

For the item below, let $I \in \text{Int}(\mathbf{Z}^2)$. Notice the following:

- (ii) Since $\min(I)$ is a finite antichain, we can list the elements of $\min(I)$ in ascending order of their x -coordinates, i.e. $\min(I) := \{p_0, \dots, p_k\}$ and such that for each $i = 0, \dots, k$, the x -coordinate of p_i is less than that of p_{i+1} . Similarly, let $\max(I) := \{q_0, \dots, q_\ell\}$ be ordered in ascending order of q_j 's x -coordinates. We have that $p_0 \leq q_0$ (Figure 2).

► **Definition 17** (Lower and upper zigzags of an interval). *Let I , $\min(I)$, and $\max(I)$ be as in Remark 16 ii. We define the following two zigzag posets (Figure 2):*

$$\begin{aligned} \min_{\mathbf{ZZ}}(I) &:= \{p_0 < (p_0 \vee p_1) > p_1 < (p_1 \vee p_2) > \dots < (p_{k-1} \vee p_k) > p_k\} \\ &= \min(I) \cup \{p_i \vee p_{i+1} : i = 0, \dots, k-1\}, \end{aligned} \tag{5}$$

$$\begin{aligned} \max_{\mathbf{ZZ}}(I) &:= \{q_0 > (q_0 \wedge q_1) < q_1 > (q_1 \wedge q_2) < \dots > (q_{\ell-1} \wedge q_\ell) > q_\ell\} \\ &= \max(I) \cup \{q_i \wedge q_{i+1} : i = 0, \dots, \ell-1\}. \end{aligned} \tag{6}$$

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Note that $\min_{\mathbf{ZZ}}(I)$ and $\max_{\mathbf{ZZ}}(I)$ are lower and upper fences of I respectively.

For $p, q \in P$, let us write $p \triangleleft q$ if $p < q$ and there is no $r \in P$ such that $p < r < q$. Similarly, we write $p \triangleright q$ if $p > q$ and there is no $r \in P$ such that $p > r > q$.

► **Definition 18.** Given a poset P , a **path** Γ between two points $p, q \in P$ is a sequence of points $p = p_0, \dots, p_k = q$ in P such that either $p_i \leq p_{i+1}$ or $p_i \geq p_{i+1}$ for every $i \in [1, k-1]$ (in particular, there can be a pair $i \neq j$ such that $p_i = p_j$). The path Γ is said to be **monotonic** if $p_i \leq p_{i+1}$ for each i . The path Γ is called **faithful** if either $p_i \triangleleft p_{i+1}$ or $p_i \triangleright p_{i+1}$ for each i .

► **Definition 19** (Boundary cap of an interval). We define the **boundary cap** ∂I of $I \in \text{Int}(\mathbf{Z}^2)$ as the path obtained by concatenating $\min_{\mathbf{ZZ}}(I)$ and $\max_{\mathbf{ZZ}}(I)$ in Eqs. (5) and (6).

$$\partial I := \underbrace{p_k < (p_k \vee p_{k-1}) > p_{k-1} < \cdots > p_0}_{2k+1 \text{ terms from } \min_{\mathbf{ZZ}}(I)} \leq \underbrace{q_0 > (q_0 \wedge q_1) < q_1 > \cdots < q_\ell}_{2\ell+1 \text{ terms from } \max_{\mathbf{ZZ}}(I)}, \quad (7)$$

We remark that ∂I can contain multiple copies of the same point. Namely, there can be $i \in [0, k]$ and $j \in [0, \ell]$ such that either $p_i = q_j$ (Figure 2 (A)), $p_i = q_j \wedge q_{j+1}$ (Figure 2 (C)), $p_i \vee p_{i+1} = q_j$ (Figure 2 (C)), or $p_i \vee p_{i+1} = q_j \wedge q_{j+1}$ (Figure 2 (D)).

Consider the following zigzag poset of the same length as ∂I :

$$\mathbf{ZZ}_{\partial I} : \underbrace{\bullet_1 < \bullet_2 > \bullet_3 < \cdots > \bullet_{2k+1}}_{2k+1} < \underbrace{\circ_1 > \circ_2 < \circ_3 > \cdots < \circ_{2\ell+1}}_{2\ell+1}. \quad (8)$$

Still using the notation in Eqs. (7) we have the following order-preserving map

$$\iota_I : \mathbf{ZZ}_{\partial I} \rightarrow I \quad (9)$$

whose image is ∂I : \bullet_1 is sent to p_k , \bullet_2 is sent to $p_k \vee p_{k-1}$, ..., and $\circ_{2\ell+1}$ is sent to q_ℓ .

3.3 Generalized rank invariant via boundary zigzags

The goal of this section is to establish Theorem 24.

