An Approximation Algorithm for Maximum Stable Matching with Ties and Constraints

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Abstract

We present a polynomial-time $\frac{3}{2}$-approximation algorithm for the problem of finding a maximum-cardinality stable matching in a many-to-many matching model with ties and laminar constraints on both sides. We formulate our problem using a bipartite multigraph whose vertices are called workers and firms, and edges are called contracts. Our algorithm is described as the computation of a stable matching in an auxiliary instance, in which each contract is replaced with three of its copies and all agents have strict preferences on the copied contracts. The construction of this auxiliary instance is symmetric for the two sides, which facilitates a simple symmetric analysis. We use the notion of matroid-kernel for computation in the auxiliary instance and exploit the base-orderability of laminar matroids to show the approximation ratio.

In a special case in which each worker is assigned at most one contract and each firm has a strict preference, our algorithm defines a $\frac{3}{2}$-approximation mechanism that is strategy-proof for workers.

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1 Introduction

The *college admission problem* (CA) is a many-to-one generalization of the well-known *stable marriage problem* [18,32,34], introduced by Gale and Shapley [16]. An instance of CA involves two disjoint agent sets called students and colleges. Each agent has a strict linear order of preference over agents on the opposite side, and each college has an upper quota for the number of assigned students. It is known that any instance of CA has a stable matching, we can find it efficiently, and all stable matchings have the same cardinality.

Recently, matching problems with constraints have been studied extensively [6,9,15,27,28]. Motivated by the matching scheme used in the higher education sector in Hungary, Biró et al. [4] studied CA *with common quotas*. In this problem, in addition to individual colleges, certain subsets of colleges, called *bounded sets*, have upper quotas. Such constraints are also called *regional caps* or *distributional constraints*, and they have been studied in [17,29]. Meanwhile, motivated by academic hiring, Huang [21] introduced the *classified stable matching problem*. This is an extension of CA in which each individual college has quotas for subsets of students, called *classes*. Its many-to-many generalizations have been studied in [14,44].[1]

[1] In [14,17,21,44], not only upper quotas but also lower quotas are considered. With lower quotas, the existence of stable matching is not guaranteed. In this paper, we consider only upper quotas.

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For these models, the laminar structure of constraints is commonly found to be the key to the existence of a stable matching. A family $\mathcal{L}$ of sets is called *laminar* if any $L, L' \in \mathcal{L}$ satisfy $L \subseteq L'$ or $L \supseteq L'$ or $L \cap L' = \emptyset$ (also called *nested* or *hierarchical*). In [4, 21], the authors showed that a stable matching exists in their models if regions or classes form laminar families, whereas the existence is not guaranteed in the general case. Furthermore, in the laminar case, a stable matching can be found efficiently, and all stable matchings have the same cardinality. Applications with laminar constraints have been discussed in [29].

The purpose of this paper is to introduce ties to a matching model with laminar constraints. In the previous studies described above, the preferences of agents were assumed to be strictly ordered. However, ties naturally arise in real problems. Matching models with ties have been studied widely in the literature [18, 23, 34], where the preference of an agent is said to contain a *tie* if she is indifferent between two or more agents on the opposite side. When ties are allowed, the existence of a stable matching is maintained; however, stable matchings vary in cardinalities. As it is desirable to produce a large matching in practical applications, we consider the problem of finding a maximum-cardinality stable matching.

Such a problem is known to be difficult even in the simple matching model without constraints. The problem of finding a maximum stable matching in the setting of *stable marriage with ties and incomplete lists*, called MAX-SMTI, is NP-hard [24, 35], as is obtaining an approximation ratio within $\frac{33}{29}$ [43]. For its approximability, several algorithms with improved approximation ratios have been proposed [25, 26, 30, 31, 36, 38]. The current best ratio is $\frac{3}{2}$ by a polynomial-time algorithm proposed by McDermid [36] as well as linear-time algorithms proposed by Paluch [38] and Király [31]. The $\frac{3}{2}$-approximability extends to the settings of CA with ties [31] and the student-project allocation problem with ties [8].

Our Contribution. We present a polynomial-time $\frac{3}{2}$-approximation algorithm for the problem of finding a maximum-cardinality stable matching in a many-to-many matching model with ties and laminar constraints on both sides. We call this problem MAX-SMTI-LC and formulate it using a bipartite multigraph, where we call the two vertex sets *workers* and *firms*, respectively, and each edge a *contract*. Each agent has upper quotas on a laminar family defined on incident contracts. Our formulation can deal with each agent's constraints, such as *classified stable matching*. Furthermore, distributional constraints such as CA *with common quotas* can be handled by considering a dummy agent that represents a consortium of the agents on one side (see the remark at the end of Section 2). Our algorithm runs in $O(k \cdot |E|^2)$ time, where E is the set of contracts and k is the maximum level of nesting of laminar constraints. The *level of nesting* of a laminar family $\mathcal{L}$ is the maximum length of a chain $L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_k$ of members of $\mathcal{L}$; hence, $k \leq |E|$.

Our algorithm is described as the computation of a stable matching in an auxiliary instance. Here, we explain the ideas underlying the construction of the auxiliary instance, which is inspired by the algorithms of Király [31] and Hamada, Miyazaki, and Yanagisawa [19].

First, we briefly explain Király's $\frac{3}{2}$-approximation algorithm for MAX-SMTI [31]. In this algorithm, each worker makes proposals from top to bottom in her list sequentially, as with the worker-oriented Gale–Shapley algorithm. A worker rejected by all firms is given a second chance for proposals. Each firm prioritizes a worker in the second cycle over a worker in the first cycle if they are tied in its preference list. This idea of *promotion* is used to handle ties in firms' preference lists. To handle ties in workers' lists, Király's algorithm lets each worker prioritize a currently unmatched firm over a currently matched firm if they are tied in her preference list. This priority rule depends on the states of firms at each moment, which makes the algorithm complicated when we introduce constraints on both sides.

Then, we introduce the idea of the algorithm of Hamada et al. [19], who proposed a worker-strategy-proof algorithm for MAX-SMTI that attains the $\frac{3}{2}$-approximation ratio when ties appear only in workers' lists. They modified Király's algorithm such that each worker's proposal order is predetermined and is not affected by the history of the algorithm. Their algorithm can be seen as a Gale–Shapley-type algorithm in which each worker makes proposals twice to each firm in a tie before proceeding to the next tie, and each firm prioritizes second proposals over first proposals regardless of its preference. By combining their algorithm with the promotion operation of Király's algorithm, we obtain a Gale–Shapley-type algorithm in which each worker makes at most three proposals to each firm.

Based on these observations, we propose a method for transforming a MAX-SMTI-LC instance I into an auxiliary instance I^*, which is also a MAX-SMTI-LC instance. Each contract e_i in I is replaced with three copies x_i, y_i, z_i in I^*. Each agent has a strict preference on the copied contracts, which reflects the priority rules in the algorithms of Király and Hamada et al. The instance I^* has an upper bound 1 for each triple $\{x_i, y_i, z_i\}$ and also has constraints corresponding to those in I. The construction of I^* is completely symmetric for workers and firms. We show that, for any stable matching M^* of I^*, its projection $M := \{\, e_i \mid \{x_i, y_i, z_i\} \cap M^* \neq \emptyset \,\}$ is a $\frac{3}{2}$-approximate solution for I. Both the stability and the approximation ratio of M are implied by the stability of M^* in I^*, and the process of computing M^* is irrelevant. Thus, our method enables us to conduct a symmetric and static analysis even with constraints.

Because the auxiliary instance I^* has no ties, we can find a stable matching of I^* efficiently by using the matroid framework of Fleiner [12, 13]. In the analysis of the approximation ratio, we exploit the fact that the family of feasible sets defined by laminar constraints forms a matroid with a property called *base-orderability*.

In the last section, we show that the result of Hamada et al. [19] mentioned above is generalized to a many-to-one matching setting with laminar constraints on the firm side. In other words, if we restrict MAX-SMTI-LC such that each worker is assigned at most one contract and each firm has a strict preference, then we can provide a worker-strategy-proof mechanism that returns a $\frac{3}{2}$-approximate solution. We obtain this conclusion using the strategy-proofness result of Hatfield and Milgrom [20].

Paper Organization. The remainder of this paper is organized as follows. Section 2 formulates our matching model, while Section 3 describes our algorithm. Section 4 presents a lemma on base-orderable matroids that is the key to our proof of the approximation ratio. Sections 5 and 6 are devoted to the proofs of correctness and time complexity, respectively. Section 7 investigates strategy-proof approximation mechanisms for our model.

Throughout the paper, we denote the set of non-negative integers by $\mathbf{Z}_+$. For a subset $S \subseteq E$ and an element $e \in E$, we denote $S + e := S \cup \{e\}$ and $S - e := S \setminus \{e\}$.

