

Revisiting Parameter Synthesis for One-Counter Automata

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Abstract

We study the synthesis problem for one-counter automata with parameters. One-counter automata are obtained by extending classical finite-state automata with a counter whose value can range over non-negative integers and be tested for zero. The updates and tests applicable to the counter can further be made parametric by introducing a set of integer-valued variables called parameters. The synthesis problem for such automata asks whether there exists a valuation of the parameters such that all infinite runs of the automaton satisfy some ω -regular property. Lechner showed that (the complement of) the problem can be encoded in a restricted one-alternation fragment of Presburger arithmetic with divisibility. In this work (i) we argue that said fragment, called $\forall\exists_{\mathbb{R}}\text{PAD}^+$, is unfortunately undecidable. Nevertheless, by a careful re-encoding of the problem into a decidable restriction of $\forall\exists_{\mathbb{R}}\text{PAD}^+$, (ii) we prove that the synthesis problem is decidable in general and in 2NEXP for several fixed ω -regular properties. Finally, (iii) we give polynomial-space algorithms for the special cases of the problem where parameters can only be used in counter tests.

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1 Introduction

Our interest in one-counter automata (OCA) with parameters stems from their usefulness as models of the behaviour of programs whose control flow is determined by *counter variables*.

```
1 def funprint(x):
2   i = 0
3   i += x
4   while i >= 0:
5     if i == 0:
6       print("Hello")
7     if i == 1:
8       print("world")
9     if i >= 2:
10      assert(False)
11     i -= 1
12 # end program
```



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Indeed, the executions of such a program can be over-approximated by its control-flow graph (CFG) [1]. The CFG can be leveraged to get a *conservative response* to interesting questions about the program, such as: “is there a value of x such that the false assertion is avoided?” The CFG abstracts away all variables and their values (see Figure 1) and this introduces non-determinism. Hence, the question becomes: “is it the case that all paths from the initial vertex avoid the one labelled with 10?” In this particular example, the abstraction is too *coarse* and thus we obtain a false negative. In such cases, the abstraction of the program should be refined [9]. A natural refinement of the CFG in this context is obtained by tracking the value of i (cf. program graphs in [2]). The result is an OCA with parameters such that: For $x \in \{0, 1\}$ it has no run that reaches the state labelled with 10. This is an instance of a *safety (parameter) synthesis problem* for which the answer is positive.



■ **Figure 1** On the left, the CFG with vertex labels corresponding to source code line numbers; on the right, the CFG extended by tracking the value of i .

In this work, we focus on the parameter synthesis problems for given OCA with parameters and do not consider the problem of obtaining such an OCA from a program (cf. [10]).

Counter automata [29] are a classical model that extend finite-state automata with integer-valued counters. These have been shown to be useful in modelling complex systems, such as programs with lists and XML query evaluation algorithms [5, 8]. Despite their usefulness as a modelling formalism, it is known that two counters suffice for counter automata to become Turing powerful. In particular, this means that most interesting questions about them are undecidable [29]. To circumvent this, several restrictions of the model have been studied in the literature, e.g. reversal-bounded counter automata [18] and automata with a single counter. In this work we focus on an extension of the latter: OCA with parametric updates and parametric tests.

An existential version of the synthesis problems for OCA with parameters was considered by Göller et al. [12] and Bollig et al. [4]. They ask whether there exist a valuation of the parameters and a run of the automaton which satisfies a given ω -regular property. This is in contrast to the present problem where we quantify runs universally. (This is required for the conservative-approximation use case described in the example above.) We note that, of those two works, only [12] considers OCA with parameters allowed in both counter updates and counter tests while [4] studies OCA with parametric tests only. In this paper, unless explicitly stated otherwise, we focus on OCA with parametric tests and updates like in [12]. Further note that the model we study has an asymmetric set of tests that can be applied to the counter: lower-bound tests, and equality tests (both parametric and non-parametric). The primary reason for this is that adding upper-bound tests results in a model for which even the decidability of the (arguably simpler) existential reachability synthesis problem is a long-standing open problem [7]. Namely, the resulting model corresponds to Ibarra’s *simple programs* [17].

In both [12] and [4], the synthesis problems for OCA with parameters were stated as open. Later, Lechner [21] gave an encoding for the complement of the synthesis problems into a one-alternation fragment of Presburger arithmetic with divisibility (PAD). Her encoding relies



■ **Figure 2** Syntactical fragments of PAD ordered w.r.t. their language (of sentences).

■ **Table 1** Known and new complexity bounds for parameter synthesis problems.

	Lower bound	Upper bound
LTL	PSPACE -hard [32]	in 3NEXP (Cor. 17)
Reachability	coNP -hard (Prop. 23)	in 2NEXP (Thm. 10)
Safety, Büchi, coBüchi	NP^{NP} -hard [21, 22]	

on work by Haase et al. [16], which shows how to compute a linear-arithmetic representation of the reachability relation of OCA (see [24] for an implementation). In the same work, Haase et al. show that the same can be achieved for OCA with parameters using the divisibility predicate. In [21], Lechner goes on to consider the complexity of (validity of sentences in) the language corresponding to the one-alternation fragment her encoding targets. An earlier paper [6] by Bozga and Iosif argues that the fragment is decidable and Lechner carefully repeats their argument while leveraging bounds on the bitsize of solutions of existential PAD formulas [23] to argue the complexity of the fragment is **co2NEXP**. For ω -regular properties given as a linear temporal logic (LTL) formula, her encoding is exponential in the formula and thus it follows that the LTL synthesis problem is decidable and in **3NEXP**.

Problems in the literature. *Presburger arithmetic* is the first-order theory of $\langle \mathbb{Z}, 0, 1, +, \langle \rangle \rangle$. *Presburger arithmetic with divisibility (PAD)* is the extension of PA obtained when we add a binary *divisibility predicate*. The resulting language is undecidable [30]. In fact, a single quantifier alternation already allows to encode general multiplication, thus becoming undecidable [27]. However, the purely existential (Σ_0) and purely universal (Π_0) fragments have been shown to be decidable [3, 26].

The target of Lechner’s encoding is $\forall\exists_R\text{PAD}^+$, a subset of all sentences in the Π_1 -fragment of PAD. Such sentences look as follows: $\forall \mathbf{x} \exists \mathbf{y} \bigvee_{i \in I} \bigwedge_{j \in J_i} f_j(\mathbf{x}) \mid g_j(\mathbf{x}, \mathbf{y}) \wedge \varphi_i(\mathbf{x}, \mathbf{y})$ where φ is a quantifier-free PAD formula without divisibility. Note that all divisibility constraints appear in positive form (hence the $^+$) and that, within divisibility constraints, the existentially-quantified variables y_i appear only on the right-hand side (hence the \exists_R). In [6], the authors give a quantifier-elimination procedure for sentences in a further restricted fragment we call the Bozga-Iosif-Lechner fragment (BIL) that is based on “symbolically applying” the generalized Chinese remainder theorem (CRT) [20]. Their procedure does not eliminate all quantifiers but rather yields a sentence in the Π_0 -fragment of PAD. (Decidability of the BIL language would then follow from the result of Lipshitz [26].) Then, they *briefly argue* how the algorithm generalizes to $\forall\exists_R\text{PAD}^+$. There are two crucial problems in the argument from [6] that we have summarized here (and which were reproduced in Lechner’s work): First, the quantifier-elimination procedure of Bozga and Iosif does not directly work for BIL. Indeed, not all BIL sentences satisfy the conditions required for the CRT to be applicable as used in their algorithm. Second, there is no way to generalize their algorithm to $\forall\exists_R\text{PAD}^+$ since the language is undecidable. Interestingly, undecidability follows directly from other results in [6, 21]. In Lechner’s thesis [22], the result from [6] was stated as being under review. Correspondingly, the decidability of the synthesis problems for OCA with parameters was only stated conditionally on $\forall\exists_R\text{PAD}^+$ being decidable.

