Brief Announcement: Local Certification of Graph Decompositions and Applications to Minor-Free Classes

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Abstract

Local certification consists in assigning labels to the nodes of a network to certify that some given property is satisfied, in such a way that the labels can be checked locally. In the last few years, certification of graph classes received a considerable attention. The goal is to certify that a graph G belongs to a given graph class $\mathcal{G}$. Such certifications with labels of size $O(\log n)$ (where n is the size of the network) exist for trees, planar graphs and graphs embedded on surfaces. Feuilloley et al. ask if this can be extended to any class of graphs defined by a finite set of forbidden minors.

In this paper, we develop new decomposition tools for graph certification, and apply them to show that for every small enough minor H, H-minor-free graphs can indeed be certified with labels of size $O(\log n)$. We also show matching lower bounds with a new simple proof technique.

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1 Introduction

Local certification is an active field of research in the theory of distributed computing. On a high level it consists in certifying global properties in such a way that the verification can be done locally. More precisely, for a given property, a local certification consists of a labeling (called a *certificate assignment*), and of a local verification algorithm. If the configuration of the network is correct, then there should exist a labeling of the nodes that is accepted by the verification algorithm, whereas if the configuration is incorrect no labeling should make the verification algorithm accept.

Local certification originates from self-stabilization, and was first concerned with certifying that a solution to an algorithmic problem is correct. However, it is also important to understand how to certify properties of the network itself, that is, to find locally checkable proofs that the network belongs to some graph class. There are several reasons for that. First, because certifying some solutions can be hard in general graphs, while they become simpler on more restricted classes. To make use of this fact, it is important to be able to certify that the network does belong to the restricted class. Second, because some distributed algorithms work only on some specific graph classes, and we need a way to ensure that the network does belong to the class, before running the algorithm. Third, the distinction between certifying solutions and network properties is rather weak, in the sense that the techniques are basically the same. So we should take advantage of the fact that a lot is known about graph classes to learn more about certification.

In the domain of graph classes certification, there have been several results on various classes such as trees [10], bipartite graphs [9] or graphs of bounded diameter [3], but until two years ago little was known about essential classes, such as planar graphs, H-free or H-minor-free graphs. Recently, it has been shown that planar graphs and graphs of bounded genus can be certified with $O(\log n)$-bit labels [7, 8, 5]. This size, $O(\log n)$, is the gold standard of certification, in the sense that little can be achieved with $o(\log n)$ bits, thus $O(\log n)$ is often the best we can hope for. It happens that planar and bounded-genus graphs are classic examples of graphs classes defined by forbidden minors, which naturally raises the following question.

▶ **Question 1** ([8, 6]). *Can any graph class defined by a finite set of forbidden minors be certified with $O(\log n)$-bit certificates?*

This open question is quite challenging: there are as many good reasons to believe that the answer is positive as negative.

First, the literature provides some reasons to believe that the conjecture is true. Properties that are known to be hard to certify, that is, that are known to require large certificates, are very different from minor-freeness. Specifically, all these properties (*e.g.* small diameter [3], non-3-colorability [9], having a non-trivial automorphism [9]) are non-hereditary. That is, removing a node or an edge may yield a graph that is not in the class. Intuitively, hereditary properties might be easier to certify in the sense that one does not need to encode information about every single edge or node, as the class is stable by removal of edges and nodes. Minor-freeness is a typical example of hereditary property. Moreover, this property, that has been intensively studied in the last decades, is known to carry a lot of structure, which is an argument in favor of the existence of a compact certification (that is a certification with $O(\log n)$-bit labels).

On the other hand, from a graph theory perspective, it might be surprising that a general compact certification existed for minor-free graphs. Indeed, for the known results, obtaining a compact certification is tightly linked to the existence of a precise constructive characterization of the class (*e.g.* a planar embedding for planar graphs [7, 5], or a canonical path to the root for trees [10]). While such a characterization is known for some restricted minor-closed classes, we are far from having such a characterization for every minor-closed class. Note that there are a lot of combinatorial and algorithmic results on H-minor free graphs, but they actually follow from properties satisfied by H-minor free graphs, not from exact characterizations of such graphs. For certification, we need to rule out the graphs that do not belong to the class, hence a characterization is somehow necessary.