2 Problem Formulation

An instance of the *stable matching with ties and laminar constraints*, which we call SMTI-LC, is a tuple $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$ defined as follows. Let W and F be disjoint finite sets called *workers* and *firms*, respectively. We call $a \in W \cup F$ an *agent* when we do not distinguish between workers and firms. We are provided a set E of *contracts*. Each contract $e \in E$ is associated with one worker and one firm, denoted by $\partial_W(e)$ and $\partial_F(e)$, respectively. Multiple contracts are allowed to exist between a worker–firm pair. Then, $(W, F; E)$ is represented as a bipartite multigraph in which W and F are vertex sets, and each $e \in E$ is an edge connecting $\partial_W(e)$ and $\partial_F(e)$. For each $a \in W \cup F$, we denote the set of associated contracts by E_a, i.e.,

$$E_w \coloneqq \{\, e \in E \mid \partial_W(e) = w \,\} \quad (w \in W), \qquad E_f \coloneqq \{\, e \in E \mid \partial_F(e) = f \,\} \quad (f \in F).$$

Then, the family $\{\, E_w \mid w \in W \,\}$ forms a partition of E, as does $\{\, E_f \mid f \in F \,\}$.

Each agent $a \in W \cup F$ has a laminar family $\mathcal{L}_a$ of subsets of E_a and a quota function $q_a \colon \mathcal{L}_a \to \mathbf{Z}_+$. For any subset $M \subseteq E$ of contracts and an agent $a \in W \cup F$, we denote by $M_a \coloneqq M \cap E_a$ the set of contracts assigned to a. We say that M is *feasible* for $a \in W \cup F$ if

$$\forall L \in \mathcal{L}_a : |M_a \cap L| \leq q_a(L).$$

A set $M \subseteq E$ is called a *matching* if it is feasible for all agents in $W \cup F$.

Each agent $a \in W \cup F$ has a preference list P_a that consists of all elements in E_a and may contain ties. In this paper, a preference list is written in one row, from left to right according to preference, where two or more contracts with equal preference are included in the same parentheses. For example, if the preference list P_a of an agent $a \in W \cup F$ is represented as

$$P_a : \; e_2 \;\; (\; e_1 \;\; e_4 \;) \;\; e_3,$$

then e_2 is a's top choice, e_1 and e_4 are the second choices with equal preference, and e_3 is the last choice. For contracts $e, e' \in E_a$, we write $e \succ_a e'$ if a prefers e to e'. Furthermore, we write $e \succeq_a e'$ if $e \succ_a e'$ or a is indifferent between e and e' (including the case $e = e'$).

For a matching $M \subseteq E$, a contract $e \in E \setminus M$, and an associated agent $a \in \{\partial_W(e), \partial_F(e)\}$, we say that e is *free for a* in M if

- $M_a + e$ is feasible for a, or
- there is $e' \in M_a$ such that $e \succ_a e'$ and $M_a + e - e'$ is feasible for a.

In other words, a contract e is free for an agent a if a has an incentive to add e to the current assignment possibly at the expense of some less preferred contract e'. A contract $e \in E \setminus M$ *blocks* M if e is free for both $\partial_W(e)$ and $\partial_F(e)$. A matching M is *stable* if there is no contract in $E \setminus M$ that blocks M.

The goal of our problem MAX-SMTI-LC is to find a maximum-cardinality stable matching for a given SMTI-LC instance. Because MAX-SMTI-LC is a generalization of the NP-hard problem MAX-SMTI, we consider the approximability. Similarly to the case of MAX-SMTI, for the problem MAX-SMTI-LC, a 2-approximate solution can be easily obtained using an arbitrary tie-breaking method (see the full version [45] for the proof). In the next section, we present a $\frac{3}{2}$-approximation algorithm.

▶ Remark. We demonstrate that SMTI-LC includes several models investigated in previous works, which implies that our algorithm finds $\frac{3}{2}$-approximate solutions for the problems of finding maximum-cardinality stable matchings in those models with ties.

First, SMTI and the stable b-matching problem are special cases such that $E \subseteq W \times F$ and $\mathcal{L}_a = \{E_a\}$ for every $a \in W \cup F$. Furthermore, the two-sided laminar classified stable matching problem [14, 21], if lower quotas are absent, is a special case with $E \subseteq W \times F$.

To represent CA with laminar common quotas [4], let W be the set of students and let $F \coloneqq \{f\}$, where f is regarded as a consortium of all colleges in C. The set of contracts is defined by $E \coloneqq \{\, (w, f, c) \mid \text{a college } c \in C \text{ is acceptable for a student } w \in W \,\}$, where $\partial_W(e) = w$, $\partial_F(e) = f$ for any $e = (w, f, c)$. Note that $E = E_f$. A quota for a region $C' \subseteq C$ is then represented as a quota for the set $\{\, (w, f, c) \in E \mid c \in C' \,\} \subseteq E_f$. Thus, laminar common quotas can be represented as constraints on a laminar family on E_f.

For the student-project allocation problem [8], let W and F be the sets of students and lecturers, respectively, and $E \coloneqq \{(w, f, p) \mid \text{a project } p \text{ acceptable for } w \in W \text{ is offered by } f \in F\}$. Let $E_{f,p} \subseteq E_f$ be the set of contracts associated with a project p offered by a lecturer f. Then, the lecturer's upper quota and projects' upper quotas define two-level laminar constraints on the family $\mathcal{L}_f = \{E_f\} \cup \{\, E_{f,p} \mid p \text{ is offered by } f \,\}$.

For the above-mentioned settings, we can appropriately set the preferences of agents such that the stability in the previous works coincides with the stability in SMTI-LC.

3 Algorithm

Our approximation algorithm for MAX-SMTI-LC consists of three steps: (i) construction of an auxiliary instance, (ii) computation of any stable matching of this auxiliary instance, and (iii) mapping the obtained matching to a matching of the original instance. In what follows, we describe how to construct an auxiliary instance I^* from a given instance I and how to map a matching of I^* to that of I.

Let $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$ be an instance of MAX-SMTI-LC, where the set E of contracts is represented as $E = \{\, e_i \mid i = 1, 2, \ldots, n \,\}$. We construct an auxiliary instance $I^* = (W, F, E^*, \{\mathcal{L}^*_a, q^*_a, P^*_a\}_{a \in W \cup F})$, which is also an SMTI-LC instance; however, each preference list P^*_a does not contain ties.

The sets of workers and firms in I^* are the same as those in I. The set E^* of contracts in I^* is given as $E^* = \{\, x_i, y_i, z_i \mid i = 1, 2, \ldots, n \,\}$, where x_i, y_i, and z_i are copies of e_i; hence, $\partial_W(x_i) = \partial_W(y_i) = \partial_W(z_i) = \partial_W(e_i)$ and $\partial_F(x_i) = \partial_F(y_i) = \partial_F(z_i) = \partial_F(e_i)$. We define a mapping $\pi : 2^{E^*} \to 2^E$ by $\pi(S^*) = \{\, e_i \mid \{x_i, y_i, z_i\} \cap S^* \neq \emptyset \,\}$ for any $S^* \subseteq E^*$.

For any agent $a \in W \cup F$, the laminar family $\mathcal{L}^*_a$ and the quota function $q^*_a : \mathcal{L}^*_a \to \mathbf{Z}_+$ are defined as follows. For each $e_i \in E_a$, we have $\{x_i, y_i, z_i\} \in \mathcal{L}^*_a$ and $q^*_a(\{x_i, y_i, z_i\}) = 1$. For each $L \in \mathcal{L}_a$, we have $L^* \coloneqq \{\, x_i, y_i, z_i \mid e_i \in L \,\} \in \mathcal{L}^*_a$ and $q^*_a(L^*) = q_a(L)$. These are all that $\mathcal{L}^*_a$ contains. Then, for any set $M^* \subseteq E^*$ of contracts, we see that M^* is feasible for a in I^* if and only if M^* contains at most one copy of each $e_i \in E_a$ and the set $\pi(M^*)$ is feasible for a in I.