► **Definition 20.** Let P be a poset. Let $\Gamma : p_0, \dots, p_k$ be a path in P . A $(k+1)$ -tuple $\mathbf{v} \in \bigoplus_{i=0}^k M_{p_i}$ is called the **section of M along Γ** if $\mathbf{v}_{p_i} \sim \mathbf{v}_{p_{i+1}}$ for each i (Notation 12).

Note that \mathbf{v} is not necessarily a section of the restriction $M|_{\{p_0, \dots, p_k\}}$ of M to the subposet $\{p_0, \dots, p_k\} \subseteq I$. Furthermore, Γ can contain multiple copies of the same point in P .

► **Example 21.** Consider $M : \{(1, 1), (1, 2), (2, 2), (2, 1)\} \subset \mathbf{Z}^2 \rightarrow \mathbf{vec}$ given as follows.

$$\begin{array}{ccc} M_{(1,2)} & \longrightarrow & M_{(2,2)} \\ \uparrow & & \uparrow \\ M_{(1,1)} & \longrightarrow & M_{(2,1)} \end{array} = \begin{array}{ccc} \mathbb{F} & \xrightarrow{1} & \mathbb{F} \\ \uparrow 1 & & \uparrow (1 \ 1) \\ \mathbb{F} & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \mathbb{F}^2 \end{array}$$

Consider the path $\Gamma : (1, 1), (1, 2), (2, 2), (2, 1)$ which contains all points in the indexing poset. Then, $\mathbf{v} := (1, 1, 1, (0, 1)) \in M_{(1,1)} \oplus M_{(1,2)} \oplus M_{(2,2)} \oplus M_{(2,1)}$ is a section of M along Γ , while \mathbf{v} is *not* a section of M itself, i.e. $\mathbf{v} \notin \varprojlim M$.

By Proposition 13 (ii), we directly have:

► **Proposition 22.** *Let $p, q \in P$. For any vectors $v_p \in M_p$ and $v_q \in M_q$ $[v_p] = [v_q]$ in⁴ the colimit $\varinjlim M$ if and only if there exist a path $\Gamma : p = p_0, p_1, \dots, p_n = q$ in P and a section \mathbf{v} of M along Γ such that $\mathbf{v}_p = v_p$ and $\mathbf{v}_q = v_q$.*

The map $\iota_I : \mathbf{ZZ}_{\partial I} \rightarrow I$ in Eq.s. (9) induces a bijection between the sections of $M_{\partial I}$ and the sections of M along ∂I in a canonical way. Hence:

► **Setup 2.** In the rest of §3.3, we fix both $I \in \mathbf{Int}(\mathbf{Z}^2)$ and a functor $M : I \rightarrow \mathbf{vec}$. Each element in $\varinjlim M_{\partial I}$ is identified with the corresponding section of M along ∂I . Also, we identify points in (7) and (8) via ι_I .

► **Definition 23** (Zigzag module along ∂I). *Define the zigzag module $M_{\partial I} : \mathbf{ZZ}_{\partial I} \rightarrow \mathbf{vec}$ by $(M_{\partial I})_x := M_{\iota_I(x)}$ for $x \in \mathbf{ZZ}_{\partial I}$ and $\varphi_{M_{\partial I}}(x, y) := \varphi_M(\iota_I(x), \iota_I(y))$ for $x \leq y$ in $\mathbf{ZZ}_{\partial I}$.*

One of our main results is the following.

► **Theorem 24.** *$\text{rank}(M)$ is equal to the multiplicity of the full interval in $\text{barc}(M_{\partial I})$.*

Proof. By Theorem 6, it suffices to show that

$$\text{rank}(\psi_M : \varinjlim M \rightarrow \varinjlim M) = \text{rank}(\psi_{M_{\partial I}} : \varinjlim M_{\partial I} \rightarrow \varinjlim M_{\partial I}).$$

Let $L := \min_{\mathbf{ZZ}}(I)$ and $U := \max_{\mathbf{ZZ}}(I)$ which are lower and upper fences of I respectively. Let us define the maps e, r, i and ξ as described in the paragraph after Proposition 15. Then, by Proposition 15 and the commutative diagram in (4), it suffices to prove that the rank of ξ equals the rank of $\psi_{M_{\partial I}}$. To this end, we show that there exist a surjective linear map $f : \varinjlim M_{\partial I} \rightarrow \varinjlim M|_L$ and an injective linear map $g : \varinjlim M|_U \rightarrow \varinjlim M_{\partial I}$ such that $\psi_{M_{\partial I}} = g \circ \xi \circ f$. We define f as the canonical section restriction $(\mathbf{v}_q)_{q \in \partial I} \mapsto (\mathbf{v}_q)_{q \in L}$. We define g as the canonical map, i.e. $[v_q] \mapsto [v_q]$ for any $q \in U$ and any $v_q \in M_q$. By Proposition 22 and by construction of $M_{\partial I}$, the map g is well-defined.