Our contribution. In Section 2, using developments from [6, 21], we argue that $\forall\exists_R\text{PAD}^+$ is undecidable (Theorem 2). Then, in the same section, we “fix” the definition of the BIL fragment by adding to it a necessary constraint so that the quantifier-elimination procedure from [6] works correctly. For completeness, and to clarify earlier mistakes in the literature, we recall Lechner’s analysis of the algorithm and conclude, just as she did, that the complexity of BIL is in **co2NEXP** [21] (Theorem 4). After some preliminaries regarding OCA with parameters in Section 3, we re-establish decidability of various synthesis problems in Section 4 (Theorem 10 and Corollary 17, see Table 1 for a summary). To do so, we follow Lechner’s original idea from [21] to encode them into $\forall\exists_R\text{PAD}^+$ sentences. However, to ensure we obtain a BIL sentence, several parts of her encoding have to be adapted. Finally, in Section 5 we make small modifications to the work of Bollig et al. [4] to give more efficient algorithms that are applicable when only tests have parameters (Theorem 18 and Corollary 19).

2 Presburger Arithmetic with divisibility

Presburger arithmetic (PA) is the first-order theory over $\langle \mathbb{Z}, 0, 1, +, < \rangle$ where $+$ and $<$ are the standard addition and ordering of integers. *Presburger arithmetic with divisibility (PAD)* is the extension of PA obtained when we add the binary divisibility predicate $|$, where for all $a, b \in \mathbb{Z}$ we have $a | b \iff \exists c \in \mathbb{Z} : b = ac$. Let X be a finite set of first-order variables. A *linear polynomial* over $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ is given by the syntax rule: $p(\mathbf{x}) ::= \sum_{1 \leq i \leq n} a_i x_i + b$, where the a_i, b and the first-order variables from \mathbf{x} range over \mathbb{Z} . In general, quantifier-free PAD formulas have the grammar: $\varphi ::= \varphi_1 \wedge \varphi_2 | \neg\varphi | f(\mathbf{x}) P g(\mathbf{x})$, where P can be the order predicate $<$ or the divisibility predicate $|$, and f, g are linear polynomials. We define the standard Boolean abbreviation $\varphi_1 \vee \varphi_2 \iff \neg(\neg\varphi_1 \wedge \neg\varphi_2)$. Moreover we introduce the abbreviations $f(x) \leq g(x) \iff f(x) < g(x) + 1$ and $f(x) = g(x) \iff f(x) \leq g(x) \wedge g(x) \leq f(x)$.

The *size* $|\varphi|$ of a PAD formula φ is defined by structural induction over $|\varphi|$: For a linear polynomial $p(\mathbf{x})$ we define $|p(\mathbf{x})|$ as the number of symbols required to write it if the coefficients are given in binary. Then, we define $|\varphi_1 \wedge \varphi_2| \stackrel{\text{def}}{=} |\varphi_1| + |\varphi_2| + 1$, $|\neg\varphi| \stackrel{\text{def}}{=} |\varphi| + 1$, $|f(\mathbf{x}) P g(\mathbf{x})| \stackrel{\text{def}}{=} |f(\mathbf{x})| + |g(\mathbf{x})| + 1$.

2.1 Allowing one restricted alternation

We define the language $\forall\exists_R\text{PAD}$ of all PAD sentences allowing a universal quantification over some variables, followed by an existential quantification over variables that may not appear on the left-hand side of divisibility constraints. Formally, $\forall\exists_R\text{PAD}$ is the set of all PAD sentences of the form: $\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \varphi(\mathbf{x}, \mathbf{y})$ where φ is a quantifier-free PAD formula and all its divisibility constraints are of the form $f(\mathbf{x}) | g(\mathbf{x}, \mathbf{y})$.

Positive-divisibility fragment. We denote by $\forall\exists_R\text{PAD}^+$ the subset of $\forall\exists_R\text{PAD}$ sentences φ where the negation operator can only be applied to the order predicate $<$ and the only other Boolean operators allowed are conjunction and disjunction. In other words, $\forall\exists_R\text{PAD}^+$ is a restricted negation normal form in which divisibility predicates cannot be negated. Lechner showed in [21] that all $\forall\exists_R\text{PAD}$ sentences can be translated into $\forall\exists_R\text{PAD}^+$ sentences.

► **Proposition 1** (Lechner’s trick [21]). *For all φ_1 in $\forall\exists_R\text{PAD}$ one can compute φ_2 in $\forall\exists_R\text{PAD}^+$ such that φ_1 is true if and only if φ_2 is true.*

2.2 Undecidability of both one-alternation fragments

We will now prove that the language $\forall\exists_R\text{PAD}^+$ is undecidable, that is, to determine whether a given sentence from $\forall\exists_R\text{PAD}^+$ is true is an undecidable problem.

► **Theorem 2.** *The language $\forall\exists_R\text{PAD}^+$ is undecidable.*

From Proposition 1 it follows that arguing $\forall\exists_R\text{PAD}$ is undecidable suffices to prove the theorem. The latter was proven in [6]. More precisely, they show the complementary language is undecidable. Their argument consists in defining the least-common-multiple predicate, the squaring predicate, and subsequently integer multiplication. Undecidability thus follows from the MRDP theorem [28] which states that satisfiability for such equations (i.e. Hilbert's 10th problem) is undecidable. Hence, Theorem 2 is a direct consequence of the following result.

► **Proposition 3** (From [6]). *The language $\forall\exists_R\text{PAD}$ is undecidable.*

2.3 The Bozga-Iosif-Lechner fragment

The Bozga-Iosif-Lechner (BIL) fragment is the set of all $\forall\exists_R\text{PAD}^+$ sentences of the form:

$$\forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m (\mathbf{x} < 0) \vee \bigvee_{i \in I} \bigwedge_{j \in J_i} (f_j(\mathbf{x}) \mid g_j(\mathbf{x}, \mathbf{y}) \wedge f_j(\mathbf{x}) > 0) \wedge \varphi_i(\mathbf{x}) \wedge \mathbf{y} \geq \mathbf{0}$$

where $I, J_i \subseteq \mathbb{N}$ are all finite index sets, the f_j and g_j are linear polynomials and the $\varphi_i(\mathbf{x})$ are quantifier-free PA formulas over the variables \mathbf{x} . Note that, compared to $\forall\exists_R\text{PAD}^+$, BIL sentences only constraint non-negative values of \mathbf{x} . (This technicality is necessary due to our second constraint below.) For readability, henceforth, we omit $(\mathbf{x} < 0)$ and just assume the \mathbf{x} take non-negative integer values, i.e. from \mathbb{N} . Additionally, it introduces the following three important constraints:

1. The \mathbf{y} variables may only appear on the right-hand side of divisibility constraints.
2. All divisibility constraints $f_j(\mathbf{x}) \mid g_j(\mathbf{x}, \mathbf{y})$ are conjoined with $f_j(\mathbf{x}) > 0$.
3. The \mathbf{y} variables are only allowed to take non-negative values.

It should be clear that the first constraint is necessary to avoid undecidability. Indeed, if the \mathbf{y} variables were allowed in the PA formulas $\varphi_i(\mathbf{x})$ then we could circumvent the restrictions of where they appear in divisibilities by using equality constraints. The second constraint is similar in spirit. Note that if $a = 0$ then $a \mid b$ holds if and only if $b = 0$ so if the left-hand side of divisibility constraints is allowed to be 0 then we can encode PA formulas on \mathbf{x} and \mathbf{y} as before. Also, the latter (which was missing in [6, 21]) will streamline the application of the generalized Chinese remainder theorem in the algorithm described in the sequel. While the third constraint is not required for decidability, it is convenient to include it for Section 4, where we encode instances of the synthesis problem into the BIL fragment.

In the rest of this section, we recall the decidability proof by Bozga and Iosif [6] and refine Lechner's analysis [21] to obtain the following complexity bound.

► **Theorem 4.** *The BIL-fragment language is decidable in co2NEXP .*

The idea of the proof is as follows: We start from a BIL sentence. First, we use the *generalized Chinese remainder theorem* (CRT, for short) to replace all of the existentially quantified variables in it with a single universally quantified variable. We thus obtain a sentence in $\forall\text{PAD}$ (i.e. the Π_0 -fragment of PAD) and argue that the desired result follows from the bounds on the bitsize of satisfying assignments for existential PAD formulas [23].

