Matchings, Critical Nodes, and Popular Solutions

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Abstract

We consider a matching problem in a marriage instance G. Every node has a strict preference order ranking its neighbors. There is a set C of prioritized or critical nodes and we are interested in only those matchings that match as many critical nodes as possible. Such matchings are useful in several applications and we call them *critical matchings*. A stable matching need not be critical. We consider a well-studied relaxation of stability called *popularity*. Our goal is to find a popular critical matching, i.e., a weak Condorcet winner within the set of critical matchings where nodes are voters. We show that popular critical matchings always exist in G and min-size/max-size such matchings can be efficiently computed.

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1 Introduction

We consider a matching problem in a bipartite graph $G = (A \cup B, E)$ on n nodes and m edges where every node ranks its neighbors in a strict order of preference. Such a graph is also called a *marriage instance*. We seek an optimal matching in G and the classical notion of optimality for matchings in such an instance is *stability* introduced by Gale and Shapley [9] in 1962. A matching M is stable if there is no edge that *blocks* M where an edge (a, b) is said to block M if a and b prefer each other to their respective assignments in M.

Stable matchings always exist in a marriage instance and the Gale-Shapley algorithm finds one in linear time. The Gale-Shapley algorithm and its many-to-one generalization have been used to match students to schools and colleges [1, 2, 17] and graduating medical students to hospitals [4, 21]. All stable matchings in G match the same set of nodes [10]. As discussed in [3], in the medical matching scheme in Scotland, a stable matching left several students unmatched. There was a matching that matched all the students, however this matching admitted some blocking edges. Thus there are real-world applications where the size of the matching is more important than the absence of blocking edges.

More generally, there are applications where certain nodes are prioritized or *critical* and the number of critical nodes that get matched is of primary importance. One such application is the assignment of sailors to billets in the US Navy [22, 26]. Here every sailor has to be matched to a billet and some critical billets cannot be left vacant. So such billets and all the sailors are the critical nodes here. Allocation problems in humanitarian organizations constitute more such applications, see e.g., [24, 25].

Motivated by such applications, we consider the following model where we are given a marriage instance $G = (A \cup B, E)$ along with a set $C \subseteq A \cup B$ of critical nodes. The number of critical nodes that get matched is the most important attribute of a matching. An admissible or critical matching is one that matches as many critical nodes as possible.

▶ **Definition 1.** *A matching M in G is critical if there is no matching in G that matches more critical nodes than M.*

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A stable matching need not be critical. When stable matchings are not critical, a natural alternative is to seek a critical matching that admits the least number of blocking edges. However this is an NP-hard problem [3]. It was shown there that finding a maximum matching (so every node is critical here) that admits the minimum number of blocking edges is NP-hard; moreover, this is NP-hard to approximate within $n^{1-\varepsilon}$, for any $\varepsilon > 0$. This motivates relaxing the problem of finding a critical matching with the least number of blocking edges to finding one that satisfies a more relaxed variant of stability. *Popularity* is a natural relaxation of stability that captures welfare in a collective sense.

We say node v prefers matching M to matching N if v prefers its partner in M to its partner in N and being left unmatched is the worst choice for any node. We can compare any pair of matchings M and N by holding an election between them where every node casts a vote for the matching in $\{M, N\}$ that it prefers and it abstains from voting if it is indifferent between M and N. Let $\phi(M, N)$ (resp., $\phi(N, M)$) be the number of votes for M (resp., N) in the M versus N election. Matching N is more popular than matching M if $\phi(N, M) > \phi(M, N)$.

▶ **Definition 2.** *A matching M is popular if $\Delta(N, M) \leq 0$ for all matchings N in G, where $\Delta(N, M) = \phi(N, M) - \phi(M, N)$.*

Thus a matching M is popular if there is no matching that is more popular than M. The notion of popularity was introduced in 1975 by Gärdenfors [11] where he observed that every stable matching is popular. It is easy to decide if there is a popular matching that is also critical – it is known that any node that is matched in some popular matching has to be matched in any max-size popular matching [12]. A max-size popular matching can be computed in linear time [14]. However as was the case with stable matchings, it can be the case that no popular matching is critical. Consider the following example where $A = \{a_0, a_1, a_2\}$ and $B = \{b_0, b_1, b_2\}$. Node preferences are described below.

$a_0 : b_1$ $\qquad$ $a_1 : b_1 \succ b_2 \succ b_0$ $\qquad$ $a_2 : b_1 \succ b_2$

$b_0 : a_1$ $\qquad$ $b_1 : a_1 \succ a_2 \succ a_0$ $\qquad$ $b_2 : a_1 \succ a_2$

The node a_0 has only one neighbor b_1. The node a_1 regards b_1 as its top choice, b_2 as its second choice, and b_0 as its third choice. The node a_2 regards b_1 as its top choice and b_2 as its second choice. The preferences of nodes in B are symmetric to those in A.

The above instance has only one stable matching $S = \{(a_1, b_1), (a_2, b_2)\}$. This instance has one more popular matching $P = \{(a_1, b_2), (a_2, b_1)\}$. Suppose $C = \{a_0, a_1\}$ is the set of critical nodes. Then neither S nor P is critical. Here $M_0 = \{(a_0, b_1), (a_1, b_2)\}, M_1 = \{(a_0, b_1), (a_1, b_0)\}$, and $M_2 = \{(a_0, b_1), (a_1, b_0), (a_2, b_2)\}$ are the critical matchings. Thus there need not exist any popular matching that is critical.

A natural alternative is to ask for a critical matching M such that there is no *critical* matching more popular than M. Given that the number of critical nodes that get matched is more important than node preferences, elections that involve non-critical matchings are not relevant since by the definition of our setting, any critical matching is better than any non-critical matching. So the desired matchings are the critical ones and any pair of critical matchings can be compared by holding an election between them. Thus we are only interested in elections between pairs of critical matchings.

▶ **Definition 3.** *A critical matching M is a popular critical matching in G if $\Delta(N, M) \leq 0$ for any critical matching N.*

A popular critical matching is a weak Condorcet winner [5, 18] in the voting instance where every critical matching is a candidate and nodes are voters. The relation "more popular than" is not transitive, i.e., there may be cycles with respect to this relation, so weak Condorcet winners need not exist in every voting instance. It might be the case that for any critical matching, there is a "more popular" critical matching. Interestingly, it was shown in [14] that popular *maximum* matchings (i.e., $C = A \cup B$) always exist in G. Does this positive result hold for every $C \subset A \cup B$? So the following questions are relevant:

- For any $C \subset A \cup B$, does a popular critical matching always exist in G?
- Is it easy to find one?
- Is it easy to find a max-size popular critical matching?

In this paper we show positive answers to all the above questions. Recall that $|E| = m$.

▶ **Theorem 4.** *For any $C \subset A \cup B$, popular critical matchings always exist in $G = (A \cup B, E)$ and a max-size such matching can be computed in $O(|C|m + m)$ time.*

We first show the following result. Then we extend this algorithm to show Theorem 4.

▶ **Theorem 5.** *Given a marriage instance $G = (A \cup B, E)$ along with a subset C of critical nodes, a min-size popular critical matching in G can be computed in $O(|C|m + m)$ time.*

1.1 Background and related results

Algorithmic questions in popular matchings have been well-studied during the last decade and we refer to [6] for a survey. Popular matchings always exist in a marriage instance and efficient algorithms are known to find min-size/max-size popular matchings in a marriage instance [9, 13, 14]. A size-popularity trade-off was shown in [14] to efficiently find matchings whose *unpopularity* is bounded from above and size is bounded from below. As shown there, this implies that a maximum matching that is popular within the set of maximum matchings always exists and can be efficiently computed. So $C = A \cup B$ in [14] while $C = \emptyset$ in the Gale-Shapley algorithm. Thus for the two extreme cases of C, it was known that popular critical matchings always exist and can be efficiently computed.

A related problem is the hospital-residents problem with lower quotas. This is a many-to-one matching problem where every node has a strict preference order over its neighbors and every hospital has a capacity; moreover certain hospitals have lower quotas which denotes the minimum number of residents that have to be matched to this hospital in any feasible matching. It was shown in [19] that whenever feasible matchings exist, a matching that is popular among feasible matchings always exists and a max-size such matching can be computed in polynomial time. Very recently and independent of our work, the above result was generalized in [20] to the setting where certain residents are marked and every marked resident has to be matched in any feasible matching.

Hardness results for "almost stable" critical matchings. Several hardness results for finding *almost* stable maximum matchings (so every vertex is critical) in a marriage instance were shown in [3]. It was shown there that even if all preference lists were restricted to be of length at most 3, finding a maximum matching that admits the minimum number of blocking edges is NP-hard. An alternative approach is to count the number of nodes that are involved in blocking edges [8, 23]. The problem of finding a maximum matching that minimizes this number is also NP-hard to compute/approximate, as shown in [3].

1.2 Techniques

We use the machinery of stable matchings and LP-duality to show our results. We construct a new marriage instance $G' = (A' \cup B', E')$ on $O(|C|n + n)$ nodes and $O(|C|m + m)$ edges such that any stable matching in G' corresponds to a popular critical matching in G. The instance G' resembles instances used in [7, 14, 16] to compute max-size popular matchings and popular maximum matchings.

We now give a quick overview of the popular maximum matching algorithm from [14]. This algorithm partitions the node set $A \cup B$ into *levels* so that any stable matching in this "graph with levels" corresponds to a popular maximum matching in G. To begin with, all nodes are in some level ℓ and the Gale-Shapley algorithm is run on this instance. If the stable matching leaves some nodes in A unmatched then all unmatched nodes in A are *promoted* to level $\ell + 1$. Once promoted to level $\ell + 1$, each such node starts proposing all over again – it will be the case that every node in B prefers higher level neighbors to lower level neighbors. So some of these promoted nodes may find partners.

