Stackelberg-Pareto Synthesis

Véronique Bruyère
Université de Mons (UMONS), Mons, Belgium

Jean-François Raskin
Université libre de Bruxelles (ULB), Brussels, Belgium

Clément Tamines
Université de Mons (UMONS), Mons, Belgium

Abstract

In this paper, we study the framework of two-player Stackelberg games played on graphs in which Player 0 announces a strategy and Player 1 responds rationally with a strategy that is an optimal response. While it is usually assumed that Player 1 has a single objective, we consider here the new setting where he has several. In this context, after responding with his strategy, Player 1 gets a payoff in the form of a vector of Booleans corresponding to his satisfied objectives. Rationality of Player 1 is encoded by the fact that his response must produce a Pareto-optimal payoff given the strategy of Player 0. We study the Stackelberg-Pareto Synthesis problem which asks whether Player 0 can announce a strategy which satisfies his objective, whatever the rational response of Player 1. For games in which objectives are either all parity or all reachability objectives, we show that this problem is fixed-parameter tractable and NEXPTIME-complete. This problem is already NP-complete in the simple case of reachability objectives and graphs that are trees.

2012 ACM Subject Classification Software and its engineering → Formal methods; Theory of computation → Logic and verification; Theory of computation → Solution concepts in game theory

Keywords and phrases Stackelberg non-zero sum games played on graphs, synthesis, parity objectives

Digital Object Identifier 10.4230/LIPIcs.CONCUR.2021.27

Related Version *Full Version*: `https://arxiv.org/abs/2102.08925` [9]

Funding This work is partially supported by the PDR project Subgame perfection in graph games (F.R.S.-FNRS), the ARC project Non-Zero Sum Game Graphs: Applications to Reactive Synthesis and Beyond (Fédération Wallonie-Bruxelles), the EOS project Verifying Learning Artificial Intelligence Systems (F.R.S.-FNRS and FWO), and the COST Action 16228 GAMENET (European Cooperation in Science and Technology).

1 Introduction

Two-player zero-sum infinite-duration games played on graphs are a mathematical model used to formalize several important problems in computer science, such as *reactive system synthesis*. In this context, see e.g. [26], the graph represents the possible interactions between the system and the environment in which it operates. One player models the system to synthesize, and the other player models the (uncontrollable) environment. In this classical setting, the objectives of the two players are opposite, that is, the environment is *adversarial*. Modelling the environment as fully adversarial is usually a *bold abstraction* of reality as it can be composed of one or several components, each of them having their own objective.

In this paper, we consider the framework of *Stackelberg games* [31], a richer non-zero-sum setting, in which Player 0 (the system) called *leader* announces his strategy and then Player 1 (the environment) called *follower* plays rationally by using a strategy that is an optimal response to the leader's strategy. This framework captures the fact that in practical applications, a strategy for interacting with the environment is committed before the interaction actually happens. The goal of the leader is to announce a strategy that

32nd International Conference on Concurrency Theory (CONCUR 2021).
Editors: Serge Haddad and Daniele Varacca; Article No. 27; pp. 27:1–27:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

guarantees him a payoff at least equal to some given threshold. In the specific case of Boolean objectives, the leader wants to see his objective being satisfied. The concept of leader and follower is also present in the framework of *rational synthesis* [17, 24] with the difference that this framework considers several followers, each of them with their own Boolean objective. In that case, rationality of the followers is modeled by assuming that the environment settles to an equilibrium (e.g. a Nash equilibrium) where each component (composing the environment) is considered to be an *independent selfish individual*, excluding cooperation scenarios between components or the possibility of coordinated rational multiple deviations. Our work proposes a novel and natural *alternative* in which the single follower, modeling the environment, has several objectives that he wants to satisfy. After responding to the leader with his own strategy, Player 1 receives a vector of Booleans which is his payoff in the corresponding outcome. Rationality of Player 1 is encoded by the fact that he only responds in such a way to receive *Pareto-optimal payoffs*, given the strategy announced by the leader. This setting encompasses scenarios where, for instance, several components can collaborate and agree on trade-offs. The goal of the leader is therefore to announce a strategy that guarantees him to satisfy his own objective, whatever the response of the follower which ensures him a Pareto-optimal payoff. The problem of deciding whether the leader has such a strategy is called the *Stackelberg-Pareto Synthesis problem* (SPS problem).

Contributions. In addition to the definition of the new setting, our main contributions are the following ones. We consider the general class of ω-regular objectives modelled by *parity* conditions and also consider the case of *reachability* objectives for their simplicity[1]. We provide a thorough analysis of the complexity of solving the SPS problem for both objectives. Our results are interesting and singular both from a theoretical and practical point of view.

First, we show that the SPS problem is *fixed-parameter tractable* (FPT) for reachability objectives when the number of objectives of the follower is a parameter and for parity objectives when, in addition, the maximal priority used in each priority function is also a parameter of the complexity analysis (Theorem 3). These are important results as it is expected that, in practice, the *number* of objectives of the environment is limited to a few. To obtain these results, we develop a reduction from our non-zero-sum games to a zero-sum game in which the protagonist, called *Prover*, tries to show the existence of a solution to the problem, while the antagonist, called *Challenger*, tries to disprove it. This zero-sum game is defined in a *generic way*, independently of the actual objectives used in the initial game, and can then be easily adapted according to the case of reachability or parity objectives.

Second, we prove that the SPS problem is NEXPTIME-complete for both reachability and parity objectives (Theorem 6 and Theorem 9), and that it is already NP-complete in the simple setting of reachability objectives and graphs that are trees (Theorem 7). To the best of our knowledge, this is the first NEXPTIME-completeness result for a natural class of games played on graphs. To obtain the hardness for NEXPTIME, we present a natural *succinct version* of the set cover problem that is complete for this class (Theorem 11), a result of potential independent interest. We then show how to reduce this problem to the SPS problem. To obtain the NEXPTIME-membership of the SPS problem, we have shown that exponential-size solutions exist for positive instances of the SPS problem and this allows us to design a nondeterministic exponential-time algorithm. Unfortunately, it was not possible to use the FPT algorithm mentioned above to show this membership due to its too high time complexity; conversely, our NEXPTIME algorithm is not FPT.

[1] Indeed, in the classical context of two-player zero-sum games, solving reachability games is in P whereas solving parity games is only known to be in NP $\cap$ co-NP, see e.g. [18].

