On Doctrines and Cartesian Bicategories

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Abstract

We study the relationship between cartesian bicategories and a specialisation of Lawvere's hyperdoctrines, namely elementary existential doctrines. Both provide different ways of abstracting the structural properties of logical systems: the former in algebraic terms based on a string diagrammatic calculus, the latter in universal terms using the fundamental notion of adjoint functor. We prove that these two approaches are related by an adjunction, which can be strengthened to an equivalence by imposing further constraints on doctrines.

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1 Introduction

In [21, 22, 23] Lawvere introduced the notion of *hyperdoctrine* in an effort to capture the *universal* content of logical theories, and first-order logic in particular. Here, by universal, we intend by means of *universal properties* in category theory. The starting point is the notion of Lawvere theory [20], the universal way of capturing the notion of algebraic theory – where the universal property is that of cartesian categories, namely categories with finite products. In terms of logical content, Lawvere theories provide the notion of *term*. Now, a hyperdoctrine is a certain contravariant functor P from the Lawvere theory of terms to a posetal 2-category, e.g. lattices or Heyting algebras. The basic, high-level idea is that the functor takes us from terms to *formulas*; more precisely, the objects of the Lawvere theories, which can be thought of as variable contexts, are taken to the Lindenbaum-Tarski algebra of formulas over these contexts. In this way, the concept of *quantifier* can be captured by means of a universal property – the existence of left-adjoints (existential quantification) and right-adjoints (universal quantification) to the image along P of the projections.

In recent years, there has been a large number [1, 2, 5, 10, 12, 13, 17, 28, 30] of contributions that use string diagrams in order to model computational phenomena of different kinds. Typically, the languages come with an equational theory, which can be used to reason about systems via diagrammatic reasoning. Interestingly, the *same* algebraic structures seem to appear in many different contexts, e.g. commutative monoids and comonoids, Frobenius algebras, Hopf algebras, etc. These applications, while using the language and tools of (monoidal) category theory, are of a rather different nature than the more established "universal" approaches, such as Lawvere theories and hyperdoctrines sketched in the previous paragraph. A bridge between the universal and algebraic worlds is given by a theorem of Fox [15] that characterises cartesian categories as symmetric monoidal categories where each object is equipped with a well-behaved commutative comonoid structure. This means that any Lawvere theory can be seen concretely as a string diagrammatic language (see e.g., [6]). More recently, the notion of discrete cartesian restriction category was characterised in a similar way [11], with partial Frobenius algebras taking the place of commutative comonoids. This raises a natural question: can we capture the universal content of logical theories algebraically in a similar way? In other words, what are the "Fox theorems" for logic?

In this paper we turn our attention to the regular fragment of first-order logic with equality: formulas are built up from terms and the equality relation using the existential quantifier and conjunction. There has been much work on categorifying this fragment, notably the significant corpus of work on allegories [16]. More relevant to our story, we focus on the contrast between universal and algebraic approaches. A universal treatment, the notion of *elementary existential doctrine*, was introduced in [23] and studied extensively in [26]. The basic setup is the same as for Lawvere's hyperdoctrines, but one asks only for the left adjoints, which, as we have previously mentioned, are the universal explanation for existential quantifiers. On the algebraic side, the concept that stands out is that of Carboni and Walters' cartesian bicategories (of relations) [9], which are poset-enriched symmetric monoidal categories where objects are equipped with a special Frobenius algebra and a lax-natural commutative comonoid structures. While Carboni and Walters emphasised the relational algebraic aspects, they were certainly aware of the logical connections. In fact, some recent works [4, 14, 29] exploited various ramifications of the correspondence between cartesian bicategories and regular logic.

Our goal for this paper is a "Fox theorem" for the regular fragment, connecting the universal and the algebraic approaches. Our starting observation is that, given a cartesian bicategory $(\mathbb{B}, \otimes, I)$, one obtains an elementary existential doctrine by restricting the hom functor $\mathrm{Hom}_{\mathbb{B}}(-, I)\colon \mathbb{B}^{\mathrm{op}} \to \mathsf{Set}$ to the (cartesian) category of *maps* of $\mathbb{B}$. In Remark 2.5 of [27], it is mentioned that the other direction is also possible: given an elementary existential doctrine one can construct a cartesian bicategory. We explore the ramifications of this remark in detail. We show that these two translations are functorial and, actually, that they form an adjunction. More precisely, it turns out that the category of cartesian bicategories is a reflective subcategory of the category of doctrines.

The adjunction, however, is *not* an equivalence. We prove this with a counterexample that captures the crux of the matter: there are doctrines $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ where the indexing categories of terms $\mathbb{C}$ are not tailored to the represented logics. In doctrines-as-logical-theories, roughly speaking, equality can come from two places: implicitly, from the indexing term category, and explicitly, via logical equivalence. Doctrines, therefore, have an additional degree of intensionality: doctrines that "substantially represent" the same logic may have distinct index categories and thus not be isomorphic. This issue does not arise in cartesian bicategories where the role of $\mathbb{C}$ is played by the subcategory of maps: maps are arrows satisfying certain properties, rather than given a priori in a fixed index category.

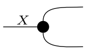
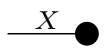
We conclude by observing that, by adding further constraints to the notion of elementary existential doctrine, namely comprehensive diagonals and the Rule of Unique Choice from [27], it is possible to exclude such problematic doctrines. By doing so, we restrict the adjunction to an equivalence, thus obtaining a satisfactory "Fox theorem".

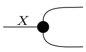
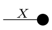
▶ Notation. Given $f\colon X \to Y$ and $g\colon Y \to Z$ morphisms in some category, we denote their composite as $f\,;g$, $g \circ f$ or gf. We write P_f for the action of a doctrine P on a morphism f.

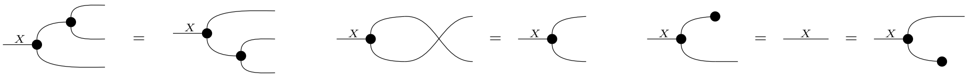
2 Cartesian Categories

The starting point of our exposition is the definition of cartesian category that, thanks to the results of Fox [15], can be given in the following form, which is particularly convenient for the purposes of this paper.

▶ **Definition 1.** *A cartesian category is a symmetric monoidal category* $(\mathbb{C}, \otimes, I)$ *where every object* $X \in \mathbb{C}$ *is equipped with morphisms*

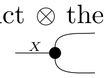
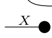
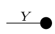
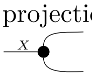
 $: X \to X \otimes X$ *and*  $: X \to I$ *such that*

1.  *and*  *form a cocommutative comonoid, that is they satisfy*



2. *Each morphism* $f\colon X \to Y$ *is a comonoid homomorphism, that is*

3. *The choice of comonoid on every object is coherent with the monoidal structure in the sense that*

Indeed, given a category $\mathbb{C}$ with finite products, one can construct a monoidal category as in the definition above, by taking as monoidal product $\otimes$ the categorical product and as its unit I the terminal object; for every object X,  is given by the pairing $\langle \mathrm{id}_X, \mathrm{id}_X\rangle\colon X \to X \otimes X$, hereafter denoted by Δ_X, and  by the unique morphism $!_X\colon X \to 1$. Conversely, given a symmetric monoidal category $(\mathbb{C}, \otimes, I)$ as in Definition 1, $\otimes$ forms a categorical product where projections $\pi_X : X \otimes Y \to X$ are given as $id_X \otimes$  and the pairing $\langle f, g\rangle\colon X \to Y \otimes Z$ as  $;(f \otimes g)$ for all arrows $f\colon X \to Y$ and $g\colon X \to Z$.

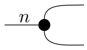
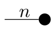
Hereafter, we will use $\times$ to denote both the cartesian product of sets and the categorical product in an arbitrary cartesian category. It is worth to remark that while Set (sets and functions) and Rel (sets and relations) are both cartesian categories, the categorical product in Set is indeed the cartesian product, while in Rel it is actually the disjoint union.

▶ **Example 2.** Another cartesian category that will play an important role is L_Σ, the *Lawvere theory* [20] generated by a cartesian signature Σ (a set of symbols f equipped with some arity $ar(f) \in \mathbb{N}$). In L_Σ, objects are natural numbers and arrows are tuples of terms over a

$$\frac{i \le n}{x_i \colon (n,1)}(V) \qquad \frac{f \in \Sigma \quad ar(f)=m \quad \langle t_1, \dots t_m\rangle \colon (n,m)}{f\langle t_1, \dots, t_m\rangle \colon (n,1)}(\Sigma) \qquad \frac{t_1 \colon (n,1) \quad \langle t_2, \dots, t_m\rangle \colon (n,m-1)}{\langle t_1, \dots, t_m\rangle \colon (n,m)}\langle \dots \rangle \qquad \frac{}{\langle\rangle \colon (n,0)}\langle\rangle$$

$$\frac{}{\top \colon (n,0)}(\top) \qquad \frac{P \in \mathbb{P} \quad ar(P)=m \quad \langle t_1, \dots t_m\rangle \colon (n,m)}{P\langle t_1, \dots, t_m\rangle \colon (n,0)}(\mathbb{P}) \qquad \frac{\phi \colon (n+1,0)}{\exists x_{n+1}.\phi \colon (n,0)}(\exists) \qquad \frac{\langle t_1, t_2\rangle \colon (n,2)}{t_1 = t_2 \colon (n,0)}(=) \qquad \frac{\phi \colon (n,0) \quad \psi \colon (n,0)}{\phi \wedge \psi \colon (n,0)}(\wedge)$$

Figure 1 Sort inference rules for L_Σ (top line) and for formulas in regular logic (bottom line).

countable set of variables $V = \{x_1, x_2, \dots\}$. More precisely, arrows from n to m are tuples $\langle t_1, \dots, t_m\rangle$ of sort (n,m) as defined by the inference rules in the first line of Figure 1. It is easy to check that $\langle t_1, \dots, t_m\rangle$ has sort (n,m) if each term t_i has variables in $\{x_1, \dots, x_n\}$. Composition is defined by (simultaneous) substitution: the composition of $\langle t_1, \dots, t_m\rangle \colon (n,m)$ with $\langle s_1, \dots, s_l\rangle \colon (m,l)$ is the tuple $\langle u_1, \dots, u_l\rangle \colon (n,l)$ where $u_i = s_i[^{t_1 \dots t_m}/_{x_1 \dots x_m}]$ for all $i = 1, \dots, l$. One can readily check that L_Σ is a symmetric monoidal category having $(\mathbb{N}, +, 0)$ as the monoid of objects, i.e., it is a *prop* (product and permutation category, see [19, 24]). Identities id_n and symmetries $\sigma_{n,m}$ are defined as expected;  $: n \to n+n$ is the tuple $\langle x_1, \dots, x_n, x_1, \dots, x_n\rangle$ thus acting as a *duplicator* of variables;  $: n \to 0$ is the empty tuple $\langle\rangle$, acting as a *discharger*.