This may "un-match" some nodes in A initially matched in level ℓ. These nodes continue proposing as per the Gale-Shapley algorithm and any node in A that is unsuccessful in finding a partner in level ℓ gets promoted to level $\ell + 1$. Any node in A that does not find a partner even as a level $\ell + 1$ node gets promoted to level $\ell + 2$ and so on. It was shown in [14] that $|A|$ levels suffice to construct a maximum matching that is popular within the set of maximum matchings.

Our algorithms. If all the critical nodes are in A then the above algorithm easily generalizes to solving the popular critical matching problem by promoting only critical nodes in A to higher levels and non-critical nodes in A will always remain in level ℓ. However we need to deal with critical nodes in the set B as well. For this, our new idea is the following: critical nodes in B that are left unmatched in the Gale-Shapley algorithm in level ℓ get *demoted* to level $\ell - 1$. It will be the case that every node in A prefers *lower level* neighbors to *higher level* neighbors. So in fact, the Gale-Shapley algorithm should begin by nodes in A proposing to lower level neighbors first (before the ones in level ℓ).

Thus the main difference between our instance G' and the earlier instance from [14] (explicitly described in [16]) is that there is non-uniformity among the nodes now. All the nodes in $A \cup B$ are permitted in only one intermediate level, i.e., level ℓ. Non-critical nodes in A are excluded from levels higher than ℓ and non-critical nodes in B are excluded from levels lower than ℓ. We show that any stable matching in G' corresponds to a min-size popular critical matching in G. We construct another instance G'' such that the entire node set $A \cup B$ is permitted in *two* levels: level ℓ and level $\ell + 1$. We show that any stable matching in G'' corresponds to a max-size popular critical matching in G. When $C = \emptyset$, the instance G'' is the same as the instance from [7] whose stable matchings correspond to max-size popular matchings in G.

Our proofs of correctness. We prove the correctness of our algorithms via the LP method by constructing witnesses that certify "popularity within the set of critical matchings" for our matchings. These witnesses are solutions to certain linear programs. Such witnesses are known for popular matchings [15] and popular maximum matchings [16]. Our witnesses are a little more complicated since our primal LP involves more constraints (due to criticality) and so the dual LP has more variables.

The dual LP solutions that we show (see Lemma 11 and Lemma 16) allow us to give simple proofs of correctness and enable us to show (using complementary slackness) that our two algorithms respectively compute min-size and max-size popular critical matchings in G.

By contrast, the proof of correctness of the popular maximum matching algorithm in [14] was combinatorial; popular maximum matchings were characterized in terms of forbidden alternating paths and cycles and it was shown that there was no forbidden alternating path or cycle with respect to the matching returned.

Organization of the paper. Section 2 describes our witness for a popular critical matching. The min-size and max-size popular critical matching algorithms are given in Section 3 and Section 4, respectively.

2 A witness for a popular critical matching

Our input consists of a marriage instance $G = (A \cup B, E)$ with strict preferences and a set $C \subseteq A \cup B$ of critical nodes. We first characterize critical matchings.

▶ **Lemma 6.** *A matching M in G is critical if and only if there is no alternating path p with respect to M that satisfies either of the conditions given below:*

1. *p is an augmenting path with respect to M and at least one endpoint of p is in C.*
2. *p has even length with exactly one endpoint in C and this node is left unmatched in M.*

Proof. Let M be a matching with an alternating path p such that either (i) p is an augmenting path wrt M and at least one endpoint of p is in C or (ii) p has even length with exactly one endpoint in C and this node is left unmatched in M. Then $M \oplus p$ matches at least one more critical node than M. Thus M cannot be a critical matching.

Conversely, suppose M is not a critical matching. Let N be a critical matching. Consider $M \oplus N$. Since N matches more critical nodes than M, there has to be an alternating path p in $M \oplus N$ where N matches more critical nodes than M. So p has an endpoint in C that is matched in N and not in M. If the other endpoint of p is unmatched in M then p is an augmenting path wrt M; else the other endpoint is matched in M and this endpoint is not in C since N matches more critical nodes than M in the alternating path p.

So either (i) p is an augmenting path wrt M and at least one endpoint of p is in C or (ii) p has even length with exactly one endpoint in C, which is left unmatched in M. ◀

Let M be any critical matching in G. Let k_A (resp., k_B) denote the number of nodes in $C_A = C \cap A$ (resp., $C_B = C \cap B$) that are matched in M. The following lemma will be very useful to us.

▶ **Lemma 7.** *Every matching in G matches at most k_A nodes in C_A and at most k_B nodes in C_B.*

Proof. Suppose not. Let N be a matching in G that matches more than k_A nodes in C_A. Then there is an alternating path p in $M \oplus N$ where N matches more nodes of C_A than the critical matching M. If the length of p is odd then p is an augmenting path wrt M whose at least one endpoint is in C. But this is a forbidden structure for any critical matching (by Lemma 6).

So the length of p is even. Then the other endpoint of p (the one matched in M and unmatched in N) is in A, call this node v. Since N matches more nodes of C_A than M in the path p, the node v cannot be in C_A. Hence p is an even length alternating path with exactly one endpoint in C and this node is left unmatched in M. This is again a forbidden structure for any critical matching (by Lemma 6). Thus we get a contradiction. The proof when N matches more than k_B nodes in C_B is analogous. ◀

A linear program for popular critical matchings. It will be convenient to assume that each node considers itself as its last choice neighbor. Let $\tilde{G}$ denote the graph G augmented with self-loops. Any matching M in G can be regarded as a perfect matching $\tilde{M}$ in $\tilde{G}$ by augmenting M with appropriate self-loops. Corresponding to M, an edge weight function wt_M in the graph $\tilde{G}$ can be defined. Let $\mathsf{wt}_M(u,u) = 0$ if u is left unmatched in M, i.e., if (u,u) is in $\tilde{M}$; else $\mathsf{wt}_M(u,u) = -1$. For any edge $(a,b) \in E$:

$$\text{let } \mathsf{wt}_M(a,b) = \begin{cases} 2 & \text{if } (a,b) \text{ blocks } M; \\ -2 & \text{if both } a \text{ and } b \text{ prefer their respective partners in } M \text{ to each other;} \\ 0 & \text{otherwise.} \end{cases}$$

For any $e \in E$, note that $\mathsf{wt}_M(e)$ is the sum of votes of the endpoints of e for each other versus their respective partners in $\tilde{M}$; each vote is in $\{\pm 1, 0\}$ where 1 is "more preferred to" and so on. For any node u, let $\delta(u)$ be the set of edges incident to u in G.

Consider the following linear program (LP1). Note that Lemma 7 implies that all critical matchings in G match k_A nodes in C_A and k_B nodes in C_B. This is used in constraint (2).

$$\text{maximize} \sum_{e \in \tilde{E}} \mathsf{wt}_M(e) \cdot x_e \tag{LP1}$$

subject to

$$\sum_{e \in \delta(u) \cup \{u,u\}} x_e = 1 \quad \forall u \in A \cup B \tag{1}$$

$$\sum_{a \in C_A} \sum_{e \in \delta(a)} x_e = k_A \quad \text{and} \quad \sum_{b \in C_B} \sum_{e \in \delta(b)} x_e = k_B \tag{2}$$

$$x_e \geq 0 \quad \forall e \in E \cup \{(u,u) : u \in A \cup B\}. \tag{3}$$

We know from Lemma 7 that $\sum_{a \in C_A} \sum_{e \in \delta(a)} x_e \leq k_A$ and $\sum_{b \in C_B} \sum_{e \in \delta(b)} x_e \leq k_B$ are valid inequalities for the matching polytope of $\tilde{G}$. So the feasible region of (LP1) defines a face of the perfect matching polytope of $\tilde{G}$ and hence it is integral. Every integral point in this face corresponds to a critical matching and every critical matching (augmented with self-loops at unmatched nodes) belongs to this face. Thus (LP1) computes a max-weight matching $\tilde{N}$, where N is a critical matching in G.

Consider the dual LP. This is (LP2) given below. The dual variables are y_u for $u \in A \cup B$ along with z_A and z_B.

$$\text{minimize} \sum_{u \in A \cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B) \tag{LP2}$$

subject to

$$y_a + y_b \geq \mathsf{wt}_M(a,b) \quad \forall (a,b) \in E \text{ where } a \notin C_A, b \notin C_B \tag{4}$$

$$y_a + y_b + z_A \geq \mathsf{wt}_M(a,b) \quad \forall (a,b) \in E \text{ where } a \in C_A, b \notin C_B \tag{5}$$

$$y_a + y_b + z_B \geq \mathsf{wt}_M(a,b) \quad \forall (a,b) \in E \text{ where } a \notin C_A, b \in C_B \tag{6}$$

$$y_a + y_b + z_A + z_B \geq \mathsf{wt}_M(a,b) \quad \forall (a,b) \in E \text{ where } a \in C_A, b \in C_B \tag{7}$$

$$y_u \geq \mathsf{wt}_M(u,u) \quad \forall u \in A \cup B. \tag{8}$$

▶ **Proposition 8.** *Let M be a critical matching such that the optimal value of* (LP2) *is at most 0. Then M is a popular critical matching.*

Proof. The optimal value of (LP1) is $\max_N \mathsf{wt}_M(\tilde{N})$, where N is a critical matching in G. It follows from the definition of the function wt_M that $\mathsf{wt}_M(\tilde{N}) = \phi(N, M) - \phi(M, N) = \Delta(N, M)$ for any matching N in G. Thus the optimal value of (LP1) is $\max_N \Delta(N, M)$, where N is a critical matching. If the optimal value of (LP2) is at most 0 then the optimal value of (LP1) is also at most 0 (by weak duality). This means $\Delta(N, M) \leq 0$ for every critical matching N. ◀

We will use Proposition 8 to prove the correctness of our algorithms in Section 3 and Section 4. That is, we will construct matchings M such that there exist feasible solutions $(\vec{y}, \vec{z})$ to (LP2) with $\sum_{u \in A \cup B} y_u \ + \ (k_A \cdot z_A) \ + \ (k_B \cdot z_B) = 0$.