Related Work. Rational synthesis is introduced in [17] for ω-regular objectives in a setting where the followers are cooperative with the leader, and later in [24] where they are adversarial. Precise complexity results for various ω-regular objectives are established in [13] for both settings. Those complexities differ from the ones of the problem studied in this paper. Indeed, for reachability objectives, adversarial rational synthesis is PSPACE-complete, while for parity objectives, its precise complexity is not settled (the problem is PSPACE-hard and in NEXPTIME). Extension to non-Boolean payoffs, like mean-payoff or discounted sum, is studied in [19, 20] in the cooperative setting and in [1, 16] in the adversarial setting.

When several players (like the followers) play with the aim to satisfy their objectives, several solution concepts exist such as Nash equilibrium [25], subgame perfect equilibrium [27], secure equilibria [11, 12], or admissibility [2, 4]. The constrained existence problem, close to the cooperative rational synthesis problem, is to decide whether there exists a solution concept such that the payoff obtained by each player is larger than some threshold. Let us mention [13, 29, 30] for results on the constrained existence for Nash equilibria and [5, 6, 28] for such results for subgame perfect equilibria. Rational verification is studied in [21, 22]. This problem (which is not a synthesis problem) is to decide whether a given LTL formula is satisfied by the outcome of all Nash equilibria (resp. some Nash equilibrium). The interested reader can find more pointers to works on non-zero-sum games for reactive synthesis in [3, 7].

Structure. The paper is structured as follows. In Section 2, we introduce the class of Stackelberg-Pareto games and the SPS problem. We show in Section 3 that the SPS problem is in FPT for reachability and parity objectives. The complexity class of this problem is studied in Section 4 where we prove that it is NEXPTIME-complete and NP-complete in case of reachability objectives and graphs that are trees. In Section 5, we provide a conclusion and discuss future work. Detailed proofs of our results can be found in the full version of this paper.

2 Preliminaries and Stackelberg-Pareto Synthesis Problem

This section introduces the class of two-player Stackelberg-Pareto games in which the first player has a single objective and the second has several. We present a decision problem on those games called the Stackelberg-Pareto Synthesis problem, which we study in this paper.

2.1 Preliminaries

Game Arena. A *game arena* is a tuple $G = (V, V_0, V_1, E, v_0)$ where (V, E) is a finite directed graph such that: *(i)* V is the set of vertices and (V_0, V_1) forms a partition of V where V_0 (resp. V_1) is the set of vertices controlled by Player 0 (resp. Player 1), *(ii)* $E \subseteq V \times V$ is the set of edges such that each vertex v has at least one successor v', i.e., $(v, v') \in E$, and *(iii)* $v_0 \in V$ is the initial vertex. We call a game arena a *tree arena* if it is a tree in which every leaf vertex has itself as its only successor. A *sub-arena* G' with a set $V' \subseteq V$ of vertices and initial vertex $v'_0 \in V'$ is a game arena defined from G as expected.

Plays. A *play* in a game arena G is an infinite sequence of vertices $\rho = v_0 v_1 \ldots \in V^\omega$ such that it starts with the initial vertex v_0 and $(v_j, v_{j+1}) \in E$ for all $j \in \mathbb{N}$. *Histories* in G are finite sequences $h = v_0 \ldots v_j \in V^+$ defined similarly. A history is *elementary* if it contains no cycles. We denote by Plays_G the set of plays in G. We write Hist_G (resp. $\mathsf{Hist}_{G,i}$) the set of histories (resp. histories ending with a vertex in V_i). We use the notations Plays, Hist, and Hist_i when G is clear from the context. We write $\mathsf{Occ}(\rho)$ the set of vertices occurring in ρ and $\mathsf{Inf}(\rho)$ the set of vertices occurring infinitely often in ρ.

Strategies. A *strategy* σ_i for Player i is a function $\sigma_i \colon \mathsf{Hist}_i \to V$ assigning to each history $hv \in \mathsf{Hist}_i$ a vertex $v' = \sigma_i(hv)$ such that $(v, v') \in E$. It is *memoryless* if $\sigma_i(hv) = \sigma_i(h'v)$ for all histories $hv, h'v$ ending with the same vertex $v \in V_i$. More generally, it is *finite-memory* if it can be encoded by a Moore machine $\mathcal{M}$ [18]. The *memory size* of σ_i is the number of memory states of $\mathcal{M}$. In particular, σ_i is memoryless when it has a memory size of one.

Given a strategy σ_i of Player i, a play $\rho = v_0 v_1 \dots$ is *consistent* with σ_i if $v_{j+1} = \sigma_i(v_0 \dots v_j)$ for all $j \in \mathbb{N}$ such that $v_j \in V_i$. Consistency is naturally extended to histories. We denote by $\mathsf{Plays}_{\sigma_i}$ (resp. Hist_{σ_i}) the set of plays (resp. histories) consistent with σ_i. A *strategy profile* is a tuple $\sigma = (\sigma_0, \sigma_1)$ of strategies, one for each player. We write $\mathsf{out}(\sigma)$ the unique play consistent with both strategies and we call it the *outcome* of σ.

Objectives. An *objective* for Player i is a set of plays $\Omega \subseteq \mathsf{Plays}$. A play ρ *satisfies* the objective Ω if $\rho \in \Omega$. In this paper, we focus on the two following ω-regular objectives. Let $T \subseteq V$ be a subset of vertices called a *target set*, the *reachability* objective $\mathsf{Reach}(T) = \{\rho \in \mathsf{Plays} \mid \mathsf{Occ}(\rho) \cap T \neq \emptyset\}$ asks to visit at least one vertex of T. Let $c : V \to \mathbb{N}$ be a function called a *priority function* which assigns an integer to each vertex in the arena, the *parity* objective $\mathsf{Parity}(c) = \{\rho \in \mathsf{Plays} \mid \min_{v \in \mathsf{Inf}(\rho)}(c(v)) \text{ is even}\}$ asks that the minimum priority visited infinitely often be even.

2.2 Stackelberg-Pareto Synthesis Problem

Stackelberg-Pareto Games. A *Stackelberg-Pareto game* (SP game) $\mathcal{G} = (G, \Omega_0, \Omega_1, \dots, \Omega_t)$ is composed of a game arena G, an objective Ω_0 for Player 0 and $t \geq 1$ objectives $\Omega_1, \dots, \Omega_t$ for Player 1. In this paper, we focus on SP games where the objectives are either all reachability or all parity objectives and call such games *reachability* (resp. *parity*) *SP games*.