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► **Theorem 5** (Generalized Chinese remainder theorem [20]). *Let $m_i \in \mathbb{N}_{>0}$, $a_i, r_i \in \mathbb{Z}$ for $1 \leq i \leq n$. Then, there exists $x \in \mathbb{Z}$ such that $\bigwedge_{i=1}^n m_i \mid (a_i x - r_i)$ if and only if:*

$$\bigwedge_{1 \leq i, j \leq n} \gcd(a_i m_j, a_j m_i) \mid (a_i r_j - a_j r_i) \wedge \bigwedge_{i=1}^n \gcd(a_i, m_i) \mid r_i.$$

The solution for x is unique modulo $\text{lcm}(m'_1, \dots, m'_n)$, where $m'_i = m_i / \gcd(a_i, m_i)$.

From a BIL sentence, we apply the CRT to the rightmost existentially quantified variable and get a sentence with one less existentially quantified variable and with gcd-expressions. Observe that the second restriction we highlighted for the BIL fragment (the conjunction with $f_j(\mathbf{x}) > 0$) is necessary for the correct application of the CRT. We will later argue that we can remove the gcd expressions to obtain a sentence in $\forall\text{PAD}$.

► **Example 6.** Consider the sentence:

$$\forall x \exists y_1 \exists y_2 \bigvee_{i \in I} \bigwedge_{j \in J_i} (f_j(x) \mid g_j(x, \mathbf{y}) \wedge f_j(x) > 0) \wedge \varphi_i(x) \wedge \mathbf{y} \geq \mathbf{0}.$$

Let α_j denote the coefficient of y_2 in $g_j(x, \mathbf{y})$ and $r_j(x, y_1) \stackrel{\text{def}}{=} -(g_j(x, \mathbf{y}) - \alpha_j y_2)$. We can rewrite the above sentence as $\forall x \exists y_1 \bigvee_{i \in I} \psi_i(x, y_1) \wedge \varphi'_i(x) \wedge y_1 \geq 0$ where:

$$\begin{aligned} \psi_i(x, y_1) &= \exists y_2 \bigwedge_{j \in J_i} (f_j(x) \mid (\alpha_j y_2 - r_j(x, y_1))) \wedge y_2 \geq 0, \text{ and} \\ \varphi'_i(x) &= \varphi_i(x) \wedge \bigwedge_{j \in J_i} f_j(x) > 0. \end{aligned}$$

Applying the CRT, $\psi_i(x, y_1)$ can equivalently be written as follows:

$$\bigwedge_{j, k \in J_i} \gcd(\alpha_k f_j(x), \alpha_j f_k(x)) \mid (\alpha_j r_k(x, y_1) - \alpha_k r_j(x, y_1)) \wedge \bigwedge_{j \in J_i} \gcd(\alpha_j, f_j(x)) \mid r_j(x, y_1).$$

Note that we have dropped the $y_2 \geq 0$ constraint without loss of generality since the CRT states that the set of solutions forms an arithmetic progression containing infinitely many positive (and negative) integers. This means the constraint will be trivially satisfied for any valuation of x and y_1 which satisfies $\psi_i(x, y_1) \wedge \varphi_i(x) \wedge y_1 \geq 0$ for some $i \in I$. Observe that y_1 only appears in polynomials on the right-hand side of divisibilities.

The process sketched in the example can be applied in general to BIL sentences sequentially starting from the rightmost quantified y_i . At each step, the size of the formula is at most squared. In what follows, it will be convenient to deal with a single polyadic gcd instead of nested binary ones. Thus, using associativity of gcd and pushing coefficients inwards – i.e. using the equivalence $a \cdot \gcd(x, y) \equiv \gcd(ax, ay)$ for $a \in \mathbb{N}$ – we finally obtain a sentence:

$$\forall x_1 \dots \forall x_n \bigvee_{i \in I} \bigwedge_{j \in L_i} (\gcd(\{f'_{j,k}(\mathbf{x})\}_{k=1}^{K_j} \mid g'_j(\mathbf{x})) \wedge \varphi'_i(\mathbf{x})) \quad (1)$$

where $|L_i|$, $|K_j|$, and the coefficients may all be doubly-exponential in the number m of removed variables, due to iterated squaring.

Eliminating the gcd operator. In this next step, our goal is to obtain an \forall PAD sentence from Equation (1). Recall that \forall PAD “natively” allows for negated divisibility constraints. (That is, without having to encode them using Lechner’s trick.) Hence, to remove expressions in terms of gcd from Equation (1), we can use the following identity:

$$\text{gcd}(f_1(\mathbf{x}), \dots, f_n(\mathbf{x})) \mid g(\mathbf{x}) \iff \forall d \left(\bigwedge_{i=1}^n d \mid f_i(\mathbf{x}) \right) \rightarrow d \mid g(\mathbf{x}).$$

This substitution results in a constant blowup of the size of the sentence. The above method gives us a sentence $\forall \mathbf{x} \forall d \psi(\mathbf{x}, d)$, where $\psi(\mathbf{x}, d)$ is a quantifier-free PAD formula. To summarize:

► **Lemma 7.** *For any BIL sentence $\varphi = \forall x_1 \dots \forall x_n \exists y_1 \dots \exists y_m \bigvee_{i \in I} \varphi_i(\mathbf{x}, \mathbf{y})$ we can construct an \forall PAD sentence $\psi = \forall x_1 \dots \forall x_n \forall d \bigvee_{i \in I} \psi_i(\mathbf{x}, d)$ such that: φ is true if and only if ψ is true and for all $i \in I$, $|\psi_i| \leq |\varphi_i|^{2^m}$. The construction is realizable in time $\mathcal{O}(|\varphi|^{2^m})$.*

To prove Theorem 4, the following small-model results for purely existential PAD formulas and BIL will be useful.

► **Theorem 8** ([23, Theorem 14]). *Let $\varphi(x_1, \dots, x_n)$ be a \exists PAD formula. If φ has a solution then it has a solution $(a_1, \dots, a_n) \in \mathbb{Z}^n$ with the bitsize of each a_i bounded by $|\varphi|^{\text{poly}(n)}$.*

► **Corollary 9.** *Let $\forall x_1 \dots \forall x_n \varphi(x_1, \dots, x_n)$ be a BIL sentence. If $\neg \varphi$ has a solution then it has a solution $(a_1, \dots, a_n) \in \mathbb{Z}^n$ with the bitsize of each a_i bounded by $|\varphi|^{2^m \text{poly}(n+1)}$.*

Proof. Using Lemma 7, we translate the BIL sentence to $\forall x_1 \dots \forall x_n \forall d \psi(\mathbf{x}, d)$, where the latter is an \forall PAD sentence. Then, using Theorem 8, we get that the \exists PAD formula $\neg \psi(\mathbf{x}, d)$ admits a solution if and only if it has one with bitsize bounded by $|\psi|^{\text{poly}(n+1)}$. Now, from Lemma 7 we have that $|\psi|$ is bounded by $|\varphi|^{2^m}$. Hence, we get that the bitsize of a solution is bounded by: $|\varphi|^{2^m \text{poly}(n+1)}$. ◀

We are now ready to prove the theorem.

Proof of Theorem 4. As in the proof of Corollary 9, we translate the BIL sentence to $\forall x_1 \dots \forall x_n \forall d \psi(\mathbf{x}, d)$. Note that our algorithm thus far runs in time: $\mathcal{O}(|\varphi|^{2^m})$. By Corollary 9, if $\neg \psi(\mathbf{x}, d)$ has a solution then it has one encodable in binary using a doubly exponential amount of bits with respect to the size of the input BIL sentence. The naive guess-and-check decision procedure applied to $\neg \psi(\mathbf{x}, d)$ gives us a **co2NEXP** algorithm for BIL sentences. Indeed, after computing $\psi(\mathbf{x}, d)$ and guessing a valuation, checking it satisfies $\neg \psi$ takes polynomial time in the bitsize of the valuation and $|\psi|$, hence doubly exponential time in $|\varphi|$. ◀

3 Succinct One-Counter Automata with Parameters

We now define OCA with parameters and recall some basic properties. The concepts and observations we introduce here are largely taken from [16] and the exposition in [22].