▶ **Definition 3.** *A morphism of cartesian categories is a strict monoidal functor preserving the chosen comonoid structures.*

▶ **Example 4.** Let Σ_1 be the cartesian signature consisting of a single symbol f with arity 1, and Σ_2 be the signature with two symbols, g_1 and g_2, both of arity 1. Consider the corresponding Lawvere theories L_{Σ_1} and L_{Σ_2}. The assignment $f \mapsto g_1$ induces a morphism of cartesian categories, hereafter denoted by $F_1 \colon \mathsf{L}_{\Sigma_1} \to \mathsf{L}_{\Sigma_2}$. Similarly, let $F_2 \colon \mathsf{L}_{\Sigma_1} \to \mathsf{L}_{\Sigma_2}$ denote the morphism of cartesian categories where f is mapped to g_2. Finally, there is a unique morphism of cartesian categories $Q \colon \mathsf{L}_{\Sigma_2} \to \mathsf{L}_{\Sigma_1}$ mapping g_1 and g_2 to f.

3 Elementary Existential Doctrines

Recall that an *inf-semilattice* is a partially ordered set with all finite infima, including a top element $\top$. We denote the category of inf-semilattices and inf-preserving functions by InfSL.

The following definition is taken almost verbatim from [27]. The difference is that there the base category $\mathbb{C}$ only needs binary products, whereas we also require a terminal object.

▶ **Definition 5.** *Let $\mathbb{C}$ be a cartesian category. An elementary existential doctrine is given by a functor $P \colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ that is:*

- Elementary, *namely for every object A in $\mathbb{C}$ there is an element $\delta_A \in P(A \times A)$ such that for every map $e = id_X \times \Delta_A \colon X \times A \to X \times A \times A$, the function $P_e \colon P(X \times A \times A) \to P(X \times A)$ has a left adjoint $\exists_e \colon P(X \times A) \to P(X \times A \times A)$ defined by the assignment*

$$\exists_e(\alpha) = P_{\langle \pi_1, \pi_2\rangle \colon X \times A \times A \to X \times A}(\alpha) \wedge P_{\langle \pi_2, \pi_3\rangle \colon X \times A \times A \to A \times A}(\delta_A). \tag{1}$$

- Existential, *namely for every $A_1, A_2 \in \mathbb{C}$ and projection $\pi_i \colon A_1 \times A_2 \to A_i$ with $i \in \{1,2\}$, the function $P_{\pi_i} \colon P(A_i) \to P(A_1 \times A_2)$ has a left-adjoint $\exists_{\pi_i}$ that satisfies*

- *the* Beck-Chevalley condition*: for any projection* $\pi\colon X \times A \to A$ *and any pullback*

$$\begin{array}{ccc} X' & \xrightarrow{\pi'} & A' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X \times A & \xrightarrow{\pi} & A \end{array} \quad \textit{it holds that } \exists_{\pi'}(P_{f'}(\beta)) = P_f(\exists_\pi(\beta)) \textit{ for any } \beta \in P(X \times A). \tag{2}$$

- Frobenius reciprocity*: for any projection* $\pi\colon X \times A \to A$*,* $\alpha \in P(A)$ *and* $\beta \in P(X)$*, it holds that* $\exists_\pi(P_\pi(\alpha) \wedge \beta) = \alpha \wedge \exists_\pi(\beta)$*.*

▶ Remark 6. Taking X in (1) to be the terminal object of $\mathbb{C}$, one obtains that the function $\exists_{\Delta_A}\colon P(A) \to P(A \times A)$ given by

$$\exists_{\Delta_A}(\alpha) = P_{\pi_1}(\alpha) \wedge \delta_A \tag{3}$$

is left-adjoint to $P_{\Delta_A}\colon P(A \times A) \to P(A)$. This condition, which appears in [27], is therefore redundant when $\mathbb{C}$ has terminal object.

▶ Remark 7. In any cartesian category, the diagram below is a pullback of $f\colon A' \to A$ with a projection $\pi\colon X \times A \to A$.

$$\begin{array}{ccc} X \times A' & \xrightarrow{\pi_2} & A' \\ {\scriptstyle \mathrm{id}_X \times f}\downarrow & & \downarrow{\scriptstyle f} \\ X \times A & \xrightarrow{\pi} & A \end{array}$$

Therefore, given a functor $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$, to check that P satisfies the Beck-Chevalley condition it suffices to show that (2) holds for $X' = X \times A'$, $\pi' = \pi_2$ and $f' = \mathrm{id}_X \times f$.

The contravariant powerset $\mathcal{P}$, while often seen as an endofunctor on Set, can also be seen as a functor $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{InfSL}$. It is the classical example of an elementary existential doctrine. Recall that, for $f\colon Y \to X$ in Set, $\mathcal{P}_f\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is $\mathcal{P}_f(Z) = \{y \in Y \mid f(y) \in Z\}$. Given a projection $\pi\colon X \times A \to A$, $\mathcal{P}_\pi$ and its left adjoint $\exists_\pi$ are as follows:

$$\begin{array}{ccc} \mathcal{P}(A) & \xrightarrow{\mathcal{P}_\pi} & \mathcal{P}(X \times A) \\ B & \longmapsto & \{(x,b) \in X \times A \mid b \in B\} \end{array} \qquad \begin{array}{ccc} \mathcal{P}(X \times A) & \xrightarrow{\exists_\pi} & \mathcal{P}(A) \\ S & \longmapsto & \{a \in A \mid \exists x \in X.\, (x,a) \in S\} \end{array}$$

For every set A, δ_A is fixed to be $\{(a,a) \mid a \in A\} \in \mathcal{P}(A \times A)$. With this information, it is easy to check that $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{InfSL}$ satisfies the conditions in Definition 5. The reader can find the worked out details in [3].

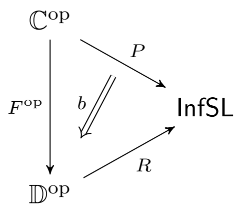
▶ **Example 8.** Let Σ and $\mathbb{P}$ be signatures of function and predicate symbols respectively. The regular fragment of first order logic consists of formulas built from conjunction $\wedge$, true $\top$, existential quantification $\exists$, equality $t_1 = t_2$ and atoms $P\langle t_1, \dots t_m\rangle$ where $P \in \mathbb{P}$ and t_i are terms over Σ. Formulas are sorted according to the rules in Figure 1: ϕ has sort $(n, 0)$ if the free variables of ϕ are in $\{x_1, \dots, x_n\}$.

The indexed Lindenbaum-Tarski algebra functor $LT\colon \mathsf{L}_\Sigma^{\mathrm{op}} \to \mathsf{InfSL}$ assigns to each $n \in \mathbb{N}$ the set of formulas of sort $(n, 0)$ modulo logical equivalence (defined in the usual way). These form a semilattice with top given by $\top$ and meet by $\wedge$, where $\phi \leq \psi$ if and only if ψ is a logical consequence of ϕ. To the arrow $\langle t_1, \dots, t_m\rangle\colon n \to m$, LT assigns the substitution mapping each $\phi\colon (m, 0)$ to $\phi[{}^{t_1,\dots,t_m}/{}_{x_1,\dots,x_m}]\colon (n, 0)$. In particular, for the projection $\pi\colon n + m \to n$ (that is $\langle x_1, \dots, x_n\rangle\colon n + m \to n$) LT_π maps a formula of sort

$(n,0)$ to the same formula but with sort $(n+m,0)$[1]. Its left adjoint $\exists_\pi$ maps a formula $\phi\colon (n+m,0)$ to the formula $\exists x_{n+1}.\dots\exists x_{n+m}.\,\phi\colon (n,0)$. The Beck-Chevalley condition asserts that "substitution commutes with quantification": if ϕ is a formula with at most $n+1$ free variables and t is a term that does not contain x_{n+1}, then $\exists x_{n+1}.\,(\phi[^t/_{x_i}]) = (\exists x_{n+1}.\,\phi)[^t/_{x_i}]$. Frobenius reciprocity states that $\exists x_i.\,(\phi \wedge \psi) = \phi \wedge (\exists x_i.\,\psi)$ if x_i is not a free variable of ϕ. For all $n \in \mathbb{N}$, δ_n is the formula $(x_1 = x_{n+1}) \wedge (x_2 = x_{n+2}) \wedge \cdots \wedge (x_n = x_{n+n})$.