Payoffs in SP Games. The *payoff* of a play $\rho \in \mathsf{Plays}$ corresponds to the vector of Booleans $\mathsf{pay}(\rho) \in \{0,1\}^t$ such that for all $i \in \{1, \dots, t\}$, $\mathsf{pay}_i(\rho) = 1$ if $\rho \in \Omega_i$, and $\mathsf{pay}_i(\rho) = 0$ otherwise. Note that we omit to include Player 0 when discussing the payoff of a play. Instead we say that a play ρ is *won* by Player 0 if $\rho \in \Omega_0$ and we write $\mathsf{won}(\rho) = 1$, otherwise it is *lost* by Player 0 and we write $\mathsf{won}(\rho) = 0$. We write $(\mathsf{won}(\rho), \mathsf{pay}(\rho))$ the *extended payoff* of ρ. Given a strategy profile σ, we write $\mathsf{won}(\sigma) = \mathsf{won}(\mathsf{out}(\sigma))$ and $\mathsf{pay}(\sigma) = \mathsf{pay}(\mathsf{out}(\sigma))$. For reachability SP games, since reachability objectives are prefix-dependant and given a history $h \in \mathsf{Hist}$, we also define $\mathsf{won}(h)$ and $\mathsf{pay}(h)$ as done for plays.

We introduce the following partial order on payoffs. Given two payoffs $p = (p_1, \dots, p_t)$ and $p' = (p'_1, \dots, p'_t)$ such that $p, p' \in \{0,1\}^t$, we say that p' is *larger* than p and write $p \leq p'$ if $p_i \leq p'_i$ for all $i \in \{1, \dots, t\}$. Moreover, when it also holds that $p_i < p'_i$ for some i, we say that p' is *strictly larger* than p and we write $p < p'$. A subset of payoffs $P \subseteq \{0,1\}^t$ is an *antichain* if it is composed of pairwise incomparable payoffs with respect to $\leq$.

Stackelberg-Pareto Synthesis Problem. Given a strategy σ_0 of Player 0, we consider the set of payoffs of plays consistent with σ_0 which are *Pareto-optimal*, i.e., maximal with respect to $\leq$. We write this set $P_{\sigma_0} = \max\{\mathsf{pay}(\rho) \mid \rho \in \mathsf{Plays}_{\sigma_0}\}$. Notice that it is an antichain. We say that those payoffs are *σ_0-fixed Pareto-optimal* and write $|P_{\sigma_0}|$ the number of such payoffs. A play $\rho \in \mathsf{Plays}_{\sigma_0}$ is called σ_0-fixed Pareto-optimal if its payoff $\mathsf{pay}(\rho)$ is in P_{σ_0}.

The problem studied in this paper asks whether there exists a strategy σ_0 for Player 0 such that every play in $\mathsf{Plays}_{\sigma_0}$ which is σ_0-fixed Pareto-optimal satisfies the objective of Player 0. This corresponds to the assumption that given a strategy of Player 0, Player 1 will play *rationally*, that is, with a strategy σ_1 such that $\mathsf{out}((\sigma_0, \sigma_1))$ is σ_0-fixed Pareto-optimal. It is therefore sound to ask that Player 0 wins against such rational strategies.

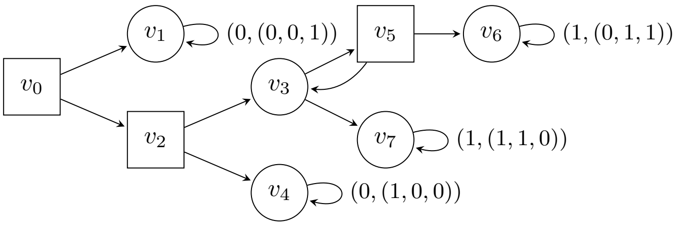

Figure 1 A reachability SP game.

▶ **Definition 1.** *Given an SP game, the* Stackelberg-Pareto Synthesis problem *(SPS problem) is to decide whether there exists a strategy* σ_0 *for Player* 0 *(called a* solution*) such that for each strategy profile* $\sigma = (\sigma_0, \sigma_1)$ *with* $\mathsf{pay}(\sigma) \in P_{\sigma_0}$*, it holds that* $\mathsf{won}(\sigma) = 1$.

Witnesses. Given a strategy σ_0 that is a solution to the SPS problem and any payoff $p \in P_{\sigma_0}$, for each play ρ consistent with σ_0 such that $\mathsf{pay}(\rho) = p$ it holds that $\mathsf{won}(\rho) = 1$. For each $p \in P_{\sigma_0}$, we arbitrarily select such a play which we call a *witness* (of p). We denote by Wit_{σ_0} the set of all witnesses, of which there are as many as payoffs in P_{σ_0}. In the sequel, it is useful to see this set as a tree composed of $|\mathsf{Wit}_{\sigma_0}|$ branches. Additionally for a given history $h \in \mathsf{Hist}$, we write $\mathsf{Wit}_{\sigma_0}(h)$ the set of witnesses for which h is a prefix, i.e., $\mathsf{Wit}_{\sigma_0}(h) = \{\rho \in \mathsf{Wit}_{\sigma_0} \mid h \text{ is prefix of } \rho\}$. Notice that $\mathsf{Wit}_{\sigma_0}(h) = \mathsf{Wit}_{\sigma_0}$ when $h = v_0$ and that $\mathsf{Wit}_{\sigma_0}(h)$ decreases as h increases, until it contains a single value or becomes empty.

▶ **Example 2.** Consider the reachability SP game with arena G depicted in Figure 1 in which Player 1 has $t = 3$ objectives. The vertices of Player 0 (resp. Player 1) are depicted as ellipses (resp. rectangles)[2]. Every objective in the game is a reachability objective defined as follows: $\Omega_0 = \mathsf{Reach}(\{v_6, v_7\})$, $\Omega_1 = \mathsf{Reach}(\{v_4, v_7\})$, $\Omega_2 = \mathsf{Reach}(\{v_3\})$, $\Omega_3 = \mathsf{Reach}(\{v_1, v_6\})$. The extended payoff of plays reaching vertices from which they can only loop is displayed in the arena next to those vertices, and the extended payoff of play $v_0v_2(v_3v_5)^\omega$ is $(0,(0,1,0))$.

Consider the memoryless strategy σ_0 of Player 0 such that he chooses to always move to v_5 from v_3. The set of payoffs of plays consistent with σ_0 is $\{(0,0,1),(0,1,0),(1,0,0),(0,1,1)\}$ and the set of those that are Pareto-optimal is $P_{\sigma_0} = \{(1,0,0),(0,1,1)\}$. Notice that play $\rho = v_0v_2(v_4)^\omega$ is consistent with σ_0, has payoff $(1,0,0)$ and is lost by Player 0. Strategy σ_0 is therefore not a solution to the SPS problem. In this game, there is only one other memoryless strategy for Player 0, where he chooses to always move to v_7 from v_3. One can verify that it is again not a solution to the SPS problem.

