

Partitioning H -Free Graphs of Bounded Diameter

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Abstract

A natural way of increasing our understanding of NP-complete graph problems is to restrict the input to a special graph class. Classes of H -free graphs, that is, graphs that do not contain some graph H as an induced subgraph, have proven to be an ideal testbed for such a complexity study. However, if the forbidden graph H contains a cycle or claw, then these problems often stay NP-complete. A recent complexity study (MFCS 2019) on the k -COLOURING problem shows that we may still obtain tractable results if we also bound the diameter of the H -free input graph. We continue this line of research by initiating a complexity study on the impact of bounding the diameter for a variety of classical vertex partitioning problems restricted to H -free graphs. We prove that bounding the diameter does not help for INDEPENDENT SET, but leads to new tractable cases for problems closely related to 3-COLOURING. That is, we show that NEAR-BIPARTITENESS, INDEPENDENT FEEDBACK VERTEX SET, INDEPENDENT ODD CYCLE TRANSVERSAL, ACYCLIC 3-COLOURING and STAR 3-COLOURING are all polynomial-time solvable for chair-free graphs of bounded diameter. To obtain these results we exploit a new structural property of 3-colourable chair-free graphs.

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1 Introduction

Many well-known graph problems are NP-complete in general but become polynomial-time solvable under input restrictions. We focus on problems that partition the vertex set V of a graph G into sets V_1, \dots, V_k such that each V_i satisfies some property π_i and where V_1 might have the extra condition of being large or being small. For instance, the k -COLOURING problem is to decide if V can be partitioned into sets V_1, \dots, V_k , called *colour classes*, such that each V_i is an independent set. To give another example, the INDEPENDENT SET problem is to decide if V can be partitioned into sets V_1 and V_2 where V_1 is independent and $|V_1| \geq p$ for some given integer p . Our underlying goal is to understand which graph properties ensure tractability of these problems and which properties cause the computational hardness. In the literature, input is restricted in various ways. In particular, *hereditary* graph classes have been considered.

Hereditary graph classes are the classes of graphs closed under vertex deletion. They form a natural and rich framework that cover many well-known graph classes (see, for example, [7]). Moreover, they enable a systematic study on the computational complexity of graph problems



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under input restrictions. The reason is that a graph class \mathcal{G} is hereditary if and only if \mathcal{G} can be characterized by a set $\mathcal{F}_{\mathcal{G}}$ of forbidden induced subgraphs; we also say that \mathcal{G} is $\mathcal{F}_{\mathcal{G}}$ -free. A natural starting point for a systematic study is the case where $\mathcal{F}_{\mathcal{G}}$ has size 1, say $\mathcal{F}_{\mathcal{G}} = \{H\}$ for some graph H . In this case the graphs in \mathcal{G} are said to be H -free. In other words, no graph $G \in \mathcal{G}$ can be modified into H by a sequence of vertex deletions.

In the literature, there are extensive studies on H -free graphs; for example, on bull-free graphs [11] and claw-free graphs [12, 17]. There also exist several surveys on graph problems or graph parameters for hereditary graph classes that are characterized by a small set of forbidden induced subgraphs, for example, on COLOURING [16, 27] and clique-width [13].

A well-known dichotomy on COLOURING restricted to H -free graphs is due to Král', Kratochvíl, Tuza, and Woeginger [19]. Namely, COLOURING on H -free graphs is polynomial-time solvable if H is an induced subgraph of P_4 (the 4-vertex path) or of $P_1 + P_3$ (the disjoint union of P_1 and P_3) and it is NP-complete otherwise. Recently, similar but almost-complete dichotomies (up to one missing case each) were established for ACYCLIC COLOURING, STAR COLOURING and INJECTIVE COLOURING [4]. In particular, all these problems stay NP-complete if the forbidden induced subgraph H has a cycle or claw (the 4-vertex star $K_{1,3}$). Moreover, the latter holds even if the number of colours k is fixed, i.e., not part of the input.

Several other vertex partitioning problems on H -free graphs stay NP-complete as well if H has a cycle or claw. Examples of such problems include (INDEPENDENT) FEEDBACK VERTEX SET [6, 23, 26], (INDEPENDENT) ODD CYCLE TRANSVERSAL [6, 10] and EVEN CYCLE TRANSVERSAL [24]. Hence, for all these problems, if H is a cycle or claw, then we need to add more structure to the class of input graphs in order to find tractable results for H -free graphs. One way of doing this is to bound the *diameter* of the input graph G for some problem. Our research question then becomes:

Does bounding the diameter of an H -free graph lead to new tractability results?

We note that graph classes of diameter at most d are hereditary if and only if $d \leq 1$. Many graph problems, such as COLOURING, ACYCLIC COLOURING, STAR COLOURING, CLIQUE and INDEPENDENT SET stay NP-complete even for graphs of diameter 2. The reason is that we can take an arbitrary graph G from such a problem and add a dominating vertex: the graph G is a yes-instance if and only if the new graph G' is a yes-instance.

This approach of adding a dominating vertex does not work if we consider 3-COLOURING. Mertzios and Spirakis [22] proved in a highly nontrivial way that 3-COLOURING is NP-complete even for triangle-free graphs of diameter at most 3. However, determining the complexity of 3-COLOURING for graphs of diameter 2 is a notoriously open problem (see [3, 9, 20, 22, 25]); we refer to [22] and [14] for subexponential-time algorithms for LIST 3-COLOURING on graphs of diameter at most 2. It is also known that ACYCLIC 3-COLOURING and STAR COLOURING, restricted to graphs of diameter at most d , are polynomial-time solvable if $d = 2$ or $d = 3$, respectively, but NP-complete if $d = 5$ or $d = 8$, respectively [8]. Moreover, the related problems NEAR BIPARTITENESS and INDEPENDENT FEEDBACK VERTEX SET, which we define below, are polynomial-time solvable for $d = 2$ but NP-complete for $d = 3$ [5]. These results also show that bounding the diameter on its own (without forbidding any induced graph H) does not suffice.

We refer to [20, 21] for a number of results on 3-COLOURING and LIST 3-COLOURING for H -free graphs of bounded diameter, where H is a cycle or a *polyad*, which is a tree where exactly one vertex has degree at least 3 (polyads are also known as *subdivided stars*). One crucial observation in [20], based on an application of Ramsey's Theorem, was the starting point of this investigation: for all integers d, k , the k -COLOURING problem is constant-time solvable on claw-free graphs of diameter at most d . In the same paper [20], this result was generalized to the case where H is the *chair*, which is the graph obtained from the claw after subdividing exactly one of its edges. The chair is also known as the *fork*.

