Localisable Monads

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Abstract

Monads govern computational side-effects in programming semantics. A collection of monads can be combined together in a local-to-global way to handle several instances of such effects. Indexed monads and graded monads do this in a modular way. Here, instead, we start with a single monad and equip it with a fine-grained structure by using techniques from tensor topology. This provides an intrinsic theory of local computational effects without needing to know how constituent effects interact beforehand.

Specifically, any monoidal category decomposes as a sheaf of local categories over a base space. We identify a notion of localisable monads which characterises when a monad decomposes as a sheaf of monads. Equivalently, localisable monads are formal monads in an appropriate presheaf 2-category, whose algebras we characterise. Three extended examples demonstrate how localisable monads can interpret the base space as locations in a computer memory, as sites in a network of interacting agents acting concurrently, and as time in stochastic processes.

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1 Introduction

The computation of some desired value may influence parts of the environment in which the computation occurs that are separate from the value itself. Rather than being accidental byproducts, several modern programming platforms harness such *computational side-effects* to structure computations in a modular way [31, 30]. The most well-known use is via *monads* [28, 29], which let one analyse a computational effect apart from the rest of the computation.

A computation may use more than one effect. The corresponding monads can then be combined using *distributive laws* into a single monad [17, 3, 38]. This combination uses the fact that the base category on which the monad lives is highly structured; usually it is a cartesian category of presheaves. It may involve other formalisms such as Lawvere theories [33, 32], but we focus on monads here. An especially interesting case is when many

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instances of effects of the same kind are in play [34]. A related use of monads is to have several layers of granularity to an effect. Indexed monads and *graded monads* then model for example different levels of access to a computational effect [12, 26]. Here, as in the previous case, this is usually conceived of in a local-to-global fashion, where one specifies the behaviour at each level and then adds interplay between the levels.

In this article we take a dual approach and start with a single monad on a category with some structure. We then ask when and how that monad is the combination of constituent monads. This work is a first step towards an *intrinsic* theory of computational effects, one that doesn't need to specify in detail how constituent effects have to interact in advance. In particular, we do not postulate that the base category consists of presheaves, which is a consequence rather than an assumption.

To do so, we follow the programme of *tensor topology*, by observing that any monoidal category comes equipped with a notion of base space over which the category decomposes [10, 2, 9, 14]. This "spatial" aspect can be cleanly separated: any monoidal category embeds into a category of global sections of a sheaf of so-called local monoidal categories (see Theorems 10 and 11 below). This is recalled in Section 2.

Our main question is when and how a monad on a monoidal category respects this decomposition in the sense that it corresponds to a sheaf of monads on the local categories. The answer is a *localisable monad*, discussed in Section 3. To connect back to the local-to-global approach, we then characterise such monads as *formal monads* [35] in a (pre)sheaf category in Section 4. This opens a way to analyse the (Kleisli) algebras for localisable monads, which we do in Section 6. The breadth of this approach is demonstrated in Section 5, where we work out three extended examples. They show a range of how localisable monads may interpret the base space: as locations in a computer memory governed by a *local state* monad; as sites in a network of interacting agents governed by a monad inspired by the *pi calculus*; and as moments in time governed by a monad of *stochastic processes*. Section 7 concludes. Some proofs can be found in the extended version of this paper [5].

2 Tensor topology

This section summarises necessary notions from tensor topology. We have to be brief, and for more details we refer the reader to [10, 2, 14, 9]. To save space we will not use the graphical calculus for monoidal categories [16], but will not be careful in denoting coherence isomorphisms in this section. The following notions and results hold for arbitrary monoidal categories, but for simplicity we deal here with the symmetric monoidal case only.

▶ **Definition 1.** *A* central idempotent *in a symmetric monoidal category is a morphism* $u\colon U \to I$ *such that* $\rho_U \circ (U \otimes u) = \lambda_U \circ (u \otimes U)\colon U \otimes U \to U$ *and this map is invertible. We identify two central idempotents* $u\colon U \to I$ *and* $v\colon V \to I$ *when there is an isomorphism* $m\colon U \to V$ *satisfying* $u = v \circ m$*. Write* $\mathrm{ZI}(\mathbf{C})$ *for the collection of central idempotents of* $\mathbf{C}$*.*

A central idempotent $u\colon U \to I$ is completely determined by its domain U. The central idempotents always form a (meet-)semilattice. The order is defined by $u \leq v$ if and only if $u = v \circ m$ for some morphism $m\colon U \to V$. The meet is given $u \wedge v = \lambda_I \circ (u \otimes v)\colon U \otimes V \to I$. The largest central idempotent is the identity $1\colon I \to I$.

▶ **Example 2.** Consider a (meet-)semilattice $(L, \wedge, 1)$ as a symmetric monoidal category $\mathbf{C}$: objects of $\mathbf{C}$ are elements of L, there is a morphism $u \to v$ if and only if $u \leq v$, and $u \otimes v = u \wedge v$. Then $\mathrm{ZI}(\mathbf{C}) \simeq L$. In fact, ZI is a functor that is right adjoint to the inclusion of the category of semilattices into the category of symmetric monoidal categories.

▶ **Example 3.** If $\mathbf{C}$ is cartesian – that is, tensor products are in fact categorical products – then central idempotents are exactly subterminal objects: objects U whose unique morphism $!\colon U \to 1$ to the terminal object is monic. In particular, if X is any topological space, the category of sheaves over X has as central idempotent semilattice the collection of open sets $U \subseteq X$ under intersection.

▶ **Example 4.** If X is a locally compact Hausdorff topological space, the category of Hilbert modules over $C_0(X)$ is symmetric monoidal. It is equivalent to the category of fields of Hilbert spaces over X, and its central idempotents correspond to open subsets $U \subseteq X$.

Because of the previous examples, we can think of central idempotents as open subsets of a hidden base space that any symmetric monoidal category comes equipped with. Tensor topology develops general accompanying notions of locality, restriction, and support. For example, we can restrict attention to the "part of the category that lives over an open set", as follows.

▶ **Proposition 5.** *For every central idempotent u in a symmetric monoidal category $\mathbf{C}$, there is a symmetric monoidal category $\mathbf{C}\|_u$ where:*

- *objects are as in $\mathbf{C}$;*
- *morphisms $A \to B$ are morphisms $A \otimes U \to B$ in $\mathbf{C}$;*
- *composition of $f\colon A\otimes U \to B$ and $g\colon B\otimes U \to C$ is $g\circ(f\otimes U)\circ(A\otimes U\otimes u)^{-1}\colon A\otimes U \to C$;*
- *the identity on A is given by $A \otimes u$;*
- *tensor product of objects is as in $\mathbf{C}$;*
- *tensor product of morphisms $f\colon A \otimes U \to B$ and $f'\colon A' \otimes U \to B'$ is $(f \otimes f') \circ (A \otimes \sigma_{A',U} \otimes U) \circ (A \otimes A' \otimes U \otimes u)^{-1}\colon A \otimes A' \otimes U \to B \otimes B'$.* ◀

▶ Remark 6. In $\mathbf{C}\|_u$, any object A is isomorphic to $A \otimes U$: the isomorphism and its inverse are given by the identity $A \otimes U \to A \otimes U$ in $\mathbf{C}$ and $A \otimes u \otimes u\colon A \otimes U \otimes U \to A$.

▶ **Example 7.** In the category $\mathbf{C}$ of sheaves over a topological space X, central idempotents u correspond to open subsets $U \subseteq X$ as in Example 3. The category $\mathbf{C}\|_u$ is then equivalent to the category of sheaves over U.

The intuition of a category $\mathbf{C}$ "living over" open subsets is further strengthened by the following lemma, that says we can pass between the part of a category living over a larger open subset and the part living over a smaller open subset.

▶ **Lemma 8.** *If $u \leq v$ are central idempotents in $\mathbf{C}$, with $u = v \circ m$, there is an adjunction:*

$$\begin{tikzcd}
\mathbf{C}\|_u \arrow[r, bend left, "\mathbf{C}\|^{u\leq v}"] \arrow[r, phantom, "\perp"] & \mathbf{C}\|_v \arrow[l, bend left, "\mathbf{C}\|_{u\leq v}"]
\end{tikzcd}$$

The functor $\mathbf{C}\|_{u\leq v}$ is given by $A \mapsto A$ and $f \mapsto f \circ (A \otimes m)$ and is strict monoidal. The functor $\mathbf{C}\|^{u\leq v}$ is given by $A \mapsto A \otimes U$ and $f \mapsto (f \otimes U) \circ (A \otimes u \otimes U)^{-1} \circ (A \otimes U \otimes v)$ and is oplax monoidal. The unit of the adjunction is an isomorphism.