The preference list P^*_w of each worker $w \in W$ is defined as follows. Take a tie $(e_{i_1} e_{i_2} \cdots e_{i_\ell})$ in P_w. We replace it with a strict linear order of 2ℓ contracts $x_{i_1} x_{i_2} \cdots x_{i_\ell} y_{i_1} y_{i_2} \cdots y_{i_\ell}$. Apply this operation to all the ties in P_w, where we regard a contract not included in any tie as a tie of length one. Next, at the end of the resultant list, append the original list P_w with each e_i replaced with z_i and all the parentheses omitted. Here is a demonstration. If the preference list of a worker w is

$$P_w : \ (\ e_2 \ \ e_6\ ) \ \ e_1 \ \ (\ e_3 \ \ e_4\ ),$$

then her list in I^* is

$$P^*_w : \ \ x_2\ x_6\ y_2\ y_6\ x_1\ y_1\ x_3\ x_4\ y_3\ y_4\ z_2\ z_6\ z_1\ z_3\ z_4.$$

The preference list P^*_f of each firm $f \in F$ is defined in the same manner, where the roles of x_i and z_i are interchanged. For example, if the preference list of a firm f is

$$P_f : \ \ e_3 \ \ (\ e_2 \ \ e_4 \ \ e_7\ ) \ \ e_5,$$

then its list in I^* is

$$P^*_f : \ \ z_3\ y_3\ z_2\ z_4\ z_7\ y_2\ y_4\ y_7\ z_5\ y_5\ x_3\ x_2\ x_4\ x_7\ x_5.$$

Thus, we have defined the auxiliary instance I^*. As this is again an SMTI-LC instance, a stable matching of I^* is defined as before. The existence of a stable matching of I^* is guaranteed by the existing framework of Fleiner [12, 13], as will be explained in Section 6. Here is the main theorem of this paper, which states that any stable matching of I^* defines a $\frac{3}{2}$-approximate solution for I.

▶ **Theorem 1.** *For a stable matching M^* of I^*, let $M := \pi(M^*)$. Then, M is a stable matching of I with $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$, where M_{OPT} is a maximum-cardinality stable matching of I.*

We prove Theorem 1 in Section 5. This theorem guarantees the correctness of Algorithm 1.

Algorithm 1 $\frac{3}{2}$-approximation algorithm for MAX-SMTI-LC.

Input: An instance $I = (W, F, E, \{\mathcal{L}_a, q_a, P_a\}_{a \in W \cup F})$.
Output: A stable matching M with $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$, where M_{OPT} is an optimal solution.
1: Construct an auxiliary instance I^*.
2: Find any stable matching M^* of I^*.
3: Let $M = \pi(M^*)$ and return M.

Clearly, the first and third steps of Algorithm 1 can be performed efficiently. Furthermore, the second step can be executed in polynomial time by applying the generalized Gale–Shapley algorithm of Fleiner [12, 13]. In Section 6, we will explain this more precisely and present the time complexity represented in the following theorem.

▶ **Theorem 2.** *One can find a stable matching M of I with $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$ in $O(k \cdot |E|^2)$ time, where M_{OPT} is a maximum-cardinality stable matching and k is the maximum level of nesting of laminar families $\mathcal{L}_a$ $(a \in W \cup F)$.*

4 Base-orderable Matroids

For the proofs of Theorems 1 and 2, we introduce some concepts related to matroids (see, e.g., Oxley [37] for more information on matroids).

For a finite set E and a family $\mathcal{I} \subseteq 2^E$, a pair $(E, \mathcal{I})$ is called a *matroid* if the following three conditions hold: (I1) $\emptyset \in \mathcal{I}$, (I2) $S \subseteq T \in \mathcal{I}$ implies $S \in \mathcal{I}$, and (I3) for any $S, T \in \mathcal{I}$ with $|S| < |T|$, there exists $e \in T \setminus S$ such that $S + e \in \mathcal{I}$.

For a matroid $(E, \mathcal{I})$, each member of $\mathcal{I}$ is called an *independent set*. An independent set is called a *base* if it is inclusion-wise maximal in $\mathcal{I}$. We denote the family of all bases by $\mathcal{B}$. By the matroid axiom (I3), it follows that $|B_1| = |B_2|$ holds for any bases $B_1, B_2 \in \mathcal{B}$.

▶ **Definition 3** (Base-orderable Matroid)**.** *A matroid $(E, \mathcal{I})$ is called* base-orderable *if for any two bases $B_1, B_2 \in \mathcal{B}$, there exists a bijection $\varphi\colon B_1 \to B_2$ with the property that, for every $e \in B_1$, both $B_1 - e + \varphi(e)$ and $B_2 + e - \varphi(e)$ are bases.*

A class of base-orderable matroids includes *gammoids* (see [7] and [42, Theorem 42.12]), and gammoids include laminar matroids described below (see [10] and [11, Section 2.3.1]).

▶ **Example 4** (Laminar Matroid)**.** For a laminar family $\mathcal{L}$ on E and a function $q\colon \mathcal{L} \to \mathbf{Z}_+$, define $\mathcal{I} = \{\, S \subseteq E \mid \forall L \in \mathcal{L} : |S \cap L| \leq q(L) \,\}$. Then, $(E, \mathcal{I})$ is a base-orderable matroid.

A matroid is *laminar* if it can be defined in the above-mentioned manner for some $\mathcal{L}$ and q.

Base-orderability is known to be closed under the following operations (see, e.g., [5, 22]).

Contraction.[2] For a matroid $(E,\mathcal{I})$ and any $S \in \mathcal{I}$, define $\mathcal{I}_S := \{\, T \subseteq E \setminus S \mid S \cup T \in \mathcal{I} \,\}$. Then, $(E \setminus S, \mathcal{I}_S)$ is a matroid. If $(E,\mathcal{I})$ is base-orderable, then so is $(E \setminus S, \mathcal{I}_S)$.

Truncation. For a matroid $(E,\mathcal{I})$ and any integer $p \in \mathbf{Z}_+$, define $\mathcal{I}_p := \{\, S \in \mathcal{I} \mid |S| \leq p \,\}$. Then, $(E,\mathcal{I}_p)$ is a matroid. If $(E,\mathcal{I})$ is base-orderable, then so is $(E,\mathcal{I}_p)$.

Direct Sum. For matroids $(E_j,\mathcal{I}_j)$ $(j = 1, 2, \dots, \ell)$ such that E_j are all pairwise disjoint, let $E := E_1 \cup E_2 \cup \cdots \cup E_\ell$ and $\mathcal{I} := \{\, S_1 \cup S_2 \cup \cdots \cup S_\ell \mid S_j \in \mathcal{I}_j \ (j = 1, 2, \dots, \ell) \,\}$. Then, $(E,\mathcal{I})$ is a matroid. If all $(E_j,\mathcal{I}_j)$ are base-orderable, then so is $(E,\mathcal{I})$.

On the intersection of two base-orderable matroids, we show the following property, which plays a key role in proving the $\frac{3}{2}$-approximation ratio of our algorithm. This generalizes the fact that, if (one-to-one) bipartite matchings M and N satisfy $|M| < \frac{2}{3}|N|$, then $M \triangle N$ contains a connected component that forms an alternating path of length at most three.

▶ **Lemma 5.** *For base-orderable matroids $(E,\mathcal{I}_1)$ and $(E,\mathcal{I}_2)$, suppose that $S, T \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|S| < \frac{2}{3}|T|$. If $S + e \notin \mathcal{I}_1 \cap \mathcal{I}_2$ for every $e \in T \setminus S$, then there exist distinct elements e_i, e_j, e_k such that $e_i, e_k \in T \setminus S$, $e_j \in S \setminus T$, and the following conditions hold:*

- $S + e_i \in \mathcal{I}_1$,
- *both* $S + e_i - e_j$ *and* $T - e_i + e_j$ *belong to* $\mathcal{I}_2$,
- *both* $S - e_j + e_k$ *and* $T + e_j - e_k$ *belong to* $\mathcal{I}_1$,
- $S + e_k \in \mathcal{I}_2$.

Proof. By the matroid axiom (I3), there is a subset $A_1 \subseteq T \setminus S$ such that $|A_1| = |T| - |S|$ and $S_1 := S \cup A_1 \in \mathcal{I}_1$. Then, $|S_1| = |T|$; hence, $|S_1 \setminus T| = |T \setminus S_1|$. Let $(E', \mathcal{I}'_1)$ be a matroid obtained from $(E,\mathcal{I}_1)$ by contracting $S_1 \cap T$ and truncating with size $|S_1 \setminus T|$, i.e., $E' = E \setminus (S_1 \cap T)$ and $\mathcal{I}'_1 := \{\, R \subseteq E' \mid R \cup (S_1 \cap T) \in \mathcal{I}_1, |R| \leq |S_1 \setminus T| \,\}$. Then, $S_1 \setminus T$ and $T \setminus S_1$ are bases of $(E', \mathcal{I}'_1)$. As $(E', \mathcal{I}'_1)$ is base-orderable, there is a bijection $\varphi_1 \colon S_1 \setminus T \to T \setminus S_1$ such that both $(S_1 \setminus T) - e + \varphi_1(e)$ and $(T \setminus S_1) + e - \varphi_1(e)$ are bases of $(E', \mathcal{I}'_1)$ for every $e \in S_1 \setminus T$. By the definition of $\mathcal{I}'_1$, this implies that both $S - e + \varphi_1(e)$ and $T + e - \varphi_1(e)$ belong to $\mathcal{I}_1$ for every $e \in S_1 \setminus T$. By the same argument, there exists $A_2 \subseteq T \setminus S$ such that $|A_2| = |T| - |S|$ and $S_2 := S \cup A_2 \in \mathcal{I}_2$, and there exists a bijection $\varphi_2 \colon S_2 \setminus T \to T \setminus S_2$ such that both $S - e + \varphi_2(e)$ and $T + e - \varphi_2(e)$ belong to $\mathcal{I}_2$ for every $e \in S_2 \setminus T$.