Our Results. We first consider the INDEPENDENT SET problem. For this problem we will prove new NP-hardness results that show that bounding the diameter does not help. To explain this, let the graph $S_{h,i,j}$, for $1 \leq h \leq i \leq j$, be the *subdivided claw*, which is the tree with one vertex x of degree 3 and exactly three leaves, which are at distance h , i and j from x , respectively. Note that $S_{1,1,1}$ is the claw $K_{1,3}$, the graph $S_{1,1,2}$ is the chair and that every subdivided claw is a polyad. Let \mathcal{S} be the set of graphs, each connected component of which is a subdivided claw or a path. Alekseev [1] proved that for every finite set of graphs \mathcal{F} , if no graph from \mathcal{F} belongs to \mathcal{S} , then INDEPENDENT SET is NP-complete for the class of \mathcal{F} -free graphs. In Section 2, we show that exactly the same NP-completeness result holds for the class of \mathcal{F} -free graphs of diameter 2 if and only if $|\mathcal{F}| = 1$.

We then turn to the class of H -free graphs where H is a polyad. First, we focus on the case where H is the chair. In Section 3, we prove that for every integer $d \geq 1$, a number of vertex partitioning problems that require yes-instances to be 3-colourable become polynomial-time solvable on chair-free graphs of diameter at most d . The problems are ACYCLIC 3-COLOURING, STAR 3-COLOURING, NEAR-BIPARTITENESS, INDEPENDENT FEEDBACK VERTEX SET and INDEPENDENT ODD CYCLE TRANSVERSAL. We define these problems below.

Our proof is based on a common strategy. Namely, we determine the following for every chair-free 3-colourable non-bipartite input graph G of bounded diameter: either G has a constant number of 3-colourings or there exists a set S such that $G - S$ has this property. We prove that we can let S be the set of private neighbours of some vertex u of a triangle on vertices u, v, w , that is, the vertices of S are adjacent to neither v nor w . We then consider each constructed 3-colouring c and determine in polynomial time if we can extend c to a solution for the vertex partitioning problem under consideration.

In Section 4 we prove that there is little hope of a full extension from the chair to arbitrary polyads H . To be more precise, we prove that for ACYCLIC 3-COLOURING, STAR 3-COLOURING and INDEPENDENT ODD CYCLE TRANSVERSAL, there exists a polyad H and a constant d such that each of these problems is NP-complete for the class of H -free graphs of diameter at most d . In the same section we give some relevant open problems.

Additional Terminology. A graph is *acyclic 3-colourable* or *star 3-colourable* if it is 3-colourable and every two colour classes induce a forest or a star forest, respectively (in this context, the P_1 and P_2 are seen as stars). The corresponding decision problems are ACYCLIC 3-COLOURING and STAR 3-COLOURING. A graph G is *near-bipartite* if its vertex set can be partitioned into an independent set I and a forest F ; we also say that I is an *independent feedback vertex set* of G . The problems NEAR-BIPARTITENESS and INDEPENDENT FEEDBACK VERTEX SET are to decide if a graph is near-bipartite or has an independent feedback vertex set of size at most k for some given integer k . A subset $S \subseteq V$ of a graph $G = (V, E)$ is an *independent odd cycle transversal* if S is independent and $G - S$ is bipartite. Note that a graph is 3-colourable if and only if it has an independent odd cycle transversal. The INDEPENDENT ODD CYCLE TRANSVERSAL problem is to decide if a given graph has an independent odd cycle transversal of size at most k for some given integer k .

Let C_r , P_r and K_r be the cycle, path and complete graph on r vertices. The graph $G + H = (V(G) \cup V(H), E(G) \cup E(H))$ is the disjoint union of graphs G and H , and sG is the disjoint union of s copies of G . A graph G is \mathcal{H} -free if G is H -free for every $H \in \mathcal{H}$.

2 Independent Set

We let \mathcal{S} denote the set of graphs, each connected component of which is either a subdivided claw or a path. The following well-known result is due to Alekseev.

► **Theorem 1** ([1]). *Let \mathcal{F} be a finite set of graphs. If no graph from \mathcal{F} belongs to \mathcal{S} , then INDEPENDENT SET is NP-complete for \mathcal{F} -free graphs.*

We strengthen Theorem 1 to \mathcal{F} -free graphs of diameter 2 if $|\mathcal{F}| = 1$. We need two lemmas. We sketch the proof of the first lemma.

► **Lemma 2.** *For every $r \geq 3$, INDEPENDENT SET is NP-complete for C_r -free graphs of diameter 2.*

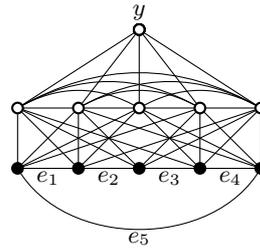
Proof. First suppose that $r = 3$. By Theorem 1, INDEPENDENT SET is NP-complete for C_3 -free graphs. Let (G, k) be an instance of INDEPENDENT SET, where G is an n -vertex C_3 -free graph. We may assume without loss of generality that $n \geq 2$ and $k \geq 2$; otherwise, the problem is trivial. We also assume that G has no dominating vertex. Otherwise, if u is a dominating vertex, then G has an independent set of size at least $k \geq 2$ if and only if $G - u$ has an independent set of size at least k . Hence, by the iterative deletion of universal vertices, we either solve the problem or obtain an equivalent instance without universal vertices. From G we now construct a graph G' as follows:

- Construct a copy of G .
- For every pair $\{u, v\}$ of nonadjacent vertices of G ,
 - construct an arbitrary inclusion maximal independent set I_{uv} of G containing u and v ,
 - construct a vertex x_{uv} and make x_{uv} adjacent to every vertex of I_{uv} .
 Denote by X the set of all vertices x_{uv} constructed for the pairs of nonadjacent u and v .
- Construct an independent set of n^2 vertices Y and make every vertex of Y adjacent to every vertex of X .