3 An algorithm for a popular critical matching

Let $G = (A \cup B, E)$ be the given marriage instance and let $C \subseteq A \cup B$ be the set of critical nodes. Recall the overview of our algorithm given in Section 1.2. We want to partition the node set $A \cup B$ into *levels* so that any stable matching in this new graph corresponds to a popular critical matching in G.

Recall that we use $C_A = C \cap A$ (resp., $C_B = C \cap B$) to denote the set of critical nodes in A (resp., B). Let $|C_A| = \alpha$ and $|C_B| = \beta$. There will be $\alpha + \beta + 1$ levels indexed $0, \ldots, \alpha + \beta$.

A new instance $G' = (A' \cup B', E')$. We now describe a new instance G' whose stable matchings will map to popular critical matchings in G. The set A' is described below.

- For every $a \in C_A$, the set A' has $\alpha + \beta + 1$ copies of a: call these nodes $a_0, a_1, \ldots, a_{\alpha+\beta}$.
- For every $a \in A \setminus C_A$, the set A' has $\beta + 1$ copies of a: call these nodes $a_0, a_1, \ldots, a_\beta$.

Thus $A' = \cup_{a \in C_A} \{a_0, a_1, \ldots, a_{\alpha+\beta}\} \cup_{a \in A \setminus C_A} \{a_0, a_1, \ldots, a_\beta\}$. Define the set B' as follows. $B' = \{b' : b \in B\} \cup_{a \in C_A} \{d_1(a), \ldots, d_{\alpha+\beta}(a)\} \cup_{a \in A \setminus C_A} \{d_1(a), \ldots, d_\beta(a)\}$.

The set $\{b' : b \in B\}$ is a copy of the set B. Along with nodes in $\{b' : b \in B\}$, the set B' contains *dummy* nodes (the d-nodes). Such dummy nodes were first used in [7] and they make it easy for us to describe "promotions" from one level to another.

When $a \in C_A$, there are $\alpha + \beta + 1$ copies of a in A' and the set B' has $d_1(a), \ldots, d_{\alpha+\beta}(a)$. We will set preferences such that in any stable matching in G', $\alpha + \beta$ copies of a have to be matched to these dummy nodes. Similarly, when $a \in A \setminus C_A$, there are $\beta + 1$ copies of a in A' and the set B' has $d_1(a), \ldots, d_\beta(a)$. We will set preferences such that in any stable matching in G', β copies of a have to be matched to these dummy nodes. Thus in any stable matching in G', for each $a \in A$, at most *one* node among all a_i's is "free" to be matched to a neighbor in $\{b' : b \in B\}$.

The edge set. Corresponding to each $(a, b) \in E$, we will have the following edges in E'. There are four cases here depending on whether a is in C_A or not and b is in C_B or not.

1. $a \notin C_A$ and $b \notin C_B$: there is exactly one edge (a_β, b') in E' that corresponds to (a, b).
2. $a \notin C_A$ and $b \in C_B$: there are $\beta + 1$ edges (a_i, b') in E' where $0 \leq i \leq \beta$.
3. $a \in C_A$ and $b \notin C_B$: there are $\alpha + 1$ edges (a_i, b') in E' where $\beta \leq i \leq \alpha + \beta$.
4. $a \in C_A$ and $b \in C_B$: there are $\alpha + \beta + 1$ edges (a_i, b') in E' where $0 \leq i \leq \alpha + \beta$.

For each $a \in A$, the set E' also has the following edges:

- if $a \in C_A$ then $(a_{i-1}, d_i(a))$ and $(a_i, d_i(a))$ for $1 \leq i \leq \alpha + \beta$;
- if $a \in A \setminus C_A$ then $(a_{i-1}, d_i(a))$ and $(a_i, d_i(a))$ for $1 \leq i \leq \beta$.

For any i, the preference order of $d_i(a)$ is $a_{i-1} \succ a_i$.

Preference orders. Consider $a \in A$. Let a's preference order in G be $b_1 \succ \cdots \succ b_k$. Suppose $\{c_1, \ldots, c_r\} = \{b_1, \ldots, b_k\} \cap C$. That is, $c_1, \ldots, c_r$ are a's critical neighbors. Let a's preference order among these nodes be $c_1 \succ \cdots \succ c_r$.

- a_0's preference order in G' is $c'_1 \succ \cdots \succ c'_r \succ d_1(a)$.
- For $1 \le i \le \beta - 1$, a_i's preference order is $d_i(a) \succ c'_1 \succ \cdots \succ c'_r \succ d_{i+1}(a)$.
- For $a \notin C_A$: the preference order of a_β is $d_\beta(a) \succ b'_1 \succ \cdots \succ b'_k$.
- For $a \in C_A$:
  - for $\beta \le i \le \alpha + \beta - 1$, the preference order of a_i is $d_i(a) \succ b'_1 \succ \cdots \succ b'_k \succ d_{i+1}(a)$.
  - the preference order of $a_{\alpha+\beta}$ is $d_{\alpha+\beta}(a) \succ b'_1 \succ \cdots \succ b'_k$.

For $a \in A$, other than the dummy nodes, observe that it is only copies of critical neighbors that are present in the preference list of a_i for $0 \le i \le \beta - 1$.

For $a \notin C_A$, observe that copies of all neighbors of a, i.e., $b'_1, \ldots, b'_k$, are present only in the preference list of a_β. For $a \in C_A$, copies of all neighbors of a_i are present in the preference list of a_i for $\beta \le i \le \alpha + \beta$.

Consider any $b \in B$. Let b's preference order in G be $a \succ \cdots \succ z$. Let $\{a', \ldots, z'\} = \{a, \ldots, z\} \cap C$. Let b's preference order among its critical neighbors be $a' \succ \cdots \succ z'$. Suppose $b \notin C_B$. Then the preference order of b' in G' is:

$$\underbrace{a'_{\alpha+\beta} \succ \cdots \succ z'_{\alpha+\beta}}_{\text{level } \alpha+\beta \text{ neighbors}} \succ \cdots \succ \underbrace{a'_{\beta+1} \succ \cdots \succ z'_{\beta+1}}_{\text{level } \beta+1 \text{ neighbors}} \succ \underbrace{a_\beta \succ \cdots \succ z_\beta}_{\text{level } \beta \text{ neighbors}}$$

So b' prefers any subscript or level i neighbor to any level j neighbor for $i > j$. Note that copies of only critical neighbors are present in level i for $\beta + 1 \le i \le \alpha + \beta$ and copies of all neighbors of b, i.e., $a, \ldots, z$, are present only in level β.

Suppose $b \in C_B$. Then the preference order of b' in G' is:

$$\underbrace{a'_{\alpha+\beta} \succ \cdots \succ z'_{\alpha+\beta}}_{\text{level } \alpha+\beta \text{ neighbors}} \succ \cdots \succ \underbrace{a'_{\beta+1} \succ \cdots \succ z'_{\beta+1}}_{\text{level } \beta+1 \text{ neighbors}} \succ \underbrace{a_\beta \succ \cdots \succ z_\beta}_{\text{level } \beta \text{ neighbors}} \succ \cdots \succ \underbrace{a_0 \succ \cdots \succ z_0}_{\text{level } 0 \text{ neighbors}}$$

Note that copies of only critical neighbors are present in level i for $\beta + 1 \le i \le \alpha + \beta$ and copies of all neighbors of b are present in level i for $0 \le i \le \beta$.

The matching M. For any stable matching M' in G', define $M \subseteq E$ to be the set of edges obtained by deleting edges in M' that are incident to dummy nodes and replacing any edge $(a_i, b') \in M'$ with the original edge $(a, b) \in E$.

For any $a \in A$ and all $i \ge 1$, the dummy node $d_i(a)$ is the top choice neighbor for a_i, hence the stable matching M' has to match all dummy nodes. Thus at most one node among all the a_i's can be matched in M' to a neighbor in $\{b' : b \in B\}$. So M is a matching in G. Theorem 9 (proved below) is our main theorem in this section.

▶ **Theorem 9.** *For any stable matching M' in G', the corresponding matching M is a min-size popular critical matching in G.*

Since a stable matching always exists in G', popular critical matchings always exist in G. Thus the first part of Theorem 4 follows. The time taken to construct G' and to compute a stable matching in G' is $O(|C|m + m)$. Thus Theorem 5 follows from Theorem 9.

We will prove Theorem 9 now. As done in [14], it will be useful to partition the set $A \cup B$ into subsets as described below (see Fig. 1). We will partition the set of all nodes in A that are matched in M into $A_0 \cup \cdots \cup A_{\alpha+\beta}$ where for $0 \le i \le \alpha + \beta$: $A_i = \{a \in A : (a_i, b') \in M'$

for some $b \in B\}$, i.e., A_i is the collection of those a's such that a_i is matched in M' to a neighbor in $\{b' : b \in B\}$. Add unmatched nodes in C_A to $A_{\alpha+\beta}$ and add unmatched nodes in $A \setminus C_A$ to A_β.