▶ **Definition 9** (Cf. [27]). *The category* EED *consists of the following data.*

- *Objects are elementary existential doctrines* $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$.
- *Morphisms from* $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ *to* $R\colon \mathbb{D}^{\mathrm{op}} \to \mathsf{InfSL}$ *are pairs* (F,b)*, where* $F\colon \mathbb{C} \to \mathbb{D}$ *is a strict cartesian functor while* $b\colon P \to R \circ F^{\mathrm{op}}$ *is a natural transformation*



that preserves equalities and existential quantifiers, that is $b_{A\times A}(\delta^P_A) = \delta^R_{F(A)}$ *for all* A *in* $\mathbb{C}$ *and, for any projection* $\pi\colon X \times A \to A$ *in* $\mathbb{C}$*, the following diagram commutes.*

$$\begin{array}{ccc} P(X\times A) & \xrightarrow{\exists^P_\pi} & P(A) \\ {\scriptstyle b_{X\times A}}\downarrow & & \downarrow{\scriptstyle b_A} \\ RF(X\times A) & \xrightarrow[\exists^R_{F(\pi)}]{} & RF(A) \end{array} \tag{4}$$

- *Composition of* $(F,b)\colon P \to R$ *as above with* $(G,c)\colon R \to S$ *is given by* $(GF, cF \circ b)$.

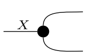
▶ Remark 10. In [27], morphisms of elementary existential doctrines $P \to R$ are pairs (F,b) where $F\colon \mathbb{C} \to \mathbb{D}$ is a functor between the base categories that preserves binary products merely up to isomorphism, so $F(X) \xleftarrow{F(\pi_1)} F(X\times Y) \xrightarrow{F(\pi_2)} F(Y)$ is a product diagram of $F(X)$ and $F(Y)$ in $\mathbb{D}$ but it might not coincide with the *chosen* product of $\mathbb{D}$. For this reason, preservation of equality for b in [27] means that $b_{A\times A}(\delta^P_A) = R_{\langle F(\pi_1),F(\pi_2)\rangle}(\delta^R_{F(A)})$.

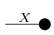
▶ Remark 11. Let $P\colon \mathbb{D}^{\mathrm{op}} \to \mathsf{InfSL}$ be an elementary existential doctrine and $F\colon \mathbb{C} \to \mathbb{D}$ a strict cartesian functor. Then $P \circ F^{\mathrm{op}}\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ is again an elementary existential doctrine, where $\delta^{PF^{\mathrm{op}}}_A = \delta^P_{F(A)}$ for all A in $\mathbb{B}$ and, for very $\pi_i\colon X_1 \times X_2 \to X_i$, $\exists^{PF^{\mathrm{op}}}_{\pi_i} = \exists^P_{F(\pi_i)}$.

4 Cartesian Bicategories

In this section we recall from [9] definitions and properties of cartesian bicategories.

▶ **Definition 12.** *A cartesian bicategory is a symmetric monoidal category* $(\mathbb{B}, \otimes, I)$ *enriched over the category of posets (that is every hom-set is a partial order and both composition and* $\otimes$ *are monotonous operations) where every object* $X \in \mathbb{B}$ *is equipped with morphisms*



$: X \to X \otimes X$ *and* 

$: X \to I$ *such that*

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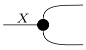
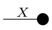
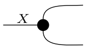
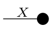
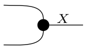
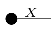
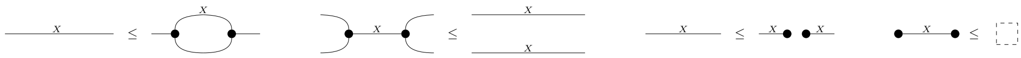
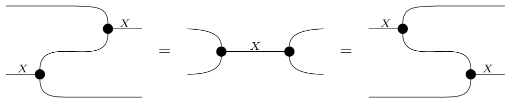
[1] The second projection $\pi\colon n+m \to m$ requires the reindexing of variables. We do not discuss this case in order to keep the presentation simpler.

1.  *and*  *form a cocommutative comonoid, as in Definition 1.1*
2.  *and*  *have right adjoints*  *and*  *respectively, that is*

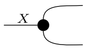
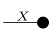
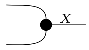
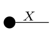


3. *The Frobenius law holds, that is*

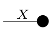
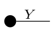


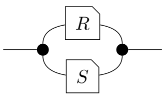
4. *Each morphism $R\colon X \to Y$ is a* lax *comonoid homomorphism, that is*

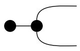
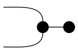
5. *The choice of comonoid is coherent with the monoidal structure*[2]*, as in Definition 1.3.*

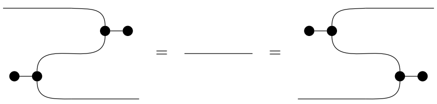
The archetypal example of a cartesian bicategory is the category of sets and relations Rel, with cartesian product of sets as monoidal product and $1 = \{\bullet\}$ as unit I. To be precise, Rel has sets as objects and relations $R \subseteq X \times Y$ as arrows $X \to Y$. Composition and monoidal product are defined as expected: $R\,;S = \{(x,z) \mid \exists y \text{ s.t. } (x,y) \in R \text{ and } (y,z) \in S\}$ and $R \otimes S = \{\big((x_1,x_2)\,,\,(y_1,y_2)\big) \mid (x_1,y_1) \in R \text{ and } (x_2,y_2) \in S\}$. For each set X, the comonoid structure is given by the diagonal function $\Delta_X\colon X \to X \times X$ and the unique function $!_X\colon X \to 1$, considered as relations, that is  $= \{\big(x,(x,x)\big) \mid x \in X\}$ and  $= \{(x,\bullet) \mid x \in X\}$. Their right adjoints are given by their opposite relations:  $= \{\big((x,x),x\big) \mid x \in X\}$ and  $= \{(\bullet,x) \mid x \in X\}$. Following the analogy with Rel, we will often call "relations" arbitrary morphisms of a cartesian bicategory.

One of the fundamental properties of cartesian bicategories that follows from the existence of right adjoints (Property 2 in Definition 12) is that every local poset $\mathrm{Hom}_{\mathbb{B}}(X,Y)$ allows to take the intersection of relations and has a top element.

▶ **Lemma 13.** *Let $\mathbb{B}$ be a cartesian bicategory and $X, Y \in \mathbb{B}$. The poset* $\mathrm{Hom}_{\mathbb{B}}(X,Y)$ *has a top element given by*   *and the meet of relations $R, S\colon X \to Y$ is given by*



The Frobenius law (Property 3) gives a compact closed structure – in other words, it allows us to bend wires around. The cup of this compact closed structure is , the cap analogously  and the Frobenius law implies the snake equations:



(5)

To obtain an intuition for the lax comonoid homomorphism condition (Property 4), it is useful to spell out its meaning in Rel: in the first inequality, the left and the right-hand side are,

[2] In the original definition of [9] this property is replaced by requiring the uniqueness of the comonoid/-monoid. However, as suggested in [29], coherence seems to be the property of primary interest.

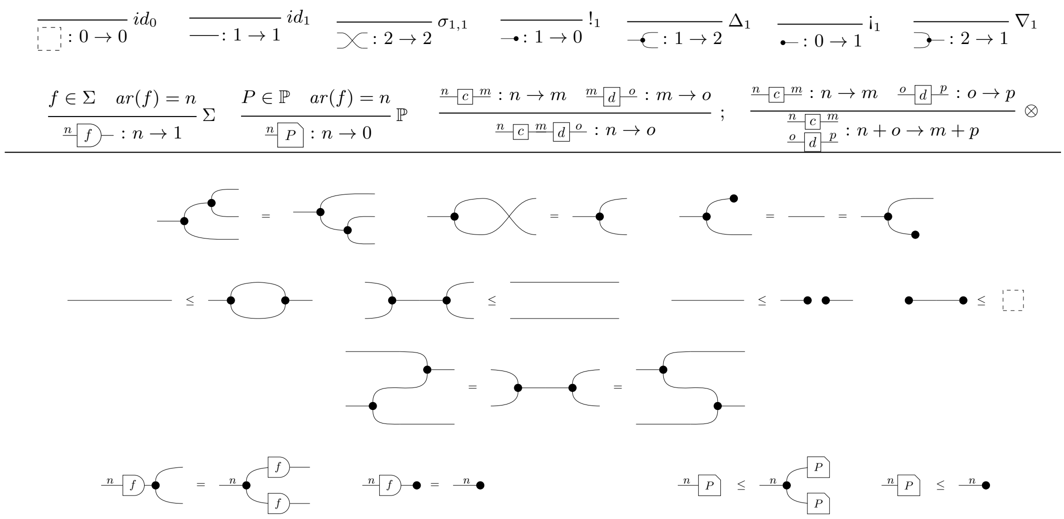


Figure 2 Sort inference rules (top) and axioms (bottom) for $\mathsf{CB}_{\Sigma,\mathbb{P}}$.

respectively, the relations $\{(x,(y,y)) \mid (x,y) \in R\}$ and $\{(x,(y,z)) \mid (x,y) \in R \text{ and } (x,z) \in R\}$, while in the second inequality, they are the relations $\{(x,\bullet) \mid \exists y \in Y \text{ s.t. } (x,y) \in R\}$ and $\{(x,\bullet) \mid x \in X\}$. It is immediate to see that the two left-to-right inclusions hold for any relation $R \subseteq X \times Y$, while the right-to-left inclusions hold for exactly those relations that are (graphs of) functions: the inclusions specify, respectively, single-valuedness and totality.