A *succinct parametric one-counter automaton (SOCAP)* is a tuple $\mathcal{A} = (Q, T, \delta, X)$, where Q is a finite set of states, X is a finite set of parameters, $T \subseteq Q \times Q$ is a finite set of transitions and $\delta : T \rightarrow Op$ is a function that associates an operation to every transition. The set $Op = CU \uplus PU \uplus ZT \uplus PT$ is the union of: *Constant Updates* $CU \stackrel{\text{def}}{=} \{+a : a \in \mathbb{Z}\}$, *Parametric Updates* $PU \stackrel{\text{def}}{=} \{Sx : S \in \{+1, -1\}, x \in X\}$, *Zero Tests* $ZT \stackrel{\text{def}}{=} \{= 0\}$, and

Parametric Tests $PT \stackrel{\text{def}}{=} \{= x, \geq x : x \in X\}$. We denote by “= 0” or “= x ” an *equality test* between the value of the counter and zero or the value of x respectively; by “ $\geq x$ ”, a *lower-bound test* between the values of the counter and x . A *valuation* $V : X \rightarrow \mathbb{N}$ assigns to every parameter a natural number. We assume CU are encoded in binary, hence the S in SOCAP. We omit “parametric” if $X = \emptyset$ and often write $q \xrightarrow{op} q'$ to denote $\delta(q, q') = op$.

A *configuration* is a pair (q, c) where $q \in Q$ and $c \in \mathbb{N}$ is the *counter value*. Given a valuation $V : X \rightarrow \mathbb{N}$ and a configuration (q_0, c_0) , a *V-run from* (q_0, c_0) is a sequence $\rho = (q_0, c_0)(q_1, c_1) \dots$ such that for all $i \geq 0$ the following hold: $q_i \xrightarrow{op_{i+1}} q_{i+1}$; $c_i = 0$, $c_i = V(x)$, and $c_i \geq V(x)$, if $\delta(q_i, q_{i+1})$ is “= 0”, “= x ”, and “ $\geq x$ ”, respectively; and c_{i+1} is obtained from c_i based on the counter operations. That is, c_{i+1} is c_i if $\delta(q_i, q_{i+1}) \in (ZT \cup PT)$; $c_i + a$ if $\delta(q_i, q_{i+1}) = +a$; $c_i + S \cdot V(x)$ if $\delta(q_i, q_{i+1}) = Sx$. We say ρ *reaches* a state $q_f \in Q$ if there exists $j \in \mathbb{N}$, such that $q_j = q_f$. Also, ρ reaches or visits a set of states $F \subseteq Q$ iff ρ reaches a state $q_f \in F$. If V is clear from the context we just write run instead of V -run.

The *underlying (directed) graph* of \mathcal{A} is $G_{\mathcal{A}} = (Q, T)$. A V -run $\rho = (q_0, c_0)(q_1, c_1) \dots$ in \mathcal{A} induces a path $\pi = q_0 q_1 \dots$ in $G_{\mathcal{A}}$. We assign weights to $G_{\mathcal{A}}$ as follows: For $t \in T$, $\text{weight}(t)$ is 0 if $\delta(t) \in ZT \cup PT$; a if $\delta(t) = +a$; and $S \cdot V(x)$ if $\delta(t) = Sx$. We extend the weight function to finite paths in the natural way. Namely, we set $\text{weight}(q_0 \dots q_n) \stackrel{\text{def}}{=} \sum_{i=0}^{n-1} \text{weight}(q_i, q_{i+1})$.

Synthesis problems. *The synthesis problem* asks, given a SOCAP \mathcal{A} , a state q and an ω -regular property p , whether there exists a valuation V such that all infinite V -runs from $(q, 0)$ satisfy p . We focus on the following classes of ω -regular properties. Given a set of target states $F \subseteq Q$ and an infinite run $\rho = (q_0, c_0)(q_1, c_1) \dots$ we say ρ satisfies:

- the *reachability* condition if $q_i \in F$ for some $i \in \mathbb{N}$;
- the *Büchi* condition if $q_i \in F$ for infinitely many $i \in \mathbb{N}$;
- the *coBüchi* condition if $q_i \in F$ for finitely many $i \in \mathbb{N}$ only;
- the *safety* condition if $q_i \notin F$ for all $i \in \mathbb{N}$;
- the *linear temporal logic* (LTL) formula φ over a set of atomic propositions P – and with respect to a labelling function $f : Q \rightarrow 2^P$ – if $f(q_0)f(q_1) \dots \models \varphi$.¹

We will decompose the synthesis problems into reachability sub-problems. It will thus be useful to recall the following connection between reachability (witnesses) and graph flows.

Flows. For a directed graph $G = (V, E)$, we denote the set of immediate successors of $v \in V$ by $vE := \{w \in V \mid (v, w) \in E\}$ and the immediate predecessors of v by Ev , defined analogously. An *s-t flow* is a mapping $f : E \rightarrow \mathbb{N}$ that satisfies flow conservation: $\forall v \in V \setminus \{s, t\} : \sum_{u \in Ev} f(u, v) = \sum_{u \in vE} f(v, u)$. That is, the total incoming flow equals the total outgoing flow for all but the source and the target vertices. We then define the *value* of a flow f as: $|f| \stackrel{\text{def}}{=} \sum_{v \in sE} f(s, v) - \sum_{u \in Es} f(u, s)$. We denote by *support*(f) the set $\{e \in E \mid f(e) > 0\}$ of edges with non-zero flow. A cycle in a flow f is a cycle in the sub-graph induced by $\text{support}(f)$. For weighted graphs, we define $\text{weight}(f) \stackrel{\text{def}}{=} \sum_{e \in E} f(e) \text{weight}(e)$.

Path flows. Consider a path $\pi = v_0 v_1 \dots$ in G . We denote by f_π its Parikh image, i.e. f_π maps each edge e to the number of times e occurs in π . A flow f is called a *path flow* if there exists a path π such that $f = f_\pi$. Finally, we observe that an *s-t path flow* f in G induces a *t-s path flow* f' with $f'(u, v) = f(v, u)$, for all $(u, v) \in E$, in the skew transpose of G .

¹ See, e.g., [2] for the classical semantics of LTL.

4 Encoding Synthesis Problems into the BIL Fragment

In this section, we prove that all our synthesis problems are decidable. More precisely, we establish the following complexity upper bounds.

► **Theorem 10.** *The reachability, Büchi, coBüchi, and safety synthesis problems for succinct one-counter automata with parameters are all decidable in 2NEXP .*

The idea is as follows: we focus on the coBüchi synthesis problem and reduce its complement to the truth value of a BIL sentence. To do so, we follow Lechner’s encoding of the complement of the Büchi synthesis problem into $\forall\exists_R\text{PAD}^+$ [21]. The encoding heavily relies on an encoding for (existential) reachability from [16]. We take extra care to obtain a BIL sentence instead of an $\forall\exists_R\text{PAD}^+$ one as Lechner originally does.

It can be shown that the other synthesis problems reduce to the coBüchi one in polynomial time. The corresponding bounds thus follow from the one for coBüchi synthesis. The proof of the following lemma is given in the long version of the paper.

► **Lemma 11.** *The reachability, safety and the Büchi synthesis problems can be reduced to the coBüchi synthesis problem in polynomial time.*

Now the cornerstone of our reduction from the complement of the coBüchi synthesis problem to the truth value of a BIL sentence is an encoding of *reachability certificates* into $\forall\exists_R\text{PAD}$ formulas which are “almost” in BIL. In the following subsections we will focus on a SOCAP $\mathcal{A} = (Q, T, \delta, X)$ with $X = \{x_1, \dots, x_n\}$ and often write \mathbf{x} for (x_1, \dots, x_n) . We will prove that the existence of a V -run from (q, c) to (q', c') can be reduced to the satisfiability problem for such a formula.