Proof. See [2, Lemmas 5.4 and 5.5]. ◀

To make the intuition built up so far completely rigorous, we now summarise a series of results saying that any symmetric monoidal category may be regarded as a sheaf of monoidal categories over a base topological space. To state them, we need to introduce mild conditions on the central idempotents being respected by tensor products.

▶ **Definition 9.** *A symmetric monoidal category* $\mathbf{C}$ *is called* stiff *when the diagram on the left below is a pullback for any object* A *and central idempotents* u *and* v*.*

$$\begin{tikzcd}
A \otimes U \otimes V \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & A \otimes V \arrow[d, "A \otimes v"] \\
A \otimes U \arrow[r, "A \otimes u"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes U \otimes V \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, very near start, "\lrcorner"] & A \otimes V \arrow[d, tail] \\
A \otimes U \arrow[r, tail] \arrow[ur, phantom, very near end, "\ulcorner"] & A \otimes (U \vee V)
\end{tikzcd}$$

We say $\mathbf{C}$ *has* finite universal joins *of central idempotents when it has an initial object* 0 *satisfying* $A \otimes 0 \simeq 0$ *for all objects* A*, and* $\mathrm{ZI}(\mathbf{C})$ *has binary joins such that the diagram on the right above is a pullback and a pushout for all objects* A *and central idempotents* u *and* v*.*

The following theorem says that any stiff monoidal category can be freely completed with universal finite joins of central idempotents [2, Theorem 12.8]. Finally, Theorem 11 [2, Theorem 8.6] says that any symmetric monoidal category $\mathbf{C}$ with universal finite joins has a particularly nice form. It considers the semilattice of central idempotents $\mathrm{ZI}(\mathbf{C})$ as the basic opens of a topological space X by taking its Zariski spectrum [2, Section 4].

▶ **Theorem 10.** *Any stiff symmetric monoidal category allows a strict monoidal full embedding into a symmetric monoidal category with finite universal joins of central idempotents.*

▶ **Theorem 11.** *Any symmetric monoidal category* $\mathbf{C}$ *with universal finite joins of central idempotents is monoidally equivalent to a category of global sections of a sheaf* $u \mapsto \mathbf{C}\|_u$ *of local monoidal categories over* $\mathrm{ZI}(\mathbf{C})$*.*

Here, a monoidal category $\mathbf{C}$ is called local when $u \vee v = 1$ implies $u = 1$ or $v = 1$ in $\mathrm{ZI}(\mathbf{C})$. When $\mathrm{ZI}(\mathbf{C})$ is the opens of a topological space, that means there is a single focal point that all nets in the topological space converge to – intuitively, $\mathbf{C}$ is local when it has no nontrivial central idempotents. Being a sheaf of local monoidal categories means that the stalks $\mathbf{C}\|_x = \mathrm{colim}_{x \in u} \mathbf{C}\|_u$ over points $x \in X$ are local monoidal categories.

It follows that any stiff symmetric monoidal category embeds into such a category of global sections. This makes precise the intuition that a symmetric monoidal category continuously varies over its base space of central idempotents.

3 Localisable monads

The previous section showed how any symmetric monoidal category $\mathbf{C}$ may be regarded as a sheaf $\mathbf{C}\|_u$ of local ones. In this section, we work out when a monad on $\mathbf{C}$ corresponds to a sheaf of monads on $\mathbf{C}\|_u$. The crucial definition is as follows.

▶ **Definition 12.** *A monad* T *on a monoidal category* $\mathbf{C}$ *is called* localisable *when there are morphisms* $\mathrm{st}_{A,U} \colon T(A) \otimes U \to T(A \otimes U)$ *for each object* A *and central idempotent* $u \colon U \to I$ *satisfying:*

$$T(\rho_A) \circ \mathrm{st}_{A,I} = \rho_{T(A)} \tag{1}$$

$$T(\alpha_{A,U,V}) \circ \mathrm{st}_{A,U\otimes V} = \mathrm{st}_{A\otimes U,V} \circ (\mathrm{st}_{A,U} \otimes V) \circ \alpha_{TA,U,V} \tag{2}$$

$$\eta_{A\otimes U} = \mathrm{st}_{A,U} \circ (\eta_A \otimes U) \tag{3}$$

$$\mu_{A\otimes U} \circ T(\mathrm{st}_{A,U}) \circ \mathrm{st}_{T(A),U} = \mathrm{st}_{A,U} \circ (\mu_A \otimes U) \tag{4}$$

$$\mathrm{st}_{A,V} \circ (T(A) \otimes m) = T(A \otimes m) \circ \mathrm{st}_{A,U} \tag{5}$$

$$\mathrm{st}_{B,U} \circ \big(T(f) \otimes U\big) = T(f \otimes U) \circ \mathrm{st}_{A,U} \tag{6}$$

for any morphism $f \colon A \to B$ *and central idempotents* $u \colon U \to I$ *and* $v \colon V \to I$*, and where* $m \colon U \to V$ *in* (5) *satisfies* $u = v \circ m$*.*

▶ **Example 13.** Consider a semilattice $(L, \wedge, 1)$ as a symmetric monoidal category $\mathbf{C}$ as in Example 2. A monad on $\mathbf{C}$ then is exactly a closure operator on L, that is, a function $\overline{(-)}\colon L \to L$ satisfying $u \leq \overline{u} = \overline{\overline{u}}$ and $u \leq v \implies \overline{u} \leq \overline{v}$. This monad is localisable if and only if $\overline{u} \wedge v \leq \overline{u \wedge v}$ for all $u, v \in L$. This is for example the case when L is the powerset of a set X, and $\overline{U}$ is the closure of $U \subseteq X$ in a fixed topology on X.

▶ **Example 14.** Strong monads [21, 18] are localisable: axioms (1)–(4) are a special case of the axioms for a strong monad; and axioms (5)–(6) follow from naturality of strength. Hence a monad T on a symmetric monoidal closed category is localisable if $T(U \multimap A) \simeq T(U) \multimap T(A)$, namely with $\mathrm{st}_{A,U}$ as follows (where coev denotes the curry of the identity on $A \otimes U$)

$$\begin{tikzcd}
T(A) \otimes U \arrow[r, "T(\mathrm{coev}) \otimes \eta"] & T\big(U \multimap (A \otimes U)\big) \otimes T(U) \arrow[d, equal] \\
T(A \otimes U) & \big(T(U) \multimap T(A \otimes U)\big) \otimes T(U) \arrow[l, "\mathrm{ev}"]
\end{tikzcd}$$

▶ **Example 15.** It follows from Example 14 and [20] that a monad T on a cartesian closed category is localisable as soon as $T(A \times B) \simeq T(A) \times T(B)$. In particular, this applies for any monad on the category of sheaves over a topological space X as in Example 3.

We will work out more examples in Section 5 below. Next we consider the main consequence of a monad on $\mathbf{C}$ being localisable: it restricts to the categories $\mathbf{C}\|_u$.