▶ **Definition 14.** *A map in a cartesian bicategory is an arrow f that is a comonoid homomorphism, i.e. for which the equalities in Definition 1.2 hold. For $\mathbb{B}$ a cartesian bicategory, its category of maps,* $\mathrm{Map}(\mathbb{B})$*, has the same objects of $\mathbb{B}$ and as morphisms the maps of $\mathbb{B}$.*

▶ **Lemma 15.** *A morphism* —f— *is a map if and only if it has a right adjoint – a morphism R such that* —R—f— $\leq$ —— *and* —— $\leq$ —f—R— *.*

As expected, maps in Rel are precisely functions. Thus, $\mathrm{Map}(\mathsf{Rel})$ is exactly Set. For maps it makes sense to imagine a flow of information from left to right. We will therefore draw —f— to denote a map f. Note that we use lower-case letters for maps and upper-case for arbitrary morphisms. Since X —•< and X —• are maps by Lemma 15 and since, by definition, every map respects them, we have that $\mathrm{Map}(\mathbb{B})$, which inherits the monoidal product from $\mathbb{B}$, has the structure of a cartesian category (Definition 1).

▶ **Lemma 16.** *For a cartesian bicategory $\mathbb{B}$, the monoidal product $\otimes$ is a product on* $\mathrm{Map}(\mathbb{B})$*, and the monoidal unit I is terminal. In other words,* $(\mathrm{Map}(\mathbb{B}), \otimes, I)$ *is a cartesian category.*

▶ **Definition 17.** *A morphism of cartesian bicategories is a strict monoidal functor that preserves the ordering, the chosen monoid and the comonoid structures. Cartesian bicategories and their morphisms form a category* CBC.

▶ **Proposition 18.** *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a morphism of cartesian bicategories. Then restricting its domain to* $\mathrm{Map}(\mathbb{A})$ *yields a strict cartesian functor* $F\restriction_{\mathrm{Map}(\mathbb{A})}\colon \mathrm{Map}(\mathbb{A}) \to \mathrm{Map}(\mathbb{B})$.

The remaining sections focus on the relationship between cartesian bicategories and elementary existential doctrines. First, a little taste of the similarity between them.

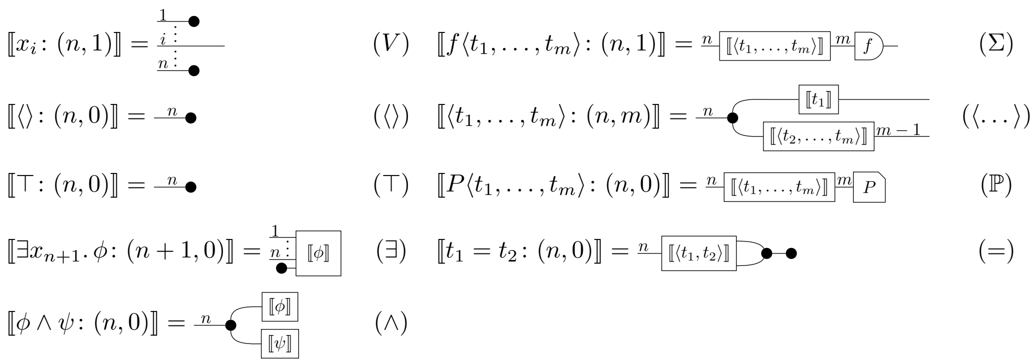
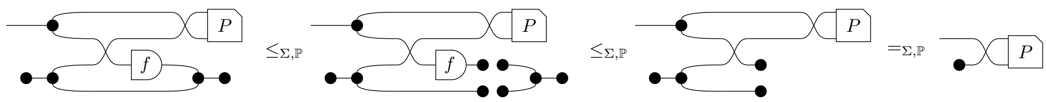


Figure 3 Translation $[\![-]\!]$ from sorted formulas (Figure 1) to string diagrams (Figure 2).

▶ **Example 19.** In Example 8, we outlined the Lindenbaum-Tarski doctrine for the regular fragment of first order logic. We now introduce a cartesian bicategory, denoted by $\mathsf{CB}_{\Sigma,\mathbb{P}}$, that provides a string diagrammatic calculus for this fragment. Like Lawvere theories, $\mathsf{CB}_{\Sigma,\mathbb{P}}$ has $(\mathbb{N},+,0)$ as monoid of objects. Arrows are (equivalence classes of) string diagrams [33] generated according to the rules in Figure 2 (top). For all $n,m \in \mathbb{N}$, identities $\overset{n}{\text{---}}\colon n \to n$, symmetries $\colon n{+}m \to m{+}n$, duplicators $\colon n \to n{+}n$, dischargers $\colon n \to 0$ and their adjoints can be constructed from the basic diagrams in the first row. Observe that function symbols f with arity n are depicted with coarity 1, while predicate symbols $P \in \mathbb{P}$ with arity n as with coarity 0.

Figure 2 (bottom) illustrates the axioms for the calculus: in the last row, function symbols $\colon n \to 1$ are forced to be comonoid homomorphism, while predicate symbols $\colon n \to 0$ are just lax; the first three rules impose properties 1, 2 and 3 of Definition 12 on the generating (co)monoid (the laws for arbitrary n follow from these). Let $\leq_{\Sigma,\mathbb{P}}$ be the precongruence with respect to ; and $\otimes$ generated by the axioms and $=_{\Sigma,\mathbb{P}}$ the corresponding equivalence, i.e., $\leq_{\Sigma,\mathbb{P}} \cap \geq_{\Sigma,\mathbb{P}}$. Now $\mathsf{CB}_{\Sigma,\mathbb{P}}[n,m]$ is exactly the set of $=_{\Sigma,\mathbb{P}}$-equivalence classes of diagrams $d\colon n \to m$ ordered by $\leq_{\Sigma,\mathbb{P}}$. Simple inductions suffice to check that $\mathsf{CB}_{\Sigma,\mathbb{P}}$ is indeed a cartesian bicategory and that, moreover, $\mathrm{Map}(\mathsf{CB}_{\Sigma,\mathbb{P}})$ is isomorphic to L_Σ.

In Figure 3 we introduce a function $[\![-]\!]$ that translates sorted formulas $\phi\colon (n,0)$ into string diagrams of type $n \to 0$. From a general result in [32], it follows that ϕ is a logical consequence of ψ if and only if $[\![\psi]\!] \leq_{\Sigma,\mathbb{P}} [\![\phi]\!]$. For an example take $\psi \equiv \exists x_2.\, P(x_2,x_1) \wedge f(x_1) = x_2 \colon (1,0)$ and $\phi \equiv \exists x_2.\, P(x_2,x_1)\colon (1,0)$. Then $[\![\psi]\!]$ is the leftmost string diagram below, while $[\![\phi]\!]$ is the rightmost one. The following derivation proves that $\exists x_2.\, P(x_2,x_1)$ is a logical consequence of $\exists x_2.\, P(x_2,x_1) \wedge f(x_1) = x_2$.



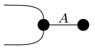
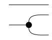
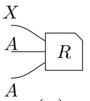
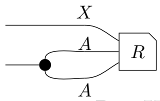
5 From Cartesian Bicategories to Doctrines

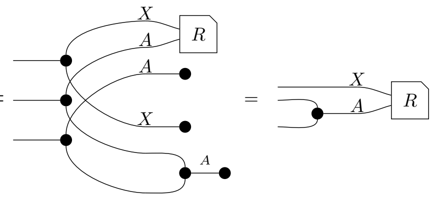
In this section we illustrate how a cartesian bicategory $\mathbb{B}$ gives rise to an elementary existential doctrine. The starting observation is that, using the conclusion of Lemma 13, the functor $\mathrm{Hom}_{\mathbb{B}}(-,I)\colon \mathbb{B}^{\mathrm{op}} \to \mathsf{Set}$ sends objects X to Inf-semilattices. However, for an arbitrary morphism $R\colon X \to Y$, $\mathrm{Hom}_{\mathbb{B}}(R, I)\colon \mathrm{Hom}_{\mathbb{B}}(Y, I) \to \mathrm{Hom}_{\mathbb{B}}(X, I)$ may *not* be an inf-preserving function. A sufficient condition is to require R to be a map; indeed, in that case it is immediate to see that infima, as defined in Lemma 13, are preserved. Thus, by restricting the domain of the Hom-functor to $\mathrm{Map}(\mathbb{B})^{\mathrm{op}}$, one obtains a contravariant functor from the cartesian category $\mathrm{Map}(\mathbb{B})$ (Lemma 16) to InfSL:

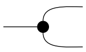
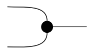
$$\mathcal{R}(\mathbb{B}) = \mathrm{Hom}_{\mathbb{B}}(-, I)\colon \mathrm{Map}(\mathbb{B})^{\mathrm{op}} \to \mathsf{InfSL}\,. \tag{6}$$

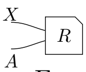
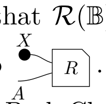
▶ **Theorem 20.** *The functor $\mathcal{R}(\mathbb{B})$ in* (6) *is an elementary existential doctrine.*

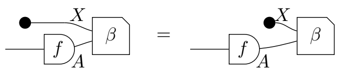
Proof. We need to show that $\mathcal{R}(\mathbb{B})$ is elementary and existential.