► **Proposition 12.** *Given states q, q' , one can construct in deterministic exponential time in $|\mathcal{A}|$ a PAD formula: $\varphi_{\text{reach}}^{(q, q')}(\mathbf{x}, a, b) = \exists \mathbf{y} \bigvee_{i \in I} \varphi_i(\mathbf{x}, \mathbf{y}) \wedge \psi_i(\mathbf{y}, a, b) \wedge \mathbf{y} \geq \mathbf{0}$ such that $\forall \mathbf{x} \exists \mathbf{y} \bigvee_{i \in I} \varphi_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{y} \geq \mathbf{0}$ is a BIL sentence, the $\psi_i(\mathbf{y}, a, b)$ are quantifier-free PA formulas, and additionally:*

- a valuation V of $X \cup \{a, b\}$ satisfies $\varphi_{\text{reach}}^{(q, q')}$ iff there is a V -run from $(q, V(a))$ to $(q', V(b))$;
- the bitsize of constants in $\varphi_{\text{reach}}^{(q, q')}$ is of polynomial size in $|\mathcal{A}|$;
- $|\varphi_{\text{reach}}^{(q, q')}|$ is at most exponential with respect to $|\mathcal{A}|$; and
- the number of \mathbf{y} variables is polynomial with respect to $|\mathcal{A}|$.

Below, we make use of this proposition to prove Theorem 10. Then, we prove some auxiliary results in Section 4.1 and, in Section 4.2, we present a sketch of our proof of Proposition 12.

We will argue that $\forall \mathbf{x} \exists a \exists b \varphi_{\text{reach}}^{(q, q')}(\mathbf{x}, a, b)$ can be transformed into an equivalent BIL sentence. Note that for this to be the case it suffices to remove the $\psi_i(\mathbf{y}, a, b)$ subformulas. Intuitively, since these are quantifier-free PA formulas, their set of satisfying valuations is *semi-linear* (see, for instance, [15]). Our intention is to remove the $\psi_i(\mathbf{y}, a, b)$ and replace the occurrences of \mathbf{y}, a, b in the rest of $\varphi_{\text{reach}}^{(q, q')}(\mathbf{x}, a, b)$ with linear polynomials “generating” their set of solutions. This is formalized below.

Affine change of variables. Let $\mathbf{A} \in \mathbb{Z}^{m \times n}$ be an integer matrix of size $m \times n$ of rank r , and $\mathbf{b} \in \mathbb{Z}^m$. Let $\mathbf{C} \in \mathbb{Z}^{p \times n}$ be an integer matrix of size $p \times n$ such that $\begin{pmatrix} \mathbf{A} \\ \mathbf{C} \end{pmatrix}$ has rank s , and $\mathbf{d} \in \mathbb{Z}^p$. We write μ for the maximum absolute value of an $(s-1) \times (s-1)$ or $s \times s$ sub-determinant of the matrix $\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{C} & \mathbf{d} \end{pmatrix}$ that incorporates at least r rows from $\begin{pmatrix} \mathbf{A} & \mathbf{b} \end{pmatrix}$.

► **Theorem 13** (From [33]). *Given integer matrices $\mathbf{A} \in \mathbb{Z}^{m \times n}$ and $\mathbf{C} \in \mathbb{Z}^{p \times n}$, integer vectors $\mathbf{b} \in \mathbb{Z}^m$ and $\mathbf{d} \in \mathbb{Z}^p$, and μ defined as above, there exists a finite set I , a collection of $n \times (n-r)$ matrices $\mathbf{E}^{(i)}$, and $n \times 1$ vectors $\mathbf{u}^{(i)}$, indexed by $i \in I$, all with integer entries bounded by $(n+1)\mu$ such that: $\{\mathbf{x} \in \mathbb{Z}^n : \mathbf{A}\mathbf{x} = \mathbf{b} \wedge \mathbf{C}\mathbf{x} \geq \mathbf{d}\} = \bigcup_{i \in I} \{\mathbf{E}^{(i)}\mathbf{y} + \mathbf{u}^{(i)} : \mathbf{y} \in \mathbb{Z}^{n-r}, \mathbf{y} \geq \mathbf{0}\}$.*

We are now ready to prove Theorem 10.

Proof of Theorem 10. We will first prove that the complement of the coBüchi synthesis problem can be encoded into a BIL sentence. Recall that the complement of the coBüchi synthesis problem asks: given a SOCAP \mathcal{A} with parameters X , for all valuations does there exist an infinite run from a given configuration $(q, 0)$, that visits the target set F infinitely many times. Without loss of generality, we assume that the automaton has no parametric tests as they can be simulated using parametric updates and zero tests.

The idea is to check if there exists a reachable “pumpable cycle” containing one of the target states. Formally, given the starting configuration $(q, 0)$, we want to check if we can reach a configuration (q_f, k) , where $q_f \in F$ and $k \geq 0$ and then we want to reach q_f again via a pumpable cycle. This means that starting from (q_f, k) we reach the configuration (q_f, k) again or we reach a configuration (q_f, k') with $k' \geq k$ without using zero-test transitions. Note that reachability while avoiding zero tests is the same as reachability in the sub-automaton obtained after deleting all the zero-test transitions. We write $\varphi_{\text{reach-nt}}$ for the φ_{reach} formula constructed for that sub-automaton as per Proposition 12. The above constraints can be encoded as a formula $\varphi_{\text{Büchi}}(\mathbf{x}) = \exists k \exists k' \bigvee_{q_f \in F} \zeta(\mathbf{x}, k, k')$ where the subformula ζ is: $(k \leq k') \wedge \varphi_{\text{reach}}^{(q, q_f)}(\mathbf{x}, 0, k) \wedge \left(\varphi_{\text{reach-nt}}^{(q_f, q_f)}(\mathbf{x}, k, k') \vee \varphi_{\text{reach}}^{(q_f, q_f)}(\mathbf{x}, k, k) \right)$. Finally, the formula $\varphi_{\text{Büchi}}(\mathbf{x})$ will look as follows:

$$\exists \mathbf{y} \exists k \exists k' \bigvee_{i \in I} \bigwedge_{j \in J_i} (f_j(\mathbf{x}) \mid g_j(\mathbf{x}, \mathbf{y})) \wedge \varphi_i(\mathbf{x}) \wedge \psi_i(\mathbf{y}, k, k') \wedge \mathbf{y} \geq \mathbf{0}$$

where, by Proposition 12, the $\varphi_i(\mathbf{x})$ are quantifier-free PA formulas over \mathbf{x} constructed by grouping all the quantifier-free PA formulas over \mathbf{x} . Similarly, we can construct $\psi_i(\mathbf{y}, k, k')$ by grouping all the quantifier free formulas over \mathbf{y}, k and k' . Now, we use the affine change of variables to remove the formulas $\psi_i(\mathbf{y}, k, k')$. Technically, the free variables from the subformulas ψ_i will be replaced in all other subformulas by linear polynomials on newly introduced variables \mathbf{z} . Hence, the final formula $\varphi_{\text{Büchi}}(\mathbf{x})$ becomes:

$$\exists \mathbf{z} \bigvee_{i \in I'} \bigwedge_{j \in J_i} (f_j(\mathbf{x}) \mid g_j(\mathbf{x}, \mathbf{z})) \wedge \varphi_i(\mathbf{x}) \wedge \mathbf{z} \geq \mathbf{0}.$$

Note that, after using the affine change of variables, the number of \mathbf{z} variables is bounded by the number of old existentially quantified variables (\mathbf{y}, k, k') . However, we have introduced exponentially many new disjuncts.²