We now show that $\psi_{M_{\partial I}} = g \circ \xi \circ f$. Let $\mathbf{v} := (\mathbf{v}_q)_{q \in \partial I} \in \varinjlim M_{\partial I}$. Then, by definition of $\psi_{M_{\partial I}}$ (Setup 1), the image of \mathbf{v} via $\psi_{M_{\partial I}}$ is $[v_{q_0}]$ where $q_0 \in U$ is defined as in Remark 16 ii. Also, we have

$$\mathbf{v} \xrightarrow{f} (\mathbf{v}_q)_{q \in L} \xrightarrow{\xi} [v_{q_0}] (\in \varinjlim M|_U) \xrightarrow{g} [v_{q_0}] (\in \varinjlim M_{\partial I}),$$

which proves the equality $\psi_{M_{\partial I}} = g \circ \xi \circ f$.

We claim that f is surjective. Let $r' : \varinjlim M \rightarrow \varinjlim M_{\partial I}$ be the canonical section restriction map $(\mathbf{v}_q)_{q \in I} \mapsto (\mathbf{v}_q)_{q \in \partial I}$. Then, the restriction $r : \varinjlim M \rightarrow \varinjlim M|_L$, can be seen as the composition of two restrictions $r = f \circ r'$. Since r is the inverse of the isomorphism e in diagram (4), r is surjective and thus so is f .

Next we claim that g is injective. Let $i' : \varinjlim M_{\partial I} \rightarrow \varinjlim M$ be defined by $[v_q] \mapsto [v_q]$ for any $q \in \partial I$ and any $v_q \in M_q$. By Proposition 22 and by construction of $M_{\partial I}$, the map i' is well-defined. Then, for the isomorphism i in diagram (4), we have $i = i' \circ g$. This implies that g is injective. ◀

⁴ For simplicity, we write $[v_p]$ and $[v_q]$ instead of $[j_p(v_p)]$ and $[j_q(v_q)]$ respectively where $j_p : M_p \rightarrow \bigoplus_{r \in P} M_r$ and $j_q : M_q \rightarrow \bigoplus_{r \in P} M_r$ are the canonical inclusion maps.

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► **Remark 25.** In Definition 19 one may consider the “lower” boundary cap $\widehat{\partial I}$, as an alternative to ∂I :

$$\widehat{\partial I} : p_0 < p_0 \vee p_1 > p_1 < \cdots > p_k \leq q_\ell > q_\ell \wedge q_{\ell-1} < q_{\ell-1} > \cdots < q_0.$$

The value $\text{rank}(M)$ also equals the multiplicity of the full interval in the barcode of the zigzag module induced over $\widehat{\partial I}$.

By Theorem 24, we can utilize algorithms for zigzag persistence in order to compute the generalized rank invariant and the generalized persistence diagram of any \mathbf{Z}^2 -module that is obtained by applying the homology functor to a finite simplicial bifiltration consisting of $O(t)$ simplices over an index set of size $O(t)$. For this, we complete the boundary cap of a given interval to a faithful path (i.e. we put the missing monotonic paths between every pair of consecutive points) and then simply run a zigzag persistence algorithm, say the $O(t^\omega)$ algorithm of Milosavljevic et al. [34], on the filtration restricted to this path.

► **Remark 26.** To compute $\text{dgm}_{\mathbb{I}}(M)(I)$ by the formula in (1), one needs to consider terms whose number depends exponentially on the number of neighbors of I . However, for any interval that has at most $O(\log t)$ neighbors, we have $2^{O(\log t)} = t^c$ terms for some constant $c > 0$. It follows that using $O(t^\omega)$ zigzag persistence algorithm for computing generalized ranks, we obtain an $O(t^{\omega+c})$ algorithm for computing generalized persistence diagrams of intervals that have at most $O(\log t)$ neighbors.

4 Computing intervals and detecting interval decomposability

When a persistence module M admits a summand N that is isomorphic to an interval module, N will be called an **interval summand** of M . In this section, we apply Theorem 24 for computing generalized rank via zigzag to different problems that ask to find interval summands of an input finite \mathbf{Z}^2 -module: Problems I, II, and III.

Let K be a finite abstract simplicial complex and let $\text{sub}(K)$ be the poset of all subcomplexes of K , ordered by inclusion. Given any poset P , an order-preserving map $\mathcal{F} : P \rightarrow \text{sub}(K)$ is called a simplicial filtration (of K).