1. First we prove that $\mathcal{R}(\mathbb{B})$ is elementary. We fix $\delta_A =$  $\in \mathrm{Hom}(A \otimes A, I)$. The string diagram for $e = id_X \otimes \Delta_A\colon X \otimes A \to X \otimes A \otimes A$ is . The function $\mathcal{R}(\mathbb{B})_e$ maps  to . The function $\exists_e\colon \mathcal{R}(\mathbb{B})(A \otimes X) \to \mathcal{R}(\mathbb{B})(A \otimes X \otimes X)$ defined in (1) maps every $R \in \mathrm{Hom}(X \otimes A, I)$ to

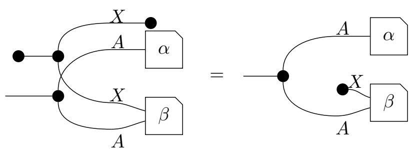
$$\exists_e(R) = \mathcal{R}(\mathbb{B})_{\langle \pi_1, \pi_2 \rangle}(R) \wedge \mathcal{R}(\mathbb{B})_{\langle \pi_2, \pi_3 \rangle}(\delta_A) =$$ 

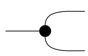
   Now $\exists_e$ is left-adjoint to $\mathcal{R}(\mathbb{B})_e$ because  is left-adjoint to .

2. To show that $\mathcal{R}(\mathbb{B})$ is existential, let $\pi\colon X \otimes A \to A$ be a projection. Then $\exists_\pi$ maps  to . This is left-adjoint to $\mathcal{R}(\mathbb{B})_\pi$ since is left-adjoint to .
   - For the Beck-Chevalley condition, consider the diagram in Remark 7. We need to show that $\exists_{\pi_2}(\mathcal{R}(\mathbb{B})_{\mathrm{id}_X \otimes f}(\beta)) = \mathcal{R}(\mathbb{B})_f(\exists_\pi(\beta))$. Translated to diagrams,

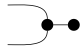
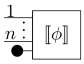
     which holds trivially.
   - For Frobenius reciprocity, take a projection $\pi\colon X \otimes A \to A$, $\alpha \in \mathcal{R}(\mathbb{B})(A)$ and $\beta \in \mathcal{R}(\mathbb{B})(X \otimes A)$. We need that $\exists_\pi(\mathcal{R}(\mathbb{B})_\pi(\alpha) \wedge \beta) = \alpha \wedge \exists_\pi(\beta)$, which translates to

     This holds by naturality of the symmetry and since is the counit of . ◀

By applying $\mathcal{R}$ to the cartesian bicategory Rel, one obtains $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{InfSL}$, in the sense that $\mathcal{P} \cong \mathcal{R}(\mathsf{Rel})$ in EED. The isomorphism is the pair (Γ, b), where $\Gamma\colon \mathsf{Set} \to \mathrm{Map}(\mathsf{Rel})$ is the strict cartesian functor computing the graph of a function and $b\colon \mathcal{P} \to \mathcal{R}(\mathsf{Rel}) \circ \Gamma^{\mathrm{op}}$ is the natural transformation defined for all sets A and $S \in \mathcal{P}(A)$ as $b_A(S) = \{(s, \bullet) \mid s \in S\}$.

▶ **Example 21.** Consider $\mathsf{CB}_{\Sigma,P}$ of Example 19: then $\mathcal{R}(\mathsf{CB}_{\Sigma,P})$ is isomorphic to the doctrine $LT\colon \mathsf{L}_\Sigma^{\mathrm{op}} \to \mathsf{InfSL}$ of Example 8. Here is why: the first two rows in Figure 3 define a cartesian isomorphism $F_{[\![\cdot]\!]}\colon \mathsf{L}_\Sigma \to \mathrm{Map}(\mathsf{CB}_{\Sigma,\mathbb{P}})$. For all $n \in \mathbb{N}$, we define $b_n\colon LT(n) \to$

$\mathrm{Hom}(F_{[\![\cdot]\!]}(n), 0) = \mathrm{Hom}(n, 0)$ as the function mapping $\phi\colon (n, 0)$ to $[\![\phi]\!] : n \to 0$. This give rise to a natural isomorphism $b\colon LT \to \mathcal{R}(\mathsf{CB}_{\Sigma,\mathbb{P}}) \circ F^{\mathrm{op}}_{[\![\cdot]\!]}$. To show that b preserves equalities and existential quantifiers (Definition 9) it suffices to note that $[\![x_1 = x_2]\!] =$  and $[\![\exists x_{n+1}.\, \phi]\!] =$ . The pair $(F_{[\![\cdot]\!]}, b)$ witnesses the isomorphism of LT and $\mathcal{R}(\mathsf{CB}_{\Sigma,\mathbb{P}})$.

▶ **Proposition 22.** *Assigning doctrines to cartesian bicategories as in (6) extends to a functor $\mathcal{R}\colon \mathsf{CBC} \to \mathsf{EED}$. For a morphism of cartesian bicategories $F\colon \mathbb{A} \to \mathbb{B}$ in CBC, $\mathcal{R}(F) = (F{\restriction_{\mathrm{Map}(\mathbb{A})}}, b^F)$ where $b^F\colon \mathcal{R}(\mathbb{A}) \to \mathcal{R}(\mathbb{B}) \circ (F{\restriction_{\mathrm{Map}(\mathbb{A})}})^{\mathrm{op}}$ is defined as*

$$b^F_A(U) = F(U) \in \mathrm{Hom}_{\mathbb{B}}(F(A), I) \quad \textit{for all } A \in \mathbb{A} \textit{ and } U \in \mathrm{Hom}_{\mathbb{A}}(A, I). \tag{7}$$

Proof. Let $F\colon \mathbb{A} \to \mathbb{B}$ be a morphism in CBC. By Proposition 18, we have that $F{\restriction_{\mathrm{Map}(\mathbb{A})}}$ is a strict cartesian functor. Regarding b^F, its naturality is ensured by (in fact, equivalent to) the functoriality of F, while the preservation of equalities and existential quantifiers of $\mathcal{R}(\mathbb{A})$ follows from the fact that F preserves the structure of cartesian bicategory of $\mathbb{A}$, which is used to define the structure of elementary existential doctrine of $\mathcal{R}(\mathbb{A})$. Therefore $\mathcal{R}(F)$ is indeed a morphism in EED. Preservation of compositions and identities is straightforward. ◀

6 From Doctrines to Cartesian Bicategories

Given an elementary existential doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$, we can form a category $\mathscr{A}_P$ whose objects are the same as $\mathbb{C}$ and whose morphisms are given by the elements of $P(X \times Y)$, intuitively seen as *relations*, as observed in [27]. Inspired by the calculus of ordinary relations on sets, using the structure of P we can define a notion of composition and tensor product of these inner relations, and endow each object with a comonoid structure that makes $\mathscr{A}_P$ a cartesian bicategory. Here we recall the essential definitions, while the proof of the fact that $\mathscr{A}_P$ actually satisfies Definition 12 is rather laborious and therefore omitted here: the interested reader can find it in all details in [3].

Since $\mathrm{Hom}_{\mathscr{A}_P}(X, Y) = P(X \times Y)$, we get that $\mathscr{A}_P$ is poset-enriched. Given that P is elementary, the obvious candidate for identity on X is $\delta_X \in \mathrm{Hom}_{\mathscr{A}_P}(X, X)$. Composition works as follows: let $f \in \mathrm{Hom}_{\mathscr{A}_P}(X, Y) = P(X \times Y)$ and $g \in \mathrm{Hom}_{\mathscr{A}_P}(Y, Z) = P(Y \times Z)$.

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow^{\pi_Z} & \downarrow{\scriptstyle \pi_Y} & \searrow^{\pi_X} & \\ X \times Y & & X \times Z & & Y \times Z \end{array}$$

Consider the projections above. Then the composite $f \,;\, g$ is defined as

$$f \,;\, g = \exists_{\pi_Y}(P_{\pi_Z}(f) \wedge P_{\pi_X}(g)).$$

The monoidal structure of $\mathscr{A}_P$ is very straightforward: on objects, the monoidal product $\otimes$ is given by the cartesian product in $\mathbb{C}$. On morphisms, for $f \in \mathrm{Hom}_{\mathscr{A}_P}(A, B) = P(A \times B)$ and $g \in \mathrm{Hom}_{\mathscr{A}_P}(C, D) = P(C \times D)$, consider the projections

$$\begin{array}{ccccc} & & A \times C \times B \times D & & \\ & \swarrow^{\langle \pi_1, \pi_3 \rangle} & & \searrow^{\langle \pi_2, \pi_4 \rangle} & \\ A \times B & & & & C \times D \end{array}$$

Then let

$$f \otimes g = P_{\langle \pi_1, \pi_3 \rangle}(f) \wedge P_{\langle \pi_2, \pi_4 \rangle}(g).$$

This makes $\mathscr{A}_P$ a monoidal, poset-enriched category. The rest of the structure of cartesian bicategory is inherited from $\mathbb{C}$ by means of a crucial tool: the *graph functor* of P, hereafter denoted by $\Gamma_P\colon \mathbb{C} \to \mathscr{A}_P$. It is the identity on objects and sends arrows $f\colon X \to Y$ in $\mathbb{C}$ to

$$\Gamma_P(f) = P_{f\times \mathrm{id}_Y}(\delta_Y) \in P(X\times Y) = \mathrm{Hom}_{\mathscr{A}_P}(X,Y).$$

▶ **Proposition 23.** *Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ be an elementary, existential doctrine. Then:*

- *Γ_P is strict monoidal,*
- *$\Gamma_P(f)$ has a right adjoint, namely $P_{\mathrm{id}_Y\times f}(\delta_Y)$, for every $f\colon X\to Y$ in $\mathbb{C}$. Therefore, by Lemma 15, it corestricts to a strict cartesian functor $\Gamma_P\colon \mathbb{C} \to \mathrm{Map}(\mathscr{A}_P)$.*

Consider for instance the powerset doctrine $\mathcal{P}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{InfSL}$: the elements of $\mathcal{P}(X\times Y)$ are precisely the relations from X to Y, while composition and monoidal product in $\mathscr{A}_{\mathcal{P}}$ coincide with the usual composition and product of relations, see § 4. In other words, $\mathscr{A}_{\mathcal{P}} = \mathsf{Rel}$. The functor $\Gamma_{\mathcal{P}}$ calculates graphs of functions, which are exactly the maps in Rel.