Observe that for every pair $\{u, v\}$ of nonadjacent vertices of G , I_{uv} can be constructed by the straightforward greedy procedure starting from the set $\{u, v\}$. This implies that G can be constructed in polynomial time. Our construction immediately implies that G' is triangle free, because the initial graph G is triangle-free and the neighbourhood of every vertex from $X \cup Y$ is an independent set. We claim that the diameter of G' is at most 2, and moreover that G has an independent set of size at least k if and only if G' has an independent set of size at least k' . Proof details are omitted. ◀

► **Lemma 3.** *INDEPENDENT SET is NP-complete for $K_{1,4}$ -free graphs of diameter 2.*

Proof. By Theorem 1, INDEPENDENT SET is NP-hard for $(C_3, K_{1,4})$ -free graphs. Let (G, k) be an instance of INDEPENDENT SET, where G is a $(C_3, K_{1,4})$ -free graph of order n and size m . We may assume without loss of generality that G is connected. Note that G is subcubic, that is, has maximum degree at most 3. We may assume without loss of generality that G has no vertices of degree 1 (as we can pick such vertices to be in the independent set and remove their neighbours from G until this operation can no longer be applied). To G , we add for every pair of edges e_1 and e_2 of G that do not share an end-vertex, a new vertex x_{e_1, e_2} and an edge between x_{e_1, e_2} and the two end-vertices of both e_1 and e_2 . We also add a new vertex y and all edges such that $\{x_{e_1, e_2} : e_1, e_2\} \cup \{y\}$ induces a complete subgraph. Let G' be the resulting graph and let $X := \{x_{e_1, e_2} : e_1, e_2\} \cup \{y\}$. See also Fig. 1.



■ **Figure 1** The graph G' in the proof of Lemma 3 if $G = C_5$. The black vertices are those of G and the white vertices in the middle are $x_{e_1e_3}, x_{e_2e_5}, x_{e_1e_4}, x_{e_3e_5},$ and $x_{e_2e_4}$ from left to right.

For a contradiction, assume that there is some induced $K_{1,4}$ in G' , say z is its centre vertex and z_1, z_2, z_3, z_4 are its leaves. Since $N_{G'}(x)$ can be partitioned into at most 3 cliques for each $x \in X$, we have $z \in V(G)$. Furthermore, since G is subcubic and X is a clique, X contains at least and at most one vertex of $\{z_1, z_2, z_3, z_4\}$, respectively. As we see now, $N_G(z) \subseteq \{z_1, z_2, z_3, z_4\}$. However, each vertex of X that is adjacent to z , has a neighbour in $N_G(z)$, which contradicts our supposition that z is the centre vertex of an induced $K_{1,4}$.

To show that G' has diameter 2, first consider an arbitrary vertex u of G . As G is connected, there is a vertex $v \in N_G(u)$. As G has minimum degree at least 2 and G is triangle-free, there are vertices $u' \in N_G(u) \setminus \{v\}$ and $v' \in N_G(v) \setminus \{u, u'\}$. For $e_1 = uu'$ and $e_2 = vv'$, e_1 and e_2 do not have a common end-vertex, $u, x \in N_{G'}[x_{e_1, e_2}]$, and $dist_{G'}(u, x) \leq 2$ for each $x \in X$. Additionally, if $w \in V(G) \setminus \{u\}$ is a vertex with $dist_G(u, w) \geq 3$, then, since G is connected, there are edges $e_3, e_4 \in E(G)$ such that u is incident to e_3 , w is incident to e_4 , and the two edges e_3, e_4 have no common incident vertex. Thus, $u, w \in N_G(x_{e_3, e_4})$ and so $dist_{G'}(u, w) \leq 2$. As X is a clique, we conclude that G' has diameter 2.

We observe that $I \cup \{y\}$ is an independent set of size $k + 1$ in G' if I is an independent set of size k in G . Vice versa, given an independent set I' of size $k + 1$ in G' , at most one of its vertices belongs to Y and $I' \setminus X$ is an independent set of size at least k in G .

The graph G' can be constructed in time $\mathcal{O}(m^2)$ and has at most $n + m^2$ vertices. Vice versa, given an independent set I' in G' , the set $I' \setminus X$ can be constructed in time $\mathcal{O}(m^2)$. ◀

We now strengthen Theorem 1, but can only do this for the case where $|\mathcal{F}| = 1$. For example, if $\mathcal{F} = \{C_3, K_{1,4}\}$, every \mathcal{F} -free graph has maximum degree 3. Then every \mathcal{F} -free graph with bounded diameter has constant size. Hence, INDEPENDENT SET is NP-complete for \mathcal{F} -free graphs by Theorem 1 but constant-time solvable for \mathcal{F} -free graphs of bounded diameter.

► **Theorem 4.** *Let H be a graph. If $H \notin \mathcal{S}$, then INDEPENDENT SET is NP-complete for H -free graphs of diameter at most 2. If $H \in \mathcal{S}$, then INDEPENDENT SET for H -free graphs is polynomially equivalent to INDEPENDENT SET for H -free graphs of diameter at most 2.*

Proof. Let H be a graph. First assume that $H \notin \mathcal{S}$. If H contains an induced cycle C_r for some $r \geq 3$, then the class of H -free graphs contains the class of C_r -free graphs, and we use Lemma 2. Otherwise H is a forest with either an induced $K_{1,4}$ or a connected component that has at least two vertices of degree 3. In the first case, we use Lemma 3. In the second case, we reduce from INDEPENDENT SET for H -free graphs, which is NP-complete by Theorem 1. Let (G, k) be an instance of INDEPENDENT SET, where G is an H -free graph. We add a dominating vertex to G to obtain a graph G' . As H has no dominating vertex, G' is H -free. We also note that (G, k) and (G', k) are equivalent instances.

Now assume that $H \in \mathcal{S}$. Any polynomial-time algorithm for INDEPENDENT SET on H -free graphs can be used on H -free graphs of diameter 2. As INDEPENDENT SET is polynomial-time solvable for $K_{1,3}$ -free graphs [28], we may assume that $H \notin \{K_{1,3}, P_1, P_2, P_3\}$. Any polynomial-time algorithm for INDEPENDENT SET on H -free graphs of diameter 2 can be used on H -free graphs of arbitrary diameter as follows. To the H -free input graph G we add a dominating vertex. As $H \in \mathcal{S} \setminus \{K_{1,3}, P_1, P_2, P_3\}$, this yields an H -free graph G' of diameter 2. We then observe that G has an independent set of size at least k if and only if G' has an independent set of size at least k . ◀

3 Chair-Free Graphs of Bounded Diameter

It follows from Ramsey's Theorem that every k -colourable $K_{1,r}$ -free graph of diameter at most d has order bounded by a function in d, k, r ; see [20] for a proof of this observation. As a consequence, every problem that has the property that all its yes-instances are k -colourable for some constant k is constant-time solvable on $K_{1,r}$ -free graphs of diameter at most d . We aim to extend the above observation to H -free graphs of bounded diameter when H is obtained from a star after at least one edge subdivision. In [20], a number of results are given for 3-COLOURING for such graph classes. We consider a variety of problems that require all the yes-instances to be 3-colourable. We focus on the first interesting case which is where H is the chair $S_{1,1,2}$ (recall that the chair is obtained from the claw after subdividing one edge).