► **Setup 3.** Throughout §4, \mathcal{F} denotes a bifiltration of a simplicial complex K defined over an interval $P \in \mathbf{Int}(\mathbf{Z}^2)$. Let $t := \max(|K|, |P|)$ denote the maximum of the number of simplices in K and the number of points in P . By $M_{\mathcal{F}} : P \rightarrow \mathbf{vec}$ we denote the module induced by \mathcal{F} through the homology functor with coefficients in the field \mathbb{F} .

Computing the dimension function. In all algorithms below, we utilize a subroutine $\text{DIM}(\mathcal{F}, P)$, which computes the dimension of the vector space $(M_{\mathcal{F}})_p$ for every $p \in P$.

► **Proposition 27.** $\text{DIM}(\mathcal{F}, P)$ can be executed in $O(t^3)$ time (proof in the full version).

4.1 Detecting interval modules

We consider the following problem.

► **Problem I.** Determine whether $M_{\mathcal{F}}$ is isomorphic to the direct sum of a certain number of copies of \mathbb{I}_P and if so, report the number of such copies.

Algorithm `ISINTERVAL` solves Problem I. The correctness of the algorithm follows from Proposition 28. Below, for an interval $I \in \mathbf{Int}(\mathbf{Z}^2)$ and for $m \in \mathbf{Z}_{\geq 0}$ we define $\mathbb{I}_I^m := \underbrace{\mathbb{I}_I \oplus \mathbb{I}_I \oplus \dots \oplus \mathbb{I}_I}_m$. In particular, \mathbb{I}_I^0 is defined to be the trivial module. Let us recall that $(M_{\mathcal{F}})_{\partial P}$ denotes the zigzag module along the boundary cap ∂P (Definition 23).

■ **Algorithm 1** `ISINTERVAL`(\mathcal{F}, P).

-
- Step 1. Compute zigzag barcode $\text{barc}((M_{\mathcal{F}})_{\partial P})$ and let m be the multiplicity of the full interval.
 - Step 2. Call `DIM`(\mathcal{F}, P) (Computes $\dim(M_{\mathcal{F}})_p$ for every $p \in P$)
 - Step 3. If $\dim(M_{\mathcal{F}})_p = m$ for each point $p \in P$ return m , otherwise return 0 indicating $M_{\mathcal{F}}$ has a summand which is not an interval module supported over P .
-

► **Proposition 28.** Assume that a given $M : P \rightarrow \mathbf{vec}$ has the indecomposable decomposition $M \cong \bigoplus_{i=1}^m M_i$. Then, every summand M_i is isomorphic to the interval module \mathbb{I}_P if and only if $\text{rk}_{\mathbb{I}}(M)(P) = \dim M_p = m$ for all $p \in P$ (proof in the full version).

► **Proposition 29.** Algorithm `ISINTERVAL` can be run in $O(t^3)$ time (proof in the full version).

4.2 Interval decomposable modules and its summands

Setup 3 still applies in §4.2. Next, we consider the problem of computing all indecomposable summands of $M_{\mathcal{F}}$ under the assumption that $M_{\mathcal{F}}$ is interval decomposable (Definition 3).

► **Problem II.** Assume that $M_{\mathcal{F}} : P \rightarrow \mathbf{vec}$ is interval decomposable. Find $\text{barc}(M_{\mathcal{F}})$.

We present algorithm `INTERVAL` to solve Problem II in $O(t^{\omega+2})$ time. This algorithm is eventually used to detect whether a given module is interval decomposable or not (Problem III). Before describing `INTERVAL`, we first describe another algorithm `TRUEINTERVAL`. The outcomes of both `INTERVAL` and `TRUEINTERVAL` are the same as the barcode of $M_{\mathcal{F}}$ in Problem II (Propositions 33 and 37). Whereas `TRUEINTERVAL` is more intuitive, real implementation is accomplished via `INTERVAL`.

► **Definition 30.** Let $\mathcal{I}(M_{\mathcal{F}}) := \{I \in \mathbf{Int}(P) : \text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I) > 0\}$. We call $I \in \mathcal{I}(M_{\mathcal{F}})$ maximal if there is no $J \supsetneq I$ in $\mathbf{Int}(P)$ such that $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(J)$ is nonzero.

► **Proposition 31.** Assume that $M_{\mathcal{F}}$ is interval decomposable and let $I \in \mathcal{I}(M_{\mathcal{F}})$ be maximal. Then, I belongs to $\text{barc}(M_{\mathcal{F}})$ and the multiplicity of I in $\text{barc}(M_{\mathcal{F}})$ is equal to $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I)$.