▶ **Proposition 16.** *If T is a localisable monad on $\mathbf{C}$ and u a central idempotent, the following defines a monad $T\|_u$ on $\mathbf{C}\|_u$:*

$$\begin{aligned} T\|_u(A) &= T(A) & \qquad (\eta\|_u)_A &= \eta_A \otimes u \\ T\|_u\big(f\colon A \otimes U \to B\big) &= T(f) \circ \mathrm{st}_{A,U} & (\mu\|_u)_A &= \mu_A \otimes u \end{aligned}$$

Proof. This is mainly a matter of unwinding definitions and being careful in which category compositions are taken. For example, the unit law $(\mu\|_u)_A \circ (\eta\|_u)_{T\|_u(A)} = T(A)$ in $\mathbf{C}\|_u$ comes down to the following diagram commuting in $\mathbf{C}$:

$$\begin{tikzcd}
T(A) \otimes U \otimes U \arrow[r, "\eta_{TA} \otimes u \otimes U"] & T^2(A) \otimes U \arrow[d, "\mu_A \otimes u"] \\
T(A) \otimes U \arrow[u, "T(A) \otimes (u \otimes U)^{-1}"] \arrow[ur, "\eta_{TA} \otimes U"] \arrow[r, "T(A) \otimes u"'] & T(A)
\end{tikzcd}$$

Similarly, naturality of $\eta\|_u$, which is $T\|_u(f) \circ (\eta\|_u)_A = (\eta\|_u)_B \circ f$ in $\mathbf{C}\|_u$, comes down to commutativity of the following diagram in $\mathbf{C}$:

$$\begin{tikzcd}
A \otimes U \otimes U \arrow[r, "\eta_A \otimes U \otimes u"] \arrow[d, equal] & T(A) \otimes U \otimes I \arrow[r, "\rho_{T(A) \otimes U}"] \arrow[d, "\mathrm{st}_{A,U} \otimes I"'] & T(A) \otimes U \arrow[d, "\mathrm{st}_{A,U}"] \\
A \otimes U \otimes U \arrow[r] \arrow[d, "f \otimes U"'] & T(A \otimes U) \otimes I \arrow[r, "\rho_{T(A \otimes U)}"'] \arrow[d, "T(f) \otimes I"'] & T(A \otimes U) \arrow[d, "T(f)"] \\
B \otimes U \arrow[r, "\eta_B \otimes u"'] & T(B) \otimes I \arrow[r, "\rho_{T(B)}"'] & T(B)
\end{tikzcd}$$

Here the upper left square follows from (3), the right squares are naturality of unitors, and the lower left square is naturality of η in $\mathbf{C}$. The other laws are verified similarly. ◀

▶ **Example 17.** Consider a closure operator $T(u) = \overline{u}$ on a semilattice $\mathbf{C} = L$ as in Example 13. Then $T_u(a)$ is simply $\overline{a}$. This is a well-defined closure operator on the pre-order $\mathbf{C}\|_u$: if $a \wedge u \leq b$, then $\overline{a} \wedge u \leq \overline{a \wedge u} \leq \overline{b}$ because T is localisable. Collapsing the pre-order $\mathbf{C}\|_u$ to a partially ordered semilattice as in Remark 6 simply gives the downset ${\downarrow}u = \{a \in L \mid a \leq u\}$ of u in L, and T_u just becomes the restriction of the closure operator to ${\downarrow}u$.

Recall that a *(lax) monad morphism* [35] from a monad (S, η^S, μ^S) on $\mathbf{C}$ to a monad (T, η^T, μ^T) on $\mathbf{D}$ consists of a functor $F\colon \mathbf{C} \to \mathbf{D}$ and a natural transformation $\varphi\colon T \circ F \Rightarrow F \circ S$ making the following two diagrams commute:

$$\begin{tikzcd}
F \arrow[rr, Rightarrow, "\eta^T_F"] \arrow[dr, Rightarrow, "F\eta^S"'] & & T \circ F \arrow[dl, Rightarrow, "\varphi"] \\
& F \circ S &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 \circ F \arrow[r, Rightarrow, "T\varphi"] \arrow[d, Rightarrow, "\mu^T_F"'] & T \circ F \circ S \arrow[r, Rightarrow, "\varphi S"] & F \circ S^2 \arrow[d, Rightarrow, "F\mu^S"] \\
T \circ F \arrow[rr, Rightarrow, "\varphi"'] & & F \circ S
\end{tikzcd} \tag{7}$$

Monads on $\mathbf{C}$ and their (lax) morphisms form a category $\mathbf{Monad}(\mathbf{C})$. An *oplax monad morphism* has $\psi\colon F \circ S \Rightarrow T \circ F$ that respects units and multiplication instead of φ.

▶ **Lemma 18.** *Let T be a localisable monad on $\mathbf{C}$. If $u \leq v$ are central idempotents, then the functor $\mathbf{C}\|_{u \leq v}$ from Lemma 8 is a (lax) monad morphism $T\|_v \to T\|_u$ with $\varphi_A = T(A) \otimes u$.*

Proof. Here we need to show the naturality of φ and the commutativity of the diagrams (7). These directly follow from (5), bifunctoriality of the tensor product and a few commuting diagrams that can be found in the extended version of this paper [5]. ◀

If $F\colon \mathbf{C} \to \mathbf{D}$ with $\varphi\colon T \circ F \Rightarrow F \circ S$ is a (lax) monad morphism between localisable monads S and T, and F is a (lax) monoidal functor with $\theta_{A,B}\colon F(A) \otimes F(B) \to F(A \otimes B)$, we say (F, φ, θ) is a (lax) *morphism of localisable monads* when the following diagram commutes:

$$\begin{tikzcd}
TF(A) \otimes F(U) \arrow[r, "\mathrm{st}_{FA,FU}"] \arrow[d, "\varphi_A \otimes F(U)"'] & T\big(F(A) \otimes F(U)\big) \arrow[r, "T(\theta_{A,U})"] & TF(A \otimes U) \arrow[d, "\varphi_{A,U}"] \\
FS(A) \otimes F(U) \arrow[r, "\theta_{S(A),U}"'] & F\big(S(A) \otimes U\big) \arrow[r, "\mathrm{st}_{A,U}"'] & FS(A \otimes U)
\end{tikzcd}$$

In this sense, the monad morphism $T\|_v \to T\|_u$ of Lemma 18 is localisable.

▶ **Corollary 19.** *If T is a localisable monad on $\mathbf{C}$, and $u \leq v$ are central idempotents, then the functor $\mathbf{C}\|^{u \leq v}$ from Lemma 8 is an oplax monad morphism $T\|_u \to T\|_v$ with $\psi_A = \mathrm{st}_{A,U}$.*

Proof. Applying [35, Theorem 9] to Lemmas 8 and 18, we can compute ψ as follows. By the adjunction, $\varphi_A\colon T\|_u(\mathbf{C}\|_{u \leq v}(\mathbf{C}\|^{u \leq v}(A))) \to \mathbf{C}\|_{u \leq v}(T\|_v(\mathbf{C}\|^{u \leq v}(A)))$ corresponds to a morphism

$$\mathbf{C}\|^{u \leq v}(T\|_u(\mathbf{C}\|_{u \leq v}(\mathbf{C}\|^{u \leq v}(A)))) \to T\|_v(\mathbf{C}\|^{u \leq v}(A))$$

and $\psi_A\colon \mathbf{C}\|^{u \leq v}(T\|_u(A)) \to T\|_v(\mathbf{C}\|^{u \leq v}(A))$ is obtained by precomposing this morphism with the unit $A \to \mathbf{C}\|_{u \leq v}(\mathbf{C}\|^{u \leq v}(A))$ of the adjunction. Starting with $\varphi_A = T(A) \otimes u$, this gives exactly $\psi_A = \mathrm{st}_{A,U}$. ◀

▶ Remark 20. If T is a localisable monad on a stiff symmetric monoidal category $\mathbf{C}$, and x is a point of $\mathrm{ZI}(\mathbf{C})$ regarded as a topological space, we can go further and define a monad $T\|_x$ on the stalk $\mathbf{C}\|_x$. The stalk $\mathbf{C}\|_x$ is defined as the colimit of the diagram $\mathbf{C}\|_{u \leq v}\colon \mathbf{C}\|_v \to \mathbf{C}\|_u$

ranging over all central idempotents $u \leq v$ containing the point x, taken in the category of symmetric monoidal categories. Accordingly, $T\|_x$ is the colimit over the same diagram, but now taken in the category of localisable monads. Using the concrete description in [2, Definition 7.1] of these stalks, we can compute:

$$\begin{aligned} T\|_x(A) &= T(A) & \qquad (\eta\|_x)_A &= [1, \eta_A \circ \rho_A] \\ T\|_x\big([u, f\colon A \otimes U \to B]\big) &= [u, T(f) \circ \mathrm{st}_{A,U}] & \qquad (\mu\|_x)_A &= [1, \mu_A \circ \rho_{T^2(A)}] \end{aligned}$$

If $x \in u$ there is a localisable monad morphism $T\|_u \to T\|_x$ formed by the functor $\mathbf{C}\|_{x\in u}\colon \mathbf{C}\|_u \to \mathbf{C}\|_x$ given by $\mathbf{C}\|_{x\in u}(A) = A$ and $\mathbf{C}\|_{x\in u}(f\colon A \to B) = [u, f]$ with the identity natural transformation $\varphi\colon T\|_u \circ \mathbf{C}\|_{x\in u} \Rightarrow \mathbf{C}\|_{x\in u} \circ T\|_x$.