We need the following characterization of bipartite chair-free graphs, due to Alekseev. A *complex* is a bipartite graph that can be obtained by removing the edges of a possibly empty matching from a complete bipartite graph.

► **Theorem 5** ([2]). *If G is a connected bipartite chair-free graph, then G is a cycle or a path or a complex.*

It is well-known (cf. [18]) that finding the components of a graph by breadth-first search takes $O(n + m)$ time. Let p be the number of components of a graph G , n be its order, and m be its size. Then G is a forest if and only if $p = n - m$, which implies the following result.

► **Observation 6.** *If G is a graph, then we can decide if G is a forest in $O(n + m)$ time.*

If T is a tree of order n , then its diameter is at most 2 if and only if its maximum degree equals $n - 1$. Therefore, we can decide whether a given graph is a forest each component of which is of diameter at most 2 in $O(n + m)$ time. When working with vertex labellings, our findings imply the following observation.

► **Observation 7.** *If G is a graph and ℓ is a vertex labelling of G with labels 1, 2, and 3, then we can decide whether ℓ is a 3-colouring, star 3-colouring, or acyclic 3-colouring of G in $O(n + m)$ time.*

It is also well-known that we can use breadth-first search for deciding whether a given graph G is bipartite and, if so, we are in a position to determine its parts in the same time. By this fact, we obtain the following result.

► **Observation 8.** *If G is a graph, k is an integer, and ℓ is a vertex labelling of G with labels 1, 2, and 3, then we can decide whether one colour class of ℓ is an independent feedback vertex set or an independent odd cycle transversal (of size at most k) in $O(n + m)$ time.*

To prove our results we need some more terminology. A *list assignment* of a graph G is a function L that gives each vertex $u \in V(G)$ a (finite) *list of admissible colours* $L(u) \subseteq \{1, 2, \dots\}$. A colouring c *respects* L if $c(u) \in L(u)$ for every $u \in V(G)$. If $|L(u)| \leq 2$

for each $u \in V(G)$, then L is a 2-list assignment. The 2-LIST COLOURING problem is to decide if a graph G with a 2-list assignment L has a colouring that respects L . We use the following result.

► **Lemma 9** ([15]). *The 2-LIST COLOURING problem is solvable in $O(n + m)$ time on graphs with n vertices and m edges.*

Let G be a graph of diameter d for some $d \geq 1$ and S be some set of vertices of G . A vertex $u \notin S$ is a *private neighbour* of a vertex $v \in S$ with respect to S if u is adjacent to v but non-adjacent to any other vertex of S . We let $P(v)$ be the set of private neighbours of v with respect to S . Let $N_0 = S$ and, for $i \in \{1, \dots, d\}$, let N_i be the set of vertices that do not belong to $N_0 \cup N_1 \cup \dots \cup N_{i-1}$ but do have a neighbour in N_{i-1} . As G has diameter at most d , we partition $V(G)$ into the sets N_0, N_1, \dots, N_d (where some sets might be empty). We say that we *partition $V(G)$ from S* . Note that this partitioning takes $O(n + m)$ time by breadth-first search on the graph G' which is obtained from G by adding a new vertex u and all edges from u to every vertex of S .

In [20], it was shown that 3-COLOURING is polynomial-time solvable for chair-free graphs of bounded diameter. The first statement of Theorem 10 below is proven by similar but more precise arguments. The second statement is a new result that requires new arguments.

► **Theorem 10.** *Let $d \geq 1$ be an integer and G be a chair-free non-bipartite graph of diameter d with n vertices and m edges.*

1. *We can decide whether G is 3-colourable in $O(n + m)$ time.*
2. *If G is 3-colourable, then we find in $O(n + m)$ time either all 3-colourings of G , or a triangle $xyzx$ in G with exactly one vertex, say x , that has private neighbours and all 3-colourings of $G - P(x)$ that can be extended to 3-colourings of G . In both cases, we find at most $3^{9 \cdot 2^d + 8}$ 3-colourings.*

Proof. We first check in constant time whether G is of order at most $2d + 1$. If so, then we can determine in constant time all 3-colourings of G and these are at most 3^{2d+1} . Note that $3^{2d+1} < 3^{9 \cdot 2^d + 8}$. We proceed by assuming that G is of order at least $2d + 2$ and claim that G contains a triangle. We prove this claim by contradiction: assume that G is triangle-free. As G is not bipartite, there is an odd cycle in G . Let $x_1 x_2 \dots x_p x_1$ be a shortest one. As G is triangle-free and of diameter d , we find $5 \leq p \leq 2d + 1$, respectively. Moreover, as G is of order at least $2d + 2$, there is some vertex outside this cycle that has a neighbour on this cycle. Without loss of generality let us assume y with $y \notin \{x_1, x_2, \dots, x_p\}$ is adjacent to x_1 . As G is triangle-free, y does not have two consecutive neighbours on $x_1 x_2 \dots x_p x_1$. As G is chair-free and y is neither adjacent to x_2 nor to x_p , we find that y must be adjacent to x_3 . We repeat this argument and obtain that y is adjacent to x_{2q+1} for every $0 \leq q \leq \lfloor \frac{p}{2} \rfloor$. In particular y is adjacent to the two consecutive vertices x_1 and x_p , a contradiction. We conclude that our assumption is false and that G contains a triangle.