▶ **Example 24.** Recall the doctrine $LT\colon \mathsf{L}_\Sigma \to \mathsf{InfSL}$ and the cartesian bicategory $\mathsf{CB}_{\Sigma,\mathbb{P}}$ from Examples 8 and 19. The functor $\Gamma_{LT}\colon \mathsf{L}_\Sigma \to \mathrm{Map}(\mathscr{A}_{LT})$ is inductively defined as the unique cartesian functor mapping each $f\in\Sigma$ with arity $ar(f)=n$ to the formula $f(x_1,\dots,x_n) = x_{n+1}\colon (n+1,0)$.

Now, since $\mathbb{C}$ is a cartesian category, every object is canonically equipped with a natural and coherent comonoid structure. We can use the strict monoidal functor $\Gamma_P\colon \mathbb{C}\to\mathscr{A}_P$ to transport this comonoid to $\mathscr{A}_P$, and by Proposition 23 we have that copying and discarding in $\mathscr{A}_P$ both have right adjoints. Finally, one can prove that every morphism in $\mathscr{A}_P$ is a lax-comonoid homomorphism and that the Frobenius law holds.

▶ **Theorem 25.** *Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ be an elementary existential doctrine. Then $\mathscr{A}_P$ is a cartesian bicategory. Moreover, the assignment $P \mapsto \mathscr{A}_P$ extends to a functor $\mathcal{L}\colon \mathsf{EED} \to \mathsf{CBC}$ as follows: for P as above, $R\colon \mathbb{D}^{\mathrm{op}} \to \mathsf{InfSL}$ and $(F,b)\colon P \to R$ in EED,*

$$\begin{array}{rcl}
\mathscr{A}_P & \xrightarrow{\mathcal{L}(F,b)} & \mathscr{A}_R \\
X & \longmapsto & FX \\
P(X\times Y)\ni r \Big\downarrow & \longmapsto & \Big\downarrow b_{X\times Y}(r)\in R(F(X)\times F(Y)) \\
Y & \longmapsto & FY
\end{array}$$

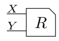
7 An Adjunction

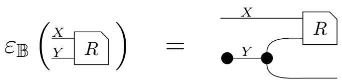
We saw in Sections 5 and 6 that using $\mathcal{L}$ and $\mathcal{R}$ one can pass, in a functorial way, between the worlds of cartesian bicategories and elementary existential doctrines. Here we show that, in fact, they define an adjunction $\mathcal{L} \dashv \mathcal{R}$. For this we need natural transformations

$$\eta\colon \mathrm{id}_{\mathsf{EED}} \to \mathcal{R}\mathcal{L}, \qquad \varepsilon\colon \mathcal{L}\mathcal{R} \to \mathrm{id}_{\mathsf{CBC}}$$

that make the following triangles commute for every $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ in EED and $\mathbb{B}$ in CBC.

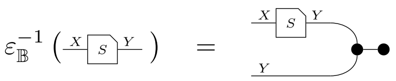
$$\begin{array}{ccc}
\mathcal{L}(P) & \xrightarrow{\mathcal{L}(\eta_P)} & \mathcal{L}\mathcal{R}\mathcal{L}(P) \\
& {}_{\mathrm{id}_{\mathcal{L}(P)}}\searrow & \downarrow{\scriptstyle \varepsilon_{\mathcal{L}(P)}} \\
& & \mathcal{L}(P)
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{R}(\mathbb{B}) & \xrightarrow{\eta_{\mathcal{R}(\mathbb{B})}} & \mathcal{R}\mathcal{L}\mathcal{R}(\mathbb{B}) \\
& {}_{\mathrm{id}_{\mathcal{R}(\mathbb{B})}}\searrow & \downarrow{\scriptstyle \mathcal{R}(\varepsilon_{\mathbb{B}})} \\
& & \mathcal{R}(\mathbb{B})
\end{array}
\qquad (8)$$

Let us start with ε. Recall that $\mathcal{R}(\mathbb{B}) = \mathrm{Hom}_{\mathbb{B}}(-, I) \colon \mathrm{Map}(\mathbb{B})^{\mathrm{op}} \to \mathsf{InfSL}$ and that, for $f \colon X \to Y$ in $\mathrm{Map}(\mathbb{B})$ and $U \in \mathrm{Hom}_{\mathbb{B}}(Y, I)$, $\mathcal{R}(\mathbb{B})_f(U)$ is equal to $U \circ f \colon X \to I$. By definition then, $\mathcal{LR}(\mathbb{B}) = \mathscr{A}_{\mathcal{R}(\mathbb{B})}$ has the same objects of $\mathbb{B}$ while a morphism $X \to Y$ in $\mathcal{LR}(\mathbb{B})$ is an element of $\mathrm{Hom}_{\mathbb{B}}(X \otimes Y, I)$. Hence, $\varepsilon_{\mathbb{B}}$ has to take a  in $\mathbb{B}$ and produce a morphism $\varepsilon_{\mathbb{B}}(R) \colon X \to Y$ in $\mathbb{B}$. We can do so by "bending" the Y string, defining:



(9)

In fact, by the snake equation in (5), $\varepsilon_{\mathbb{B}}$ has an inverse: define, for $S \colon X \to Y$ in $\mathbb{B}$,



and it is immediate to see that $\varepsilon_{\mathbb{B}}^{-1}\varepsilon_{\mathbb{B}}(R) = R$ and $\varepsilon_{\mathbb{B}}\varepsilon_{\mathbb{B}}^{-1}(S) = S$.

▶ **Example 26.** Recall from Example 21 that $\mathcal{R}(\mathsf{CB}_{\Sigma,P}) \cong LT$. Applying $\mathcal{L}$ to both, one gets $\mathcal{LR}(\mathsf{CB}_{\Sigma,P}) \cong \mathcal{L}(LT) = \mathscr{A}_{LT}$, hence using ε we have that $\mathsf{CB}_{\Sigma,P} \cong \mathscr{A}_{LT}$. In particular, given a formula $\phi \colon (n+m, 0)$, one obtains a morphism $n \to m$ in $\mathsf{CB}_{\Sigma,P}$ by first translating ϕ to the string diagram $[\![\phi]\!] : n + m \to 0$ (which is a morphism in $\mathcal{LR}(\mathsf{CB}_{\Sigma,P})$ of type $n \to m$), and then bending the last m inputs using ε as illustrated in (9).

▶ Remark 27. $\varepsilon_{\mathbb{B}}^{-1}$ coincides with $\Gamma_{\mathcal{R}(\mathbb{B})}$ on $\mathrm{Map}(\mathbb{B})$, since $\varepsilon_{\mathbb{B}}^{-1}(f) = \delta_Y^{\mathcal{R}(\mathbb{B})} \circ (f \otimes \mathrm{id}_Y) = \Gamma_{\mathcal{R}(\mathbb{B})}(f)$ when $f \colon X \to Y$ is a map. In other words, $\varepsilon_{\mathbb{B}}^{-1}$ is an extension of Γ_P to the whole of $\mathbb{B}$.

Regarding η, we have $\mathcal{L}(P) = \mathscr{A}_P$, whose objects are the objects of $\mathbb{C}$, and hom-sets are $\mathscr{A}_P(X, Y) = P(X \times Y)$. This means that

$$\mathcal{RL}(P) = \mathrm{Hom}_{\mathscr{A}_P}(-, I) = P(- \times I) \colon \mathrm{Map}(\mathscr{A}_P)^{\mathrm{op}} \to \mathsf{InfSL}\,.$$

To give a morphism $\eta_P \colon P \to \mathcal{RL}(P)$ in EED means therefore to give a functor $F \colon \mathbb{C} \to \mathrm{Map}(\mathscr{A}_P)$ and a natural transformation $b \colon P \to P(F(-) \times I)$ satisfying certain conditions. We have a natural candidate for F: Proposition 23 tells us that Γ_P is a functor whose image, in fact, is included in $\mathrm{Map}(\mathscr{A}_P)$ and, moreover, it is cartesian. Being the identity on objects, the natural transformation part of the definition of η_P must have components $b_X \colon P(X) \to P(X \times I)$: it is clear then that the natural transformation

$$P\rho = \big(P_{\rho_X} \colon P(X) \to P(X \times I)\big)_{X \in \mathbb{C}},$$

obtained by whiskering P with the right unitor $\rho_X \colon X \times I \to X$ of $\mathbb{C}$, is a sensible choice for b. In short, we define

$$\eta = \big((\Gamma_P, P\rho) \colon P \to \mathcal{RL}(P)\big)_{P \in \mathsf{EED}}. \tag{10}$$

The interested reader can find a proof of the fact that η and ε are well-defined natural transformations that satisfy the triangular equalities (8) in [3].

▶ **Theorem 28.** *The functors $\mathcal{L}$ and $\mathcal{R}$ form an adjunction*

$$\mathsf{EED} \underset{\mathcal{R}}{\overset{\mathcal{L}}{\rightleftarrows}} \mathsf{CBC} \qquad (\bot) \tag{11}$$

whose unit is η (10) and whose counit, which is a natural isomorphism, is ε (9).

Since the counit ε of the adjunction $\mathcal{L} \dashv \mathcal{R}$ is actually a natural isomorphism, the functor $\mathcal{R}\colon \mathsf{CBC} \to \mathsf{EED}$ is full and faithful. It turns out, however, that the adjunction $\mathcal{L} \dashv \mathcal{R}$ is not an equivalence, because $\mathcal{L}$ is not faithful. The following example shows why.