The representation of Theorems 10 and 11 is in fact functorial [2, Section 11]: a (lax) monoidal functor $T\colon \mathbf{C} \to \mathbf{C}$ corresponds to a family of stalk functors $T\|_x\colon \mathbf{C}\|_x \to \mathbf{C}\|_x$ that are continuous in a certain sense. However, this notion of continuity is quite involved, and we will not pursue it further here.

4 Formal monads, graded monads, and indexed monads

This section characterises localisable monads as formal monads in a certain presheaf category, and connects to graded monads and indexed monads.

4.1 Formal monads

We will characterise localisable monads as formal monads in the 2-category $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$ with functors $\mathrm{ZI}(\mathbf{C})^{\mathrm{op}} \to \mathbf{Cat}$ as 0-cells, *natural* transformations as 1-cells, and modifications as 2-cells [35, 25]. More precisely, we will define a formal monad *on* the sheaf $\overline{\mathbf{C}}\colon \mathrm{ZI}(\mathbf{C})^{\mathrm{op}} \to \mathbf{Cat}$ that maps a central idempotent u to the category $\mathbf{C}\|_u$ and morphisms $u \leq v$ to the functors $\mathbf{C}\|_{u\leq v}\colon \mathbf{C}\|_v \to \mathbf{C}\|_u$ of Lemma 8. A formal monad then consists of a natural transformation $\overline{T}\colon \overline{\mathbf{C}} \Rightarrow \overline{\mathbf{C}}$ and two modifications $\mu\colon \overline{T}\overline{T} \Rrightarrow \overline{T}$ and $\eta\colon \mathrm{id}_{\overline{\mathbf{C}}} \Rrightarrow \overline{T}$ satisfying the usual monad laws. More precisely, the data of this formal monad consists of:

- monads $(T\|_u, \mu\|_u, \eta\|_u)$ on $\mathbf{C}\|_u$ for every central idempotent u in $\mathbf{C}$;
- functors $\mathbf{C}\|_{u\leq v}\colon \mathbf{C}\|_v \to \mathbf{C}\|_u$ for central idempotents $u \leq v$ in $\mathbf{C}$;

such that the following equations hold in $\mathbf{C}\|_u$:

$$\begin{tikzcd}
\mathbf{C}\|_{u\leq v}(A) \arrow[r, "\mathbf{C}\|_{u\leq v}((\eta\|_v)_A)"] \arrow[dr, "(\eta\|_v)_{\mathbf{C}\|_{u\leq v}(A)}"'] & \mathbf{C}\|_{u\leq v}(T\|_v(A)) \arrow[d, equal] & T_u^2(\mathbf{C}\|_{u\leq v}(A)) \arrow[r, "(\mu\|_u)_{\mathbf{C}\|_{u\leq v}(A)}"] \arrow[d, equal] & T_u(\mathbf{C}\|_{u\leq v}(A)) \arrow[d, equal] \\
& T\|_u(\mathbf{C}\|_{u\leq v}(A)) & \mathbf{C}\|_{u\leq v}(T_v^2(A)) \arrow[r, "\mathbf{C}\|_{u\leq v}((\mu\|_v)_A)"'] & \mathbf{C}\|_{u\leq v}(T_v(A))
\end{tikzcd} \tag{8}$$

Moreover $\overline{T}$ is natural, meaning that if $u = v \circ m$ then for any $f\colon A \to B$ in $\mathbf{C}\|_v$:

$$T\|_u(\mathbf{C}\|_{u\leq v}A) = \mathbf{C}\|_{u\leq v}T\|_v(A) \tag{9}$$

$$T\|_u\big(\mathbf{C}\|_{u\leq v}f\big) = \mathbf{C}\|_{u\leq v}T\|_v(f). \tag{10}$$

Given the definition of $\mathbf{C}\|_{u\leq v}$, the first equation simply reads $T\|_u(A) = T\|_v(A)$. The following two lemmas follow from the definition of the adjoint functors $\mathbf{C}\|^{u\leq v} \dashv \mathbf{C}\|_{u\leq v}$.

▶ **Lemma 21.** *There is a comonad $- \otimes U$ on $\mathbf{C}$ for any central idempotent u of $\mathbf{C}$. More generally, there is a comonad $- \otimes U$ on $\mathbf{C}\|_v$ for any central idempotents $u \leq v$ of $\mathbf{C}$.*

▶ **Lemma 22.** *The category $\mathbf{C}\|_u$ is the co-Kleisli category of the comonad $- \otimes U$ on $\mathbf{C}\|_v$.*

It follows from Lemma 22 that there is a canonical adjunction between the co-Kleisli category $\mathbf{C}\|_u$ and category $\mathbf{C}\|_v$ (or the base category $\mathbf{C}$ for $v = 1$) given by adjoint functors $\mathbf{C}\|^{u\leq v} \dashv \mathbf{C}\|_{u\leq v}$ such that $-\otimes U = \mathbf{C}\|^{u\leq v} \circ \mathbf{C}\|_{u\leq v}$. These correspond to the adjoint functors defined in Lemma 8. Further than Lemma 8, observe the following decomposition.

▶ **Lemma 23.** *If $u \leq v \leq w$ are central idempotents in $\mathbf{C}$, the functors of Lemma 8 satisfy:*

$$\mathbf{C}\|^{u\leq w} = \mathbf{C}\|^{v\leq w} \circ \mathbf{C}\|^{u\leq v}$$

$$\mathbf{C}\|_{u\leq w} = \mathbf{C}\|_{u\leq v} \circ \mathbf{C}\|_{v\leq w}$$

$$\mathbf{C}\|^{u\leq v} = \mathbf{C}\|_{v\leq w} \circ \mathbf{C}\|^{u\leq w}$$

$$\begin{tikzcd}
\mathbf{C}\|_v \arrow[dr, bend left, "\mathbf{C}\|^{v\leq w}"] \arrow[dd, bend left, "\mathbf{C}\|_{u\leq v}"] & \\
 & \mathbf{C}\|_w \arrow[ul, "\mathbf{C}\|_{v\leq w}"'] \arrow[dl, "\mathbf{C}\|_{u\leq w}"] \\
\mathbf{C}\|_u \arrow[uu, bend left, "\mathbf{C}\|^{u\leq v}"] \arrow[ur, bend right, "\mathbf{C}\|^{u\leq w}"'] &
\end{tikzcd}$$

Proof. This follows directly from the definition of the functors. ◀

▶ **Proposition 24.** *Let $\mathbf{C}$ be a stiff category. Let $(\overline{T}, \overline{\mu}, \overline{\eta})$ be a formal monad in $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$ above $\overline{\mathbf{C}}$ and let $u \leq v$ be central idempotents. Then the monad $T\|_v$ is a localisable monad with the strength $\mathrm{st}_{A,U} \colon T\|_v(A) \otimes U \to T\|_v(A \otimes U)$ defined as the following composition in $\mathbf{C}\|_v$ for any object A in $\mathbf{C}\|_v$:*

$$\begin{array}{c}
T\|_v(A) \otimes U = \mathbf{C}\|^{u\leq v}\mathbf{C}\|_{u\leq v}T\|_v A = \mathbf{C}\|^{u\leq v}T\|_u\mathbf{C}\|_{u\leq v}A \\
\Big\downarrow {\scriptstyle \mathbf{C}\|^{u\leq v}T\|_u\eta^{u\leq v}_{\mathbf{C}\|_{u\leq v}A}} \\
\mathbf{C}\|^{u\leq v}T\|_u\mathbf{C}\|_{u\leq v}\mathbf{C}\|^{u\leq v}\mathbf{C}\|_{u\leq v}A = \mathbf{C}\|^{u\leq v}\mathbf{C}\|_{u\leq v}T\|_v\mathbf{C}\|^{u\leq v}\mathbf{C}\|_{u\leq v}A \\
\Big\downarrow {\scriptstyle \varepsilon^{u\leq v}_{T\|_v\mathbf{C}\|^{u\leq v}\mathbf{C}\|_{u\leq v}A}} \\
T\|_v\mathbf{C}\|^{u\leq v}\mathbf{C}\|_{u\leq v}A = T\|_v(A \otimes U)
\end{array} \tag{11}$$

where $\eta^{u\leq v}$ and $\varepsilon^{u\leq v}$ are the unit and counit of adjunction $\mathbf{C}\|^{u\leq v} \dashv \mathbf{C}\|_{u\leq v}$.