We continue and show that we can compute a triangle, say T , of G in $O(n + m)$ time. Let u be a vertex of G . We partition $V(G)$ from $\{u\}$ and note that breadth-first search computes a breadth-first tree F , that is, F is a spanning tree of G such that each vertex of N_i has distance i to u in F for any i . As G is not bipartite, there has to be an edge e and an integer i such that e is incident to two vertices of N_i . We can compute such an edge that additionally minimizes i in $O(n + m)$ time. By adding this edge to F , we find an odd cycle C in G . As F is of diameter at most $2d$, we find that C has at most $2d + 1$ vertices. Hence, we can determine in constant time a shortest induced odd cycle, say C' , in $G[V(C)]$. We check in constant time whether C' is a triangle. If not, then C' is of order at least 5. As

G is of order at least $2d + 2$, there is a vertex outside C' that has a neighbour on C' . We compute such a vertex, say y , in $O(n + m)$ time. As shown above, y has two consecutive neighbours on C' . As C' has at most $2d + 1$ vertices, we can find such two vertices, and thus a triangle in G , in constant time.

Let $\{x, y, z\}$ be the vertex set of the triangle T . We partition $V(G)$ from $V(T)$ in $O(n + m)$ time. We additionally determine all private neighbours of the vertices of T and all vertices of N_1 that are adjacent to all vertices of T in linear time. If there is a vertex of the latter type, then G is not 3-colourable. Thus, we focus on the case where each vertex of N_1 is adjacent to at most two vertices of T . We compute in linear time the set N_1^* of all vertices of N_1 that have two neighbours of T . Clearly, $S = N_1 \setminus N_1^*$ consists of all private neighbours of the vertices of T , and its computation takes linear time. We proceed by considering $G - N_1$. We check in linear time if this graph has at most $9 \cdot 2^d + 2$ vertices.

Let us consider the subcase where $G - N_1$ has at least $9 \cdot 2^d + 3$ vertices. We claim that G is not 3-colourable and prove this claim by contradiction: assume that G is 3-colourable. Hence, G is K_4 -free. Recall that every vertex of N_1 has at most two neighbours on T . Let $i \geq 1$ and u be a vertex of N_i . As G is chair-free, the neighbours of u in N_{i+1} form a clique. As G is K_4 -free, we obtain that u has at most 2 neighbours in N_{i+1} . It follows that

$$3 + 9 \cdot 2^d \leq |N_0| + |N_2| + |N_3| + \dots + |N_d| \leq 3 + |N_2| \cdot \sum_{i=2}^d 2^{i-2} < 3 + |N_2| \cdot 2^{d-1}.$$

Hence, $|N_2| > 18$. We let N_2^* be the neighbours of N_1^* in N_2 . Consider the set N_{xy} of common neighbours of x and y in N_1^* . The set N_{xy} is an independent set as G is K_4 -free. Every vertex $u \in N_2^*$ with a neighbour v in N_{xy} must be adjacent to every vertex in N_{xy} , as G is chair-free. For the same reason, no vertex of N_1^* has two non-adjacent neighbours in N_2^* . As G is K_4 -free, this means that there are at most two vertices in N_2^* that are adjacent to the vertices of N_{xy} . By applying the same reasoning for every other pair of vertices of T , we find that N_2^* has size at most 6. Thus, $|N_2 \setminus N_2^*| > 12$. As every vertex of N_1 has at most two neighbours in N_2 , it follows that $|S| > 6$. We consider the subcase where at least two vertices, say x and y , of T have a private neighbour. Assume that x has two non-adjacent private neighbours u and v in S . Then these three vertices, together with y and a private neighbour $w \in S$ of y induce a chair unless w is adjacent to at least one of u and v . If w is adjacent to u but not to v , then $\{u, v, w, x, z\}$ induces a chair. Hence, w is adjacent to both u and v , but then $\{u, v, w, y, z\}$ induces a chair. Therefore, the private neighbours of every vertex of T form a clique. As G is 3-colourable, we find $|S| \leq 6$ if at least two vertices of T have private neighbours. Thus, we obtain that all vertices of S are adjacent to a common vertex, say x , of T . As G is 3-colourable, we find that $G[S]$ is bipartite. Therefore, we partition S into two independent sets A and B (one of these two sets might be empty). As G is chair-free and as A is independent, the vertices of A share the same set of neighbours in N_2 . Similarly, the vertices of B share the same set of neighbours in N_2 . As G is chair-free and K_4 -free, the neighbourhood of every vertex of $A \cup B$ is a clique of size at most 2. We conclude that the total number of vertices in N_2 with a neighbour in S is at most 4, a contradiction as $|N_2 \setminus N_2^*| > 12$. We find that G is not 3-colourable and proceed by assuming that $G - N_1$ has at most $9 \cdot 2^d + 2$ vertices.

We consider every vertex labelling of $G - N_1$ with labels 1, 2, 3 and determine in $O(n + m)$ time which ones lead to a 3-colouring of G . We discard those labellings which are not a 3-colouring of $G - N_1$. Given a 3-colouring of $G - N_1$, each vertex of N_1^* receives the remaining available label that is not used for its neighbours of T . Note that this assignment takes linear time. We discard in $O(n + m)$ time those labellings which do not lead to a

3-colouring of $G - S$. Let us take an arbitrary 3-colouring of $G - S$. We assign lists to the vertices of G as follows: we set $L(u) = \{i\}$, where i is the label of u , if $u \notin S$ and we set $L(u) = \{1, 2, 3\} \setminus \{i\}$, where i is the label of the unique neighbour of u on T , if $u \in S$. Thus, checking whether a given 3-colouring of $G - S$ leads to a 3-colouring of G takes $O(n+m)$ time as (G, L) is an instance of 2-LIST COLOURING (cf. Lemma 9). We discard those 3-colouring of $G - S$ which do not lead to a 3-colouring of G . If no 3-colouring of $G - S$ lead to a 3-colouring of G , then G is not 3-colourable. Hence, we proceed by assuming that at least one does, and so we find that G is 3-colourable. As there are at most $3^{9 \cdot 2^d + 2}$ vertex labellings of $G - N_1$, we can determine all 3-colourings of $G - S$ that can be extended to 3-colourings of G in $O(n+m)$ time, and there are at most $3^{9 \cdot 2^d + 2}$ such colourings of $G - S$.

If $S = \emptyset$, then $G - S$ equals G . We consider the subcase where at least two vertices of T have a private neighbour. As shown above, the private neighbours of every vertex of T form a clique. If $|S| > 6$, which we check in constant time, then G is not 3-colourable. Otherwise, as we have at most $3^{9 \cdot 2^d + 2}$ 3-colourings of $G - S$, we find at most $3^{9 \cdot 2^d + 8}$ 3-colourings of G and their computation takes $O(n+m)$ time. We finally consider the subcase where all vertices of S are adjacent to a single vertex, say x , of T . We conclude that $S = P(x)$, which completes our proof. \blacktriangleleft

We are now in a position to prove our main result of this section.