By construction, for a valuation V there is an infinite V -run in \mathcal{A} from $(q, 0)$ that visits the target states infinitely often iff $\varphi_{\text{Büchi}}(V(\mathbf{x}))$ is true. Hence, $\forall \mathbf{x} (\mathbf{x} < \mathbf{0} \vee \varphi_{\text{Büchi}}(\mathbf{x}))$ precisely encodes the complement of the coBüchi synthesis problem. Also, note that it is a BIL sentence since the subformulas (and in particular the divisibility constraints) come from our usage of Proposition 12. Now, the number of \mathbf{z} variables, say m , is bounded by the number of \mathbf{y} variables before the affine change of variables which is polynomial with respect to $|\mathcal{A}|$ from Proposition 12. Also, the bitsize of the constants in $\varphi_{\text{Büchi}}$ is polynomial in $|\mathcal{A}|$ though the size of the formula is exponential in $|\mathcal{A}|$. Now, using Lemma 7, we construct an \forall PAD sentence $\forall \mathbf{x} \forall d \psi(\mathbf{x}, d)$ from $\forall \mathbf{x} (\mathbf{x} < \mathbf{0} \vee \varphi_{\text{Büchi}}(\mathbf{x}))$. By Corollary 9, $\neg \psi$ admits a solution of bitsize bounded by: $\exp(\ln(|\varphi_{\text{Büchi}}|) 2^m \text{poly}(n+1)) = \exp(|\mathcal{A}| \cdot 2^{\text{poly}(|\mathcal{A}|)} \text{poly}(n+1))$, which is doubly exponential in the size of $|\mathcal{A}|$. As in the proof of Theorem 4, a guess-and-check algorithm for $\neg \psi$ gives us the desired **2NEXP** complexity result for the coBüchi synthesis problem. By Lemma 11, the other synthesis problems have the same complexity. \blacktriangleleft

² Indeed, because of the bounds on the entries of the matrices and vectors, the cardinality of the set I is exponentially bounded.

In the sequel we sketch our proof of Proposition 12.

4.1 Reachability certificates

We presently recall the notion of reachability certificates from [16]. Fix a SOCAP \mathcal{A} and a valuation V . A flow f in $G_{\mathcal{A}}$ is a *reachability certificate* for two configurations $(q, c), (q', c')$ in \mathcal{A} if there is a V -run from (q, c) to (q', c') that induces a path π such that $f = f_{\pi}$ and one of the following holds: (*type 1*) f has no positive-weight cycles, (*type 2*) f has no negative-weight cycles, or (*type 3*) f has a positive-weight cycle that can be taken from (q, c) and a negative-weight cycle that can be taken to (q', c') .

In the sequel, we will encode the conditions from the following result into a PAD formula so as to accommodate parameters. Intuitively, the proposition states that there is a run from (q, c) to (q', c') if and only if there is one of a special form: a decreasing prefix (type 1), a positive cycle leading to a plateau followed by a negative cycle (type 3), and an increasing suffix (type 2). Each one of the three sub-runs could in fact be an empty run.

► **Proposition 14** ([13, Lemma 4.1.14]). *If (q', c') is reachable from (q, c) in a SOCAP with $X = \emptyset$ and without zero tests then there is a run $\rho = \rho_1\rho_2\rho_3$ from (q, c) to (q', c') , where ρ_1, ρ_2, ρ_3 , each have a polynomial-size reachability certificate of type 1, 3 and 2, respectively.*

Encoding the certificates. Now, we recall the encoding for the reachability certificates proposed by Lechner [21, 22]. Then, we highlight the changes necessary to obtain the required type of formula. We begin with type-1 and type-3 certificates.

► **Lemma 15** (From [22, Lem. 33 and Prop. 36]). *Suppose \mathcal{A} has no zero tests and let $t \in \{1, 2, 3\}$. Given states q, q' , one can construct in deterministic exponential time the existential PAD formula $\Phi_t^{(q, q')}(\mathbf{x}, a, b)$. Moreover, a valuation V of $X \cup \{a, b\}$ satisfies $\Phi_t^{(q, q')}(\mathbf{x}, a, b)$ iff there is a V -run from $(q, V(a))$ to $(q', V(b))$ that induces a path π with f_{π} a type- t reachability certificate.*

The formulas $\Phi_t^{(q, q')}$ from the result above look as follows:

$$\bigvee_{i \in I} \exists \mathbf{z} \bigwedge_{j \in J_i} m_j(\mathbf{x}) \mid z_j \wedge (m_j(\mathbf{x}) > 0 \leftrightarrow z_j > 0) \wedge \varphi_i(\mathbf{x}) \wedge \psi_i(\mathbf{z}, a, b) \wedge \mathbf{z} \geq \mathbf{0}$$

where $|I|$ and the size of each disjunct are exponential.³ Further, all the φ_i and ψ_i are quantifier-free PA formulas and the $m_j(\mathbf{x})$ are all either x , $-x$, or $n \in \mathbb{N}_{>0}$.

We observe that the constraint $(m_j(\mathbf{x}) > 0 \leftrightarrow z_j > 0)$ regarding when the variables can be 0, can be pushed into a further disjunction over which subset of X is set to 0. In one case the corresponding $m_j(\mathbf{x})$'s and z_j 's are replaced by 0, in the remaining case we add to φ_i and ψ_i the constraints $z_j > 0$ and $m_j(\mathbf{x}) > 0$ respectively. We thus obtain formulas $\Psi_t^{(q, q')}(\mathbf{x}, a, b)$ with the following properties.

► **Lemma 16.** *Suppose \mathcal{A} has no zero tests and let $t \in \{1, 2, 3\}$. Given states q, q' , one can construct in deterministic exponential time a PAD formula $\Psi_t^{(q, q')}(\mathbf{x}, a, b) = \exists \mathbf{y} \bigvee_{i \in I} \varphi_i(\mathbf{x}, \mathbf{y}) \wedge \psi_i(\mathbf{y}, a, b) \wedge \mathbf{y} \geq \mathbf{0}$ s.t. $\forall \mathbf{x} \exists \mathbf{y} \bigvee_{i \in I} \varphi_i(\mathbf{x}, \mathbf{y}) \wedge \mathbf{y} \geq \mathbf{0}$ is a BIL sentence, the $\psi_i(\mathbf{y}, a, b)$ are quantifier-free PA formulas, and additionally:*

³ Lechner [21] actually employs a symbolic encoding of the Bellman-Ford algorithm to get polynomial disjuncts in her formula. However, a naïve encoding – while exponential – yields the formula we present here and streamlines its eventual transformation to BIL.

- a valuation V of $X \cup \{a, b\}$ satisfies $\Psi_t^{(q,q')}$ iff there is a V -run from $(q, V(a))$ to $(q', V(b))$ that induces a path π such that f_π is a type- t reachability certificate,
- the bitsize of constants in $\Psi_t^{(q,q')}$ is of polynomial size in $|\mathcal{A}|$,
- $|\Psi_t^{(q,q')}|$ is at most exponential with respect to $|\mathcal{A}|$, and
- the number of \mathbf{y} variables is polynomial with respect to $|\mathcal{A}|$.

4.2 Putting everything together

In this section, we combine the results from the previous subsection to construct φ_{reach} for Proposition 12. The construction, in full detail, and a formal proof that φ_{reach} enjoys the claimed properties are given in the long version of this paper. First, using Proposition 14 and the lemmas above, we define a formula $\varphi_{\text{reach-nt}}^{(q,q')}(\mathbf{x}, a, b)$ that is satisfied by a valuation V of $X \cup \{a, b\}$ iff there is a V -run from $(q, V(a))$ to $(q', V(b))$ without any zero-test transitions. To do so, we use formulas for the sub-automaton obtained by removing from \mathcal{A} all zero-test transitions. Then, the formula $\varphi_{\text{reach}}^{(q,q')}(\mathbf{x}, a, b)$ expressing general reachability can be defined by taking a disjunction over all orderings on the zero tests. In other words, for each enumeration of zero-test transitions we take the conjunction of the intermediate $\varphi_{\text{reach-nt}}$ formulas as well as $\varphi_{\text{reach-nt}}$ formulas from the initial configuration and to the final one.

Recall that for any LTL formula φ we can construct a *universal coBüchi automaton* of exponential size in $|\varphi|$ [2, 19]. (A universal coBüchi automaton accepts a word w if all of its infinite runs on w visit F only finitely often. Technically, one can construct such an automaton for φ by constructing a Büchi automaton for $\neg\varphi$ and “syntactically complementing” its acceptance condition.) By considering the product of this universal coBüchi automaton and the given SOCAP, the LTL synthesis problem reduces to coBüchi synthesis.