We represent φ_1 and φ_2 using a bipartite graph as follows. Note that, for each $\ell \in \{1, 2\}$, we have $S_\ell \setminus T = S \setminus T$ and $T \setminus S_\ell = T \setminus (S \cup A_\ell) \subseteq T \setminus S$. Let $S \setminus T$ and $T \setminus S$ be two vertex sets and let $M_\ell := \{\, (e, \varphi_\ell(e)) \mid e \in S \setminus T \,\}$ for $\ell = 1, 2$. Then, each M_ℓ is a one-to-one matching that covers $S \setminus T$ and $T \setminus (S \cup A_\ell)$. Note that the sets $A_1, A_2 \subseteq S \setminus T$ are mutually disjoint since, otherwise, some $e \in A_1 \cap A_2$ satisfies $S + e \in \mathcal{I}_1 \cap \mathcal{I}_2$, which contradicts the assumption. Then, $|T \setminus (S \cup A_1 \cup A_2)| = |T \setminus S| - |A_1| - |A_2| = |T \setminus S| - 2|T| + 2|S|$. Therefore, at most $2(|T \setminus S| - 2|T| + 2|S|)$ vertices in $S \setminus T$ are adjacent to $T \setminus (S \cup A_1 \cup A_2)$ via the edges in $M_1 \cup M_2$. Because $|S \setminus T| - 2(|T \setminus S| - 2|T| + 2|S|) = -3|S| + 2|T| + |S \cap T| > -3 \cdot \frac{2}{3}|T| + 2|T| + |S \cap T| \geq 0$, there exists $\tilde{e} \in S \setminus T$ that is not adjacent to $T \setminus (S \cup A_1 \cup A_2)$ via $M_1 \cup M_2$. This implies that $\varphi_2(\tilde{e}) \in A_1$ and $\varphi_1(\tilde{e}) \in A_2$; hence, $S + \varphi_2(\tilde{e}) \in \mathcal{I}_1$ and $S + \varphi_1(\tilde{e}) \in \mathcal{I}_2$. Let $e_i := \varphi_2(\tilde{e})$, $e_j := \tilde{e}$, and $e_k := \varphi_1(\tilde{e})$. Then, these three elements satisfy all the required conditions. ◀

[2] Contraction is defined for any subset of E [37]; however this paper uses only contraction by independent sets.

5 Correctness

This section is devoted to showing Theorem 1, which establishes the correctness of Algorithm 1.

As in Section 3, let I be an SMTI-LC instance with $E = \{ e_i \mid i = 1, 2, \ldots, n \}$ and let I^* be the auxiliary instance I^*, whose contract set is $E^* = \{ x_i, y_i, z_i \mid i = 1, 2, \ldots, n \}$.

For any agent $a \in W \cup F$, let $E_a^* = \{ x_i, y_i, z_i \mid e_i \in E_a \}$ and define families $\mathcal{I}_a$ and $\mathcal{I}_a^*$ by

$$\mathcal{I}_a = \{ S \subseteq E_a \mid \forall L \in \mathcal{L}_a : |S \cap L| \leq q_a(L) \},$$
$$\mathcal{I}_a^* = \{ S^* \subseteq E_a^* \mid \forall L^* \in \mathcal{L}_a^* : |S^* \cap L^*| \leq q_a^*(L^*) \},$$

i.e., $\mathcal{I}_a$ and $\mathcal{I}_a^*$ are the families of feasible sets in I and I^*, respectively. Then, $(E_a, \mathcal{I}_a)$ and $(E_a^*, \mathcal{I}_a^*)$ are laminar matroids and base-orderable. The definitions of $\mathcal{L}_a^*$ and q_a^* imply the following fact. Recall that $\pi : 2^{E^*} \to 2^E$ is defined by $\pi(S^*) = \{ e_i \mid \{x_i, y_i, z_i\} \cap S^* \neq \emptyset \}$.

▶ **Observation 6.** *For a set $S^* \subseteq E_a^*$, we have $S^* \in \mathcal{I}_a^*$ if and only if $|\{x_i, y_i, z_i\} \cap S^*| \leq 1$ for every $e_i \in E_a$ and $\pi(S^*) \in \mathcal{I}_a$.*

Take any stable matching M^* of I^* and let $M := \pi(M^*)$. As M^* is feasible in I^*, it contains at most one copy of each contract e_i. For any $e_i \in M$, we denote by $\pi^{-1}(e_i)$ the unique element in $\{x_i, y_i, z_i\} \cap M^*$.

By the definitions of the preference lists $\{P_a^*\}_{a \in W \cup F}$ in I^*, we can observe the following properties. For any agent $a \in W \cup F$ and contracts $e, e' \in E_a^*$, we write $e \succ_a^* e'$ if a prefers e to e' with respect to P_a^*. Recall that P_a^* does not contain ties, while P_a may contain.

▶ **Observation 7.** *For any $e_i \in E \setminus M$ and $e_j \in M$, the following conditions hold.*

- *For any agent $a \in W \cup F$, if $e_i, e_j \in E_a$ and $e_i \succ_a e_j$, then $y_i \succ_a^* \pi^{-1}(e_j)$ holds regardless of which of $\{x_i, y_i, z_i\}$ is $\pi^{-1}(e_i)$.*
- *For any worker $w \in W$, if $e_i, e_j \in E_w$ and $\pi^{-1}(e_j) \succ_w^* x_i$, then we have either $[\ \pi^{-1}(e_j) = x_j$ and $e_j \succeq_w e_i\ ]$ or $[\ \pi^{-1}(e_j) = y_j$ and $e_j \succ_w e_i\ ]$.*
- *For any firm $f \in F$, if $e_i, e_j \in E_f$ and $\pi^{-1}(e_j) \succ_f^* z_i$, then we have either $[\ \pi^{-1}(e_j) = z_j$ and $e_j \succeq_f e_i\ ]$ or $[\ \pi^{-1}(e_j) = y_j$ and $e_j \succ_f e_i\ ]$.*

First, we show the stability of M in I. For each agent $a \in W \cup F$, we write $M_a^* = M^* \cap E_a^*$, which implies that $\pi(M_a^*) = M_a$.

▶ **Lemma 8.** *The set M is a stable matching of I.*

Proof. Since M^* is feasible for all agents in I^*, Observation 6 implies that $M = \pi(M^*)$ is feasible for all agents in I, i.e., M is a matching in I.

Suppose, to the contrary, that M is not stable. Then, some contract $e_i \in E \setminus M$ blocks M. Let $w = \partial_W(e_i)$ and $f = \partial_F(e_i)$. Then, e_i is free for both w and f in M. We now show that y_i is free for both w and f in M^*, which contradicts the stability of M^*.

As e_i is free for w in I, we have (i) $M_w + e_i \in \mathcal{I}_w$ or (ii) there exists $e_j \in M_a$ such that $e_i \succ_w e_j$ and $M_a + e_i - e_j \in \mathcal{I}_w$. Note that $e_i \in E \setminus M$ implies $\{x_i, y_i, z_i\} \cap M^* = \emptyset$. In case (i), we have $\pi(M_w^* + y_i) = M_w + e_i \in \mathcal{I}_w$, which implies $M_w^* + y_i \in \mathcal{I}_w^*$; hence, y_i is free for w in M^*. In case (ii), we have $\pi(M_w^* + y_i - \pi^{-1}(e_j)) = M_w + e_i - e_j \in \mathcal{I}_w$, which implies $M_w^* + y_i - \pi^{-1}(e_j) \in \mathcal{I}_w^*$. Furthermore, as $e_i \succ_w e_j$, the first statement of Observation 7 implies $y_i \succ_w^* \pi^{-1}(e_j)$. Thus, in each case, y_i is free for w in M^*.

Similarly, we can show that y_i is free for f in M^*. Thus, y_i blocks M^*, a contradiction. ◀

Next, we show the approximation ratio using Lemma 5. Note that $\{ E_w \mid w \in W \}$ is a partition of E, as is $\{ E_f \mid f \in F \}$. Let $(E, \mathcal{I}_W)$ be the direct sum of base-orderable matroids $\{ (E_w, \mathcal{I}_w) \mid w \in W \}$ and $(E, \mathcal{I}_F)$ be the direct sum of $\{ (E_f, \mathcal{I}_f) \mid f \in F \}$. Then, they are both base-orderable matroids on E.