► Theorem 11. *If $d \geq 1$, then 3-COLOURING, ACYCLIC 3-COLOURING, STAR 3-COLOURING, INDEPENDENT ODD CYCLE TRANSVERSAL, INDEPENDENT FEEDBACK VERTEX SET, and NEAR-BIPARTITENESS can be solved in $O(n+m)$ time for chair-free graphs of diameter at most d .*

Proof. Let G be a chair-free graph of diameter at most d with n vertices and m edges. Note that G is acyclic 3-colourable or star 3-colourable only if G is 3-colourable. Moreover, if I is an independent set of G for which $G - I$ is a bipartite graph, then G is 3-colourable. Hence, our problems require all the yes-instances to be 3-colourable. If $d = 1$, then G is 3-colourable if and only if G has at most 3 vertices, and so each of our problems can be solved in constant time. We proceed by assuming $d \geq 2$ and check in $O(n+m)$ time whether G is bipartite.

Case 1. G is bipartite.

Note that G is 3-colourable, near-bipartite, and has an independent odd cycle transversal of size at most k for any integer k . We can determine the parts, say S_1 and S_2 , of G in $O(n+m)$ time. We may assume without loss of generality that $|S_1| \geq |S_2|$. We check in constant time whether $|S_1| + |S_2| \leq \max\{8, 2d\}$ and if so, then we can solve each of our problems in constant time. Otherwise, we find that $|S_1| \geq 5$. As bipartite graphs of maximum degree at most 2 and diameter at most d are paths or cycles of at most $2d$ vertices, we find that G has a vertex of degree at least 3, and so G is a complex by Theorem 5.

We first claim that in the case where G is a complex with $|S_1| \geq 5$, G is star 3-colourable if $|S_2| \leq 2$ and acyclic 3-colourable only if $|S_2| \leq 2$. Note that this claim completes the bipartite case for ACYCLIC 3-COLOURING and STAR 3-COLOURING as we can decide whether $|S_2| \leq 2$ or not in constant time and as every star 3-colouring of a graph is acyclic. We prove our claim as follows: If $|S_2| \leq 2$, then, for any $s \in S_2$, $G - s$ is a forest each component of which is of diameter at most 2, and thus G is star 3-colourable with colour classes $S_1, S_2 \setminus \{s\}$, and $\{s\}$. If $|S_2| \geq 3$, then let c be an arbitrary 3-colouring of G . By the pigeonhole principle there exists a colour class X of c that contains at least two vertices of S_1 , and so $X \cap S_2 = \emptyset$. As $|S_2| \geq 3$, there are two vertices $s_2, s'_2 \in S_2$ that are coloured alike. As $|S_1| \geq 5$, and as s_2 and s'_2 are of degree at least $|S_1| - 1$, we find that s_2 and s'_2 have at least three common

neighbours in S_1 two of which, say s_1 and s'_1 , are coloured alike. Hence, $s_1 s_2 s'_1 s'_2 s_1$ is a bichromatic 4-cycle. We conclude that every 3-colouring of G is not acyclic, which completes the proof of our claim.

It remains to consider INDEPENDENT FEEDBACK VERTEX SET for complexes with at least 9 vertices. Let k be an arbitrary integer. We claim that in the case where G is a complex with $|S_1| \geq 5$, G has an independent feedback vertex set of size at most k if and only if $k \geq |S_2| - 1$. Note that the latter can be decided in linear time. We prove our claim as follows: If $|S_2| \leq 2$, then $G - s$ is a forest for any $s \in S_2$ and G has an independent feedback vertex set of size at most k . Hence, we may assume $|S_2| \geq 3$. Let I be a minimum independent feedback vertex set in G . Such a set exists as G is bipartite. As $S_2 \setminus \{s\}$ is independent and as $G[S_1 \cup \{s\}]$ is a forest for each vertex $s \in S_2$, we find $|I| \leq |S_2| - 1$. For the sake of a contradiction, let us assume $|I| \leq |S_2| - 2$. Hence, any two vertices of $S_2 \setminus I$ have at least $|S_1| - 2$ common neighbours in S_1 , and so $|I \cap S_1| \geq |S_1| - 3 \geq 2$. Moreover, $I = I \cap S_1$ as every vertex of S_2 has a neighbour in $I \cap S_1$ and as I is independent. As I is an independent feedback vertex set with $|I| \leq |S_1| - 2$, any two vertices of $S_1 \setminus I$ do not have two common neighbours in S_2 and so $|S_2| \leq 3$. Hence, $5 \leq |S_1| \leq |I| + 3 \leq |S_2| + 1 \leq 4$, a contradiction. As $|I| = |S_2| - 1$, the proof of our claim is complete.

Case 2. G is not bipartite.

Outline. As our problems require all the yes-instances to be 3-colourable, we check first whether G is 3-colourable. If so, then we compute an induced subgraph H of G and determine the set \mathcal{C} of all its 3-colourings that can be extended to 3-colourings of G . As we compute H by applying Theorem 10, we find that $|\mathcal{C}| \leq 3^{9 \cdot 2^d + 8}$. We then distinguish some subcases. In some of them we further branch by extending our 3-colourings. However, in some of them we find that H equals G , and so Observations 7 and 8 imply that our six problems are solvable in $O(n + m)$ time as \mathcal{C} is of constant size. As an implicit step, we apply this finding whenever H is the whole graph G .

Full Proof. We first apply Theorem 10. We continue by assuming that G is 3-colourable. In fact, the only remaining case is that where the lemma provides a triangle T on vertex set $\{x, y, z\}$, a vertex x of T that has private neighbours, and the set of all 3-colourings of $G - P(x)$ that can be extended to 3-colourings of G . Note that we have at most $3^{9 \cdot 2^d + 8}$ such 3-colourings. We partition $V(G)$ from $V(T)$.

We find that $G[P(x)]$ is bipartite, as G is 3-colourable, but not necessarily connected. We extend each 3-colouring of $G - P(x)$ that can be extended to a 3-colouring of G to some vertices of $P(x)$. Let c be an arbitrary 3-colouring of $G - P(x)$ that can be extended to a 3-colouring of G . For $i \in \{0, 1, 2\}$, we compute in $O(n + m)$ time the set S_i of all vertices of $P(x)$ which have i available colours with respect to c , that is, S_i is the set of all vertices of $P(x)$ which have neighbours in $3 - i$ colours. As c can be extended to a 3-colouring of G , we find that S_0 is empty. It takes $O(n + m)$ time to determine the available colour of each vertex in S_1 . Furthermore, we can extend c by breadth-first search in the same time to the vertices of those components of $G[P(x)]$ that contain at least one vertex of S_1 .