► **Corollary 17.** *The LTL synthesis problem for succinct one-counter automata with parameters is decidable in $\mathbf{3NEXP}$.*

5 One-Counter Automata with Parametric Tests

In this section, we introduce a subclass of SOCAP where only the tests are parametric. The updates are non-parametric and assumed to be given in unary. Formally, *OCA with parametric tests (OCAPT)* allow for constant updates of the form $\{+a : a \in \{-1, 0, 1\}\}$ and zero and parametric tests. However, $PU = \emptyset$.

We consider the synthesis problems for OCAPT. Our main result in this section are better complexity upper bounds than for general SOCAP. Lemma 11 states that all the synthesis problems reduce to the coBüchi synthesis problem for SOCAP. Importantly, in the construction used to prove Lemma 11, we do not introduce parametric updates. Hence, the reduction also holds for OCAPT. This allows us to focus on the coBüchi synthesis problem – the upper bounds for the other synthesis problems follow.

► **Theorem 18.** *The coBüchi, Büchi and safety synthesis problems for OCAPT are in \mathbf{PSPACE} ; the reachability synthesis problem, in $\mathbf{NP}^{\text{coNP}} = \mathbf{NP}^{\text{NP}}$.*

To prove the theorem, we follow an idea from [4] to encode parameter valuations of OCAPT into words accepted by an alternating two-way automaton. Below, we give the proof of the theorem assuming some auxiliary results that will be established in the following subsections.

Proof. In Proposition 21, we reduce the coBüchi synthesis problem to the non-emptiness problem for alternating two-way automata. Hence, we get the \mathbf{PSPACE} upper bound. Since the Büchi and the safety synthesis problems reduce to the coBüchi one (using Lemma 11) in polynomial time, these are also in \mathbf{PSPACE} .

Next, we improve the complexity upper bound for the reachability synthesis problem from **PSPACE** to **NP^{NP}**. In Lemma 22 we will prove that if there is a valuation V of the parameters such that all infinite V -runs reach F then we can assume that V assigns to each $x \in X$ a value at most exponential. Hence, we can guess their binary encoding and store it using a polynomial number of bits. Once we have guessed V and replaced all the x_i by $V(x_i)$, we obtain a non-parametric one counter automata \mathcal{A}' with $X = \emptyset$ and we ask whether all infinite runs reach F . We will see in Proposition 23 that this problem is in **coNP**. The claimed complexity upper bound for the reachability synthesis problem follows. ◀

Using a similar idea to Corollary 17, we reduce the LTL synthesis problem to the coBüchi one and we obtain the following.

► **Corollary 19.** *The LTL synthesis problem for OCAPT is in **EXSPACE**.*

5.1 Alternating two-way automata

Given a finite set Y , we denote by $\mathbb{B}^+(Y)$ the set of positive Boolean formulas over Y , including true and false. A subset $Y' \subseteq Y$ satisfies $\beta \in \mathbb{B}^+(Y)$, written $Y' \models \beta$, if β is evaluated to true when substituting true for every element in Y' , and false for every element in $Y \setminus Y'$. In particular, we have $\emptyset \models \text{true}$.

We can now define an *alternating two-way automaton* (A2A, for short) as a tuple $\mathcal{T} = (S, \Sigma, s_{in}, \Delta, S_f)$, where S is a finite set of states, Σ is a finite alphabet, $s_{in} \in S$ is the initial state, $S_f \subseteq S$ is the set of accepting states, and $\Delta \subseteq S \times (\Sigma \cup \{\text{first?}\}) \times \mathbb{B}^+(S \times \{+1, 0, -1\})$ is the finite transition relation. The $+1$ intuitively means that the head moves to the right; -1 , that the head moves to the left; 0 , that it stays at the current position. Furthermore, transitions are labelled by Boolean formulas over successors which determine whether the current run branches off in a non-deterministic or a universal fashion.

A *run (tree)* γ of \mathcal{T} on an infinite word $w = a_0a_1 \dots \in \Sigma^\omega$ from $n \in \mathbb{N}$ is a (possibly infinite) rooted tree whose vertices are labelled with elements in $S \times \mathbb{N}$ and such that it satisfies the following properties. The root of γ is labelled by (s_{in}, n) . Moreover, for every vertex labelled by (s, m) with $k \in \mathbb{N}$ children labelled by $(s_1, n_1), \dots, (s_k, n_k)$, there is a transition $(s, \sigma, \beta) \in \Delta$ such that, the set $\{(s_1, n_1 - m), \dots, (s_k, n_k - m)\} \subseteq S \times \{+1, 0, -1\}$ satisfies β . Further $\sigma = \text{first?}$ implies $m = 0$, and $\sigma \in \Sigma$ implies $a_m = \sigma$.

A run is *accepting* if all of its infinite branches contain infinitely many labels from $S_f \times \mathbb{N}$. The *language of \mathcal{T}* is $L(\mathcal{T}) \stackrel{\text{def}}{=} \{w \in \Sigma^\omega \mid \exists \text{ an accepting run of } \mathcal{T} \text{ on } w \text{ from } 0\}$. The *non-emptiness problem for A2As* asks, given an A2A \mathcal{T} and $n \in \mathbb{N}$, whether $L(\mathcal{T}) \neq \emptyset$.

► **Proposition 20** (From [31]). *Language emptiness for A2As is in **PSPACE**.*

In what follows, from a given OCAPT \mathcal{A} we will build an A2A \mathcal{T} such that \mathcal{T} accepts precisely those words which correspond to a valuation V of X under which all infinite runs satisfy the coBüchi condition. Hence, the corresponding synthesis problem for \mathcal{A} reduces to checking non-emptiness of \mathcal{T} .

5.2 Transformation to alternating two-way automata

Following [4], we encode a valuation $V : X \rightarrow \mathbb{N}$ as an infinite *parameter word* $w = a_0a_1a_2 \dots$ over the alphabet $\Sigma = X \cup \{\square\}$ such that $a_0 = \square$ and, for every $x \in X$, there is exactly one position $i \in \mathbb{N}$ such that $a_i = x$. We write $w(i)$ to denote its prefix $a_0a_1 \dots a_i$ up to the letter a_i . By $|w(i)|_\square$, we denote the number of occurrences of \square in $a_1 \dots a_i$. (Note that we ignore a_0 .) Then, a parameter word w determines a valuation $V_w : x \mapsto |w(i)|_\square$ where $a_i = x$. We denote the set of all parameter words over X by W_X .

From a given OCAPT $\mathcal{A} = (Q, T, \delta, X)$, a starting configuration $(q_0, 0)$ and a set of target states F , we will now construct an A2A $\mathcal{T} = (S, \Sigma, s_{in}, \Delta, S_f)$ that accepts words $w \in W_X$ such that, under the valuation $V = V_w$, all infinite runs from $(q_0, 0)$ visit F only finitely many times.

► **Proposition 21.** *For all OCAPT \mathcal{A} there is an A2A \mathcal{T} with $|\mathcal{T}| = |\mathcal{A}|^{\mathcal{O}(1)}$ and $w \in L(\mathcal{T})$ if and only if all infinite V_w -runs of \mathcal{A} starting from $(q_0, 0)$ visit F only finitely many times.*

The construction is based on the A2A built in [4], although we make more extensive use of the alternating semantics of the automaton. To capture the coBüchi condition, we simulate a *safety copy* with the target states as “non-accepting sink” (states having a self-loop and no other outgoing transitions) inside \mathcal{T} . Simulated accepting runs of \mathcal{A} can “choose” to enter said safety copy once they are sure to never visit F again. Hence, for every state q in \mathcal{A} , we have two copies of the state in \mathcal{T} : $q' \in S$ representing q normally and $q'' \in S$ representing q from the safety copy. Now the idea is to encode runs of \mathcal{A} as branches of run trees of \mathcal{T} on parameter words w by letting sub-trees t whose root is labelled with (q', i) or (q'', i) correspond to the configuration $(q, |w(i)|_{\square})$ of \mathcal{A} . If t is accepting, it will serve as a witness that all infinite runs of \mathcal{A} from $(q, |w(i)|_{\square})$ satisfy the coBüchi condition.