Similarly, partition the set of all nodes in B that are matched in M into $B_0 \cup \cdots \cup B_{\alpha+\beta}$ where for $0 \le i \le \alpha+\beta$: $B_i = \{b : (a_i, b') \in M' \text{ for some } a \in A_i\}$, i.e., B_i is the collection of those b's such that the partner of b' in M' is a subscript i node. Add unmatched nodes in C_B to B_0 and add unmatched nodes in $B \setminus C_B$ to B_β.

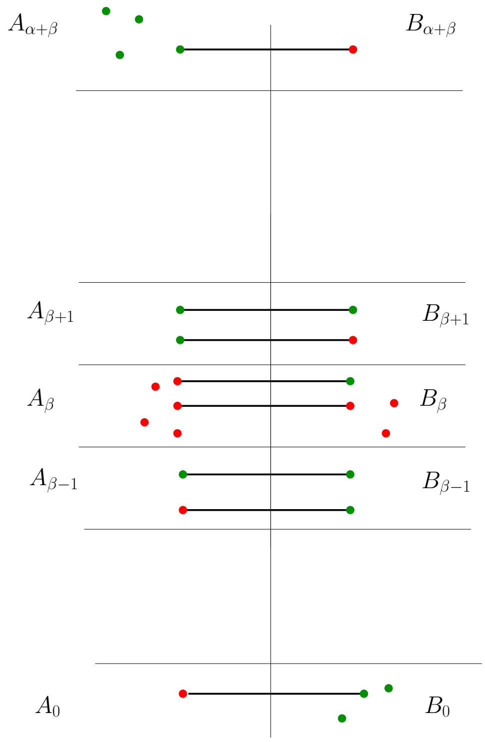

Figure 1 $A = A_0 \cup \cdots \cup A_{\alpha+\beta}$ and $B = B_0 \cup \cdots \cup B_{\alpha+\beta}$ and $M \subseteq \cup_{i=0}^{\alpha+\beta}(A_i \times B_i)$. Red nodes are outside C and green nodes are in C. All red (i.e., non-critical) nodes are in $\cup_{i \le \beta} A_i \cup_{i \ge \beta} B_i$ and unmatched red nodes are in $A_\beta \cup B_\beta$.

▶ **Lemma 10.** *M is a critical matching in G.*

The proof of Lemma 10 (given in the appendix) uses Lemma 6 and is similar to the proof that the popular maximum matching algorithm in [14] finds a maximum matching. Lemma 11 is the main technical result here.

▶ **Lemma 11.** *M is a popular critical matching in G.*

Proof. We will use Proposition 8. Let $(\vec{y}, \vec{z})$ be defined as follows.

1. Set $z_A = -2\alpha$ and $z_B = -2\beta$.
2. Set $y_u = 0$ for all unmatched nodes u. For matched nodes u, we will set y-values as follows. For $0 \le i \le \alpha+\beta$ do:
   - for $a \in A_i$: if $a \in C_A$ then set $y_a = 2\alpha + 2\beta - 2i$; else set $y_a = 2\beta - 2i$.
   - for $b \in B_i$: if $b \in C_B$ then set $y_b = 2i$; else set $y_b = 2i - 2\beta$.

▶ **Lemma 12.** *$\langle \vec{y}, \vec{z} \rangle$ defined above is a feasible solution to* (LP2).

The proof of Lemma 12 is given below (after the proof of Lemma 11). We will now show that $\sum_{u\in A\cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B) = 0$. Consider any edge $(a, b) \in M$. So there is some $i \in \{0, \ldots, \alpha+\beta\}$ such that $a \in A_i$ and $b \in B_i$.

1. If $a \notin C_A$ and $b \notin C_B$ then $y_a + y_b = (2\beta - 2i) + (2i - 2\beta) = 0$.
2. If $a \in C_A$ and $b \notin C_B$ then $y_a + y_b + z_A = (2\alpha + 2\beta - 2i) + (2i - 2\beta) - 2\alpha = 0$.
3. If $a \notin C_A$ and $b \in C_B$ then $y_a + y_b + z_B = (2\beta - 2i) + 2i - 2\beta = 0$.
4. If $a \in C_A$ and $b \in C_B$ then $y_a + y_b + z_A + z_B = (2\alpha + 2\beta - 2i) + 2i - 2\alpha - 2\beta = 0$.

Recall that k_A (resp., k_B) is the number of nodes from C_A (resp., C_B) that get matched in any critical matching. Since M is a critical matching (by Lemma 10), it matches k_A nodes from C_A and k_B nodes from C_B. So added up over all edges (a, b) in M, the left hand sides of the four equations above sum to $\sum_{u\in V} y_u + (k_A \cdot z_A) + (k_B \cdot z_B)$, where $V \subseteq A \cup B$ is the set of nodes matched in M. Since all the right hand sides are 0, this sum is 0. For any unmatched node u, we set $y_u = 0$. So $\sum_{u\in A\cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B) = 0$. Hence M is a popular critical matching in G (by Proposition 8). ◀

Proof of Lemma 12. For any node u, we claim that $y_u \geq 0$. Recall that $y_a = 2\alpha + 2\beta - 2i$ for a matched critical node $a \in A_i$ and $y_b = 2i$ for a matched critical node $b \in B_i$. Since $0 \leq i \leq \alpha + \beta$, we have $2\alpha + 2\beta - 2i \geq 0$ and $2i \geq 0$. Thus for any matched node $u \in C$, $y_u \geq 0$.

For any matched node $a \in A \setminus C_A$, observe that $a \in A_i$ for some $i \leq \beta$, so $2\beta - 2i \geq 0$. For any matched node $b \in B \setminus C_B$, observe that $b \in B_i$ for some $i \geq \beta$, so $2i - 2\beta \geq 0$. We set $y_u = 0$ for any unmatched node u. Hence $y_u \geq 0 \geq \mathsf{wt}_M(u, u)$ for all $u \in A \cup B$. Thus constraint (8) holds.

We will now show that $\langle \vec{y}, \vec{z} \rangle$ satisfies constraints (4)-(7). For any $a \in C_A$, let $y'_a = y_a + z_A$. For any $b \in C_B$, let $y'_b = y_b + z_B$. For any node $u \notin C$, let $y'_u = y_u$.

- We have $y'_a = 2\beta - 2i$ for any matched $a \in A_i$ and $y'_b = 2i - 2\beta$ for any matched $b \in B_i$.
- For any unmatched $a \in A$: $y'_a = -2\alpha$ if $a \in C_A$ and $y'_a = 0$ otherwise.
- For any unmatched $b \in B$: $y'_b = -2\beta$ if $b \in C_B$ and $y'_b = 0$ otherwise.

We will now show that $y'_a + y'_b \geq \mathsf{wt}_M(a, b)$ for all $(a, b) \in E$. Let $a \in A_i$ and $b \in B_j$. This proof is split into 4 parts: (1) $i \leq j - 1$, (2) $i = j$, (3) $i = j + 1$, and (4) $i \geq j + 2$.

1. Consider any edge (a, b) where $a \in A_i, b \in B_j$ and $i \leq j - 1$.
   - If a and b are matched nodes then $y'_a + y'_b = (2\beta - 2i) + (2j - 2\beta) = 2(j - i) \geq 2 \geq \mathsf{wt}_M(a, b)$ since $\mathsf{wt}_M(e) \in \{\pm 2, 0\}$ for all $e \in E$.
   - Suppose a is unmatched. Then $a \notin C_A$; otherwise $i = \alpha + \beta$ and so $j \geq \alpha + \beta + 1$ which is not possible. So $a \notin C_A$ and we have $y'_a = 0$ and $i = \beta$. Since $j \geq \beta + 1$, we have $y'_b = 2j - 2\beta \geq 2$. Thus $y'_a + y'_b \geq 2 \geq \mathsf{wt}_M(a, b)$.
   - Suppose b is unmatched. Then $b \notin C_B$; otherwise $j = 0$ and so $i \leq -1$ which is not possible. So $b \notin C_B$ and we have $y'_b = 0$ and $j = \beta$. Since $i \leq \beta - 1$, we have $y'_a = 2\beta - 2i \geq 2$. Thus $y'_a + y'_b \geq 2 \geq \mathsf{wt}_M(a, b)$.
2. Let $a \in A_i, b \in B_j$ where $i = j$. For any $b \in B$, within subscript i neighbors, the preference order of b' in G' is the same as b's preference order among these neighbors in G. Thus M restricted to $A_i \cup B_i$ is stable and so $\mathsf{wt}_M(a, b) \in \{-2, 0\}$.
   - If a and b are matched nodes then $y'_a + y'_b = (2\beta - 2i) + (2i - 2\beta) = 0$.
   - Suppose a is unmatched.
     - If $a \in C_A$ then $y'_a = -2\alpha$ and $i = \alpha + \beta$. So $y'_b = 2(\alpha + \beta) - 2\beta = 2\alpha$. Thus $y'_a + y'_b = -2\alpha + 2\alpha = 0$.

  - If $a \notin C_A$ then $y'_a = 0$ and $i = \beta$. The node b has to be matched since M' is stable (and thus maximal) in G'. So $y'_b = 2i - 2\beta = 0$. Thus $y'_a + y'_b = 0$.
- Suppose b is unmatched.
  - If $b \in C_B$ then $y'_b = -2\beta$ and $i = 0$. So $y'_a = 2\beta - 2i = 2\beta$. Thus $y'_a + y'_b = 2\beta - 2\beta = 0$.
  - If $b \notin C_B$ then $y'_b = 0$ and $i = \beta$. The node a has to be matched since M' is stable (and thus maximal) in G'. So $y'_a = 2\beta - 2i = 0$. Thus $y'_a + y'_b = 0$.