We can however define a finite-memory strategy σ_0' such that $\sigma_0'(v_0v_2v_3) = v_5$ and $\sigma_0'(v_0v_2v_3v_5v_3) = v_7$ and show that it is a solution to the problem. Indeed, the set of σ_0'-fixed Pareto-optimal payoffs is $P_{\sigma_0'} = \{(0,1,1),(1,1,0)\}$ and Player 0 wins every play consistent with σ_0' whose payoff is in this set. A set $\mathsf{Wit}_{\sigma_0'}$ of witnesses for these payoffs is $\{v_0v_2v_3v_5v_6^\omega, v_0v_2v_3v_5v_3v_7^\omega\}$ and is in this case the unique set of witnesses. This example shows that Player 0 sometimes needs memory in order to have a solution to the SPS problem.

3 Fixed-Parameter Complexity

In this section, we show that the SPS problem is in FPT for both cases of reachability and parity SP games. We refer the reader to [15] for the concept of fixed-parameter complexity.

[2] This convention is used throughout this paper.

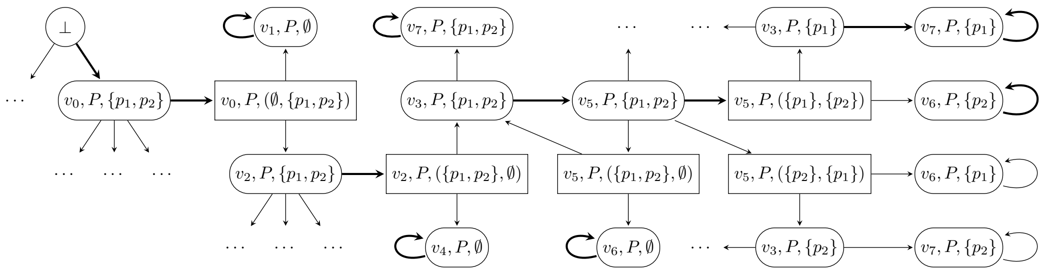

Figure 2 A part of the C-P game for Example 2 with $P = \{p_1, p_2\}, p_1 = (1, 1, 0)$, and $p_2 = (0, 1, 1)$.

▶ **Theorem 3.** *Solving the SPS problem is in* FPT *for reachability SP games for parameter t equal to the number of objectives of Player 1 and it is in* FPT *for parity SP games for parameters t and the maximal priority according to each parity objective of Player 1.*

3.1 Challenger-Prover Game

In order to prove Theorem 3, we provide a reduction to a specific two-player zero-sum game, called the *Challenger-Prover* game (C-P game). This game is a *zero-sum*[3] game played between *Challenger* (written $\mathcal{C}$) and *Prover* (written $\mathcal{P}$). We will show that Player 0 has a solution to the SPS problem in an SP game if and only if $\mathcal{P}$ has a winning strategy in the corresponding C-P game. In the latter game, $\mathcal{P}$ tries to show the existence of a strategy σ_0 that is solution to the SPS problem in the original game and $\mathcal{C}$ tries to disprove it. The C-P game is described independently of the objectives used in the SP game and its objective is described as such in a *generic way*. We later provide the proof of our FPT results by adapting it specifically for reachability and parity SP games.

Intuition on the C-P Game. Without loss of generality, the SP games we consider in this section are such that each vertex in their arena has at most *two successors*. It can be shown that any SP game $\mathcal{G}$ with n vertices can be transformed into an SP game $\bar{\mathcal{G}}$ with $\mathcal{O}(n^2)$ vertices such that every vertex has at most two successors and Player 0 has a solution to the SPS problem in $\mathcal{G}$ if and only if he has a solution to the SPS problem in $\bar{\mathcal{G}}$.

Let $\mathcal{G}$ be an SP game. The C-P game $\mathcal{G}'$ is a zero-sum game associated with $\mathcal{G}$ that intuitively works as follows. First, $\mathcal{P}$ selects a set P of payoffs which he announces as the set of Pareto-optimal payoffs P_{σ_0} for the solution σ_0 to the SPS problem in $\mathcal{G}$ he is trying to construct. Then, $\mathcal{P}$ tries to show that there exists a set of witnesses Wit_{σ_0} in $\mathcal{G}$ for the payoffs in P. After the selection of P in $\mathcal{G}'$, there is a one-to-one correspondence between plays in the arenas G and G' such that the vertices in G' are augmented with a set W which is a subset of P. Initially W is equal to P and after some history in G', W contains payoff p if the corresponding history in G is prefix of the witness with payoff p in the set Wit_{σ_0} that $\mathcal{P}$ is building. In addition, the objective $\Omega_{\mathcal{P}}$ of $\mathcal{P}$ is such that he has a winning strategy $\sigma_{\mathcal{P}}$ in $\mathcal{G}'$ if and only if the set P that he selected coincides with the set P_{σ_0} for the corresponding strategy σ_0 in $\mathcal{G}$ and the latter strategy is a solution to the SPS problem in $\mathcal{G}$. A part of the arena of the C-P game for Example 2 with a positional winning strategy for $\mathcal{P}$ highlighted in bold is illustrated in Figure 2.

[3] We assume that the reader is familiar with the concept of zero-sum games, see e.g. [18].

Arena of the C-P Game. The initial vertex $\perp$ belongs to $\mathcal{P}$. From this vertex, he selects a successor (v_0, P, W) such that $W = P$ and P is an antichain of payoffs which $\mathcal{P}$ announces as the set P_{σ_0} for the strategy σ_0 in G he is trying to construct. All vertices in plays starting with this vertex will have this same value for their P-component. Those vertices are either a triplet (v, P, W) that belongs to $\mathcal{P}$ or $(v, P, (W_l, W_r))$ that belongs to $\mathcal{C}$. Given a play ρ (resp. history h) in G', we denote by ρ_V (resp. h_V) the play (resp. history) in G obtained by removing $\perp$ and keeping the v-component of every vertex of $\mathcal{P}$ in ρ (resp. h), which we call its *projection*.