Proof. We need to prove each of the axioms of Definition 12. This consist of many commutating diagrams. The complete proof can be found in the extended version of this paper [5]. For simplicity, the proof is laid out for the case $v = 1$, but the same arguments hold for any $T\|_v$ by using the relevant strength. ◀

▶ **Proposition 25.** *A localisable monad T on a stiff category $\mathbf{C}$ induces a formal monad on $\overline{\mathbf{C}}$ in $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$. The natural transformation $\overline{T} \colon \overline{\mathbf{C}} \Rightarrow \overline{\mathbf{C}}$ has components $T\|_u$, the modification $\overline{\eta} \colon \overline{\mathbf{C}} \Rrightarrow \overline{T}$ has components $\eta\|_u$, and the modification $\overline{\mu} \colon \overline{T}^2 \Rrightarrow \overline{T}$ has components $\mu\|_u$ as in Proposition 16.*

Proof. This proof consist in verifying the naturality of $\overline{T}$, in showing that $\overline{\eta}$ and $\overline{\mu}$ are modifications (which follows directly from Lemma 18) and natural, and in proving that $\overline{\eta}$ and $\overline{\mu}$ satisfy the monad laws (which pointwise follows from Proposition 16). The complete proof is included in the extended version of this paper [5]. ◀

▶ **Theorem 26.** *For a stiff monoidal category $\mathbf{C}$ there is a bijective correspondence between localisable monads on $\mathbf{C}$ and formal monads on $\overline{\mathbf{C}}$ in $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$ (via the constructions of Propositions 24 and 25).*

Proof. Start with a localisable monad T and follow Proposition 25 to get a formal monad $\overline{T}$. Then apply Proposition 24 to get a localisable monad T' which we claim equals the original monad T. It is clear that T' equals T as a functor. It remains to check that the strength obtained this way on T' is the same as the original strength on T. To do this, note that the strength (11) from Proposition 24 can be rewritten as follows, where st denotes the original strength from the localisable monad:

$$\varepsilon_{T\|_1 FGA} \circ FT\|_u \eta^u_{GA} = (T(A \otimes U) \otimes u) \circ (\mathrm{st}_{A,U} \otimes U) \otimes (T(A) \otimes (U \otimes u)^{-1}) = \mathrm{st}_{A,U}$$

Here we use the naturality of the strength and the fact that $U \otimes u$ is an isomorphism. We prove similarly that using Proposition 25 then Proposition 24 gives us back the unit and the multiplication of the starting localisable monad. To simplify the notation we used F and G to denote $\mathbf{C}\|^{u\leq 1}$ and $\mathbf{C}\|_{u\leq 1}$.

Now start with a formal monad $\overline{T}$, turn it into a localisable monad $(\overline{T}\|_1, \mathrm{st})$, and then into a formal monad $\widetilde{T}$. Then $\widetilde{T}\|_u(A) = \overline{T}\|_u(A)$ and $\widetilde{T}\|_u$ sends a morphism $f : GA \to GB$ in $\mathbf{C}\|_u$ given by $f\colon A \otimes U \to B$ to the morphism $\overline{T}\|_u(A) \to \overline{T}\|_u(B)$ in $\mathbf{C}\|_u$ given by:

$$\overline{T}\|_1(A) \otimes U \xrightarrow{\mathrm{st}_{A,U}} \overline{T}\|_1(A \otimes U) \xrightarrow{\overline{T}\|_1(f)} \overline{T}\|_1(B)$$

We have to prove that this equals $\overline{T}\|_u(f)$. To see this, first note that by the properties of the adjunction, a map f in the coKleisi category $\mathbf{C}\|_u$ is defined in the base category as $\varepsilon \circ F(f)$, which we will denote $f^{\mathbf{C}}$. With this notation, and again using F and G to denote $\mathbf{C}\|^{u\leq 1}$ and $\mathbf{C}\|_{u\leq 1}$, we get:

$$\begin{aligned}
\overline{T}\|_1(f^{\mathbf{C}}) \circ \mathrm{st}_{A,U} &= \overline{T}\|_1\varepsilon_B \circ \overline{T}\|_1 Ff \circ \varepsilon_{\overline{T}\|_1 FGA} \circ F\overline{T}\|_u \eta_{GA} && (12)\\
&= \varepsilon_{\overline{T}\|_1 B} \circ FG\overline{T}\|_1\varepsilon_B \circ FG\overline{T}\|_1 Ff \circ F\overline{T}\|_u \eta_{GA} && (13)\\
&= \varepsilon_{\overline{T}\|_1 B} \circ F\overline{T}\|_u G\varepsilon_B \circ F\overline{T}\|_u GFf \circ F\overline{T}\|_u \eta_{GA} && (14)\\
&= \varepsilon_{\overline{T}\|_1 B} \circ F\overline{T}\|_u G\varepsilon_B \circ F\overline{T}\|_u \eta_{GB} \circ F\overline{T}\|_u f && (15)\\
&= \varepsilon_{\overline{T}\|_1 B} \circ F\overline{T}\|_u f && (16)\\
&= (\overline{T}\|_u(f))^{\mathbf{C}} && (17)
\end{aligned}$$

Line (12) follows from the definition of the strength given in Equation (11) and the definition of $f^{\mathbf{C}}$. The next three lines follow from naturality of ε used twice, Equation (10), and naturality of η respectively. Line (16) uses the property of the adjunction and the last line uses the definition of $(\overline{T}\|_u(f))^{\mathbf{C}}$.

Similarly, using Proposition 24 and then Proposition 25 gives back the unit and the multiplication of the original formal monad. ◀

4.2 Graded monads and indexed monads

We now connect these notions to the pre-existing notions of **E**-indexed monads and **E**-graded monads for a monoidal category **E**. Recall that an **E**-*graded* monad is a lax monoidal functor $\mathbf{E} \to [\mathbf{C}, \mathbf{C}]$. It consists of functors $T_u\colon \mathbf{C} \to \mathbf{C}$, a natural transformation $\eta_A\colon A \to T_I(A)$, and a transformation $\mu_{u,v,A}\colon T_u(T_v(A)) \to T_{u\otimes v}(A)$ natural in u, v, and A, satisfying some coherence diagrams [12].

On the other hand, an **E**-*indexed* monad is a functor $\mathbf{E} \to \mathbf{Monad}(\mathbf{C})$. It also consists of functors $T_u\colon \mathbf{C} \to \mathbf{C}$, but now with transformations $\eta_{u,A}\colon A \to T_u(A)$ and transformations $\mu_{u,A}\colon T_u^2(A) \to T_u(A)$ natural in u and A, such that each (T_u, η_u, μ_u) forms a monad. The formal monads on $\overline{\mathbf{C}}$ as defined in Section 4 are $\mathrm{ZI}(\mathbf{C})$-indexed monads. The next lemma provides conditions under which indexed monads induce graded monads and vice versa.

Recall that a monoidal category *has codiagonals* when there is a natural transformation $A \otimes A \to A$ that respects the coherence isomorphisms [18].

▶ **Lemma 27.** *Let* $\mathbf{E}$ *be a monoidal category. If the tensor unit is initial, then an* $\mathbf{E}$*-indexed monad induces a* $\mathbf{E}$*-graded monad. If the tensor product has codiagonals, then an* $\mathbf{E}$*-graded monad induces an* $\mathbf{E}$*-indexed monad. If* $\mathbf{E}$ *is cocartesian, there is a bijective correspondence between* $\mathbf{E}$*-graded monads and* $\mathbf{E}$*-indexed monads.*

Proof. Suppose the tensor unit 0 in $\mathbf{E}$ is initial. An $\mathbf{E}$-indexed monad (T_u, η_u, μ_u) then induces an $\mathbf{E}$-graded monad with the same T_u but $\overline{\eta}_A = \eta_{0,A}$ and $\overline{\mu}_{u,v,A}$ given by:

$$\begin{tikzcd} T_u(T_v(A)) \arrow[r, "T_{\rho^{-1}}(T_{\lambda^{-1}}(A))"] & T_{u\otimes 0}(T_{0\otimes v}(A)) \arrow[r, "T_{u\otimes !}(T_{!\otimes v}(A))"] & T^2_{u\otimes v}(A) \arrow[r, "\mu_{u\otimes v,A}"] & T_{u\otimes v}(A) \end{tikzcd}$$