It is important to note that forbidden minor characterizations are about structures that are absent from the graphs, and local certification is often about certifying the existence of some structures, which explains why it is a challenge to certify all minor-free classes with small certificates. On the other hand, as we will see later, certifying that a minor does appear in the graph is easy.

1.1 Our results

An extensive line of work in structural graph theory aims to provide characterizations of classes using so-called decomposition theorems. Amongst the most famous examples of these theorems is the proof of the 4-Color Theorem [1] or the Strong Perfect Graph Theorem [4] which consists in decomposing graphs until we reach some elementary graphs.

Our goal in this paper is to prove that many of these decomposition tools can actually be used in order to certify the fact that the graph belongs to the class. We first prove that several classic decomposition techniques existing in the literature are suitable for certification. We then apply these general tools on Question 1 to prove that several H-minor free graph classes can be certified with $O(\log n)$ bits. In particular, our results provide evidence that, if the answer to Question 1 is negative then it is certainly for large non-planar graphs H.

The decomposition tools we are able to certify are at the core of many decomposition theorems: 2-(edge-)connectivity, 3-connectivity, block-cut trees, forbidden subgraphs, and expansions of nodes or edges by new graphs. Note that these tools are also interesting by themselves, in particular, connectivity is an important measure of robustness in networks. One common challenge in the design of certification for decomposition is what we call *certificate congestion*. Consider for example a situation in which we have a certification for k graphs, and we want to merge these graphs by identifying one vertex in each of them. Then, the straightforward technique to certify the merged graph is to give to the merged node its certificate for every of these k graphs. But since k might be large, this implies a large certificate size. Since we aim for small certificates, we want to avoid such congestion. We use several solutions to cope with this problem.

Using these tools, we show that the answer to Question 1 is positive for many small graphs. These results permit to illustrate our methods with simple and compact applications of our tools. More generally, we aim at providing some evidence that graph decomposition and these tools can be successfully used in the certification setting, and it is very likely that many other minor-closed classes can be certified using our techniques.

Our main results are summarized in Figure 1, and illustrations of the corresponding minors can be found in Figure 2. These are actually the hard cases of the following theorem.

▶ **Theorem 2.** *H-minor-free classes can be certified in $O(\log n)$ bits when H has at most 4 vertices.*

We also prove a general $\Omega(\log n)$ lower bounds for H-minor-freeness for all 2-connected graphs H. This generalizes and simplifies the lower bounds of [7] which apply only to K_k and $K_{p,q}$-minor-free graphs, and use ad-hoc and more complicated techniques.

Class	Optimal size	Result
K_3-minor free	$\Theta(\log n)$	Equivalent to acyclicity [10, 9].
Diamond-minor-free	$\Theta(\log n)$	New.
K_4-minor-free	$\Theta(\log n)$	New.
$K_{2,3}$-minor-free	$\Theta(\log n)$	New.
$(K_{2,3}, K_4)$-minor-free (*i.e.* outerplanar)	$\Theta(\log n)$	New.
$K_{2,4}$-minor-free	$\Theta(\log n)$	New.

Figure 1 Our main results for the certification of minor-closed classes.

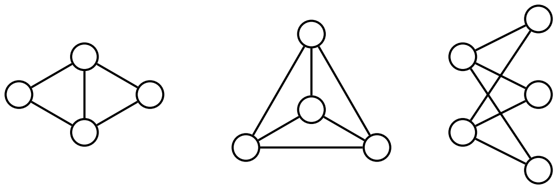

Figure 2 From left to right: the diamond, the clique on 4 vertices K_4, and the complete bipartite graph $K_{2,3}$.

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