By the definitions of $\mathcal{I}_W$ and $\mathcal{I}_F$, for any subset $N \subseteq E$, we have $N \in \mathcal{I}_W \cap \mathcal{I}_F$ if and only if $N_a := N \cap E_a$ is feasible for each $a \in W \cup F$, i.e., N is a matching. Furthermore, for any matching $N \in \mathcal{I}_W \cup \mathcal{I}_F$ and contract $e_i \in E \setminus N$, which is associated with a worker $w = \partial_W(e_i)$ (and a firm $f = \partial_F(e_i)$), the condition $N + e_i \in \mathcal{I}_W$ is equivalent to $N_w + e_i \in \mathcal{I}_w$. In addition, if $N + e_i \notin \mathcal{I}_W$, we have $N + e_i - e_j \in \mathcal{I}_W$ if and only if $e_j \in N_w$ and $N_w + e_i - e_j \in \mathcal{I}_w$. The same statements hold when w and W are replaced with f and F, respectively.

▶ **Lemma 9.** *The set M satisfies $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$, where M_{OPT} is a maximum-cardinality stable matching of I.*

Proof. Set $N := M_{\mathrm{OPT}}$ for notational simplicity. Since M and N are stable matchings, $M, N \in \mathcal{I}_W \cap \mathcal{I}_F$. In addition, $M + e_i \notin \mathcal{I}_W \cap \mathcal{I}_F$ for any $e_i \in N \setminus M$ since, otherwise, e_i blocks M. Suppose, to the contrary, that $|M| < \frac{2}{3}|N|$. Then, by Lemma 5 and the definitions of $\mathcal{I}_W$ and $\mathcal{I}_F$, there exist three contracts e_i, e_j, e_k such that $e_i, e_k \in N \setminus M$, $e_j \in M \setminus N$, and the following conditions hold:

- $M_w + e_i \in \mathcal{I}_w$,
- both $M_f + e_i - e_j$ and $N_f - e_i + e_j$ belong to $\mathcal{I}_f$,
- both $M_{w'} - e_j + e_k$ and $N_{w'} + e_j - e_k$ belong to $\mathcal{I}_{w'}$,
- $M_{f'} + e_k \in \mathcal{I}_{f'}$,

where $w = \partial_W(e_i)$, $f = \partial_F(e_i) = \partial_F(e_j)$, $w' = \partial_W(e_j) = \partial_W(e_k)$, $f' = \partial_F(e_k)$.

Since $e_i \notin M$ and $M_w + e_i \in \mathcal{I}_w$, we have $M^*_w + z_i \in \mathcal{I}^*_w$; hence, z_i is free for the worker $w = \partial_W(z_i)$ in M^*. Then, the stability of M^* implies that z_i is not free for the firm $f = \partial_F(z_i)$. Since $\pi(M^*_f + z_i - \pi^{-1}(e_j)) = M_f + e_i - e_j \in \mathcal{I}_f$ implies $M^*_f + z_i - \pi^{-1}(e_j) \in \mathcal{I}^*_f$, we should have $\pi^{-1}(e_j) \succ^*_f z_i$. Then, the third statement of Observation 7 implies that we have either $[\pi^{-1}(e_j) = z_j$ and $e_j \succeq_f e_i]$ or $[\pi^{-1}(e_j) = y_j$ and $e_j \succ_f e_i]$.

Meanwhile, since $e_k \notin M$ and $M_{f'} + e_k \in \mathcal{I}_{f'}$, we have $M^*_{f'} + x_k \in \mathcal{I}^*_{f'}$; hence, x_k is free for the firm $f' = \partial_W(x_k)$ in M^*. As M^* is stable, then x_k is not free for the worker $w' = \partial_W(x_k)$. Since $\pi(M^*_{w'} + x_k - \pi^{-1}(e_j)) = M_{w'} + e_k - e_j \in \mathcal{I}_{w'}$ implies $M^*_{w'} + x_k - \pi^{-1}(e_j) \in \mathcal{I}^*_{w'}$, we should have $\pi^{-1}(e_j) \succ^*_{w'} x_k$. Then, the second statement of Observation 7 implies that we have either $[\pi^{-1}(e_j) = x_j$ and $e_j \succeq_{w'} e_k]$ or $[\pi^{-1}(e_j) = y_j$ and $e_j \succ_{w'} e_k]$.

Because we cannot have $\pi^{-1}(e_j) = z_j$ and $\pi^{-1}(e_j) = x_j$ simultaneously, we must have $\pi^{-1}(e_j) = y_j$, $e_j \succ_f e_i$, and $e_j \succ_{w'} e_k$. As we have $N_f - e_i + e_j \in \mathcal{I}_f$ and $N_{w'} + e_j - e_k \in \mathcal{I}_{w'}$, these preference relations imply that e_j blocks N, which contradicts the stability of N. ◀

Proof of Theorem 1. Combining Lemmas 8 and 9, we obtain Theorem 1. ◀

6 Time Complexity

We explain how to implement the second step of Algorithm 1 and estimate its time complexity, which establishes Theorem 2. For this purpose, we introduce the notion of a matroid-kernel, which is a matroid generalization of a stable matching proposed by Fleiner [12, 13]. Note that it is defined not only for base-orderable matroids but for general matroids.

6.1 Matroid-kernels

A triple $\mathcal{M} = (E, \mathcal{I}, \succ)$ is called an *ordered matroid* if $(E, \mathcal{I})$ is a matroid and $\succ$ is a strict linear order on E. For an ordered matroid $\mathcal{M} = (E, \mathcal{I}, \succ)$ and an independent set $S \in \mathcal{I}$, an element $e \in E \setminus S$ is said to be *dominated* by S in $\mathcal{M}$ if $S + e \notin \mathcal{I}$ and there is no element $e' \in S$ such that $e \succ e'$ and $S + e - e' \in \mathcal{I}$.

Let $\mathcal{M}_1 = (E, \mathcal{I}_1, \succ_1)$ and $\mathcal{M}_2 = (E, \mathcal{I}_2, \succ_2)$ be two ordered matroids on the same ground set E. Then, a set $S \subseteq E$ is called an $\mathcal{M}_1\mathcal{M}_2$-*kernel* if $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ and any element $e \in E \setminus S$ is dominated by S in $\mathcal{M}_1$ or $\mathcal{M}_2$.

In [12], an algorithm for finding a matroid-kernel has been described using choice functions defined as follows. For an ordered matroid $\mathcal{M} = (E, \mathcal{I}, \succ)$, give indices of elements in E such that $E = \{e^1, e^2, \dots, e^n\}$ and $e^1 \succ e^2 \succ \cdots \succ e^n$. Define a function $C_{\mathcal{M}} : 2^E \to 2^E$ by letting $C_{\mathcal{M}}$ be the output of the following greedy algorithm for every $S \subseteq E$. Let $T^0 := \emptyset$ and define T^ℓ for $\ell = 1, 2, \dots, n$ by

$$T^\ell := \begin{cases} T^{\ell-1} + e^\ell & \text{if } \; e^\ell \in S \; \text{ and } \; T^{\ell-1} + e^\ell \in \mathcal{I}, \\ T^{\ell-1} & \text{otherwise;} \end{cases}$$

then, let ${}_{\mathcal{M}}(S) := T^n$.

Let $C_{\mathcal{M}_1}$, $C_{\mathcal{M}_2}$ be the choice functions defined from $\mathcal{M}_1 = (E, \mathcal{I}_1, \succ_1)$, $\mathcal{M}_2 = (E, \mathcal{I}_2, \succ_2)$, respectively. In [12, Theorem 2], Fleiner showed that an $\mathcal{M}_1\mathcal{M}_2$-kernel can be found using the following algorithm, which can be regarded as a generalization of the Gale–Shapley algorithm. First, set $R \leftarrow \emptyset$. Then, repeat the following three steps: (1) $S \leftarrow C_{\mathcal{M}_1}(E \setminus R)$, (2) $T \leftarrow C_{\mathcal{M}_2}(S \cup R)$, and (3) $R \leftarrow (S \cup R) \setminus T$. Stop the repetition if R is not changed at (3) and return T at that moment. In terms of the ordinary Gale–Shapley algorithm, R, S, and T correspond to the sets of contracts that are rejected by firms thus far, proposed by workers, and accepted by firms, respectively.

▶ **Theorem 10** (Fleiner [12, 13]). *For any pair of ordered matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on the same ground set E, there exists an $\mathcal{M}_1\mathcal{M}_2$-kernel. One can find an $\mathcal{M}_1\mathcal{M}_2$-kernel in $O(|E| \cdot \mathrm{EO})$ time, where* EO *is the time required to compute $C_{\mathcal{M}_1}(S)$ and $C_{\mathcal{M}_2}(S)$ for any $S \subseteq E$.*

6.2 Implementation of Our Algorithm

We show that the second step of Algorithm 1 is reduced to a computation of a matroid-kernel.