Proof. By assumption, all summands in the sum $\sum_{\substack{A \subseteq \text{nb}_{\mathbb{I}}(I) \cap P \\ A \neq \emptyset}} (-1)^{|A|} \text{rk}_{\mathbb{I}}(M_{\mathcal{F}})\left(\overline{I \cup A}\right)$ corresponding to the second term of (1) are zero. Hence, $\text{dgm}_{\mathbb{I}}(M_{\mathcal{F}})(I) = \text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I) > 0$. Since $M_{\mathcal{F}}$ is interval decomposable, by Theorem 9, $\text{dgm}_{\mathbb{I}}(M_{\mathcal{F}})(I)$ is equal to the multiplicity of I in $\text{barc}(M_{\mathcal{F}})$. Therefore, not only does I belong to $\text{barc}(M_{\mathcal{F}})$, but also the value $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I)$ is equal to the multiplicity of I in $\text{barc}(M_{\mathcal{F}})$. ◀

The following proposition is a corollary of Proposition 31.

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► **Proposition 32.** *Assume that $M_{\mathcal{F}}$ is interval decomposable and let $I \in \mathcal{I}(M_{\mathcal{F}})$ be maximal. Let $\mu_I := \text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I)$. Then, $M_{\mathcal{F}}$ admits a summand N which is isomorphic to $\mathbb{I}_I^{\mu_I}$.*

Let us now describe a procedure `TRUEINTERVAL` that outputs all indecomposable summands of a given interval decomposable module. For computational efficiency, we will implement `TRUEINTERVAL` differently. Let $M := M_{\mathcal{F}}$. First we compute $\dim M_p$ for every point $p \in P$. Iteratively, we choose a point p with $\dim M_p \neq 0$ and compute a maximal interval $I \in \mathcal{I}(M)$ containing p . Since M is interval decomposable, by Propositions 31 and 32 we have that $I \in \text{barc}(M)$ and that there is a summand $N \cong \mathbb{I}_I^{\mu_I}$ of M . Consider the quotient module $M' := M/N$. Clearly, this “peeling off” of N reduces the total dimension of the input module. Namely, $\dim M'_p = \begin{cases} \dim M_p - \mu_I, & p \in I \\ \dim M_p, & p \notin I. \end{cases}$ We continue the process by replacing M with M' until there is no point $p \in P$ with $\dim M_p \neq 0$ (note that M' is interval decomposable by Proposition 4). Since $\dim M := \sum_{p \in P} \dim M_p$ is finite, this process terminates in finitely many steps. By Propositions 4 and 32, the outcome of `TRUEINTERVAL` is a list of all intervals in $\text{barc}(M)$ with accurate multiplicities:

► **Proposition 33.** *Assume that $M_{\mathcal{F}}$ is interval decomposable. Let $I_i, i = 1, \dots, k$ be the intervals computed by `TRUEINTERVAL`. For each $i = 1, \dots, k$, let $\mu_{I_i} := \text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I_i)$. Then, we have $M_{\mathcal{F}} \cong \bigoplus_{i=1}^k \mathbb{I}_{I_i}^{\mu_{I_i}}$.*

Next, we describe an algorithm `INTERVAL` that simulates `TRUEINTERVAL` while avoiding explicit quotienting of $M_{\mathcal{F}}$ by its summands.

We associate a number $d(p)$ and a list $\text{list}(p)$ of identifiers of intervals $I \subseteq P$ to each point $p \in P$. The number $d(p)$ equals the original dimension of $(M_{\mathcal{F}})_p$ minus the number of intervals peeled off so far (counted with their multiplicities) which contained p . It is initialized to $\dim(M_{\mathcal{F}})_p$. Each time we compute a maximal interval $I \in \mathcal{I}(M_{\mathcal{F}})$ with multiplicity μ_I that contains p , we update $d(p) := d(p) - \mu_I$ keeping track of how many more intervals containing p would `TRUEINTERVAL` still be peeling off.

With each interval I that is output, we associate an identifier $\text{id}(I)$. The variable $\text{list}(p)$ maintains the set of identifiers of the intervals containing p that have been output so far. While searching for a maximal interval I , we maintain a variable list for I that contains the set of identifiers common to all points in I . Initializing list with $\text{list}(p)$ of the initial point p , we update it as we explore expanding I . Every time we augment I with a new point q , we update list by taking its intersection with the set of identifiers $\text{list}(q)$ associated with q .

We assume a routine `COUNT` that takes a list as input and gives the total number of intervals counted with their multiplicities whose identifiers are in the list. This means that if $\text{list} = \{\text{id}(I_1), \dots, \text{id}(I_k)\}$, then `COUNT(list)` returns the number $c := \sum \mu_{I_1} + \dots + \mu_{I_k}$.

Notice that, while searching for a maximal interval starting from a point, we keep considering the original given module $M_{\mathcal{F}}$ since we do not implement the true “peeling” (i.e. quotient $M_{\mathcal{F}}$ by a submodule). However, we modify the condition for checking the maximality of an interval I . We check whether $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I) > c$, that is, whether the generalized rank of $M_{\mathcal{F}}$ over I is larger than the total number of intervals containing I that would have been peeled off so far by `TRUEINTERVAL`. This idea is implemented in the following algorithm.