We present the overview of the construction below with some intuitions. A detailed proof of Proposition 21 is given in the long version of the paper.

- The constructed A2A \mathcal{T} for the given \mathcal{A} is such that for every $q \in Q$, there are two copies $q', q'' \in S$ as mentioned earlier. We also introduce new states in \mathcal{T} as required.
- The A2A includes a sub-A2A that verifies that the input word is a valid parameter word. For every x_i , a branch checks that it appears precisely once in the parameter word.
- From a run sub-tree whose root is labelled with (q', i) or (q'', i) , the A2A verifies that all runs of \mathcal{A} from $(q, |w(i)|_{\square})$ visit F only finitely many times. To do this, for all transitions δ of the form $q \xrightarrow{op} r$ in \mathcal{A} , we create a sub-A2A $\mathcal{T}_{\text{sub}}^{\delta}$ using copies of sub-A2As. For each such transition, one of two cases should hold: either the transition cannot be simulated (because of a zero test or a decrement from zero), or the transition can indeed be simulated. For the former, we add a *violation branch* to check that it is indeed the case; for the latter, a *validation branch* checks the transition can be simulated and a *simulation branch* reaches the next vertex with the updated counter value. Now if the root vertex is of the form (q', i) then the *simulation branch* could reach a vertex labelled with r' or with r'' – with the idea being that \mathcal{T} can choose to move to the safety copy or to stay in the “normal” copy of \mathcal{A} . If the root vertex is of the form (q'', i) , the simulation branch can only reach the vertex labelled with r'' with the updated counter value.
- We obtain the global A2A \mathcal{T} by connecting sub-A2As. To ensure that all runs of \mathcal{A} are simulated, we have the global transition relation Δ be a conjunction of that of the sub-A2As which start at the same state $q \in \{p', p''\}$ for some $p \in Q$. For instance, let $\delta_1 = (q, op_1, q_1)$ and $\delta_2 = (q, op_2, q_2)$ be transitions of \mathcal{A} . The constructed sub-A2As $\mathcal{T}_{\text{sub}}^{\delta_1}, \mathcal{T}_{\text{sub}}^{\delta_2}$ will contain transitions $(q, \square, \beta_1) \in \Delta_1, (q, \square, \beta_2) \in \Delta_2$ respectively. In \mathcal{T} , we instead have $(q, \square, \beta_1 \wedge \beta_2) \in \Delta$.
- Finally, the accepting states are chosen as follows: For every $q \in Q \setminus F$, we set q'' as accepting in \mathcal{T} . The idea is that if a run in \mathcal{A} satisfies the coBüchi condition then, after some point, it stops visiting target states. In \mathcal{T} , the simulated run can choose to move to the safety copy at that point and loop inside it forever thus becoming an accepting branch. On the other hand, if a run does not satisfy the condition, its simulated version cannot stay within the safety copy. (Rather, it will reach the non-accepting sink states.) Also, the violation and the validation branches ensure that the operations along the runs

have been simulated properly inside \mathcal{T} . It follows that \mathcal{T} accepts precisely those words whose run-tree contains a simulation branch where states from F have been visited only finitely many times.

5.3 An upper bound for reachability synthesis of OCAPT

Following [4], we now sketch a guess-and-check procedure using the fact that Proposition 21 implies a sufficient bound on valuations satisfying the reachability synthesis problem. Recall that, the reachability synthesis problem asks whether all infinite runs reach a target state.

► **Lemma 22** (Adapted from [4, Lemma 3.5]). *If there is a valuation V such that all infinite V -runs of \mathcal{A} reach F , there is a valuation V' such that $V'(x) = \exp(|\mathcal{A}|^{\mathcal{O}(1)})$ for all $x \in X$ and all infinite V' -runs of \mathcal{A} reach F .*

It remains to give an algorithm to verify that in the resulting non-parametric OCA (after substituting parameters with their values), all infinite runs from $(q_0, 0)$ reach F .

► **Proposition 23.** *Checking whether all infinite runs from $(q_0, 0)$ reach a target state in a non-parametric one-counter automata is **coNP**-complete.*

Before proving the claim above, we first recall a useful lemma from [21].

A path $\pi = q_0q_1 \dots q_n$ in $G_{\mathcal{A}}$ is a cycle if $q_0 = q_n$. We say the cycle is *simple* if no state (besides q_0) is repeated. A cycle *starts from a zero test* if $\delta(q_0, q_1)$ is “= 0”. A *zero-test-free cycle* is a cycle where no $\delta(q_i, q_{i+1})$ is a zero test. We define a *pumpable cycle* as being a zero-test-free cycle such that for all runs $\rho = (q_0, c_0) \dots (q_n, c_n)$ lifted from π we have $c_n \geq c_0$, i.e., the effect of the cycle is non-negative.

► **Lemma 24** (From [21]). *Let \mathcal{A} be a SOCA with an infinite run that does not reach F . Then, there is an infinite run of \mathcal{A} which does not reach F such that it induces a path $\pi_0 \cdot \pi_1^\omega$, where π_1 either starts from a zero test or it is a simple pumpable cycle.*

Sketch of proof of Proposition 23. We want to check whether all infinite runs starting from $(q_0, 0)$ reach F . Lemma 24 shows two conditions, one of which must hold if there is an infinite run that does not reach F . Note that both conditions are in fact reachability properties: a path to a cycle that starts from a zero test or to a simple pumpable cycle.

For the first condition, making the reachability-query instances concrete requires configuration a $(q, 0)$ and a state q' such that $\delta(q, q')$ is a zero test. Both can be guessed and stored in polynomial time and space. For the other condition, we can assume that π_0 does not have any simple pumpable cycle. It follows that every cycle in π_0 has a zero test or has a negative effect. Let W_{\max} be the sum of all the positive updates in \mathcal{A} . Note that the counter value cannot exceed W_{\max} along any run lifted from π_0 starting from $(q_0, 0)$. Further, since π_1 is a simple cycle the same holds for $2W_{\max}$ for runs lifted from $\pi_0\pi_1$. Hence, we can guess and store in polynomial time and space the two configurations (q, c) and (q, c') required to make the reachability-query instances concrete.

Since the reachability problem for non-parametric SOCAP is in **NP** [16], we can guess which condition will hold and guess the polynomial-time verifiable certificates. This implies the problem is in **coNP**.

For the lower bound, one can easily give a reduction from the complement of the SUBSETSUM problem, which is **NP**-complete [11]. The idea is similar to reductions used in the literature to prove **NP**-hardness for reachability in SOCAP. In the long version of the paper, the reduction is given in full detail. ◀

6 Conclusion

We have clarified the decidability status of synthesis problems for OCA with parameters and shown that, for several fixed ω -regular properties, they are in **2NEXP**. If the parameters only appear on tests, then we further showed that those synthesis problems are in **PSPACE**. Whether our new upper bounds are tight remains an open problem: neither our **coNP**-hardness result for the reachability synthesis problem nor the **PSPACE** and **NP^{NP}** hardness results known [32, 21, 22] for other synthesis problems (see Table 1) match them.

We believe the BIL fragment will find uses beyond the synthesis problems for OCA with parameters: e.g. it might imply decidability of the software-verification problems that motivated the study of $\forall\exists_R\text{PAD}^+$ in [6], or larger classes of quadratic string equations than the ones solvable by reduction to $\exists\text{PAD}$ [25]. While we have shown BIL is decidable in **2NEXP**, the best known lower bound is the trivial **coNP**-hardness that follows from encoding the complement of the SUBSETSUM problem. (Note that BIL does not syntactically include the Π_1 -fragment of PA so it does not inherit hardness from the results in [14].) Additionally, it would be interesting to reduce validity of BIL sentences to a synthesis problem. Following [16], one can easily establish a reduction to this effect for sentences of the form: $\forall\mathbf{x}\exists\mathbf{y}\bigvee_{i\in I} f_i(\mathbf{x}) \mid g(\mathbf{x}, \mathbf{y}) \wedge f_i(\mathbf{x}) > 0 \wedge \varphi_i(\mathbf{x}) \wedge \mathbf{y} \geq \mathbf{0}$ but full BIL still evades us.

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