Now suppose that $\mathbf{E}$ has codiagonals. An $\mathbf{E}$-graded monad $(T_u, \eta, \mu_{u,v})$ then induces an $\mathbf{E}$-indexed monad with the same T_u but $\overline{\eta}_{u,A} = \eta_A$ and $\overline{\mu}_{u,A}$ given by:

$$\begin{tikzcd} T^2_u(A) \arrow[r, "\mu_{u,u,A}"] & T_{u\otimes u}(A) \arrow[r, "T_{\nabla_u}(A)"] & T_u(A) \end{tikzcd}$$

If $\mathbf{E}$ is cocartesian, these two constructions are each other's inverse. For example, $\overline{\overline{\mu}}_{u,A} = \mu_{u,A}$ because:

$$\begin{tikzcd} T_{u+0}(T_{0+u}(A)) \arrow[r, "T_{u+!}(T_{!+u}(A))"] \arrow[dr, swap, "T_\rho(T_\lambda(A))"] & T^2_{u+u}(A) \arrow[r, "\mu_{u+u,A}"] \arrow[d, "T^2_{\nabla_u}(A)"] & T_{u+u}(A) \arrow[d, "T_{\nabla_u}(A)"] \\ & T^2_u(A) \arrow[r, swap, "\mu_{u,A}"] & T_u(A) \end{tikzcd}$$

Also $\overline{\overline{\eta}}_A = \eta_A$ because $!\colon 0 \to 0$ is the identity. The other properties follow from naturality in u and v. ◀

In particular, it follows that there is no difference between graded monads and indexed monads over (join-)semilattices.

5 Examples

In this section we discuss three extended examples, showing that localisable monads may interpret central idempotents as locations in a computer memory (Subsection 5.1), physical locations in a network of interacting agents (Subsection 5.2), or time in extended processes (Subsection 5.3). These examples use the following characterisation of central idempotents in functor categories.

▶ **Lemma 28.** *If* $\mathbf{C}$ *is a category and* $\mathbf{D}$ *is a symmetric monoidal category, then the functor category* $[\mathbf{C}, \mathbf{D}]$ *is again symmetric monoidal under pointwise tensor products. Regarding* $\mathrm{ZI}(\mathbf{D})$ *as a full subcategory of the slice category* $\mathbf{D}/I$*, there is an isomorphism of categories:*

$$\mathrm{ZI}[\mathbf{C}, \mathbf{D}] \simeq [\mathbf{C}, \mathrm{ZI}(\mathbf{D})]$$

Proof. Let $u\colon U \to I$ be a central idempotent in $[\mathbf{C}, \mathbf{D}]$. The functor $[\mathbf{C}, \mathbf{D}] \to \mathbf{D}$ that evaluates at a fixed object $C \in \mathbf{C}$ is strong monoidal and so preserves central idempotents. Hence each component $u_C\colon U(C) \to I$ represents a central idempotent in $\mathbf{D}$. This is functorial and gives one direction of the isomorphism.

Conversely, let $F\colon \mathbf{C} \to \mathrm{ZI}(\mathbf{D})$ be a functor. Define $U\colon \mathbf{C} \to \mathbf{D}$ by $U(C) = \mathrm{dom}(F(C))$ and $u\colon U \Rightarrow I$ by $u_C = F(C)$. This is functorial and gives the other direction of the isomorphism. It is clear that these two assignments are inverses. ◀

5.1 Quantum buffer

The (global) state monad on **Set** is a well-known monad that combines the properties of the reader and writer monads to implement computational side-effect in functional programming. It is defined as $T(-) = S \multimap (- \times S)$ for a state object $S \in \mathbf{Set}$. For example, to store one bit, take $S = \{0, 1\}$. The central idempotents of **Set** are (represented by) the empty set $\emptyset$ and the singleton set 1. It follows that the (global) state monad is trivially localisable. This example is trivial but can be expanded in several ways:

1. Expanded to the category $\mathbf{Set}^n$, whose objects are n-tuples of sets and morphisms are n-tuples of functions. The state monad on some object $A = (A_1, \dots, A_n)$ in $\mathbf{Set}^n$ is

   $$T(A_1, \dots, A_n) = (S_1, \dots, S_n) \multimap \big((A_1, \dots, A_n) \times (S_1, \dots, S_n)\big)$$

   for a chosen state object $S = (S_1, \dots, S_n) \in \mathbf{Set^n}$. For example, to store n bits, take $S_1 = \cdots = S_n = \{0, 1\}$. It follows from Lemma 28 that $\mathrm{ZI}(\mathbf{Set}^n) \simeq 2^n$. While $\mathbf{Set}^n$ is symmetric monoidal closed, the state monad does not satisfy $T(A \multimap U) = T(A) \multimap T(U)$ as in Example 14. There is still a strength, by currying the evaluation:

   $$T(A_1, \dots, A_n) \times (U_1, \dots, U_n) \times (S_1, \dots, S_n) \to (S_1, \dots, S_n) \times (A_1, \dots, A_n) \times (U_1, \dots, U_n)$$

   We have not discussed commutativity yet, but note that this strength is commutative in a sense made clear in Definition 34 below. Conceptually, this means that the computational side-effects modelled by a state monad "over" a region $(U_1, \dots, U_n)$ are independent of those modelled by $(V_1, \dots, V_n)$, assuming that $(U_1, \dots, U_n) \times (V_1, \dots, V_n) = 0$.
2. The localisable state monad of the previous point does not just work for cartesian closed categories such as $\mathbf{Set}^n$, but also for exponentiable objects in a symmetric monoidal category. For example, we can replicate it in the category **Hilb** of Hilbert spaces and completely positive linear maps used in quantum computation [16]. To store one qubit, take $S = \mathbb{C}^2$. The monad then becomes $T(-) = S^* \otimes - \otimes S$, where $S^* = \mathbf{Hilb}(S, \mathbb{C})$ is the dual Hilbert space, which is isomorphic to $T(A) = A \otimes \mathbb{M}_2$, where $\mathbb{M}_2$ is the Hilbert space of complex 2-by-2 matrices. Similarly, to store n qubits, move to $\mathbf{Hilb}^n$. We can now see a phenomenon that didn't occur for cartesian categories: rather than a quantum memory, this monad models a quantum buffer of n qubits, because there is no entanglement between the different qubits. Because $\mathrm{ZI}(\mathbf{Hilb}) = \{0, \mathbb{C}\}$, again $\mathrm{ZI}(\mathbf{Hilb}^n) \simeq 2^n$. The strength map is yet again given by the curry of the evaluation map, which makes $T(-)$ a commutative localisable monad in the sense of Definition 34 below.
3. We can also promote the (global) state monad on **Set** in another direction, namely from $n = 1$ or finite n to an arbitrary topological space X indexing the bits to be stored. Consider the category $\mathrm{Sh}(X)$ of (**Set**-valued) sheaves on X, take S to be the constant sheaf $S(U) = \{0, 1\}$, and define $T(-) = S \multimap (- \otimes S)$. As in Example 3, the central idempotents correspond to open subsets $U \subseteq X$, and this monad is still localisable. Its stalks (as discussed in Remark 20) are the simple (global) state monads on **Set** storing a single bit each.
4. Points 2 and 3 combine to model a quantum buffer over an arbitrary locally compact Hausdorff topological space X. Consider the category $\mathbf{Hilb}_{C_0(X)}$ of Hilbert modules over $C_0(X)$, take S to be Hilbert module $C_0(X, \mathbb{C}^2)$ of continuous functions $X \to \mathbb{C}^2$ that vanish at infinity, and define $T(-) = S^* \otimes - \otimes S$. As in Example 4, central idempotents are open subsets $U \subseteq X$. Again, this monad is localisable, with $T\|_U = S_u^* \otimes - \otimes S_u$ for $S_u = C_0(U, \mathbb{C}^2)$. In fact, this example is related to the one in point 3, as Hilbert modules over $C_0(X)$ correspond to a Hilbert space internal to the topos $\mathrm{Sh}(X)$ by Takahashi's Theorem [2, 15].

5.2 Concurrent processes

Suppose M_1 is a monoid of actions that some agent 1 can perform, and M_2 is a monoid of actions that an agent 2 can perform. They could, for example, be free monoids over sets of atomic actions. Then we can form the coproduct $M_1 + M_2$ of monoids, and quotient out a congruence that specifies $ab = ba$ for $a \in M_1$ and $b \in M_2$ when actions a and b are independent, to get the monoid M of Mazurkiewicz traces [8, 37]. Now M localises to M_1 by projections $M \to M_i$ that disregard actions of the other agent.