■ **Algorithm 2** INTERVAL (\mathcal{F}, P).

-
- Step 1. Call $\text{DIM}(\mathcal{F}, P)$ and set $d(p) := \dim(M_{\mathcal{F}})_p$; $\text{list}(p) := \emptyset$ for every $p \in P$
 - Step 2. While there exists a $p \in P$ with $d(p) > 0$ do
 - Step 2.1 Let $I := \{p\}$; $\text{list} := \text{list}(p)$; unmark every $q \in P$
 - Step 2.2 If there exists unmarked $q \in \text{nb}_{\mathbb{I}}(I)$ then
 - i. $\text{templist} := \text{list} \cap \text{list}(q)$; $c := \text{COUNT}(\text{templist})$
 - ii. If $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I \cup \{q\}) > c$ then⁵ mark q ; set $I := I \cup \{q\}$; $\text{list} := \text{list} \cap \text{list}(q)$
 - iii. go to Step 2.2
 - Step 2.3 Output I with multiplicity $\mu_I := \text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I) - c$
 - Step 2.4 For every $q \in I$ set $d(q) := d(q) - \mu_I$ and $\text{list}(q) := \text{list}(q) \cup \{\text{id}(I)\}$
-

The output of INTERVAL can be succinctly described as:

Output: $\{(I_i, \mu_{I_i}) : i = 1, \dots, k\}$ where $I_i \in \mathbf{Int}(P)$ and μ_{I_i} is a positive integer for each i .

► **Remark 34.** For each $p \in P$, $\dim M_p$ coincides with $\sum_{I_i \ni p} \mu_{I_i}$.

We will show that if $M_{\mathcal{F}}$ is interval decomposable, then the output of INTERVAL coincides with the barcode of $M_{\mathcal{F}}$ (Propositions 33 and 37).

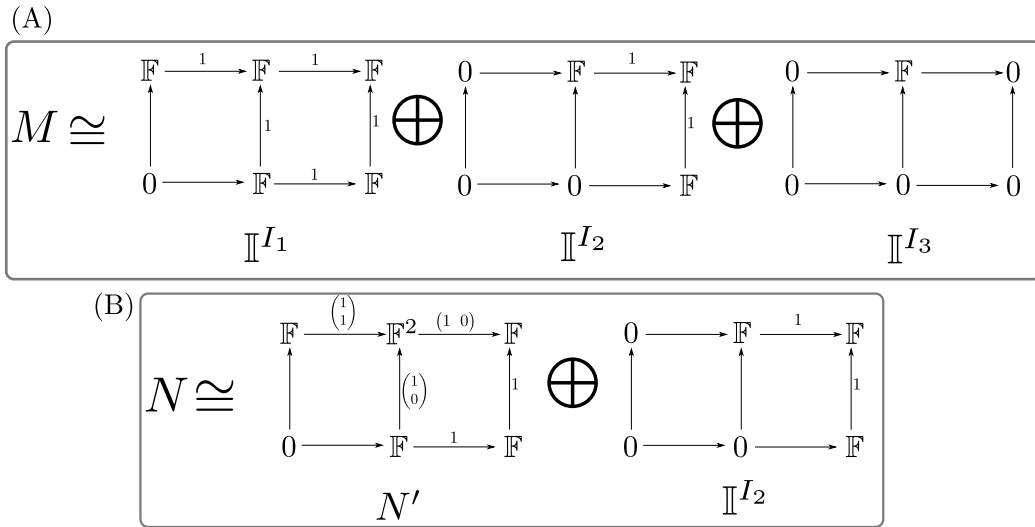
► **Example 35** (INTERVAL with interval decomposable input). Suppose that $M_{\mathcal{F}} \cong \mathbb{I}_{I_1} \oplus \mathbb{I}_{I_2} \oplus \mathbb{I}_{I_3}$ as depicted in Figure 3 (A). The algorithm INTERVAL yields $\{(I_1, 1), (I_2, 1), (I_3, 1)\}$. In particular, since $I_1 \supset I_2 \supset I_3$, INTERVAL outputs $(I_1, 1)$, $(I_2, 1)$, and $(I_3, 1)$ in order, as depicted in Figure 4 (A) (details in the full version).

► **Example 36** (INTERVAL with non-interval-decomposable input). Suppose that $N := M_{\mathcal{F}} \cong N' \oplus \mathbb{I}_{I_2}$ as depicted in Figure 3 (B). N' is an indecomposable module that is not an interval module. One possible final output of INTERVAL is $\{(J_1, 1), (J_2, 1), (J_3, 1)\}$ as depicted in Figure 4 (B). Note however that, depending on the choices of p in Step 2 and the neighbors q in Step 2.2, the final outcome can be different (details in the full version).