- After history hm such that $m = (v, P, W)$ with $v \in V_0$, $\mathcal{P}$ selects a successor v' such that $(v, v') \in E$ and vertex (v', P, W) is added to the play. This corresponds to Player 0 choosing a successor v' after history $h_V v$ in G.
- After history hm such that $m = (v, P, W)$ with $v \in V_1$, $\mathcal{P}$ selects a successor $(v, P, (W_l, W_r))$ with (W_l, W_r) a partition of W. This corresponds to $\mathcal{P}$ splitting the set W into two parts according to the two successors v_l and v_r of v. For the strategy σ_0 that $\mathcal{P}$ tries to construct and its set of witnesses Wit_{σ_0} he is building, he asserts that W_l (resp. W_r) is the set of payoffs of the witnesses in $\mathsf{Wit}_{\sigma_0}(h_V v_l)$ (resp. $\mathsf{Wit}_{\sigma_0}(h_V v_r)$).
- From a vertex $(v, P, (W_l, W_r))$, $\mathcal{C}$ can select a successor (v_l, P, W_l) or (v_r, P, W_r) which corresponds to the choice of Player 1.

Formally, the game arena of the C-P game is the tuple $G' = (V', V'_{\mathcal{P}}, V'_{\mathcal{C}}, E', \perp)$ with

- $V'_{\mathcal{P}} = \{\perp\} \cup \{(v, P, W) \mid v \in V, P \subseteq \{0,1\}^t$ is an antichain and $W \subseteq P\}$,
- $V'_{\mathcal{C}} = \{(v, P, (W_l, W_r)) \mid v \in V_1, P \subseteq \{0,1\}^t$ is an antichain and $W_l, W_r \subseteq P\}$,
- $(\perp, (v, P, W)) \in E'$ if $v = v_0$ and $P = W$,
- $((v, P, W), (v', P, W)) \in E'$ if $v \in V_0$ and $(v, v') \in E$,
- $((v, P, W), (v, P, (W_l, W_r))) \in E'$ if $v \in V_1$ and (W_l, W_r) is a partition of W,
- $((v, P, (W_l, W_r)), (v', P, W)) \in E'$ if $(v, v') \in E$ and $\{v' = v_l$ and $W = W_l\}$ or $\{v' = v_r$ and $W = W_r\}$.

In the definition of E', if v has a single successor v' in G, it is assumed to be v_l and W_r is always equal to $\emptyset$. Given the two successors v_i and v_j of v, v_i is the left successor if $i < j$.

Objective of $\mathcal{P}$ in the C-P Game. Let us now discuss the objective $\Omega_{\mathcal{P}}$ of $\mathcal{P}$. The W-component of the vertices controlled by $\mathcal{P}$ has a size that decreases along a play ρ in G'. We write $lim_W(\rho)$ the value of the W-component at the limit in ρ. Recall that with this W-component, $\mathcal{P}$ tries to construct a solution σ_0 to the SPS problem with associated sets P_{σ_0} and Wit_{σ_0}. Therefore, for him to win in the C-P game, $lim_W(\rho)$ must be a singleton or empty in every consistent play such that:

- $lim_W(\rho)$ must be a singleton $\{p\}$ with p the payoff of ρ_V in G, showing that $\rho_V \in \mathsf{Wit}_{\sigma_0}$ is a correct witness for p. In addition, it must hold that $\mathsf{won}(\rho_V) = 1$ as $p \in P$ and as $\mathcal{P}$ wants σ_0 to be a solution.
- $lim_W(\rho)$ must be the empty set such that either the payoff of ρ_V belongs to P_{σ_0} and $\mathsf{won}(\rho_V) = 1$, or the payoff of ρ_V is strictly smaller than some payoff in P_{σ_0}.

These conditions verify that the sets $P = P_{\sigma_0}$ and Wit_{σ_0} are correct and that σ_0 is indeed a solution to the SPS problem in G. They are generic as they do not depend on the actual objectives used in the SP game.

Let us give the formal definition of $\Omega_{\mathcal{P}}$. For an antichain P of payoffs, we write $\mathsf{Plays}^P_{G'}$ the set of plays in G' which start with $\bot(v_0, P, P)$ and we define the following set

$$B_P = \big\{\rho \in \mathsf{Plays}^P_{G'} \mid (lim_W(\rho) = \{p\} \wedge \mathsf{pay}(\rho_V) = p \in P \wedge \mathsf{won}(\rho_V) = 1) \vee \tag{1}$$
$$(lim_W(\rho) = \emptyset \quad \wedge \mathsf{pay}(\rho_V) \in P \qquad \wedge \mathsf{won}(\rho_V) = 1) \vee \tag{2}$$
$$(lim_W(\rho) = \emptyset \quad \wedge \exists p \in P,\ \mathsf{pay}(\rho_V) < p)\big\}. \tag{3}$$

Objective $\Omega_{\mathcal{P}}$ of $\mathcal{P}$ in $\mathcal{G}'$ is the union of B_P over all antichains P. As the C-P game is zero-sum, objective $\Omega_{\mathcal{C}}$ equals $\mathsf{Plays}_{G'} \setminus \Omega_{\mathcal{P}}$. The following theorem holds.

▶ **Theorem 4.** *Player 0 has a strategy σ_0 that is solution to the SPS problem in $\mathcal{G}$ if and only if $\mathcal{P}$ has a winning strategy $\sigma_{\mathcal{P}}$ from $\bot$ in the C-P game $\mathcal{G}'$.*

Proof. Let us first assume that Player 0 has a strategy σ_0 that is solution to the SPS problem in $\mathcal{G}$. Let P_{σ_0} be its set of σ_0-fixed Pareto-optimal payoffs and let Wit_{σ_0} be a set of witnesses. We construct the strategy $\sigma_{\mathcal{P}}$ from σ_0 such that

- $\sigma_{\mathcal{P}}(\bot) = (v_0, P, P)$ such that $P = P_{\sigma_0}$ (this vertex exists as P_{σ_0} is an antichain),
- $\sigma_{\mathcal{P}}(hm) = (v', P, W)$ if $m = (v, P, W)$ with $v \in V_0$ and $v' = \sigma_0(h_V v)$,
- $\sigma_{\mathcal{P}}(hm) = (v, P, (W_l, W_r))$ if $m = (v, P, W)$ with $v \in V_1$ and for $i \in \{l, r\}$, $W_i = \{\mathsf{pay}(\rho) \mid \rho \in \mathsf{Wit}_{\sigma_0}(h_V v_i)\}$.