Thus we have $y'_a + y'_b = 0 \geq \mathsf{wt}_M(a, b)$ in all the cases.

3. Let $a \in A_i, b \in B_j$ where $i = j + 1$. Observe that $(a_j, d_{j+1}(a)) \in M'$, i.e., a_j is matched to its least preferred neighbor $d_{j+1}(a)$. The stability of M' implies that $(u_j, b') \in M'$ for some neighbor u_j that b' prefers to a_j. Also b' prefers a_{j+1} to u_j, so a_{j+1} has to prefer $M'(a_{j+1})$ to b'. Hence both a and b are matched in M to neighbors that they prefer to each other. So $\mathsf{wt}_M(a, b) = -2$. Thus $y'_a + y'_b = (2\beta - 2(j+1)) + (2j - 2\beta) = -2 = \mathsf{wt}_M(a, b)$.
4. If $a \in A_i, b \in B_j$ where $i \geq j + 2$ then $(a_{j+1}, d_{j+2}(a)) \in M'$, i.e., a_{j+1} is matched to its least preferred neighbor $d_{j+2}(a)$. This means the edge (a_{j+1}, b') blocks M' – this is because b' prefers a_{j+1} to its assignment in M': this is either a subscript j neighbor or b' is left unmatched in M'. Since the blocking edge (a_{j+1}, b') contradicts M''s stability, there is no $(a, b) \in E$ where $a \in A_i, b \in B_j$ and $i \geq j + 2$.

Thus we have $y'_a + y'_b \geq \mathsf{wt}_M(a, b)$ for all $(a, b) \in E$. This completes the proof of Lemma 12. ◀

Min-size popular critical matching. Lemma 12 showed that $(\vec{y}, \vec{z})$ is a feasible solution to (LP2). In fact, $(\vec{y}, \vec{z})$ is an optimal solution to (LP2) since $\tilde{M}$ is a feasible solution to (LP1) and $\mathsf{wt}_M(\tilde{M}) = 0 = \sum_{u \in A \cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B)$. This will be useful in Lemma 13.

▶ **Lemma 13.** *M is a min-size popular critical matching in G.*

Proof. Let N be a critical matching of size smaller than $|M|$. Then there is some node u that is matched in M but unmatched in N. So the self-loop (u, u) is in the perfect matching $\tilde{N}$. For any node u matched in M, we have $y_u > \mathsf{wt}_M(u, u)$. This is because $y_u \geq 0$ while $\mathsf{wt}_M(u, u) = -1$. So the self-loop (u, u) is *slack* with respect to the dual optimal solution $(\vec{y}, \vec{z})$. Then complementary slackness implies that $\tilde{N}$ cannot be a primal optimal solution. The optimal value of (LP1) is 0, so this means $\mathsf{wt}_M(\tilde{N}) < 0$, i.e., $\Delta(N, M) < 0$. Hence the critical matching M is more popular than N. Thus N cannot be a popular critical matching. So M is a min-size popular critical matching in G. ◀

4 Finding a max-size popular critical matching

In this section we consider the problem of finding a *max-size* popular critical matching in $G = (A \cup B, E)$ where $C \subseteq A \cup B$ is the given critical set. We will construct a new instance $G'' = (A'' \cup B'', E'')$ which will be a minor variant of the instance G' seen in Section 3. The instance G' was motivated by considering that we ran the Gale-Shapley algorithm with all nodes in level ℓ (note that $\ell = \beta$) and promoted unmatched critical nodes in A to higher levels and demoted unmatched critical nodes in B to lower levels.

The instance G'' can be motivated by considering that we will run the max-size popular matching algorithm [14] (also called the *2-level Gale-Shapley* algorithm) with all the nodes in level β. This promotes certain nodes to level $\beta + 1$; all unmatched nodes in A are in level $\beta + 1$ and all unmatched nodes in B are in level β. Now let us promote unmatched critical nodes in A to higher levels and demote unmatched critical nodes in B downwards.

The instance G''. The instance $G'' = (A'' \cup B'', E'')$ has *one* extra level compared to G'.

- For every $a \in C_A$, the set A'' has $\alpha + \beta + 2$ copies of a: call them $a_0, a_1, \ldots, a_{\alpha+\beta+1}$.
- For every $a \in A \setminus C_A$, the set A'' has $\beta + 2$ copies of a: call them $a_0, a_1, \ldots, a_{\beta+1}$.

So $A'' = \cup_{a \in C_A}\{a_0, a_1, \ldots, a_{\alpha+\beta+1}\} \cup_{a \in A \setminus C_A} \{a_0, a_1, \ldots, a_{\beta+1}\}$. The set B'' is defined as follows. $B'' = \{b' : b \in B\} \cup_{a \in C_A} \{d_1(a), \ldots, d_{\alpha+\beta+1}(a)\} \cup_{a \in A \setminus C_A} \{d_1(a), \ldots, d_{\beta+1}(a)\}$.

As before, $\{b' : b \in B\}$ is a copy of the set B; along with nodes in $\{b' : b \in B\}$, the set B'' contains $\alpha + \beta + 1$ dummy nodes $d_1(a), \ldots, d_{\alpha+\beta+1}(a)$ for $a \in C_A$ and $\beta + 1$ dummy nodes $d_1(a), \ldots, d_{\beta+1}(a)$ for $a \in A \setminus C_A$.

The edge set. Corresponding to each $(a, b) \in E$, we have the following edges in E''. As before, there are four cases depending on whether a (similarly, b) is critical or not.

1. $a \notin C_A$ and $b \notin C_B$: there are *two* edges (a_β, b') and $(a_{\beta+1}, b')$ that correspond to (a, b).
2. $a \notin C_A$ and $b \in C_B$: there are $\beta + 2$ edges (a_i, b') where $0 \le i \le \beta + 1$.
3. $a \in C_A$ and $b \notin C_B$: there are $\alpha + 2$ edges (a_i, b') where $\beta \le i \le \alpha + \beta + 1$.
4. $a \in C_A$ and $b \in C_B$: there are $\alpha + \beta + 2$ edges (a_i, b') where $0 \le i \le \alpha + \beta + 1$.

For $a \in A \setminus C_A$, the set E'' has the edges $(a_{i-1}, d_i(a))$ and $(a_i, d_i(a))$ where $1 \le i \le \beta + 1$. For $a \in C_A$, the set E'' has the edges $(a_{i-1}, d_i(a))$ and $(a_i, d_i(a))$ where $1 \le i \le \alpha + \beta + 1$. For any $i \ge 1$, the preference order of $d_i(a)$ is $a_{i-1} \succ a_i$.

Preference orders. Let a's preference order in G be $b_1 \succ \cdots \succ b_k$. Let $\{c_1, \ldots, c_r\} = \{b_1, \ldots, b_k\} \cap C$. That is, $c_1, \ldots, c_r$ are a's critical neighbors. It will be the case that only these nodes can be neighbors of $a_0, \ldots, a_{\beta-1}$. Let a's preference order among these nodes be $c_1 \succ \cdots \succ c_r$.

- a_0's preference order is $c'_1 \succ \cdots \succ c'_r \succ d_1(a)$.
- For $1 \le i \le \beta - 1$, the preference order of a_i is $d_i(a) \succ c'_1 \succ \cdots \succ c'_r \succ d_{i+1}(a)$.
- For $a \notin C_A$:
  - the preference order of a_β is $d_\beta(a) \succ b'_1 \succ \cdots \succ b'_k \succ d_{\beta+1}(a)$;
  - the preference order of $a_{\beta+1}$ is $d_{\beta+1}(a) \succ b'_1 \succ \cdots \succ b'_k$.
- For $a \in C_A$:
  - for $\beta \le i \le \alpha + \beta$, the preference order of a_i is $d_i(a) \succ b'_1 \succ \cdots \succ b'_k \succ d_{i+1}(a)$;
  - the preference order of $a_{\alpha+\beta+1}$ is $d_{\alpha+\beta+1}(a) \succ b'_1 \succ \cdots \succ b'_k$.

Consider any $b \in B$. Let its preference order in G be $a \succ \cdots \succ z$. Let b's critical neighbors be $a', \ldots, z'$ and let b's preference order among them be $a' \succ \cdots \succ z'$.

Suppose $b \notin C_B$. Then the preference order of b' is

$$\underbrace{a'_{\alpha+\beta+1} \succ \cdots \succ z'_{\alpha+\beta+1}}_{\text{level } \alpha+\beta+1 \text{ neighbors}} \succ \cdots \succ \underbrace{a'_{\beta+2} \succ \cdots \succ z'_{\beta+2}}_{\text{level } \beta+2 \text{ neighbors}} \succ \underbrace{a_{\beta+1} \succ \cdots \succ z_{\beta+1}}_{\text{level } \beta+1 \text{ neighbors}} \succ \underbrace{a_\beta \succ \cdots \succ z_\beta}_{\text{level } \beta \text{ neighbors}}$$

Note that copies of only critical neighbors are present in level i for $\beta + 2 \le i \le \alpha + \beta + 1$ and copies of all neighbors of b, i.e., $a, \ldots, z$, are present only in levels β and $\beta + 1$.