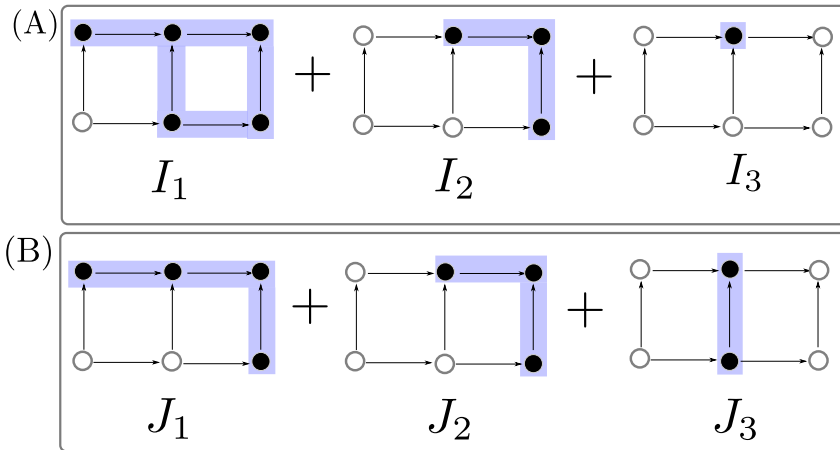
► **Proposition 37.** *If $M_{\mathcal{F}}$ is interval decomposable, INTERVAL(\mathcal{F}, P) computes an interval in $\text{barc}(M_{\mathcal{F}})$ if and only if TRUEINTERVAL(\mathcal{F}, P) computes it with the same multiplicity.*

Proof. (“if”): We induct on the list of intervals in the order they are computed by TRUEINTERVAL. We prove two claims by induction: (i) TRUEINTERVAL can be run to explore the points in P in the same order as INTERVAL while searching for maximal intervals, (ii) if $I_i, i = 1, \dots, k$, are the intervals computed by TRUEINTERVAL with this chosen order, then INTERVAL also outputs these intervals with the same multiplicities. Clearly, for $i = 1$, INTERVAL computes the maximal interval on the same input module $M_{\mathcal{F}}$ as TRUEINTERVAL does. So, clearly, TRUEINTERVAL can be made to explore P as INTERVAL does and hence their outputs are the same. Assume inductively that the hypotheses hold for $i \geq 1$. Then, TRUEINTERVAL operates next on the module $M_{i+1} := M_{\mathcal{F}} / (\mathbb{I}_{I_1}^{\mu_{I_1}} \oplus \dots \oplus \mathbb{I}_{I_i}^{\mu_{I_i}})$ (here each $\mathbb{I}_{I_i}^{\mu_{I_i}}$ stands for a summand of $M_{\mathcal{F}}$ that is isomorphic to $\mathbb{I}_{I_i}^{\mu_{I_i}}$ by Proposition 32). We let TRUEINTERVAL explore P in the same way as INTERVAL does. This is always possible because the outcome of the test for exploration remains the same in both cases as we argue.

⁵ to check $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I \cup \{q\}) > c$, we invoke Theorem 24 and run the zigzag persistence algorithm described beneath Remark 25. For efficiency, one can use zigzag update algorithm in [17].



■ **Figure 3** Modules $M, N : \{1, 2, 3\} \times \{1, 2\} \rightarrow \text{vec}$. M is interval decomposable, but N is not.



■ **Figure 4** An illustration for Examples 35 and 36.

The variable $d(p)$ at this point has the value $\dim(M_{i+1})_p$ and thus both `TRUEINTERVAL` and `INTERVAL` can start exploring from the point p if $d(p) > 0$. So, we let `TRUEINTERVAL` compute the next maximal interval I_{i+1} starting from the point p if `INTERVAL` starts from p .

Now, when `INTERVAL` tests for a point q to expand the interval I , we claim that the result would be the same if `TRUEINTERVAL` tested for q . First of all, the condition whether $I \cup \{q\}$ is an interval or not does not depend on which algorithm we are executing. Second, the list supplied to `COUNT` in Step 2.2 (i) exactly equals the list of intervals containing $I \cup \{q\}$ that `INTERVAL` has already output. By the inductive hypothesis, this list is exactly equal to the list of intervals that `TRUEINTERVAL` had already “peeled off”. Therefore, the test $\text{rk}_{\mathbb{I}}(M_{\mathcal{F}})(I \cup \{q\}) > c$ that `INTERVAL` performs in Step 2.2 (ii) is exactly the same as the test $\text{rk}_{\mathbb{I}}(M_{i+1})(I \cup \{q\}) > 0$ that `TRUEINTERVAL` would have performed for the module M_{i+1} . This establishes that `INTERVAL` computes the same interval I_{i+1} with the same multiplicity as `TRUEINTERVAL` would have computed on M_{i+1} using the same order of exploration as the inductive hypothesis claims.