It is clear that given a play ρ in G' consistent with $\sigma_{\mathcal{P}}$, the play ρ_V in G is consistent with σ_0. Let us show that $\sigma_{\mathcal{P}}$ is winning for $\mathcal{P}$ from $\bot$ in G'. Consider a play ρ in G' consistent with $\sigma_{\mathcal{P}}$. There are two possibilities. *(i)* ρ_V is a witness of Wit_{σ_0} and by construction $lim_W(\rho) = \{p\}$ with $p = \mathsf{pay}(\rho_V)$; thus $\mathsf{won}(\rho_V) = 1$ as σ_0 is a solution and ρ_V is a witness. *(ii)* ρ_V is not a witness and by construction $lim_W(\rho) = \emptyset$; as σ_0 is a solution, then $p = \mathsf{pay}(\rho_V)$ is bounded by some payoff of P_{σ_0} and in case of equality $\mathsf{won}(\rho_V) = 1$. Therefore ρ satisfies the objective B_P of $\Omega_{\mathcal{P}}$ since it satisfies condition (1) in case *(i)* and condition (2) or (3) in case *(ii)*.

Let us now assume that $\mathcal{P}$ has a winning strategy $\sigma_{\mathcal{P}}$ from $\bot$ in G'. Let P be the antichain of payoffs chosen from $\bot$ by this strategy. We construct the strategy σ_0 from $\sigma_{\mathcal{P}}$ such that $\sigma_0(h_V v) = v'$ given $\sigma_{\mathcal{P}}(hm) = (v', P, W)$ with $m = (v, P, W)$ and $v \in V_0$. Notice that this definition makes sense since there is a unique history hm ending with a vertex of $\mathcal{P}$ associated with $h_V v$ showing a one-to-one correspondence between those histories.

Let us show σ_0 is a solution to the SPS problem with P_{σ_0} being the set P. First notice that P is not empty. Indeed let ρ be a play consistent with $\sigma_{\mathcal{P}}$. As ρ belongs to $\Omega_{\mathcal{P}}$ and in particular to B_P, one can check that $P \neq \emptyset$ by inspecting conditions (1) to (3). Second notice that by definition of E', if $((v, P, W), (v, P, (W_l, W_r))) \in E'$ with $W \neq \emptyset$, then either W_l or W_r is not empty. Therefore given any payoff $p \in P$, there is a unique play ρ consistent with $\sigma_{\mathcal{P}}$ such that $lim_W(\rho) = \{p\}$. By construction of σ_0 and as $\sigma_{\mathcal{P}}$ is winning, the play ρ_V is consistent with σ_0, has payoff p, and is won by Player 0 (see (1)).

Let ρ_V be a play consistent with σ_0 and ρ be the corresponding play consistent with $\sigma_{\mathcal{P}}$. It remains to consider (2) and (3). These conditions indicate that ρ_V has a payoff equal to or strictly smaller than a payoff in P and that in case of equality $\mathsf{won}(\rho_V) = 1$. This shows that $P_{\sigma_0} = P$ and that σ_0 is a solution to the SPS problem. ◀

3.2 Proof of the FPT Results

We now sketch the proof of Theorem 3 which works by specializing the generic objective $\Omega_{\mathcal{P}}$ to handle reachability and parity SP games. We begin with reachability SP games. We extend the arena G' of the C-P game such that its vertices keep track of the objectives of $\mathcal{G}$ which are satisfied along a play. Given an extended payoff $(w, p) \in \{0,1\} \times \{0,1\}^t$ and a vertex $v \in V$, we define the *payoff update* $\mathsf{upd}(w, p, v) = (w', p')$ such that

$$w' = 1 \iff w = 1 \text{ or } v \in T_0,$$
$$p'_i = 1 \iff p_i = 1 \text{ or } v \in T_i, \quad \forall i \in \{1, \dots, t\}.$$

We obtain the extended arena G^* as follows: *(i)* its set of vertices is $V' \times \{0,1\} \times \{0,1\}^t$, *(ii)* its initial vertex is $\perp^* = (\perp, 0, (0, \dots, 0))$, and *(iii)* $((m, w, p), (m', w', p'))$ with $m' = (v', P, W)$ or $m' = (v', P, (W_l, W_r))$ is an edge in G^* if $(m, m') \in E'$ and $(w', p') = \mathsf{upd}(w, p, v')$.

We define the zero-sum game $\mathcal{G}^* = (G^*, \Omega^*_{\mathcal{P}})$ in which the three abstract conditions (1-3) detailed previously are encoded into the following Büchi objective by using the (w, p)-component added to vertices. We define $\Omega^*_{\mathcal{P}} = \mathsf{Büchi}(B^*)$ with

$$\begin{aligned} B^* = \big\{(v, P, W, w, p) \in V^*_{\mathcal{P}} \mid\ & (W = \{p\} \wedge w = 1) \vee & \text{(1')} \\ & (W = \emptyset \quad \wedge p \in P \wedge w = 1) \vee & \text{(2')} \\ & (W = \emptyset \quad \wedge \exists p' \in P,\ p < p')\big\}. & \text{(3')} \end{aligned}$$

The proof of the next proposition is a consequence of Theorem 4.

▶ **Proposition 5.** *Player 0 has a strategy σ_0 that is solution to the SPS problem in a reachability SP game $\mathcal{G}$ if and only if $\mathcal{P}$ has a winning strategy $\sigma^*_{\mathcal{P}}$ in $\mathcal{G}^*$.*

We obtain the following FPT algorithm for deciding the existence of a solution to the SPS problem in a reachability SP game $\mathcal{G}$. First, we construct the zero-sum game $\mathcal{G}^*$ whose number of vertices is linear in the number of vertices in the original game and double exponential in the number t of objectives of Player 1. Second, by Proposition 5, deciding whether there exists a solution to the SPS problem in $\mathcal{G}$ amounts to solving the zero-sum Büchi game $\mathcal{G}^*$; this can be done in quadratic time in the number of vertices of $\mathcal{G}^*$ [10]. Those two steps are in FPT for parameter t.

We now turn to parity SP games and briefly explain why solving the SPS problem in these games is in FPT, again by reduction to the C-P game. The arena G' of the C-P game remains as is and its objective $\Omega_{\mathcal{P}}$ is replaced by a *Boolean Büchi* objective $\Omega'_{\mathcal{P}}$ which encodes the three conditions for parity objectives. Boolean Büchi objectives are Boolean combinations of Büchi objectives and zero-sum games with such objectives are shown to be solvable in FPT in [8]. It follows that the SPS problem is also in FPT.

4 Complexity Class of the SPS Problem

In this section, we study the complexity class of the SPS problem and prove its NEXPTIME-completeness for both reachability and parity SP games.

4.1 NEXPTIME-Membership

We first show the membership to NEXPTIME of the SPS problem by providing a nondeterministic algorithm with time exponential in the size of the game $\mathcal{G}$. By *size*, we mean the number $|V|$ of its vertices and the number t of objectives of Player 1. Notice that the time complexity of the FPT algorithms obtained in the previous section is too high, preventing us from directly using the C-P game to show a tight membership result. Conversely, the nondeterministic algorithm provided in this section is not FPT as it is exponential in $|V|$.