Suppose $b \in C_B$. Then the preference order of b' is

$$\underbrace{a'_{\alpha+\beta+1} \succ \cdots \succ z'_{\alpha+\beta+1}}_{\text{level } \alpha+\beta+1 \text{ neighbors}} \succ \cdots \succ \underbrace{a'_{\beta+2} \succ \cdots \succ z'_{\beta+2}}_{\text{level } \beta+2 \text{ neighbors}} \succ \underbrace{a_{\beta+1} \succ \cdots \succ z_{\beta+1}}_{\text{level } \beta+1 \text{ neighbors}} \succ \cdots \succ \underbrace{a_0 \succ \cdots \succ z_0}_{\text{level } 0 \text{ neighbors}}$$

Note that copies of only critical neighbors are present in level i for $\beta + 2 \le i \le \alpha + \beta + 1$ and copies of all neighbors of b are present in level i for $0 \le i \le \beta + 1$.

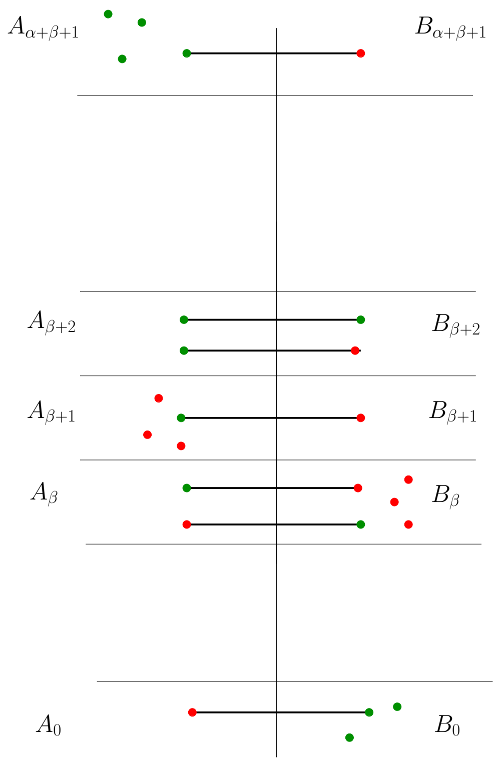

Figure 2 $A = A_0 \cup \cdots \cup A_{\alpha+\beta+1}$ and $B = B_0 \cup \cdots \cup B_{\alpha+\beta+1}$ and $M \subseteq \cup_{i=0}^{\alpha+\beta+1}(A_i \times B_i)$. Red nodes are outside C and green nodes are in C. All red (i.e., non-critical) nodes are in $\cup_{i \le \beta+1} A_i \cup_{i \ge \beta} B_i$; unmatched red nodes are in $A_{\beta+1} \cup B_\beta$.

The matching M. For any stable matching M'' in G'', define $M \subseteq E$ to be the set of edges obtained by deleting edges in M'' that are incident to dummy nodes and replacing any edge $(a_i, b') \in M''$ with the original edge $(a, b) \in E$. For each $a \in A$, the stable matching M'' matches at most one node among all a_i's to a neighbor in $\{b' : b \in B\}$ (the other a_i's have to be matched to dummy nodes). So M is a matching in G.

▶ **Theorem 14.** *For any stable matching M'' in G'', the corresponding matching M is a max-size popular critical matching in G.*

We will prove Theorem 14 by first showing that M is a critical matching (see Lemma 15), then that M is a popular critical matching (see Lemma 16), and finally that M is a max-size popular critical matching (see Lemma 19). The proof of Lemma 15 is similar to the proof of Lemma 10 and is given in the appendix.

▶ **Lemma 15.** *M is a critical matching in G.*

We will now prove that M is a popular critical matching. In order to show this, our analysis is totally analogous to our analysis in Section 3. As done there, we partition the set of all nodes in A that are matched in M into $A_0 \cup \cdots \cup A_{\alpha+\beta+1}$ where for $0 \le i \le \alpha+\beta+1$: $A_i = \{a \in A : (a_i, b') \in M''$ for some $b \in B\}$, i.e., A_i is the set of all a's in A such that a_i is matched in M'' to a neighbor in $\{b' : b \in B\}$. Add unmatched nodes in C_A to the set $A_{\alpha+\beta+1}$ and unmatched nodes in $A \setminus C_A$ to the set $A_{\beta+1}$ (see Fig. 2).

Partition the set of all nodes in B that are matched in M into $B_0 \cup \cdots \cup B_{\alpha+\beta+1}$ where for $0 \le i \le \alpha+\beta+1$: $B_i = \{b : (a_i, b') \in M''$ for some $a \in A_i\}$, i.e., b''s partner in M'' is a subscript i node. Add unmatched nodes in C_B to the set B_0 and unmatched nodes in $B \setminus C_B$ to the set B_β.

▶ **Lemma 16.** *M is a popular critical matching in G.*

Proof. We will use Proposition 8 here. Let $(\vec{y}, \vec{z})$ be defined as follows.

1. Set $z_A = -2\alpha$ and $z_B = -2\beta$. Set $y_u = 0$ for all unmatched nodes u.
2. For matched nodes u, we will set y-values as follows.
   - For $a \in A_i$: if $a \in C_A$ then set $y_a = 2\alpha + 2\beta - 2i + 1$; else set $y_a = 2\beta - 2i + 1$.
   - For $b \in B_i$: if $b \in C_B$ then set $y_b = 2i - 1$; else set $y_b = 2i - 2\beta - 1$.

▶ **Lemma 17.** *$\langle \vec{y}, \vec{z} \rangle$ defined above is a feasible solution to* (LP2).

The proof of Lemma 17 is given below. We will now show that $\sum_{u \in A\cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B) = 0$. Consider any edge $(a, b) \in M$. There is some $i \in \{0, \ldots, \alpha+\beta+1\}$ such that $a \in A_i$ and $b \in B_i$.

1. If $a \notin C_A$ and $b \notin C_B$ then $y_a + y_b = (2\beta - 2i + 1) + (2i - 2\beta - 1) = 0$.
2. If $a \in C_A$ and $b \notin C_B$ then $y_a + y_b + z_A = (2\alpha + 2\beta - 2i + 1) + (2i - 2\beta - 1) - 2\alpha = 0$.
3. If $a \notin C_A$ and $b \in C_B$ then $y_a + y_b + z_B = (2\beta - 2i + 1) + (2i - 1) - 2\beta = 0$.
4. If $a \in C_A$ and $b \in C_B$ then $y_a + y_b + z_A + z_B = (2\alpha + 2\beta - 2i + 1) + (2i - 1) - 2\alpha - 2\beta = 0$.

Recall that k_A (resp., k_B) is the number of nodes from C_A (resp., C_B) that get matched in any critical matching. Since M is a critical matching (by Lemma 15), added up over all edges (a, b) in M, the left hand sides of the four equations above sum to $\sum_{u \in V} y_u + (k_A \cdot z_A) + (k_B \cdot z_B)$, where $V \subseteq A \cup B$ is the set of nodes matched in M. Since all the right hand sides are 0, this sum is 0. For any unmatched node u, we set $y_u = 0$. Hence $\sum_{u \in A\cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B) = 0$. Thus M is a popular critical matching in G (by Proposition 8). ◀

Proof of Lemma 17. For any unmatched node u, we have $\mathsf{wt}_M(u, u) = 0$ and we set $y_u = 0$. For any matched node u, we have $\mathsf{wt}_M(u, u) = -1$ and we will now show that $y_u \ge -1$. Since $0 \le i \le \alpha+\beta+1$, we have $2\alpha + 2\beta - 2i + 1 \ge -1$ and $2i - 1 \ge -1$. Thus for any matched critical node u, we have $y_u \ge -1$.

For any matched $a \in A \setminus C_A$, observe that $a \in A_i$ for some $0 \le i \le \beta + 1$, so $y_a = 2\beta - 2i + 1 \ge -1$. For any matched $b \in B \setminus C_B$, observe that $b \in B_i$ for some $\beta \le i \le \alpha + \beta$, so $y_a = 2i - 2\beta - 1 \ge -1$. Hence $y_u \ge \mathsf{wt}_M(u, u)$ for all $u \in A \cup B$. Thus constraint (8) holds.

We will now show that $\langle \vec{y}, \vec{z} \rangle$ satisfies constraints (4)-(7). For any $a \in C_A$, let $y'_a = y_a + z_A$ and for any $b \in C_B$, let $y'_b = y_b + z_B$. For any node $u \notin C$, let $y'_u = y_u$.

- We have $y'_a = 2\beta - 2i + 1$ for any matched $a \in A$ and $y'_b = 2i - 2\beta - 1$ for any matched $b \in B$.
- For any unmatched $a \in A$: $y'_a = -2\alpha$ if $a \in C_A$ and $y'_a = 0$ otherwise.
- For any unmatched $b \in B$: $y'_b = -2\beta$ if $b \in C_B$ and $y'_b = 0$ otherwise.

We are now ready to show that $y'_a + y'_b \ge \mathsf{wt}_M(a, b)$ for all $(a, b) \in E$. Let $a \in A_i$ and $b \in B_j$. As done in the proof of Lemma 12, this proof is split into 4 parts: (1) $i \le j - 1$, (2) $i = j$, (3) $i = j + 1$, and (4) $i \ge j + 2$.

1. Consider any edge (a, b) where $a \in A_i, b \in B_j$, and $i \le j - 1$.
   - If a and b are matched nodes then $y'_a + y'_b = (2\beta - 2i + 1) + (2j - 2\beta - 1) = 2(j - i) \ge 2 \ge \mathsf{wt}_M(a, b)$.