For an auxiliary instance I^* defined in Section 2, note that $\{\, E^*_w \mid w \in W \,\}$ is a partition of E^* and let $(E^*, \mathcal{I}^*_W)$ be the direct sum of $\{(E^*_w, \mathcal{I}^*_w)\}_{w \in W}$. Furthermore, let $\succ_W$ be a strict linear order on E^* that is consistent with the workers' preferences $\{P^*_w\}_{w \in W}$ in I^*. For example, obtain $\succ_W$ by concatenating the lists P^*_w of all workers in an arbitrary order. Then, $\mathcal{M}_W = (E^*, \mathcal{I}^*_W, \succ_W)$ is an ordered matroid on the contract set E^*. As $\{\, E^*_f \mid f \in F \,\}$ is also a partition of E^*, we can define an ordered matroid $\mathcal{M}_F = (E^*, \mathcal{I}^*_F, \succ_F)$ in the same manner from $\{(E^*_f, \mathcal{I}^*_f)\}_{f \in F}$ and $\{P^*_f\}_{f \in F}$.

We show that $\mathcal{M}_W\mathcal{M}_F$-kernels are equivalent to stable matchings of I. This has already been shown in several previous works [14, 44]. We present a proof for the completeness.

▶ **Lemma 11.** *$M^* \subseteq E^*$ is a stable matching of I^* if and only if M^* is an $\mathcal{M}_W\mathcal{M}_F$-kernel.*

Proof. By the definitions of $(E^*, \mathcal{I}^*_W)$ and $(E^*, \mathcal{I}^*_f)$, a set $M^* \subseteq E^*$ is feasible for all agents in I^* if and only if $M^* \in \mathcal{I}^*_W \cap \mathcal{I}^*_F$. Recall that a contract $e \in E^* \setminus M^*$ is free for the associated worker $w := \partial_W(e)$ if $M^*_w + e \in \mathcal{I}^*_w$ or there exists $e' \in M^*_w$ such that $e \succ^*_w e'$ and $M^*_w + e - e' \in \mathcal{I}^*_w$. By the definition of $\mathcal{I}^*_W$, we have $M^*_w + e \in \mathcal{I}^*_w$ if and only if

$M^* + e \in \mathcal{I}^*_W$. In addition, if $M^*_w + e \notin \mathcal{I}^*_w$, then $M^*_w + e - e' \in \mathcal{I}^*_w$ holds for $e' \in M_w$ if and only if $M^* + e - e' \in \mathcal{I}^*_W$. Because $\succ_W$ is consistent with $\succ^*_w$, these imply that e is free for $w = \partial_W(e)$ in M^* if and only if e is not dominated by M^* in $\mathcal{M}_W$. Similarly, we can show that e is free for the associated firm $f \coloneqq \partial_F(e)$ in M^* if and only if e is not dominated by M^* in $\mathcal{M}_F$. Thus, the equivalence holds. ◀

▶ **Lemma 12.** *For any subset $S^* \subseteq E^*$, we can compute $C_{\mathcal{M}_W}(S^*)$ and $C_{\mathcal{M}_F}(S^*)$ in $O(k^* \cdot |E^*|)$ time, where k^* is the maximum level of nesting of laminar families $\mathcal{L}^*_a$ $(a \in W \cup F)$.*

Proof. We only explain the computation of $C_{\mathcal{M}_W}(S^*)$ because that of $C_{\mathcal{M}_F}(S^*)$ is similar.

Let $\mathcal{L}$ be the union of $\{\mathcal{L}^*_w\}_{w \in W}$ and define $q : \mathcal{L} \to \mathbf{Z}_+$ by setting $q(L) = q^*_w(L)$ for each $w \in W$ and $L \in \mathcal{L}^*_w$. Then, $\mathcal{L}$ is a laminar family on E^* and the matroid $(E^*, \mathcal{I}^*_W)$ is defined by $\mathcal{L}$ and q. The maximum level of nesting of $\mathcal{L}$ is again k^*.

Referring to [4], we represent $\mathcal{L}$ by a forest G whose node set is $\{\, v_L \mid L \in \mathcal{L} \,\}$. Node v_L is the parent of $v_{L'}$ in G if $L \subseteq L'$ and there is no $L'' \in \mathcal{L}$ such that $L \subsetneq L'' \subsetneq L'$. Note that $\mathcal{L}$ contains the set $\{x_i, y_i, z_i\}$ for every $e_i \in E$, which is inclusion-wise minimal in $\mathcal{L}$. Therefore, the node $v_i \coloneqq v_{\{x_i, y_i, z_i\}}$ is a leaf for any $e_i \in E$, and any leaf has this form.

We compute the sequence $T^0, T^1, \dots, T^{|E^*|}$ of sets in the definition of $C_{\mathcal{M}_W}(S^*)$ as follows. For each v_L, we store a pointer to its parent, the value of $q(L)$, and the value of $|T^{\ell-1} \cap L|$. For each $e^\ell \in E^*$, we have $T^{\ell-1} + e^\ell \in \mathcal{I}^*_W$ if and only if there is no ancestor node v_L of v_i with $q(L) = |T^{\ell-1} \cap L|$, where v_i is the leaf with $e^\ell \in \{x_i, y_i, z_i\}$. Then, we can check whether $T^{\ell-1} + e^\ell \in \mathcal{I}^*_W$ in $O(k^*)$ time by following the path of the parent pointers from v_i. When $T^\ell = T^{\ell-1} + e^\ell$, we update the stored values $|T^{\ell-1} \cap L|$ to $|T^\ell \cap L|$ for each $L \in \mathcal{L}$ with $e^\ell \in L$. This is also performed in $O(k^*)$ time by following the path of the parent pointers. ◀

Proof of Theorem 2. As we have Theorem 1, what is left is to show the time complexity. The set E^* of contracts in I^* satisfies $|E^*| = 3|E|$. The maximum level of nesting of laminar families $\mathcal{L}^*_a$ in I^* is $k+1$. By Theorem 10 and Lemmas 11 and 12, then the second step of Algorithm 1 is computed in $O((k+1) \cdot |E^*|^2) = O(k \cdot |E|^2)$ time. Since the first and third steps can be performed in $O(k \cdot |E|^2)$ time, Algorithm 1 runs in $O(k \cdot |E|^2)$ time. ◀

▶ Remark. Our analysis depends on the fact that the feasible set family defined by laminar constraints forms the independent set family of a base-orderable matroid. Actually, we can extend Theorem 1 to a setting where the family of feasible sets of each agent $a \in W \cup F$ is represented by the independent set family $\mathcal{I}_a$ of an arbitrary base-orderable matroid. To construct I^* in this case, we define E^* and $\{P^*_a\}_{a \in W \cup F}$ as in Section 3 and define the feasible set family $\mathcal{I}^*_a$ by $\mathcal{I}^*_a = \{\, S^* \subseteq E^*_a \mid |\{x_i, y_i, z_i\} \cap S^*| \le 1 \text{ for any } e_i \in E_a \text{ and } \pi(S^*) \in \mathcal{I}_a \,\}$. We can easily show that $(E^*_a, \mathcal{I}^*_a)$ is also a base-orderable matroid and apply the arguments in Sections 5 and 6, except Lemma 12. Given a membership oracle for each $\mathcal{I}_a$ available, Algorithm 1 runs in $O(\tau \cdot |E|^2)$ time in this case, where τ is the time for an oracle call.

7 Strategy-Proof Approximation Mechanisms

In this section, we investigate approximation ratios for MAX-SMTI-LC attained by strategy-proof mechanisms. First, note that our setting SMTI-LC is a generalization of the stable marriage model of Gale and Shapley [16]; hence, Roth's impossibility theorem [40] implies that there is no mechanism that returns a stable matching and is strategy-proof for agents on both sides. As with many existing works on strategy-proofness in two-sided matching models, we consider one-sided strategy-proofness in the setting of many-to-one matching. Many-to-one matching models have various applications such as assignment of residents to hospitals [39, 41] and students to high schools [1–3]. In such applications, strategy-proofness for residents or students is a desirable property preventing their strategic behavior.

7.1 Model and Definitions

We define a setting of SMTI-OLC, which is a many-to-one variant of SMTI-LC. (Here, OLC stands for "one-sided laminar constraints"). In SMTI-OLC, each worker is assigned at most one contract and hence has no laminar constraints. An instance of SMTI-OLC is described as $I = (W, F, E, \{P_w\}_{w \in W}, \{\mathcal{L}_f, q_f, P_f\}_{f \in F})$. To consider strategies of workers, we slightly change the assumption on each P_w. In Section 2, it is assumed that P_w contains all contracts in E_w. Here, we allow each worker to submit a preference list P_w that is defined on any subset of E_w and regard contracts not appearing in P_w as unacceptable for w. Let E° be the set of acceptable contracts, that is, $E^\circ = \{\, e \in E \mid e \text{ appears in } P_w, \text{ where } w = \partial_W(e) \,\}$.