▶ **Theorem 6.** *The SPS problem is in* NEXPTIME *for reachability and parity SP games.*

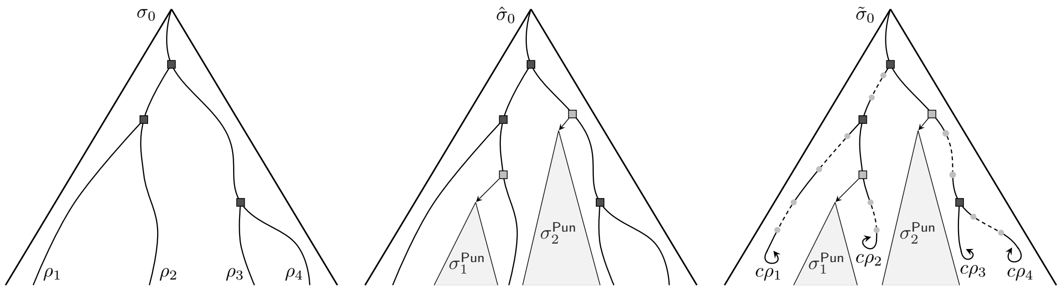

Figure 3 The creation of strategies $\hat{\sigma}_0$ and $\tilde{\sigma}_0$ from a solution σ_0 with $\mathsf{Wit}_{\sigma_0} = \{\rho_1, \rho_2, \rho_3, \rho_4\}$.

We show this membership result by proving that if Player 0 has a strategy which is a solution to the problem, he has one which is finite-memory with at most an exponential number of memory states[4]. This yields a NEXPTIME algorithm in which we nondeterministically guess such a strategy and check in exponential time that it is indeed a solution.

While our proof requires some specific arguments to treat both reachability and parity objectives, it is based on the following common principles. We first explain why, when there is a solution σ_0 to the SPS problem, there is one that is finite-memory. We consider a fixed set of witnesses Wit_{σ_0}. Figure 3 illustrates the two steps of the following construction.

- We start by showing the existence of a strategy $\hat{\sigma}_0$ constructed from σ_0, in which Player 0 follows σ_0 as long as the current consistent history is prefix of at least one witness in Wit_{σ_0}. Then when a deviation from Wit_{σ_0} occurs, Player 0 switches to a so-called *punishing strategy*. A deviation is a history that leaves the set of witnesses Wit_{σ_0} after a move of Player 1 (this is not possible by a move of Player 0). After such a deviation, $\hat{\sigma}_0$ systematically imposes that the consistent play either satisfies Ω_0 or is not σ_0-fixed Pareto-optimal, i.e., it gives to Player 1 a payoff that is strictly smaller than the payoff of a witness in Wit_{σ_0}. This makes the deviation *irrational* for Player 1. We show that this can be done, both for reachability and parity objectives, with at most exponentially many different punishing strategies, each having a size bounded exponentially in the size of the game. The strategy $\hat{\sigma}_0$ that we obtain is therefore composed of the part of σ_0 that produces Wit_{σ_0} and a punishment part whose size is at most exponential.
- Then, we show how to decompose each witness in Wit_{σ_0} into at most exponentially many *sections* that can, in turn, be compacted into finite elementary paths or lasso shaped paths of polynomial length. As Wit_{σ_0} contains exactly $|P_{\sigma_0}|$ witnesses ρ, those compact witnesses $c\rho$ can be produced by a finite-memory strategy with an exponential size for both reachability and parity objectives. This allows us to construct a strategy $\tilde{\sigma}_0$ that produces the compact witnesses and acts as $\hat{\sigma}_0$ after any deviation. This strategy is a solution of the SPS problem and has an exponential size as announced.

We can now sketch the proof of Theorem 6, again by giving arguments that work for both reachability and parity objectives. We guess a solution σ_0 to the SPS problem that we can assume to be finite-memory, that is, we guess it as a Moore machine $\mathcal{M}$ with an exponential number of memory states. We then verify that σ_0 is indeed a solution by first computing the set P_{σ_0} and then checking that every σ_0-fixed Pareto-optimal play satisfies the objective Ω_0 of Player 0. To this end, we construct the cartesian product $G \times \mathcal{M}$ which is an automaton whose infinite paths are exactly the plays consistent with σ_0. We then use classical results from automata theory about the emptiness problem for an intersection of reachability (resp. parity) objectives to get the announced exponential complexity of our verifying algorithm.

[4] Recall that to have a solution to the SPS problem, memory may be necessary as shown in Example 2.

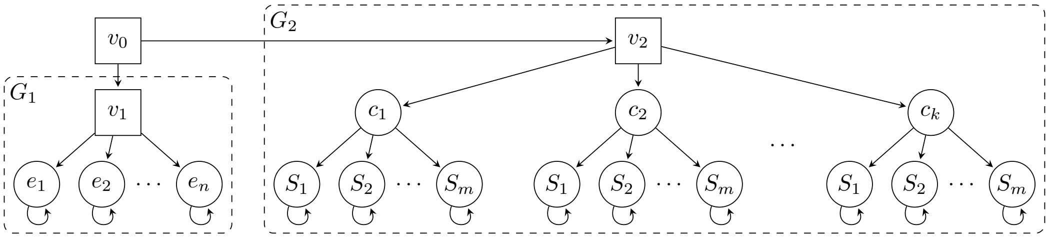

Figure 4 The tree arena used in the reduction from the SC problem.

4.2 NP-Completeness for Tree Arenas

Before turning to the NEXPTIME-hardness of the SPS problem in the next section, we first want to show that the SPS problem is already NP-complete in the simple setting of reachability objectives and arenas that are trees. To do so, we use a reduction from the Set Cover problem (SC problem) which is NP-complete [23].

▶ **Theorem 7.** *The SPS problem is* NP*-complete for reachability SP games on tree arenas.*

Notice that when the game arena is a tree, it is easy to design an algorithm for solving the SPS problem that is in NP. First, we nondeterministically guess a strategy σ_0 that can be assumed to be memoryless as the arena is a tree. Second, we apply a depth-first search algorithm from the root vertex which accumulates to leaf vertices the extended payoff of plays which are consistent with σ_0. Finally, we check that σ_0 is a solution.