- Suppose a is unmatched. Observe that $a \in A \setminus C_A$; otherwise $i = \alpha + \beta + 1$ and so $j \geq \alpha + \beta + 2$ which is not possible. Since $a \in A \setminus C_A$, we have $y'_a = 0$ and $i = \beta + 1$. Since $j \geq \beta + 2$, we have $y'_b = 2j - 2\beta - 1 \geq 3$. Thus $y'_a + y'_b \geq 3 > \mathsf{wt}_M(a, b)$.
- Suppose b is unmatched. Observe that $b \in B \setminus C_B$; otherwise $j = 0$ and so $i \leq -1$ which is not possible. Since $b \in B \setminus C_B$, we have $y'_b = 0$ and $j = \beta$. Since $i \leq \beta - 1$, we have $y'_a = 2\beta - 2i + 1 \geq 3$. Thus $y'_a + y'_b \geq 3 > \mathsf{wt}_M(a, b)$.

2. Consider any $(a, b) \in E$ where $a \in A_i$ and $b \in B_i$. For any $b \in B$, within subscript i neighbors, the preference order of b' in G'' is the same as b's preference order among these neighbors in G. Thus M restricted to $A_i \cup B_i$ is stable and so $\mathsf{wt}_M(a, b) \leq 0$.
- If a and b are matched nodes then $y'_a + y'_b = (2\beta - 2i + 1) + (2i - 2\beta - 1) = 0$.
- Suppose a is unmatched.
  - If $a \in C_A$ then $y'_a = -2\alpha$ and $i = \alpha + \beta + 1$. So $y'_b = 2(\alpha + \beta + 1) - 2\beta - 1 = 2\alpha + 1$. Thus $y'_a + y'_b = -2\alpha + 2\alpha + 1 = 1$.
  - If $a \notin C_A$ then $y'_a = 0$ and $i = \beta + 1$. So $y'_b = 2(\beta + 1) - 2\beta - 1 = 1$. Thus $y'_a + y'_b = 1$.
- Suppose b is unmatched.
  - If $b \in C_B$ then $y'_b = -2\beta$ and $i = 0$. So $y'_a = 2\beta + 1$. Thus $y'_a + y'_b = 2\beta + 1 - 2\beta = 1$.
  - If $b \notin C_B$ then $y'_b = 0$ and $i = \beta$. So $y'_a = 2\beta - 2\beta + 1 = 1$. Thus $y'_a + y'_b = 1$.

Thus we have $y'_a + y'_b \geq 0 \geq \mathsf{wt}_M(a, b)$ in all the cases.

3. Let $b \in B_j$ where $i = j + 1$. As argued in the proof of Lemma 12, case 3, for any edge (a, b) where $a \in A_{j+1}$ and $b \in B_j$, we have $\mathsf{wt}_M(a, b) = -2$. So both a and b are matched in M to neighbors they prefer to each other. So $y'_a + y'_b = (2\beta - 2i + 1) + (2(i - 1) - 2\beta - 1) = -2 = \mathsf{wt}_M(a, b)$.

4. There is no edge (a, b) where $b \in B_j$ and $i \geq j + 2$; otherwise (a_{j+1}, b') would block M'' as shown in the proof of Lemma 12, case 4.

Thus we have shown that $y'_a + y'_b \geq \mathsf{wt}_M(a, b)$ for all $(a, b) \in E$. This completes the proof of Lemma 17. ◀

Max-size popular critical matching. Observe that $(\vec{y}, \vec{z})$ is an optimal solution to (LP2) since $\tilde{M}$ is a feasible solution to (LP1) and $\mathsf{wt}_M(\tilde{M}) = 0 = \sum_{u \in A \cup B} y_u + (k_A \cdot z_A) + (k_B \cdot z_B)$. We will use the notation y'_v for $v \in A \cup B$ used in the proof of Lemma 17. Recall that for any $a \in C_A$, $y'_a = y_a + z_A$ and for any $b \in C_B$, $y'_b = y_b + z_B$. For any node $u \notin C$, $y'_u = y_u$. We will show the following claim below.

▷ Claim 18. For any edge (a, b) where a or b is unmatched, $y'_a + y'_b > \mathsf{wt}_M(a, b)$.

Proof. Consider any unmatched $a \in A$ and let $(a, b) \in E$. We already know from the proof of Lemma 17 that $y'_a + y'_b \geq \mathsf{wt}_M(a, b)$. Our goal now is to show that $y'_a + y'_b > \mathsf{wt}_M(a, b)$.

If $a \in C_A$ then $a \in A_{\alpha+\beta+1}$. Observe that $b \in B_{\alpha+\beta+1}$, otherwise the edge $(a_{\alpha+\beta+1}, b')$ would block M'. So $y'_a + y'_b = -2\alpha + 2\alpha + 1 = 1$. Since $\mathsf{wt}_M(a, b) \in \{0, \pm 2\}$, this means $y'_a + y'_b > \mathsf{wt}_M(a, b)$.

If $a \notin C_A$ then $a \in A_{\beta+1}$. Observe that $b \in \cup_{i \geq \beta+1} B_i$, otherwise the edge $(a_{\beta+1}, b')$ would block M'. If $b \in B_{\beta+1}$ then $y'_a + y'_b = 0 + 2(\beta + 1) - 2\beta - 1 = 1$ and so $y'_a + y'_b > \mathsf{wt}_M(a, b)$. If $b \in \cup_{i \geq \beta+2} B_i$ then $y'_a + y'_b \geq 0 + 2(\beta + 2) - 2\beta - 1 = 3 > \mathsf{wt}_M(a, b)$.

Consider any unmatched $b \in B$ and let $(a, b) \in E$. If $b \in C_B$ then $b \in B_0$. Observe that $a \in A_0$, otherwise the edge (a_0, b') would block M'. So $y'_a + y'_b = 2\beta + 1 - 2\beta = 1$. Since $\mathsf{wt}_M(a, b) \in \{0, \pm 2\}$, it follows that $y'_a + y'_b > \mathsf{wt}_M(a, b)$.

If $b \notin C_B$ then $b \in B_\beta$. Observe that $a \in \cup_{i \le \beta} A_i$, otherwise the edge (a_β, b') would block M'. If $a \in A_\beta$ then $y'_a + y'_b = 2\beta - 2\beta + 1 = 1$ and so $y'_a + y'_b > \mathsf{wt}_M(a,b)$. If $a \in \cup_{i \le \beta-1} A_i$ then $y'_a + y'_b \ge 2\beta - 2(\beta-1) + 1 = 3 > \mathsf{wt}_M(a,b)$.

Thus every edge incident to a node left unmatched in M is slack. ◁

Lemma 19 follows easily from Claim 18.

▶ **Lemma 19.** *M is a max-size popular critical matching in G.*

Proof. Consider any critical matching N in G such that $|N| > |M|$. So N has to match a node that is unmatched in M, i.e., N has to use a slack edge (by Claim 18). Since $(\vec{y}, \vec{z})$ is an optimal solution to (LP2), it follows from complementary slackness that the perfect matching $\tilde{N}$, which is a feasible solution to (LP1), cannot be an optimal solution.

The optimal value of (LP1) is 0, so this means $\mathsf{wt}_M(\tilde{N}) < 0$. In other words, $\Delta(N, M) < 0$, i.e., the critical matching M is more popular than N. Thus no critical matching larger than M can be a popular critical matching. Hence M is a max-size popular critical matching. ◀

The time taken to compute M is $O(|C|m + m)$, so the second part of Theorem 4 follows from Theorem 14. Recall that the first part of Theorem 4 was already shown in Section 3.

References

1 A. Abdulkadiroğlu and T. Sönmez. School choice: a mechanism design approach. *American Economic Review*, 93(3):729–747, 2003.

2 S. Baswana, P. P. Chakrabarti, S. Chandran, Y. Kanoria, and U. Patange. Centralized admissions for engineering colleges in India. *INFORMS Journal on Applied Analytics*, 49(5):338–354, 2019.

3 P. Biro, D. F. Manlove, and S. Mittal. Size versus stability in the marriage problem. *Theoretical Computer Science*, 411:1828–1841, 2010.

4 Canadian Resident Matching Service. How the matching algorithm works. `http://carms.ca/algorithm.htm`.

5 M.-J.-A.-N. de C. (Marquis de) Condorcet. *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix.* L'Imprimerie Royale, 1785.

6 Á. Cseh. Popular matchings. Trends in Computational Social Choice, Ulle Endriss (ed.), 2017.

7 Á. Cseh and T. Kavitha. Popular edges and dominant matchings. *Mathematical Programming*, 172(1):209–229, 2018.

8 K. Eriksson and O. Häggström. Instability of matchings in decentralized markets with various preference orders. *Mathematical Programming*, 36(3-4):409–420, 2008.

9 D. Gale and L.S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69(1):9–15, 1962.

10 D. Gale and M. Sotomayor. Some remarks on the stable matching problem. *Discrete Applied Mathematics*, 11(3):223–232, 1985.

11 P. Gärdenfors. Match making: assignments based on bilateral preferences. *Behavioural Science*, 20:166–173, 1975.

12 M. Hirakawa, Y. Yamauchi, S. Kijima, and M. Yamashita. On the structure of popular matchings in the stable marriage problem: Who can join a popular matching? In the 3rd International Workshop on Matching Under Preferences (MATCH-UP), 2015.

13 C.-C. Huang and T. Kavitha. Popular matchings in the stable marriage problem. *Information and Computation*, 222:180–194, 2013.

14 T. Kavitha. A size-popularity tradeoff in the stable marriage problem. *SIAM Journal on Computing*, 43(1):52–71, 2014.

15 T. Kavitha. Popular half-integral matchings. In *Proceedings of the 43rd International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 22:1–22:13, 2016.