(“only if”): See the full version. ◀

► **Proposition 38.** $\text{INTERVAL}(\mathcal{F}, P)$ runs in $O(t^{\omega+2})$ time (proof in the full version).

4.3 Interval decomposability

Setup 3 still applies in §4.3. We consider the following problem.

► **Problem III.** Determine whether the module $M_{\mathcal{F}}$ is interval decomposable or not.

If the input module $M_{\mathcal{F}}$ is interval decomposable, then the algorithm INTERVAL computes all intervals in the barcode. However, if the module $M_{\mathcal{F}}$ is not interval decomposable, then the algorithm is not guaranteed to output all interval summands. We show that INTERVAL still can be used to solve Problem III. For this we test whether each of the output intervals I with multiplicity μ_I indeed supports a summand $N \cong \mathbb{I}_I^{\mu_I}$ of $M_{\mathcal{F}}$.

To do this we run Algorithm 3 in Asashiba et al. [1] for each of the output intervals of INTERVAL . Call this algorithm TESTINTERVAL which with an input interval I , returns $\mu_I > 0$ if the module $\mathbb{I}_I^{\mu_I}$ is a summand of M and 0 otherwise.

For each of the intervals I with multiplicity μ_I returned by $\text{INTERVAL}(\mathcal{F}, P)$ we test whether $\text{TESTINTERVAL}(I)$ returns a non-zero μ_I . The first time the test fails, we declare that $M_{\mathcal{F}}$ is not interval decomposable. This gives us a polynomial time algorithm (with complexity $O(t^{3\omega+2})$) to test whether a module induced by a given bifiltration is interval decomposable or not. It is a substantial improvement over the result of Asashiba et al. [1] who gave an algorithm for tackling the same problem. Their algorithm cleverly enumerates the intervals in the poset to test, but still tests exponentially many of them and hence may run in time that is exponential in t . Because of our algorithm INTERVAL , we can do the same test but only on polynomially many intervals.

■ **Algorithm 3** $\text{ISINTERVALDECOMP}(\mathcal{F}, P)$

- Step 1. $\mathcal{I} = \{(I_i, \mu_{I_i})\} \leftarrow \text{INTERVAL}(\mathcal{F}, P)$
 - Step 2. For every $I_i \in \mathcal{I}$ do
 - Step 2.1 $\mu \leftarrow \text{TESTINTERVAL}(M_{\mathcal{F}}, I_i)$
 - Step 2.2 If $\mu \neq \mu_{I_i}$ then output false; quit
 - Step 3. output true
-

► **Proposition 39.** $\text{ISINTERVALDECOMP}(\mathcal{F}, P)$ returns true if and only if $M_{\mathcal{F}}$ is interval decomposable. It takes $O(t^{3\omega+2})$ time.

Proof. By the contrapositive of Proposition 33, if for any of the computed interval(s) $I_i, i = 1, \dots, k$ by INTERVAL , $\mathbb{I}_{I_i}^{\mu_{I_i}}$ is not a summand of $M_{\mathcal{F}}$, then $M_{\mathcal{F}}$ is not interval decomposable. On the other hand, if every such interval module is a summand of $M_{\mathcal{F}}$, then we have that $M_{\mathcal{F}} \cong \bigoplus_{i=1}^k \mathbb{I}_{I_i}^{\mu_{I_i}}$ because $\dim(M_{\mathcal{F}})_p = \sum_i^k \dim(\mathbb{I}_{I_i}^{\mu_{I_i}})_p$ for every $p \in P$.

Time complexity : By Proposition 38, Step 1 runs in time $O(t^{\omega+2})$. We claim that $\dim(M_{\mathcal{F}}) = O(t^2)$ (see Proof of Proposition 38 in the full version). Therefore, INTERVAL returns at most $O(t^2)$ intervals. According to the analysis in Asashiba et al. [1], each test in Step 2.1 takes $O(((\dim M_{\mathcal{F}})^{\omega} + t)t^{\omega}) = O(t^{3\omega})$ time and thus $O(t^{3\omega+2})$ in total over all $O(t^2)$ tests which dominates the time complexity of ISINTERVALDECOMP . ◀

5 Discussion

The algorithm INTERVAL produces all intervals of an input interval decomposable module. What happens if the input module is not interval decomposable? We can show that the algorithm still produces intervals each supporting a submodule of an indecomposable of the input module (Figure 4), see [18] for details. Some other open questions that follow are: (i) Can we generalize Theorem 24 to d -parameter persistent homology for $d > 2$? (ii) Can the complexity of the algorithms be improved? (iii) In particular, can we improve the interval testing algorithm of Asashiba et al.?

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