A set $M \subseteq E$ is called a *matching* if $M \subseteq E^\circ$, $|M_w| \leq 1$ for every worker $w \in W$, and M is feasible for every firm $f \in F$. For a matching M, a contract $e \in E \setminus M$ *blocks* M if it is free for both $\partial_W(e)$ and $\partial_F(e)$, where we say that e is *free* for the associated worker $w \coloneqq \partial_W(e)$ if $e \in E^\circ$ and either w is assigned no contract in M or prefers e to the contract assigned in M. A matching M is *stable* if there is no contract that blocks M. The auxiliary instance $I^* = (W, F, E^*, \{P^*_w\}_{w \in W}, \{\mathcal{L}^*_f, q^*_f, P^*_f\}_{f \in F})$ of I is defined similarly as in Section 3.

We remark that SMTI-OLC can be seen as a special case of SMTI-LC, although the assumption on workers' preference lists is slightly different from that of SMTI-LC. From an SMTI-OLC instance I, define $I^\circ = (W, F, E^\circ, \{\mathcal{L}^\circ_a, q^\circ_a, P^\circ_a\}_{a \in W \cup F})$ as follows. For each worker $w \in W$, set $\mathcal{L}^\circ_w = \{E^\circ_w\}$, $q^\circ_w(E^\circ_w) = 1$, and $P^\circ_w = P_w$. For each firm $f \in F$, set $\mathcal{L}^\circ_f = \{\, L \cap E^\circ \mid L \in \mathcal{L}_f \,\}$, $q^\circ_f(L \cap E^\circ) = q_f(L)$ for each $L \in \mathcal{L}_f$, and let P°_f be the restriction of P_f on E°_f (i.e., delete the elements in $E_f \setminus E^\circ_f$ from P_f). Then, I° is an instance of SMTI-LC in Section 2. By definition, we can observe that a subset $M \subseteq E$ is a stable matching of I if and only if it is a stable matching of I°. Therefore, we can apply Algorithm 1 to SMTI-OLC instances.

For subsets $M, N \subseteq E$, a worker $w \in W$, and a preference list P_w, we say that w *weakly prefers* M to N with respect to P_w if either (i) w is assigned a contract appearing in P_w only in M or (ii) w is assigned a contract appearing in P_w in both M and N and does not strictly prefer the one assigned in N with respect to P_w. A stable matching M of an SMTI-OLC instance I is *worker-optimal* if, for any other stable matching N of I, every worker w weakly prefers M to N.

A *mechanism* is a mapping from SMTI-OLC instances to matchings. Here, we define the *worker-strategy-proofness* of a mechanism. Let A be a mechanism. For any instance I and any worker w, let I' be an instance obtained from I by replacing w's list P_w with some other list P'_w. Let M and M' be the outputs of A for instances I and I', respectively. We say that A is *worker-strategy-proof* if w weakly prefers M to M' with respect to the original list P_w regardless of the choices of I, w, and P'_w.

7.2 Approximation Mechanisms

Before providing our results on SMTI-OLC, we introduce some existing results on special cases of SMTI-OLC. We first present a result on the setting without ties.

▶ **Lemma 13.** *In a restriction of* SMTI-OLC *in which all agents have strict preferences, a mechanism that returns the worker-optimal stable matching is worker-strategy-proof.*

Lemma 13 is a natural consequence of the results shown in previous works [17, 33]. For the completeness, the full version [45] provides the proof, which uses the fact that SMTI-OLC can be reduced to the model of Hatfield and Milgrom [20] if there are no ties.

Next, we introduce the results of Hamada et al. [19] on MAX-SMTI, which is a special case of MAX-SMTI-OLC in which every agent is assigned at most one contract.

▶ **Theorem 14** (Hamada et al. [19, Theorem 2]). *For* MAX-SMTI*, there is a worker-strategy-proof mechanism that returns a* 2*-approximate solution. On the other hand, for any* $\epsilon > 0$*, there is no worker-strategy-proof mechanism that returns a* $(2 - \epsilon)$*-approximate solution.*

▶ **Theorem 15** (Hamada et al. [19, Theorem 4]). *For a restriction of* MAX-SMTI *in which ties appear in only workers' preference lists, there is a worker-strategy-proof mechanism that returns a* $\frac{3}{2}$*-approximate solution. On the other hand, for any* $\epsilon > 0$*, there is no worker-strategy-proof mechanism that returns a* $(\frac{3}{2} - \epsilon)$*-approximate solution.*

The first statement of Theorem 14 is attained by a naive mechanism that first breaks ties in an increasing order of the indices and then finds the worker-optimal stable matching of the resultant instance. This method naturally extends to the setting of SMTI-OLC and yields the following theorem. See the full version [45] for the proof.

▶ **Theorem 16.** *For* SMTI-OLC*, there is a worker-strategy-proof mechanism that returns a stable matching* M *with* $|M| \geq \frac{1}{2}|M_{\mathrm{OPT}}|$ *in* $O(k \cdot |E|^2)$ *time, where* M_{OPT} *is a maximum-cardinality stable matching and* k *is the maximum level of nesting of* $\mathcal{L}_f$ $(f \in F)$*.*

Since SMTI-OLC is a generalization of SMTI, the second statement (i.e., the hardness part) of Theorem 14 immediately extends to MAX-SMTI-OLC. Therefore, for the general SMTI-OLC, there is no worker-strategy-proof mechanism with an approximation ratio better than 2.

However, in a special case in which firms' lists contain no ties, Algorithm 1 in Section 3 defines a worker-strategy-proof mechanism whose approximation ratio is $\frac{3}{2}$. That is, we can extend the first statement of Theorem 15 to the setting of SMTI-OLC. According to the second statement of Theorem 15, this is the best approximation ratio attained by a worker-strategy-proof mechanism.

▶ **Theorem 17.** *For a restriction of* SMTI-OLC *in which ties appear in only workers' lists, there is a worker-strategy-proof mechanism that returns a stable matching* M *with* $|M| \geq \frac{2}{3}|M_{\mathrm{OPT}}|$ *in* $O(k \cdot |E|^2)$ *time, where* M_{OPT} *is a maximum-cardinality stable matching and* k *is the maximum level of nesting of laminar families* $\mathcal{L}_f$ $(f \in F)$*.*

We provide a mechanism that meets the requirements in Theorem 17. Our mechanism is regarded as a possible realization of Algorithm 1. In the second step of Algorithm 1, we should choose the worker-optimal stable matching of the auxiliary instance I^*. Our mechanism is described as follows.

1. Given an instance I (in which ties appear in only workers' lists), construct I^*.
2. Find the worker-optimal stable matching M^* of I^*.
3. Let $M = \pi(M^*)$ and return M.

In the proof of Theorem 10 (Fleiner [12, p.113]), it is shown that one can find the $\mathcal{M}_1$-optimal $\mathcal{M}_1\mathcal{M}_2$-kernel in $O(|E| \cdot \mathrm{EO})$ time. The arguments in Section 6 then imply that one can find the worker-optimal stable matching of I^* in $O(k \cdot |E|^2)$ time. As we have Theorem 2, showing the strategy-proofness of the above-mentioned mechanism completes the proof of Theorem 17. To this end, we show the following lemma.

▶ **Lemma 18.** *Let* I *be an* SMTI-OLC *instance with* $E = \{\, e_i \mid i = 1, 2, \dots, n \,\}$ *and let* I^* *be the auxiliary instance. If ties appear in only workers' lists in* I*, then the worker-optimal stable matching* M^* *of* I^* *satisfies* $M^* \cap \{\, z_i \mid i = 1, 2, \dots, n \,\} = \emptyset$*.*

Proof. Suppose, to the contrary, that $z_i \in M^*$ for some index i. Then $N \coloneqq M^* - z_i + y_i$ is a matching of I^* and $w \coloneqq \partial_W(z_i) = \partial_W(y_i)$ prefers N to M^*. We intend to show that N is stable in I^*. Take any $e \in E^* \setminus N = (E^* \setminus M^*) + z_i - y_i$. If $e = z_i$, then it does not block N because $y_i \succ^*_w z_i$. If $e \neq z_i$, then the assignment of $\partial_W(e)$ does not change in M^* and N, and hence e can block N only if $f \coloneqq \partial_F(e) = \partial_F(z_i)$ and $z_i \succ^*_f e \succ^*_f y_i$. This is impossible because no contract lies between z_i and y_i in P^*_f as the list P_f of the firm f is strict. Thus, N is a stable matching of I^*, which contradicts the worker-optimality of M^*. ◀