16 T. Kavitha. Maximum matchings and popularity. In *Proceedings of the 48th International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 85:1–85:21, 2021.

17 C. Mathieu. Stable matching in practice. In the 26th Annual European Symposium on Algorithms (ESA), Keynote talk, 2018.

18 S. Merrill and B. Grofman. *A unified theory of voting: directional and proximity spatial models*. Cambridge University Press, 1999.

19 M. Nasre and P. Nimbhorkar. Popular matchings with lower quotas. In *Proceedings of the 37th Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, pages 44:1–44:15, 2017.

20 M. Nasre, P. Nimbhorkar, K. Ranjan, and A. Sarkar. Popular matchings in the hospitals-residents problem with two-sided lower quotas. In *Proceedings of the 41st Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, 2021.

21 National Resident Matching Program. Why the Match? `http://www.nrmp.org/whythematch.pdf`.

22 P. A. Robards. Applying the two-sided matching processes to the United States Navy enlisted assignment process. Master's Thesis, Naval Postgraduate School, Monterey, Canada, 2001.

23 A. E. Roth and X. Xing. Turnaround time and bottlenecks in market clearing: Decentralized matching in the market for clinical psychologists. *Journal of Political Economy*, 105(2):284–329, 1997.

24 M. Soldner. Optimization and measurement in humanitarian operations: Addressing practical needs. PhD thesis, Georgia Institute of Technology, 2014.

25 A.C. Trapp, A. Teytelboym, A. Martinello, T. Andersson, and N. Ahani. Placement optimization in refugee resettlement. Working paper, 2018.

26 W. Yang, J. A. Giampapa, and K. Sycara. Two-sided matching for the US Navy detailing process with market complication. Technical Report CMU-R1-TR-03-49, Robotics Institute, Carnegie Mellon University, 2003.

A Appendix: Missing Proofs

Before we prove Lemma 10, it will be useful to prove the following simple observation.

▶ **Observation 20.** *For any critical node left unmatched in M, all its neighbors are in $A_0 \cup B_{\alpha+\beta}$.*

Proof. If $a \in C_A$ is unmatched in M then $a_{\alpha+\beta}$ has to be unmatched in M'. This is because for $0 \le i \le \alpha+\beta-1$, the node a_i is $d_{i+1}(a)$'s top choice neighbor, hence the stable matching M' has to match a_i. If a has a neighbor b in B_i for $i \le \alpha+\beta-1$ then the edge $(a_{\alpha+\beta}, b')$ blocks M', a contradiction to its stability in G'. Thus $b \in B_{\alpha+\beta}$.

Suppose $b \in C_B$ is unmatched in M and b has a neighbor a in A_i for $i \ge 1$. This means $(a_0, d_1(a))$ is in M'. Recall that $d_1(a)$ is a_0's least preferred neighbor. So the edge (a_0, b') blocks M', a contradiction to its stability in G'. Thus $a \in A_0$. ◁

Proof of Lemma 10. We will show there is no alternating path p with respect to M such that (i) p is an augmenting path wrt M and at least one endpoint of p is in C or (ii) p has even length with exactly one endpoint in C and this node is left unmatched in M. Then it follows from Lemma 6 that M is a critical matching in G.

We will first show there is no augmenting path p wrt M with an endpoint in C_B. It follows from the definition of sets A_i and B_i that $M \subseteq \cup_{i=0}^{\alpha+\beta}(A_i \times B_i)$. An important property here is that there is no edge in $A_i \times B_j$ where $i \ge j+2$. See the proof of Lemma 12, case 4 which shows that such an edge contradicts the stability of M' in G'.

The path p starts in B_0 at an unmatched node $b \in C_B$ and all of b's neighbors are in A_0 (by Observation 20). The matched partners of b's neighbors are in B_0. The node after this can be in A_1 and its partner is in B_1 and so on. So the shortest alternating path from an unmatched $b \in B_0$ to an unmatched $a \in A$ (such a node is in $A_\beta \cup A_{\alpha+\beta}$) moves across sets as follows: [here $(A_i - B_i)$ refers to a matching edge in $A_i \times B_i$]

$$B_0 - (A_0 - B_0) - (A_1 - B_1) - (A_2 - B_2) - \cdots - (A_{\beta-1} - B_{\beta-1}) - \cdots$$

Since all nodes in sets B_i for $0 \le i \le \beta - 1$ are in C_B, this implies there are at least $\beta + 1$ nodes of C_B in p. However $|C_B| = \beta$. So there is no such augmenting path p with respect to M.

The same argument shows that the shortest even length alternating path p with an unmatched node in C_B as one endpoint and any node in $B \setminus C_B$ (such a node is in $\cup_{i \ge \beta} B_i$) as another endpoint needs to have at least $\beta + 1$ nodes of C_B in it. However $|C_B| = \beta$. So there is no such alternating path p with respect to M.

We will now show there is no augmenting path p wrt M with an endpoint in C_A. An argument analogous to the one given above shows that the shortest alternating path from an unmatched $a \in A_{\alpha+\beta}$ to an unmatched node $b \in B$ (such a node is in $B_\beta \cup B_0$) moves across sets as follows: [here $(B_i - A_i)$ refers to a matching edge in $B_i \times A_i$]

$$A_{\alpha+\beta} - (B_{\alpha+\beta} - A_{\alpha+\beta}) - (B_{\alpha+\beta-1} - A_{\alpha+\beta-1}) - \cdots - (B_{\beta+1} - A_{\beta+1}) - \cdots$$

Since all nodes in levels A_i for $\beta + 1 \le i \le \alpha + \beta$ are in C_A, this implies there are at least $\alpha + 1$ nodes of C_A in p. However $|C_A| = \alpha$. So there is no such augmenting path p with respect to M.

The same argument shows that the shortest even length alternating path p with an unmatched node in C_A as one endpoint and any node in $A \setminus C_A$ (such a node is in $\cup_{i \le \beta} A_i$) as another endpoint needs to have at least $\alpha + 1$ nodes of C_A in it. However $|C_A| = \alpha$. So there is no such alternating path p with respect to M.

Thus there is no forbidden alternating path p (as given in Lemma 6) with respect to M. Hence M is a critical matching. ◀

Proof of Lemma 15. We will use Lemma 6 to show that M is a critical matching. We will show there is no alternating path p with respect to M such that: (i) p is an augmenting path wrt M and at least one endpoint of p is in C or (ii) p has even length with exactly one endpoint in C and this node is left unmatched in M.

We will first show there is no augmenting path p wrt M with an endpoint in C_B. Every unmatched node in C_B is in B_0 and its neighbors are in A_0 (analogous to Observation 20).

It follows from the definitions of A_i and B_i that $M \subseteq \cup_{i=0}^{\alpha+\beta+1}(A_i \times B_i)$. Moreover there is no edge in $A_i \times B_j$ where $i \ge j + 2$; otherwise the edge (a_{j+1}, b') would block M''.

Thus the path p starts in B_0 at an unmatched node $b \in C_B$ and the next node is in A_0. The matched partners of b's neighbors are in B_0. The node after this can be in A_1 and its partner is in B_1 and so on. So the shortest alternating path between an unmatched node $b \in B_0$ and an unmatched node $a \in A$ (such a node is in $A_{\beta+1} \cup A_{\alpha+\beta+1}$) moves across sets as follows (see Fig. 2):

$$B_0 - (A_0 - B_0) - (A_1 - B_1) - (A_2 - B_2) - \cdots - (A_{\beta-1} - B_{\beta-1}) - \cdots$$

Since all nodes in levels B_i for $0 \le i \le \beta - 1$ are in C_B, this implies there are at least $\beta + 1$ nodes of C_B in p. However $|C_B| = \beta$. So there is no such augmenting path p with respect to M.

The same argument shows that the shortest even length alternating path p with an unmatched node in C_B (such a node is in B_0) as one endpoint and any node in $B \setminus C_B$ (such a node is in $\cup_{i \geq \beta} B_i$) as another endpoint needs to have at least $\beta + 1$ nodes of C_B in it. However $|C_B| = \beta$. So there is no such alternating path p with respect to M.

We will now show there is no augmenting path p wrt M with an endpoint in C_A. An argument analogous to the one given above shows that the shortest alternating path from an unmatched $a \in C_A$ (note that $a \in A_{\alpha+\beta+1}$) to an unmatched node in B (such a node is in $B_\beta \cup B_0$) moves across sets as follows (see Fig. 2):

$$A_{\alpha+\beta+1} - (B_{\alpha+\beta+1} - A_{\alpha+\beta+1}) - (B_{\alpha+\beta} - A_{\alpha+\beta}) - (B_{\alpha+\beta-1} - A_{\alpha+\beta-1}) - \cdots - (B_{\beta+2} - A_{\beta+2}) - \cdots$$

Since all nodes in levels A_i for $\beta + 2 \leq i \leq \alpha + \beta + 1$ are in C_A, this implies there are at least $\alpha + 1$ nodes of C_A in p. However $|C_A| = \alpha$. So there is no such augmenting path p with respect to M.

The same argument shows that the shortest even length alternating path p with an unmatched node in C_A (such a node is in $A_{\alpha+\beta+1}$) as one endpoint and any node in $A \setminus C_A$ (such a node is in $\cup_{i \leq \beta+1} A_i$) as another endpoint needs to have at least $\alpha + 1$ nodes of C_A in it. However $|C_A| = \alpha$. So there is no such alternating path p with respect to M.

Thus there is no forbidden alternating path p (as given in Lemma 6) with respect to M. Hence M is a critical matching in G. ◀