Let S_c be the set of vertices that induce those components of $G[P(x)]$ that do not contain a vertex of S_1 . Note that S_c can be computed in $O(n + m)$ time and that all neighbours of all vertices of S_c in $V(G) \setminus S_c$ are coloured alike. Moreover, every vertex of S_c has its neighbours in $[N(y) \cap N(z)] \cup N_2 \cup S_c \cup \{x\}$ by definition. As c can be extended to a 3-colouring of G , we find that our approach leads to a 3-colouring, say c' , of $G - S_c$. As there are at most $3^{9 \cdot 2^6 + 2}$ 3-colourings of $G - P(x)$, we find at most $3^{9 \cdot 2^6 + 2}$ such triples (c, c', S_c) . Furthermore, for each 3-colouring c_s of $G - P(x)$, there exists a triple (c_s, c'_s, S_{c_s}) if c_s can

be extended to a 3-colouring of G . We proceed by considering the case where $S_c \neq \emptyset$ as otherwise $G = G - S_c$. We continue by distinguishing on the problems we are considering. Recall that G is 3-colourable.

Subcase 2.1. ACYCLIC 3-COLOURING and STAR 3-COLOURING

We check whether for some triple (c, c', S_c) , the 3-colouring c' of $G - S_c$ that can be extended to an acyclic 3-colouring or star 3-colouring of G . By this approach, we clearly solve ACYCLIC 3-COLOURING and STAR 3-COLOURING.

Let (c, c', S_c) be an arbitrary triple as defined above. Recall that a star 3-colouring of a graph is acyclic. In time $O(n + m)$, we can determine the components of $G[S_c]$ and check whether $G[S_c]$ is a forest. If not, then $G[S_c \cup \{x\}]$, and thus G , is not acyclic 3-colourable. We continue and assume that $G[S_c]$ is a forest. We check in $O(n + m)$ time if a vertex of S_c has a neighbour in $N(y) \cap N(z)$. If so, say $s \in S_c$ is adjacent to $v \in N(y) \cap N(z)$, then c' cannot be extended to an acyclic 3-colouring of c as either s and x are coloured alike or one of $\{svyxs, svzxs\}$ is a bichromatic 4-cycle. We proceed by assuming that S_c has its neighbours in $N_2 \cup S_c \cup \{x\}$. As G is chair-free, every two non-adjacent vertices of S_c share the same neighbours in N_2 and, if there exists such a neighbour, then these two vertices have to be coloured differently to avoid a bichromatic 4-cycle. Therefore, in any acyclic extension of c' to G , each of the two colour classes in S_c either has size at most 1 or has no neighbour in N_2 . We check in constant time if S_c is of size at most 2. If so, then there are at most 4 possibilities to extend c' to a 3-colouring of G and for each we apply Observation 7. Hence, we may assume $|S_c| \geq 3$. We check in $O(n + m)$ time if a vertex of S_c has a neighbour in N_2 .

Let us consider the subcase where $s \in S_c$ has a neighbour, say v , in N_2 . Let G_s be the component of $G[P(x)]$ that contains s . Note that there are at most two possibilities to extend c' to the vertices of G_s . We check in linear time if $S_c \setminus V(G_s)$ is of size at least 2. If so, say $s_1, s_2 \in S_c \setminus V(G_s)$, then v is a neighbour of s, s_1 , and s_2 . Thus, xs_1vs_2x is a bichromatic 4-cycle for two vertices s'_1 and s'_2 of $\{s, s_1, s_2\}$. We conclude that c' cannot be extended to an acyclic 3-colouring of G . Hence, we may assume $|S_c \setminus V(G_s)| \leq 1$, and so there are at most four possibilities to extend c' to a 3-colouring of G each of which can be obtained in $O(n + m)$ time. We apply Observation 7 for each. We proceed by assuming that no vertex of S_c has a neighbour in N_2 . In other words, each vertex of S_c has its neighbours in $S_c \cup \{x\}$.

As x is a cut-vertex of G , any extension of c' to a 3-colouring of G is acyclic if and only if c' is acyclic. Hence, we apply Observation 7 on $G - S_c$ and c' in order to solve ACYCLIC 3-COLOURING.

We now check in $O(n + m)$ time if each component of $G[S_c]$ is of diameter at most 2. If not, then $G[S_c \cup \{x\}]$, and thus G is not star 3-colourable. Let us proceed by assuming that each component of $G[S_c]$ is of diameter at most 2. We find that every 3-colouring of $G[S_c \cup \{x\}]$ is a star 3-colouring. In other words, we can restrict ourselves to those 3-colouring extensions of c' to G that assign one colour to all vertices of S_c if S_c is independent, and an arbitrary 3-colouring extensions of c' to G if S_c is not independent. Note that we can check in $O(n + m)$ time whether S_c is independent. We find in both subcases at most two extensions of c' to G and apply Observation 7 for each in order to solve STAR 3-COLOURING.

Subcase 2.2. INDEPENDENT ODD CYCLE TRANSVERSAL

Let k be an arbitrary integer. We check whether some triple (c, c', S_c) consists of a 3-colouring c' of $G - S_c$ that can be extended to a 3-colouring of G whose one colour class is an independent odd cycle transversal of size at most k . As all the yes-instances require G to be 3-colourable, this approach clearly solves INDEPENDENT ODD CYCLE TRANSVERSAL.

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Let (c, c', S_c) be an arbitrary triple as defined above. Moreover, let X, Y, Z be the colour classes of c' with $x \in X$, $y \in Y$, and $z \in Z$. Clearly, X, Y , and Z can be computed in linear time. We decide in linear time which of $\{Y, Z\}$ is of smaller size, say $|Y| \leq |Z|$. Recall that all vertices of S_c have their neighbours in $S_c \cup X$. Note that c' can be extended to a 3-colouring of G by 2-colourings of $G[S_c]$ on the colours that c' assigns to y and z , and these are the only possibilities. We find that the smallest possible colour class of a 3-colouring of G that extends c' consists of the vertices either in X or in $Y \cup W$, where W is the smallest possible colour class of a 2-colouring of $G[S_c]$. As we can compute the components of $G[S_c]$ and its parts in $O(n + m)$ time, we find W in the same time. Hence, the smallest possible independent odd cycle transversal of G that is a colour class of an extension of c' to a 3-colouring of G is of size $\min\{|X|, |Y \cup W|\}$. We can compare the sizes of X and $Y \cup W$ with k in linear time.