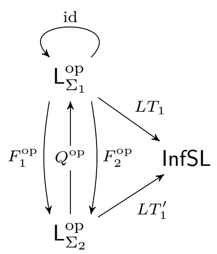
▶ **Example 29.** Let Σ_1 and Σ_2 be the signatures in Example 4 and $\mathbb{P}$ be some signature of predicate symbols. Let $LT_1\colon \mathsf{L}_{\Sigma_1}^{\mathrm{op}} \to \mathsf{InfSL}$ be the indexed Lindenbaum-Tarski algebras for Σ_1 defined as in Example 8. Recall from Example 4 the strict cartesian functor $Q\colon \mathsf{L}_{\Sigma_2} \to \mathsf{L}_{\Sigma_1}$, mapping $g_1, g_2 \in \Sigma_2$ to $f \in \Sigma_1$, and observe that

$$LT_1' = LT_1 \circ Q^{\mathrm{op}}\colon \mathsf{L}_{\Sigma_2}^{\mathrm{op}} \to \mathsf{InfSL}$$

is an elementary, existential doctrine by Remark 11. Its behaviour is somewhat peculiar: it maps $n \in \mathbb{N}$ to the set of formulas $\phi\colon (n,0)$ built from Σ_1 and $\mathbb{P}$ and to any tuple of terms $\langle t_1,\dots,t_m\rangle\colon n \to m$ in Σ_2 assigns the function mapping a formula $\phi\colon (m,0)$ to $\phi[{}^{Q(t_1),\dots,Q(t_m)}/_{x_1,\dots,x_m}]$. Observe that each $Q(t_i)$ is again a term in Σ_1: the symbols g_1 and g_2 are both translated to f. Somehow LT_1' behaves like LT_1 but they are different doctrines since their index categories, L_{Σ_2} and L_{Σ_1}, are not isomorphic. However, when transforming them into cartesian bicategories via $\mathcal{L}$, one obtains that $\mathcal{L}(LT_1') = \mathcal{L}(LT_1)$: objects are natural numbers and morphisms $n \to m$ are the elements of the set

$$LT_1'(n+m) = LT_1(Q(n+m)) = LT_1(Q(n)+Q(m)) = LT_1(n+m).$$

To formally show that $\mathcal{L}$ is not faithful we now define two morphisms in EED, both from LT_1 to LT_1', and show that their image along $\mathcal{L}$ is the same. Consider the strict cartesian functors $F_1, F_2\colon \mathsf{L}_{\Sigma_1} \to \mathsf{L}_{\Sigma_2}$ from Example 4 and observe that $QF_i = \mathrm{id}_{\mathsf{L}_{\Sigma_1}}$ for $i = 1,2$. We are in the following situation:



which means that $LT_1' \circ F_i^{\mathrm{op}} = LT_1 \circ Q^{\mathrm{op}} \circ F_i^{\mathrm{op}} = LT_1$,

thus $(F_1, \mathrm{id}_{LT_1})$ and $(F_2, \mathrm{id}_{LT_1})$ are distinct morphisms in EED from LT_1 to LT_1'. Since $\mathcal{L}(LT_1) = \mathcal{L}(LT_1')$, it follows from the definition of $\mathcal{L}$ that $\mathcal{L}(F_1,\mathrm{id}) = \mathrm{id}_{\mathscr{A}_{LT}} = \mathcal{L}(F_2,\mathrm{id})$.

8 An Equivalence

In the previous section we identified a doctrine with peculiar behaviour, and used it to show that (11) is not an equivalence. Here we characterise the additional constraints on doctrines that are needed for an equivalence.

To make the adjunction (11) an equivalence, we need its unit $\eta\colon \mathrm{id}_{\mathsf{EED}} \to \mathcal{RL}$ to be a natural isomorphism. This would mean that

$$\eta_P = (\Gamma_P, P\rho)\colon P \to \mathcal{RL}(P)$$

ought to be invertible in EED for any elementary existential doctrine $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$. Since

$$\mathcal{RL}(P) = \mathrm{Hom}_{\mathscr{A}_P}(-,I)\colon\ \mathrm{Map}(\mathscr{A}_P)^{\mathrm{op}} \to \mathsf{InfSL},$$

by definition η_P has an inverse in EED if and only if the functor $\Gamma_P\colon \mathbb{C} \to \mathrm{Map}(\mathscr{A}_P)$ is an isomorphism of cartesian categories, in which case $(\Gamma_P^{-1}, P\rho^{-1})$ would be the inverse of η_P.

But Γ_P is the identity on objects: to be an isomorphism, full and faithful suffices. That is, the maps of $\mathscr{A}_P$ must bijectively correspond with the arrows of the base category $\mathbb{C}$ of P. This is not necessarily the case, as arrows of $\mathbb{C}$ are not involved in the construction of $\mathscr{A}_P$.

The technical tools that allow us to bridge the gap between morphisms in $\mathbb{C}$ and maps in $\mathscr{A}_P$ are provided by [25]: the next two definitions are equivalent to faithfulness and, respectively, fullness of Γ_P.

▶ **Definition 30.** *Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ be an elementary existential doctrine and $\alpha \in P(A)$. A* comprehension *of α is an arrow $\{\!|\alpha|\!\}\colon X \to A$ in $\mathbb{C}$ such that $\top_{PX} \leq P_{\{\!|\alpha|\!\}}(\alpha)$ and such that for every $h\colon Z \to A$ for which $\top_{PZ} \leq P_h(\alpha)$, there is a unique arrow $h'\colon Z \to X$ such that $h = \{\!|\alpha|\!\} \circ h'$. P* has comprehensive diagonals *if every diagonal arrow $\Delta_A\colon A \to A \times A$ is the comprehension of δ^P_A.*

▶ **Example 31.** The doctrine $LT'_1\colon \mathsf{L}^{\mathrm{op}}_{\Sigma_2} \to \mathsf{InfSL}$ from Example 29 does not have comprehensive diagonal. Indeed Δ_1 is not the comprehension of δ_1: take as h the arrow $\langle g_1, g_2\rangle\colon 1 \to 2$ and observe that $LT'_{1\langle g_1,g_2\rangle}(\delta_1) = \big(f(x_1) = f(x_1)\big)\colon (1,0)$ and $\top\colon (1,0) \leq \big(f(x_1) = f(x_1)\big)\colon (1,0)$. Yet there exists no $h'\colon 1 \to 1$ in L_{Σ_2} such that $\langle g_1, g_2\rangle = \Delta_1 \circ h'$.

▶ **Definition 32.** *Let $P\colon \mathbb{C}^{\mathrm{op}} \to \mathsf{InfSL}$ be an elementary, existential doctrine. We say that P* satisfies the Rule of Unique Choice *(RUC) if for every $R \in P(X \times Y)$ which is a map in $\mathscr{A}_P$ there exists an arrow $f\colon X \to Y$ such that $\top_{PX} \leq P_{\langle \mathrm{id}_X, f\rangle}(R)$.*

▶ **Example 33.** All the doctrines considered so far satisfy RUC. For an example of a doctrine that does not satisfy it consider the composition of $F_1^{\mathrm{op}}\colon \mathsf{L}^{\mathrm{op}}_{\Sigma_1} \to \mathsf{L}^{\mathrm{op}}_{\Sigma_2}$ (Example 4) and $LT_2\colon \mathsf{L}^{\mathrm{op}}_{\Sigma_2} \to \mathsf{InfSL}$ (Example 8) that, by Remark 11, is an elementary existential doctrine. This doctrine maps n to the set of formulas $\phi\colon (n, 0)$ where terms are built from g_1 and g_2, but the index category L_{Σ_1} contains terms built from f that is translated to g_1 by F_1. Now, the formula $\phi \equiv \big(g_2(x_1) = x_2\big)\colon (2, 0)$ belongs to $LT_2 \circ F_1^{\mathrm{op}}(1 + 1)$ and gives rise to a map in $\mathscr{A}_{LT_2 \circ F_1^{\mathrm{op}}}$, but there is no arrow $t\colon 1 \to 1$ in L_{Σ_1} such that $\top \leq LT_{2\langle \mathrm{id}, F_1(t)\rangle}(\phi)$.

We denote by $\overline{\mathsf{EED}}$ the full sub-category of EED consisting only of those elementary existential doctrines with comprehensive diagonals and satisfying the Rule of Unique Choice. Conveniently, it turns out that the image of $\mathcal{R}$ is already contained in $\overline{\mathsf{EED}}$.