The following lemma engineers a single category with two central idempotents and a monoid, that localises to the given ones. The idea is to take a product of categories, but to add *silent actions*, that enforce the order in which both agents' actions occur, as in the pi calculus [27].

▶ **Lemma 29.** *Let M_1 and M_2 be monoids in symmetric monoidal categories $\mathbf{C}_1$ and $\mathbf{C}_2$ that have an initial object 0 satisfying $A \otimes 0 \simeq 0$ for all objects A. There is a symmetric monoidal category $\mathbf{C}$ with a monoid M and central idempotents u_1, u_2, that allows an isomorphism $\mathbf{C}\|_{u_i} \simeq \mathbf{C}_i$ of monoidal categories under which M_i corresponds with $\mathbf{C}\|_{u_i \leq 1}(M)$.*

If $\mathbf{C}_i$ does not yet have an initial object 0 satisfying $A \otimes 0 \simeq 0$, we may freely adjoin one to obtain a well-defined symmetric monoidal category.

Proof. First construct a new category $\mathbf{C}'$. Objects are pairs (A, B) of $A \in \mathbf{C}_1$ and $B \in \mathbf{C}_2$. Morphisms $(A, B) \to (A', B')$ include pairs (f, g) of $f \in \mathbf{C}_1(A, A')$ and $g \in \mathbf{C}_2(B, B')$, to which we freely adjoin morphisms $\tau_{A,B} \colon (A, B) \to (A, B)$ for each object (A, B). Thus morphisms are finite lists $\big((f_1, g_1), \tau_1, \ldots, \tau_{n-1}, (f_n, g_n)\big)$ where the domain of τ_n is the codomain of $f_n \otimes g_n$. Composition concatenates and then contracts:

$$\begin{aligned}&\big((f'_1, g'_1), \tau'_1, \ldots, (f'_n, g'_n)\big) \circ \big((f_1, g_1), \tau_1, \ldots, (f_m, g_m)\big)\\ &= \big((f_1, g_1), \tau_1, \ldots, (f'_1 \circ f_m, g'_1 \circ g_m), \tau, \ldots, (f'_n, g'_n)\big)\end{aligned}$$

Defining identity to be the trivial list $(\mathrm{id}[A], \mathrm{id}[B])$ makes $\mathbf{C}'$ into a well-defined category.

Next, take the free symmetric monoidal category $\mathbf{C}''$ on $\mathbf{C}'$. Objects of $\mathbf{C}''$ are finite lists of objects of $\mathbf{C}'$, and morphisms are pairs $(\pi, h_1, \ldots, h_n)$ of a permutation π of list indices and a list of morphisms in $\mathbf{C}'$; see for example [1]. Finally, consider the generalised equivalence relation [4] on $\mathbf{C}''$ generated by

$$\begin{aligned}(I, 0) \otimes \tau_{A,B} &\sim (A, 0)\\ (0, I) \otimes \tau_{A,B} &\sim (0, B)\\ \big(\pi, (f_1, g_1), (f_2, g_2)\big) &\sim (\sigma, (f_1 \otimes f_2, g_1 \otimes g_2))\end{aligned}$$

where π is the bijection $1 \mapsto 2$ and $2 \mapsto 1$ on $\{1, 2\}$. This is a symmetric monoidal congruence, so $\mathbf{C} = \mathbf{C}''/\sim$ is a well-defined symmetric monoidal category.

Because 0 is initial and $A \otimes 0 = 0$ in $\mathbf{C}_i$, the objects $(I, 0)$ and $(0, I)$ in $\mathbf{C}'$ become central idempotents u_1, u_2 in $\mathbf{C}$, and moreover $(A, B) \mapsto [A]_\sim$ is an isomorphism $\mathbf{C}\|_{u_1} \simeq \mathbf{C}_1$ and similarly for u_2. Finally, $M = (M_1, M_2)$ is a monoid in $\mathbf{C}$, that localises to M_i by construction. ◀

In the proof of the previous lemma, we could alternatively have described $\mathbf{C}$ as consisting of formal string diagrams generated by $\mathbf{C}_1 \times \mathbf{C}_2$ and the silent actions $\tau_{A,B}$ [6], or as terms in a formal syntactic language [19].

▶ **Example 30.** Let M_i be monoids in $\mathbf{C}_i = \mathbf{Set}$. They induce writer monads $T_i(A) = M_i \otimes A$ on $\mathbf{C}_i$. Now the monoid M in the category $\mathbf{C}$ of the previous lemma induces a writer monad T on $\mathbf{C}$. The monad T is localisable by Example 14, and $T\|_i$ corresponds to T_i under the isomorphism $\mathbf{C}\|_i \simeq \mathbf{C}_i$. Thus T tracks the agents' actions as side effects during a (distributed) computation.

It seems possible to extend this example to a network where the communicating agents form the points of an arbitrary topological space.

5.3 Stochastic processes

Write **Meas** for the category of measurable spaces and measurable functions. This is a symmetric monoidal category, where the tensor unit is the singleton set with its unique σ-algebra, and the tensor product of two measurable spaces is the cartesian product of the sets with the tensor product of the σ-algebras. The monoidal category **Meas** has only two central idempotents: the empty set $\emptyset$, and the tensor unit 1 itself.

Instead, consider the functor category $[\mathbb{N}, \mathbf{Meas}]$, where the partially ordered set $\mathbb{N}$ is considered as a category by having a morphism $m \to n$ if and only if $m \leq n$. Its objects are sequences $X_1, X_2, X_3, \ldots$ of measurable spaces. Lemma 28 shows that this category has many more central idempotents. It follows that central idempotents $u \colon U \Rightarrow 1$ in $[\mathbb{N}, \mathbf{Meas}]$ correspond to upward-closed subsets of $\mathbb{N} \cup \{\infty\}$, or more succinctly, to elements of $n \in \mathbb{N} \cup \{\infty\}$, by

$$U(m) = \begin{cases} \emptyset & \text{if } m < n \\ 1 & \text{if } m \geq n \end{cases}$$

The *Giry monad* $G \colon \mathbf{Meas} \to \mathbf{Meas}$ takes a measurable space to the set of probability measures on it [13]. It extends to a monad on $[\mathbb{N}, \mathbf{Meas}]$.

▶ **Example 31.** The monad $\widehat{G} = G \circ (-)$ on $[\mathbb{N}, \mathbf{Meas}]$ is localisable, where the maps $\widehat{G}(X) \otimes U \Rightarrow \widehat{G}(X \otimes U)$ can simply be taken to be identities (because $G(\emptyset) = \emptyset$). The restricted category $[\mathbb{N}, \mathbf{Meas}]\|_n$ is $[\{n, n+1, \ldots\}, \mathbf{Meas}]$, and the monad $\widehat{G}\|_n$ is simply the restriction of $\widehat{G}$ to $\{n, n+1, \ldots\}$.

The adjunction between **Meas** and the Kleisli category $\mathrm{Kl}(G)$ lifts to an adjunction between $[\mathbb{N}, \mathbf{Meas}]$ and $[\mathbb{N}, \mathrm{Kl}(G)]$. The latter is not equivalent to the Kleisli category of $\widehat{\mathbf{G}}$ because the functor $[\mathbb{N}, \mathbf{Meas}] \to [\mathbb{N}, \mathrm{Kl}(G)]$ that turns a sequence of elements of measurable spaces into a sequence of Dirac measures is not essentially surjective [36, Theorem 9].

The objects of $[\mathbb{N}, \mathbf{Meas}]$ are *stochastic processes* [24, 13, 11]. Instead of $(\mathbb{N}, \leq)$, we could equally well have taken continuous time $(\mathbb{R}^{\geq 0}, \leq)$. In fact, we could also have regarded the monoid $(\mathbb{N}, +, 0)$ or $(\mathbb{R}^{\geq 0}, +, 0)$ as a one-object category. Then $[\mathbb{N}, \mathrm{Kl}(G)]$ would consist of stationary processes, but the central idempotents would remain the same by Lemma 28: ideals of $\mathbb{N}$ or $\mathbb{R}^{\geq 0}$ under $+$ are also upward-closed subsets.