Proof of Theorem 17. As we have Theorem 2, what is left is to show that our mechanism is worker-strategy-proof. Let $I = (W, F, E, \{P_w\}_{w\in W}, \{\mathcal{L}_f, q_f, P_f\}_{f\in F})$ be an instance of the setting in the statement and let $E = \{\, e_i \mid i = 1, 2, \dots, n \,\}$. Furthermore, let I' be obtained from I by replacing P_w with some other list P'_w. Let M^* and N^* be the worker-optimal stable matchings of the auxiliary instances defined from I and I', respectively. Note that the two auxiliary instances have no ties and they differ only in the preference list of w. Then, Lemma 13 implies that w weakly prefers M^* to N^* with respect to P^*_w. In other words, either (i) w is assigned a contract on P^*_w only in M^*, or (ii) w is assigned a contract on P^*_w in both M^* and N^* and does not strictly prefer the one assigned in N^* w.r.t. P^*_w. By Lemma 18, w is not assigned a contract of type z_i in M^* or N^*. Then, the definition of P^*_w implies that w weakly prefers $\pi(M^*)$ to $\pi(N^*)$ w.r.t. P_w. Thus the mechanism is worker-strategy-proof. ◀

References

1. A. Abdulkadiroğlu, P. A. Pathak, and A. E. Roth. The New York city high school match. *American Economic Review*, 95:364–367, 2005.
2. A. Abdulkadiroğlu, P. A. Pathak, and A. E. Roth. Strategy-proofness versus efficiency in matching with indifferences: Redesigning the NYC high school match. *American Economic Review*, 99(5):1954–1978, 2009.
3. A. Abdulkadiroğlu, P. A. Pathak, A. E. Roth, and T. Sönmez. The Boston public school match. *American Economic Review*, 95:368–371, 2005.
4. P. Biró, T. Fleiner, R. W. Irving, and D. F. Manlove. The college admissions problem with lower and common quotas. *Theoretical Computer Science*, 411(34):3136–3153, 2010.
5. J. E. Bonin and T. J. Savitsky. An infinite family of excluded minors for strong base-orderability. *Linear Algebra and its Applications*, 488:396–429, 2016.
6. S. Braun, N. Dwenger, D. Kübler, and A. Westkamp. Implementing quotas in university admissions: An experimental analysis. *Games and Economic Behavior*, 85:232–251, 2014.
7. R. A. Brualdi. Induced matroids. *Proceedings of the American Mathematical Society*, 29(2):213–221, 1971.
8. F. Cooper and D. Manlove. A 3/2-approximation algorithm for the student-project allocation problem. In *Proc. 17th International Symposium on Experimental Algorithms (SEA 2018)*, volume 103, pages 8:1–8:13. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018.
9. L. Ehlers, I. E. Hafalir, M. B. Yenmez, and M. A. Yildirim. School choice with controlled choice constraints: Hard bounds versus soft bounds. *Journal of Economic Theory*, 153:648–683, 2014.
10. T. Fife and J. Oxley. Laminar matroids. *European Journal of Combinatorics*, 62:206–216, 2017.
11. L. Finkelstein. Two algorithms for the matroid secretary problem. Master's thesis, Technion-Israel Institute of Technology, Faculty of Industrial and Management Engineering, 2011.
12. T. Fleiner. A matroid generalization of the stable matching polytope. In *Proc. Eighth International Conference on Integer Programming and Combinatorial Optimization (IPCO 2001)*, volume 2081 of *LNCS*, pages 105–114. Springer-Verlag, Berlin & Heidelberg, 2001.
13. T. Fleiner. A fixed-point approach to stable matchings and some applications. *Mathematics of Operations Research*, 28(1):103–126, 2003.

14 T. Fleiner and N. Kamiyama. A matroid approach to stable matchings with lower quotas. *Mathematics of Operations Research*, 41(2):734–744, 2016.

15 D. Fragiadakis and P. Troyan. Improving matching under hard distributional constraints. *Theoretical Economics*, 12(2):863–908, 2017.

16 D. Gale and L. S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69(1):9–15, 1962.

17 M. Goto, A. Iwasaki, Y. Kawasaki, R. Kurata, Y. Yasuda, and M. Yokoo. Strategyproof matching with regional minimum and maximum quotas. *Artificial Intelligence*, 235:40–57, 2016.

18 D. Gusfield and R. W. Irving. *The Stable Marriage Problem: Structure and Algorithms*. MIT Press, Cambridge, MA, 1989.

19 K. Hamada, S. Miyazaki, and H Yanagisawa. Strategy-proof approximation algorithms for the stable marriage problem with ties and incomplete lists. In *Proc. 30th International Symposium on Algorithms and Computation (ISAAC 2019)*, volume 149 of *LIPIcs*, pages 9:1–9:14. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019.

20 J. W. Hatfield and P. R. Milgrom. Matching with contracts. *American Economic Review*, 95(4):913–935, 2005.

21 C. C. Huang. Classified stable matching. In *Proc. 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA2010)*, pages 1235–1253. SIAM, Philadelphia, 2010.

22 A. W. Ingleton. Transversal matroids and related structures. In *Higher Combinatorics* (M. Aigner eds.), pages 117–131. Reidel, Dordrecht, 1977.

23 R. W. Irving. Stable marriage and indifference. *Discrete Applied Mathematics*, 48(3):261–272, 1994.

24 K. Iwama, D. F. Manlove, S. Miyazaki, and Y. Morita. Stable marriage with incomplete lists and ties. In *Proc. 26th International Colloquium on Automata, Languages, and Programming (ICALP1999)*, pages 443–452. Springer, 1999.

25 K. Iwama, S. Miyazaki, and N. Yamauchi. A 1.875-approximation algorithm for the stable marriage problem. In *Proc. 18th annual ACM-SIAM symposium on Discrete algorithms (SODA2007)*, pages 288–297. SIAM, Philadelphia, 2007.

26 K. Iwama, S. Miyazaki, and N. Yamauchi. A $(2 - c\frac{1}{\sqrt{n}})$-approximation algorithm for the stable marriage problem. *Algorithmica*, 51(3):342–356, 2008.

27 Y. Kamada and F. Kojima. Efficient matching under distributional constraints: Theory and applications. *American Economic Review*, 105(1):67–99, 2015.

28 Y. Kamada and F. Kojima. Stability concepts in matching under distributional constraints. *Journal of Economic Theory*, 168:107–142, 2017.

29 Y. Kamada and F. Kojima. Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition. *Theoretical Economics*, 13(2):761–793, 2018.

30 Z. Király. Better and simpler approximation algorithms for the stable marriage problem. *Algorithmica*, 60(1):3–20, 2011.

31 Z. Király. Linear time local approximation algorithm for maximum stable marriage. *Algorithms*, 6(3):471–484, 2013.

32 D. E. Knuth. *Stable Marriage and Its Relation to Other Combinatorial Problems*. American Mathematical Society, Providence, 1996.

33 F. Kojima, A. Tamura, and M. Yokoo. Designing matching mechanisms under constraints: An approach from discrete convex analysis. *Journal of Economic Theory*, 176:803–833, 2018.

34 D. F. Manlove. *Algorithmics of Matching under Preferences*. World Scientific Publishing, Singapore, 2013.

35 D. F. Manlove, R. W. Irving, K. Iwama, S. Miyazaki, and Y. Morita. Hard variants of stable marriage. *Theoretical Computer Science*, 276(1-2):261–279, 2002.

36 E. Mcdermid. A 3/2-approximation algorithm for general stable marriage. In *Proc. 36th International Colloquium on Automata, Languages, and Programming (ICALP2009)*, pages 689–700. Springer, 2009.

37 J. G. Oxley. *Matroid Theory* (2nd ed.). Oxford University Press, Oxford, 2011.

38 K. Paluch. Faster and simpler approximation of stable matchings. *Algorithms*, 7(2):189–202, 2014.

39 A. E. Roth. The evolution of the labor market for medical interns and residents: A case study in game theory. *The Journal of Political Economy*, 92(6):991–1016, 1984.

40 A. E. Roth. On the allocation of residents to rural hospitals: A general property of two-sided matching markets. *Econometrica*, 54(2):425–427, 1986.

41 A. E. Roth and E. Peranson. The redesign of the matching market for american physicians: Some engineering aspects of economic design. *American economic review*, 89(4):748–780, 1999.

42 A. Schrijver. *Combinatorial Optimization: Polyhedra and Efficiency*. Springer-Verlag, Heidelberg, 2003.

43 H. Yanagisawa. *Approximation algorithms for stable marriage problems*. Ph.D. Thesis, Kyoto University, 2007.

44 Y. Yokoi. A generalized polymatroid approach to stable matchings with lower quotas. *Mathematics of Operations Research*, 42(1):238–255, 2017.

45 Y. Yokoi. An approximation algorithm for maximum stable matching with ties and constraints. *arXiv preprint*, 2021. `arXiv:2107.03076`.