▶ **Proposition 34.** *Let $\mathbb{B}$ be a cartesian bicategory. Then $\mathcal{R}(\mathbb{B})$ is in $\overline{\mathsf{EED}}$.*

Proof. It is enough to show that $\Gamma_{\mathcal{R}(\mathbb{B})}\colon \mathrm{Map}(\mathbb{B}) \to \mathrm{Map}(\mathscr{A}_{\mathcal{R}(\mathbb{B})}) = \mathrm{Map}(\mathcal{LR}(\mathbb{B}))$ is full and faithful. As noticed in Remark 27, $\Gamma_{\mathcal{R}(\mathbb{B})} = \varepsilon_{\mathbb{B}}^{-1}\restriction_{\mathrm{Map}(\mathbb{B})}$. Since $\varepsilon_{\mathbb{B}}^{-1}$ is an isomorphism in CBC, it is faithful, therefore its restriction $\Gamma_{\mathcal{R}(\mathbb{B})}$ is as well. Moreover, $\varepsilon_{\mathbb{B}}^{-1}$ is full: if $R\colon X \to Y$ in $\mathcal{LR}(\mathbb{B})$ is a map, then there exists $f\colon X \to Y$ in $\mathbb{B}$ such that $\varepsilon_{\mathbb{B}}^{-1}(f) = R$. In fact, $f = \varepsilon_{\mathbb{B}}(R)$ and since $\varepsilon_{\mathbb{B}}$ is a morphism in CBC, by Proposition 18 we have that f is a map in $\mathbb{B}$. Therefore, $\Gamma_{\mathcal{R}(\mathbb{B})}$ is full. ◀

▶ **Theorem 35.** *The categories $\overline{\mathsf{EED}}$ and CBC are equivalent via adjunction (11) where $\mathcal{L}$ and $\mathcal{R}$ are respectively restricted and corestricted to $\overline{\mathsf{EED}}$.*

9 Conclusion

We gave an exhaustive analysis of the relationship between two different categorifications of regular logic: the *universal* approach of elementary existential doctrines, and the *algebraic* approach of cartesian bicategories. We showed that cartesian bicategories give rise to elementary existential doctrines and, expanding a remark in [27], that also the other direction is possible. We proved that this correspondence is functorial, in the sense that we have a pair of functors $\mathcal{L}\colon \mathsf{EED} \to \mathsf{CBC}$ and $\mathcal{R}\colon \mathsf{CBC} \to \mathsf{EED}$ which are moreover adjoint (Theorem 28).

This adjunction can be strengthened to an equivalence provided that we refine the notion of doctrine, excluding some problematic examples (e.g. Example 29). These cases lay outside the image of $\mathcal{R}$ and thus this restriction does not affect cartesian bicategories (Theorem 35).

We hope that understanding the relationship between CBC and EED may provide some hints on the nature of the additional algebraic structure needed for cartesian bicategories to capture full first order logic, which one can do on the EED, universal side by considering Lawvere's original hyperdoctrines. It is probable that the end result will be closely related to Peirce's existential graphs [31], a 19th century proto-string-diagrammatic logical syntax. This direction has already started to be explored, from diverse perspectives, in [7, 8, 18].

References

1. John Baez and Jason Erbele. Categories in control. *Theory and Application of Categories*, 30(24):836–881, 2015. URL: `http://www.tac.mta.ca/tac/volumes/30/24/30-24abs.html`.
2. Filippo Bonchi, Joshua Holland, Robin Piedeleu, Pawel Sobocinski, and Fabio Zanasi. Diagrammatic algebra: from linear to concurrent systems. *Proc. ACM Program. Lang.*, 3(POPL):25:1–25:28, 2019. `doi:10.1145/3290338`.
3. Filippo Bonchi, Alessio Santamaria, Jens Seeber, and Paweł Sobociński. On Doctrines and Cartesian Bicategories. *arXiv:2106.08142 [cs, math]*, June 2021. `arXiv:2106.08142`.
4. Filippo Bonchi, Jens Seeber, and Pawel Sobocinski. Graphical Conjunctive Queries. In Dan Ghica and Achim Jung, editors, *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*, volume 119 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 13:1–13:23, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. `doi:10.4230/LIPIcs.CSL.2018.13`.
5. Filippo Bonchi, Pawel Sobocinski, and Fabio Zanasi. Full Abstraction for Signal Flow Graphs. In *Proceedings of the 42nd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*, POPL '15, pages 515–526, New York, NY, USA, January 2015. Association for Computing Machinery. `doi:10.1145/2676726.2676993`.
6. Filippo Bonchi, Pawel Sobocinski, and Fabio Zanasi. Deconstructing Lawvere with distributive laws. *Journal of Logical and Algebraic Methods in Programming*, 95:128–146, 2018. `doi:10.1016/j.jlamp.2017.12.002`.
7. Geraldine Brady and Todd Trimble. A string diagram calculus for predicate logic and C. S. Peirce's system beta. Unpublished, available online at `https://ncatlab.org/nlab/files/BradyTrimbleString.pdf`, 2000.
8. Geraldine Brady and Todd H. Trimble. A categorical interpretation of C.S. Peirce's propositional logic Alpha. *Journal of Pure and Applied Algebra*, 149(3):213–239, June 2000. `doi:10.1016/S0022-4049(98)00179-0`.
9. Aurelio Carboni and Robert F.C. Walters. Cartesian Bicategories I. *Journal of Pure and Applied Algebra*, 49(1):11–32, 1987. `doi:10.1016/0022-4049(87)90121-6`.
10. Bob Coecke and Ross Duncan. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, 13(4):043016, April 2011. Publisher: IOP Publishing. `doi:10.1088/1367-2630/13/4/043016`.
11. Ivan Di Liberti, Fosco Loregian, Chad Nester, and Paweł Sobociński. Functorial semantics for partial theories. In *Proceedings of the ACM on Programming Languages*, volume 5, pages 57:1–57:28, January 2021. `doi:10.1145/3434338`.
12. Brendan Fong, Paweł Sobociński, and Paolo Rapisarda. A categorical approach to open and interconnected dynamical systems. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '16, pages 495–504, New York, NY, USA, July 2016. Association for Computing Machinery. `doi:10.1145/2933575.2934556`.
13. Brendan Fong and David Spivak. String diagrams for regular logic (extended abstract). In John Baez and Bob Coecke, editors, *Applied Category Theory 2019*, volume 323 of *Electronic Proceedings in Theoretical Computer Science*, pages 196–229. Open Publishing Association, September 2020. `doi:10.4204/eptcs.323.14`.

14 Brendan Fong and David I Spivak. Graphical regular logic, 2019. `arXiv:1812.05765`.

15 Thomas Fox. Coalgebras and cartesian categories. *Communications in Algebra*, 4(7):665–667, 1976. `doi:10.1080/00927877608822127`.

16 Peter Freyd and Andre Scedrov. *Categories, Allegories*, volume 39 of *North-Holland Mathematical Library*. Elsevier B.V, 1990. URL: `https://www.sciencedirect.com/bookseries/north-holland-mathematical-library/vol/39/suppl/C`.

17 Dan R. Ghica and Achim Jung. Categorical semantics of digital circuits. In *2016 Formal Methods in Computer-Aided Design (FMCAD)*, pages 41–48, October 2016. `doi:10.1109/FMCAD.2016.7886659`.

18 Nathan Haydon and Paweł Sobociński. Compositional Diagrammatic First-Order Logic. In *Diagrammatic Representation and Inference*, Lecture Notes in Computer Science, 11th International Conference, Diagrams 2020, Tallinn, Estonia„ 2020. Springer International Publishing. `doi:10.1007/978-3-030-54249-8_32`.

19 Stephen Lack. Composing PROPs. *Theory and Application of Categories*, 13(9):147–163, 2004. URL: `http://www.tac.mta.ca/tac/volumes/13/9/13-09abs.html`.

20 F. William Lawvere. Functorial semantics of algebraic theories. *Proceedings of the National Academy of Sciences*, 50(5):869–872, 1963.

21 F. William Lawvere. Adjointness in Foundations. *Dialectica*, 23(3/4):281–296, 1969. Publisher: Wiley. `doi:10.1111/j.1746-8361.1969.tb01194.x`.

22 F. William Lawvere. Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications II*, Lecture Notes in Mathematics, pages 134–145, Berlin, Heidelberg, 1969. Springer. `doi:10.1007/BFb0080769`.

23 F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In Alex Heller, editor, *Applications of Categorical Algebra*, volume 17, pages 1–14, New York, NY, 1970. American Mathematical Society. `doi:10.1090/pspum/017`.

24 Saunders MacLane. Categorical algebra. *Bulletin of the American Mathematical Society*, 71(1):40–106, 1965. `doi:10.1090/S0002-9904-1965-11234-4`.

25 Maria Emilia Maietti, Fabio Pasquali, and Giuseppe Rosolini. Triposes, exact completions, and Hilbert's ε-operator. *Tbilisi Mathematical Journal*, 10(3):141–166, June 2017. Publisher: Tbilisi Centre for Mathematical Sciences. `doi:10.1515/tmj-2017-0106`.

26 Maria Emilia Maietti and Giuseppe Rosolini. Quotient Completion for the Foundation of Constructive Mathematics. *Logica Universalis*, 7(3):371–402, September 2013. `doi:10.1007/s11787-013-0080-2`.

27 Maria Emilia Maietti and Giuseppe Rosolini. Unifying Exact Completions. *Applied Categorical Structures*, 23(1):43–52, February 2015. `doi:10.1007/s10485-013-9360-5`.

28 Koko Muroya, Steven W. T. Cheung, and Dan R. Ghica. The geometry of computation-graph abstraction. In Anuj Dawar and Erich Grädel, editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 749–758. ACM, 2018. `doi:10.1145/3209108.3209127`.

29 Evan Patterson. Knowledge representation in bicategories of relations, 2017. `arXiv:1706.00526`.

30 Robin Piedeleu and Fabio Zanasi. A String Diagrammatic Axiomatisation of Finite-State Automata. In Stefan Kiefer and Christine Tasson, editors, *Foundations of Software Science and Computation Structures*, Lecture Notes in Computer Science, pages 469–489, Cham, 2021. Springer International Publishing. `doi:10.1007/978-3-030-71995-1_24`.

31 Ahti-Veikko Pietarinen. *The Logic of the Future*, volume 1. De Gruyter, 2019.

32 Jens Seeber. *Logical completeness for string diagrams*. PhD thesis, IMT Lucca, 2020.

33 P. Selinger. A Survey of Graphical Languages for Monoidal Categories. In B. Coecke, editor, *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 289–355. Springer, Berlin, Heidelberg, 2010. `doi:10.1007/978-3-642-12821-9_4`.