Rather than stochastic (Markov) processes, that depend on the history thus far (one time step ago only), we could have taken more interesting partially ordered sets than the totally ordered ones $\mathbb{N}$ and $\mathbb{R}^{\geq 0}$.

6 Algebras

Let $\mathbf{C}$ be a symmetric monoidal category. As we have seen in Section 4, a localisable monad $T \colon \mathbf{C} \to \mathbf{C}$ is equivalently described as a formal monad $T\|_{-}$ in the 2-category $\mathbf{K} = [\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$. What are its formal (Eilenberg-Moore) algebras?

The general answer is described in [23, 35]. The formal algebra category is an object of $\mathbf{K}$ satisfying the following. For any object $X \in \mathbf{K}$, the formal monad $T\|_-$ induces a (concrete) monad $K(X, T\|_-)$ on the category $\mathbf{K}(X, \mathbf{C}\|_-)$; this monad sends a natural transformation $\beta \colon X \Rightarrow \mathbf{C}\|_-$ to the natural transformation with components $T\|_u \circ \beta_u \colon X_u \to \mathbf{C}\|_u$. This (concrete) monad has a (concrete) Eilenberg-Moore category of algebras. Objects are pairs of a natural transformation β and a modification θ of type

$$\begin{tikzcd}
& \mathbf{C}\|_u \arrow[dr, bend left, "T\|_u"] & \\
X_u \arrow[ur, bend left, "\beta_u"] \arrow[rr, "\beta_u"'] \arrow[rr, phantom, shift left=4, "\theta_u\Downarrow"] & & \mathbf{C}\|_u
\end{tikzcd} \tag{18}$$

satisfying the algebra laws. Morphisms are modifications $\varphi \colon \beta \Rrightarrow \beta'$ satisfying:

$$\begin{tikzcd}
& & X_u \arrow[dll, bend right, "\beta_u"'] \arrow[dd, "\beta_u" description] \arrow[dd, bend left=90, "\beta'_u"] & & \\
\mathbf{C}\|_u \arrow[drr, bend right, "T\|_u"'] & \overset{\theta_u}{\Rightarrow} & & \overset{\varphi_u}{\Rrightarrow} & \\
& & \mathbf{C}\|_u & &
\end{tikzcd}
=
\begin{tikzcd}
& & X_u \arrow[dll, bend right, "\beta_u"'] \arrow[dll, bend left, "\beta'_u" description] \arrow[dd, bend left=90, "\beta'_u"] & \\
\mathbf{C}\|_u \arrow[drr, bend right, "T\|_u"'] & \overset{\varphi_u}{\Rrightarrow} & & \underset{\theta'_u}{\Rrightarrow} \\
& & \mathbf{C}\|_u &
\end{tikzcd} \tag{19}$$

This defines the object-part of a 2-functor $\mathbf{K}^{\mathrm{op}} \to \mathbf{Cat}$. Now $A \in \mathbf{K}$ is the *formal algebra* object of the formal monad $\mathbf{T}\|_-$ when this 2-functor is naturally isomorphic to $\mathbf{K}(-, A)$.

▶ **Proposition 32.** *Let T be a localisable monad on a symmetric monoidal category $\mathbf{C}$. The formal monad $T\|_-$ in $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$ has a formal algebra object $A\|_-$ where $A\|_u = \mathrm{Alg}(T\|_u)$ is the category of algebras of $T\|_u$.*

Proof. If $u \leq v$ then the monad morphism $\mathbf{C}\|_{u \leq v}$ of Lemma 18 induces a functor $A\|_v \to A\|_u$, so A is a well-defined object of $\mathbf{K} = [\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$. Now, for an object $X \in \mathbf{K}$, the hom-category $\mathbf{K}(X, A)$ has as objects natural transformations $\beta_u \colon X_u \to \mathrm{Alg}(T\|_u)$. But the objects of $\mathrm{Alg}(T\|_u)$ are themselves morphisms $\theta_u \colon T\|_u(B) \to B$ in $\mathbf{C}\|_u$, that furthermore satisfy the algebra laws. These assemble into a modification satisfying (18). It is labour-intensive but straightforward to verify that the morphisms of $\mathrm{Alg}(T\|_u)$ similarly match modifications satisfying (19), and that this in fact gives a 2-natural isomorphism to $A\|_-$. Thus $A\|_-$ is a formal algebra object. ◀

Similarly, a *formal Kleisli algebra* object of the formal monad $T\|_-$ is characterised in [23, 35] as a formal algebra object in the 2-category $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}^{\mathrm{op}}]$, where $\mathbf{Cat}^{\mathrm{op}}$ has reversed the 1-cells but not the 2-cells of $\mathbf{Cat}$.

▶ **Corollary 33.** *Let T be a localisable monad on a symmetric monoidal category $\mathbf{C}$. The formal monad $T\|_-$ in $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$ has a formal Kleisli object $K\|_-$ where $K\|_u = \mathrm{Kl}(T\|_u)$ is the Kleisli category of $T\|_u$.* ◀

A Kleisli category of a commutative monad on a symmetric monoidal category is again symmetric monoidal [7]. It would be interesting to see if there is a notion that stands to localisability as commutativity stands to strength, that guarantees that the formal Kleisli algebra object of the previous corollary is a monoid in $\mathbf{K} = [\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$. We leave this for future work, but give a tentative (re)definition now.

▶ **Definition 34.** *A localisable monad T on a symmetric monoidal category* $\mathbf{C}$ *is* commutative *when:*

$$\begin{tikzcd}
T(A) \otimes U \otimes V \arrow[r, "\mathrm{st}_{A,U} \otimes V"] \arrow[d, "T(A)\otimes\sigma_{U,V}"'] & T(A \otimes U) \otimes V \arrow[r, "\mathrm{st}_{A\otimes U,V}"] & T(A \otimes U \otimes V) \\
T(A) \otimes V \otimes U \arrow[r, "\mathrm{st}_{A,V} \otimes U"'] & T(A \otimes V) \otimes U \arrow[r, "\mathrm{st}_{A\otimes V,U}"'] & T(A \otimes V \otimes U) \arrow[u, "T(A\otimes\sigma_{V,U})"']
\end{tikzcd} \tag{20}$$

It follows from this definition that if $u \wedge v = 0$, then the computational side-effects modeled by T_u and T_v do not influence each other. Intuitively, side-effects T_u and T_v that act in disjoint areas must be independent of each other.

7 Further work

There are several interesting directions for further research.

- We have decomposed a localisable monad into monads on local monoidal categories, but can a monad on a local monoidal category be decomposed further? For example, the local state monad [29] is based on the presheaf category $[\mathbf{Inj}, \mathbf{Set}]$. Its central idempotents correspond to natural numbers, topologised by saying that a subset is open when it is upward-closed under the usual ordering of natural numbers. This topological space is already local: every net converges to the focal point 0. The "decomposition" using coends of [29] relies on the base category $[\mathbf{Inj}, \mathbf{Set}]$ having much more structure rather than just a monoidal category. The successor function of natural numbers there affords the possibility to allocate fresh locations. Our example of local states in Section 5.1 completely ignored this possibility. Can this extra structure be axiomatised – using open sets rather than points – and used for a further decomposition?
- Two monads on the same base category can be composed as soon as there is a distributive law between them [3, 38]. When does a distributive law respect the localisable nature of the monads, and how does it interact with their decomposition into monads on local monoidal categories?
- More generally than monads, when is a PROP localisable, and how does a localisable PROP decompose into local ones [22, 33]?
- Formal monads form a bridge between localisable monads and the local-to-global approach to providing a fine-grained structure over monads. Can this relationship be made more constructive? Given monads T_i on possibly different monoidal base categories $\mathbf{C}_i$, can we construct a monad T on a monoidal category $\mathbf{C}$ with central idempotents i such that $\mathbf{C}\|_i \simeq \mathbf{C}_i$ and $T\|_i \simeq T_i$? The free construction of Lemma 29 is an initial step in this direction; can it be given a more elegant concrete description, and extended to arbitrary topogical spaces?
- Is there a notion that stands to localisability as commutativity stands to strength, that guarantees that the formal Kleisli object of Corollary 33 is a monoid in $[\mathrm{ZI}(\mathbf{C})^{\mathrm{op}}, \mathbf{Cat}]$? Does it connect to partial commutativity as in the Mazurkiewicz traces of Section 5.2?

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