Subcase 2.3. INDEPENDENT FEEDBACK VERTEX SET and NEAR-BIPARTITENESS

Let k be an arbitrary integer. We check whether some triple (c, c', S_c) consists of a 3-colouring c' of $G - S_c$ that can be extended to a 3-colouring of G whose one colour class is an independent feedback vertex set (of size at most k). As all the yes-instances require G to be 3-colourable, this approach clearly solves INDEPENDENT FEEDBACK VERTEX SET and NEAR-BIPARTITENESS.

Let (c, c', S_c) be an arbitrary triple as defined above. Moreover, let X, Y, Z be the colour classes of c' with $x \in X$, $y \in Y$, and $z \in Z$. Clearly, X, Y , and Z can be computed in linear time. We check first whether $G - X$ is a forest in $O(n + m)$ time. If so, then we find that X is an independent feedback vertex set of G and we can determine its size in linear time. Hence, we proceed by assuming that $G - X$ contains a cycle or $|X| > k$. As we aim to find an extension of c' to a 3-colouring of G whose one colour class is an independent feedback vertex set (of size at most k), we find that such a set consists of the vertices of Y or of Z , and the vertices of some set $A \subseteq S_c$. Recall that all vertices of S_c have their neighbours in $[N(y) \cap N(z)] \cup N_2 \cup S_c \cup \{x\}$ and their neighbours in $[N(y) \cap N(z)] \cup N_2 \cup \{x\}$ form an independent set. Note that c' can be extended to a 3-colouring of G by 2-colourings of $G[S_c]$ on the colours that c' assigns to y and z , and these are the only possibilities. If $G[S_c]$ is connected, which can be tested in $O(n + m)$ time, then there are at most two such possibilities, and so we apply Observation 8 for each. We proceed by assuming that $G[S_c]$ is disconnected, and so $|S_c| \geq 2$.

We claim that all vertices of S_c have the same neighbours in N_2 . Let us assume that v is an arbitrary vertex of N_2 that is adjacent to some vertex of S_c . Let S_v be the set of neighbours of v in S_c . By definition, we find that S_v is non-empty. As G is chair-free, we obtain that every vertex of S_v is adjacent to every vertex of $S_c \setminus S_v$ as otherwise $\{s_1, s_2, v, x, y\}$ would induce a chair for some possible vertices $s_1 \in S_v$ and $s_2 \in S_c \setminus S_v$. As $G[S_c]$ is disconnected, we find that $S_c \setminus S_v = \emptyset$, which completes the proof of our claim as v is arbitrarily chosen.

We can check if there is a vertex in $N(y) \cap N(z)$ in $O(n + m)$ time. First assume there is such a vertex, say w . As $\{s_1, s_2, w, x, y\}$ does not induce a chair for each two vertices s_1, s_2 of an independent set I of $G[S_c]$, we find that w is adjacent to all but at most one vertex of I . As $G[S_c]$ is bipartite, it follows that w has at least $|S_c| - 2$ neighbours in S_c . For each $s \in N(w) \cap S_c$, we find $s \in A$ as $sxyws$ and $szxws$ are 4-cycles. Note that $N(w) \cap S_c$ can be computed in $O(n + m)$ time. As $|N(w) \cap S_c| \geq |S_c| - 2$, we find at most eight possibilities to extend c' to a 3-colouring of G by a 2-colouring of $G[S_c]$ in which one colour class contains all the vertices of $N(w) \cap S_c$. We apply Observation 8 for each. Hence, we may assume that $N(y) \cap N(z) = \emptyset$, and so every two vertices of S_c share the same neighbours in $V(G) \setminus S_c$.

If no vertex of N_2 has a neighbour in S_c , then x is a cut-vertex. In this case we find that G has an independent feedback vertex set of size at most k if and only if $G - S_c$ has an independent feedback vertex set (of size at most $k - |W|$, where W is the smallest possible colour class of a 2-colouring of $G[S_c]$). As W can be computed in $O(n + m)$ time, we apply Observation 8 for $G - S_c$ and c' .

We proceed by considering the situation where $v \in N_2$ has a neighbour in S_c . Recall that all vertices of S_c are adjacent to v . As xs_1vs_2x is a 4-cycle for any two vertices $s_1, s_2 \in S_c$, we find that A has size at least $|S_c| - 1$. In other words, we aim for such a 2-colouring of $G[S_c]$ whose one colour class is of size at most 1. If S_c is not independent, we have at most two such possibilities, and each leads to a 3-colouring of G . We apply Observation 8 for each. Now suppose that S_c is independent. We find that any two vertices of S_c have the same neighbours in G . Let us fix one vertex, say, s of S_c . As there is at most one vertex of S_c that is not in the independent feedback vertex set, we may assume that s is that vertex. We have four ways of colouring the vertices of S_c such that all vertices of $S_c \setminus \{s\}$ receive the same colour. It remains to apply Observation 8 for each case. ◀

4 A Final Result and Some Open Problems

We showed that bounding the diameter does not help for INDEPENDENT SET for H -free graphs. We proved that this does help for some problems related to 3-COLOURING if H is the chair. Whether these results can be extended to larger polyads H is an interesting but challenging task. For three of these problems, however, we should not seek to extend the theorems from the previous section to omission of arbitrary polyads: in our next result, we give a polyad H such that these problems are NP-complete for H -free graphs of diameter d for some constant d . We reduce from NOT-ALL-EQUAL 3-SAT, which is well-known to be NP-complete [29]. We omit the proof details.

► **Theorem 12.** *For STAR 3-COLOURING, INDEPENDENT ODD CYCLE TRANSVERSAL and ACYCLIC 3-COLOURING, there exists a polyad H and integer d so that the problem remains NP-complete on H -free graphs of diameter d .*

Finally, we ask if there exists a polyad H and an integer d such that NEAR-BIPARTITENESS and INDEPENDENT FEEDBACK VERTEX SET are NP-complete for H -free graphs of diameter at most d . Such a polyad H was already known to exist for 3-COLOURING of graphs of diameter at most 4 [20].

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