Let us explain why the SPS problem is NP-hard on tree arenas by reduction from the SC problem. We recall that an instance of the SC problem is defined by a set $C = \{e_1, e_2, \ldots, e_n\}$ of n elements, m subsets $S_1, S_2, \ldots, S_m$ such that $S_i \subseteq C$ for each $i \in \{1, \ldots, m\}$, and an integer $k \leq m$. The problem consists in finding k indexes $i_1, i_2, \ldots, i_k$ such that the union of the corresponding subsets equals C, i.e., $C = \bigcup_{j=1}^{k} S_{i_j}$.

Given an instance of the SC problem, we construct a game with an arena consisting of $n + k \cdot (m + 1) + 3$ vertices. The arena G of the game is provided in Figure 4 and can be seen as two sub-arenas reachable from the initial vertex v_0. The game is such that there is a solution to the SC problem if and only if Player 0 has a strategy from v_0 in G which is a solution to the SPS problem. The game is played between Player 0 with reachability objective Ω_0 and Player 1 with $n + 1$ reachability objectives. The objectives are defined as follows: $\Omega_0 = \mathsf{Reach}(\{v_2\})$, $\Omega_i = \mathsf{Reach}(\{e_i\} \cup \{S_j \mid e_i \in S_j\})$ for $i \in \{1, 2, \ldots, n\}$ and $\Omega_{n+1} = \mathsf{Reach}(\{v_2\})$. First, notice that every play in G_1 is consistent with any strategy of Player 0 and is lost by that player. It holds that for each $\ell \in \{1, 2, \ldots, n\}$, there is such a play with payoff $(p_1, \ldots, p_{n+1})$ such that $p_\ell = 1$ and $p_j = 0$ for $j \neq \ell$. These payoffs correspond to the elements e_ℓ which are to be covered in the SC problem. A play in G_2 visits v_2 and then a vertex c from which Player 0 selects a vertex S. Such a play is always won by Player 0 and its payoff is $(p_1, \ldots, p_{n+1})$ such that $p_{n+1} = 1$ and $p_r = 1$ if and only if the element e_r belongs to the set S. It follows that the payoff of such a play corresponds to a set of elements in the SC problem. It is easy to see that the following proposition holds.

▶ **Proposition 8.** *There is a solution to an instance of the SC problem if and only if Player* 0 *has a strategy from* v_0 *in the corresponding SP game that is a solution to the SPS problem.*

4.3 NEXPTIME-Hardness

Let us come back to regular game arenas and show the NEXPTIME-hardness result thanks to the succinct variant of the SC problem presented below.

▶ **Theorem 9.** *The SPS problem is* NEXPTIME*-hard for reachability and parity SP games.*

Succinct Set Cover Problem. The *Succinct Set Cover problem (SSC problem)* is defined as follows. We are given a Conjunctive Normal Form (CNF) formula $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_p$ over the variables $X = \{x_1, x_2, \ldots, x_m\}$ made up of p clauses, each containing some disjunction of literals of the variables in X. The set of valuations of the variables X which satisfy ϕ is written $[\![\phi]\!]$. We are also given an integer $k \in \mathbb{N}$ (encoded in binary) and an other CNF formula $\psi = D_1 \wedge D_2 \wedge \cdots \wedge D_q$ over the variables $X \cup Y$ with $Y = \{y_1, y_2, \ldots, y_n\}$, made up of q clauses. Given a valuation $val_Y : Y \to \{0,1\}$ of the variables in Y, called a *partial valuation*, we write $\psi[val_Y]$ the CNF formula obtained by replacing in ψ each variable $y \in Y$ by its valuation $val_Y(y)$. We write $[\![\psi[val_Y]]\!]$ the valuations of the remaining variables X which satisfy $\psi[val_Y]$. The SSC problem is to decide whether there exists a set $K = \big\{val_Y \mid val_Y : Y \to \{0,1\}\big\}$ of k valuations of the variables in Y such that the valuations of the remaining variables X which satisfy the formulas $\psi[val_Y]$ include the valuations of X which satisfy ϕ. Formally, we write this $[\![\phi]\!] \subseteq \bigcup_{val_Y \in K} [\![\psi[val_Y]]\!]$.

We can show that this corresponds to a set cover problem succinctly defined using CNF formulas. The set $[\![\phi]\!]$ of valuations of X which satisfy ϕ corresponds to the set of elements we aim to cover. Parameter k is the number of sets that can be used to cover these elements. Such a set is described by a formula $\psi[val_Y]$, given a partial valuation val_Y, and its elements are the valuations of X in $[\![\psi[val_Y]]\!]$. This is illustrated in the following example.

▶ **Example 10.** Consider the CNF formula $\phi = (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3)$ over the variables $X = \{x_1, x_2, x_3\}$. The set of valuations of the variables which satisfy ϕ is $[\![\phi]\!] = \{(1,1,1), (1,1,0), (1,0,1), (0,0,1)\}$. Each such valuation corresponds to one element we aim to cover. Consider the CNF formula $\psi = (y_1 \vee y_2) \wedge (x_1 \vee y_2) \wedge (x_2 \vee x_3 \vee y_1)$ over the variables $X \cup Y$ with $Y = \{y_1, y_2\}$. Given the partial valuation val_Y of the variables in Y such that $val_Y(y_1) = 0$ and $val_Y(y_2) = 1$, we get the CNF formula $\psi[val_Y] = (0 \vee 1) \wedge (x_1 \vee 1) \wedge (x_2 \vee x_3 \vee 0)$. This formula describes the contents of the set identified by the partial valuation (as a partial valuation yields a unique formula). The valuations of the variables X which satisfy $\psi[val_Y]$ are the elements contained in the set. In this case, these elements are $[\![\psi[val_Y]]\!] = \{(0,1,0), (0,0,1), (0,1,1), (1,1,0), (1,0,1), (1,1,1)\}$. We can see that this set contains the elements $\{(1,1,1), (1,1,0), (1,0,1), (0,0,1)\}$ of $[\![\phi]\!]$.

It is easy to see that the SSC problem is in NEXPTIME. Its NEXPTIME-hardness can be obtained by reduction from the Succinct Dominating Set problem, which is NEXPTIME-complete for graphs succinctly encoded using CNF formulas [14].

▶ **Theorem 11.** *The SSC problem is* NEXPTIME*-complete.*

We now describe our reduction from the SSC problem to show the NEXPTIME-hardness of solving the SPS problem for reachability SP games. The proof of this result for parity SP games uses similar arguments, adapted to the prefix-independent nature of parity objectives.

Given an instance of the SSC problem, we construct a reachability SP game with arena G consisting of a polynomial number of vertices in the number of clauses and variables in the formulas ϕ and ψ and in the length of the binary encoding of the integer k. This reduction is such that there is a solution to the SSC problem if and only if Player 0 has a strategy

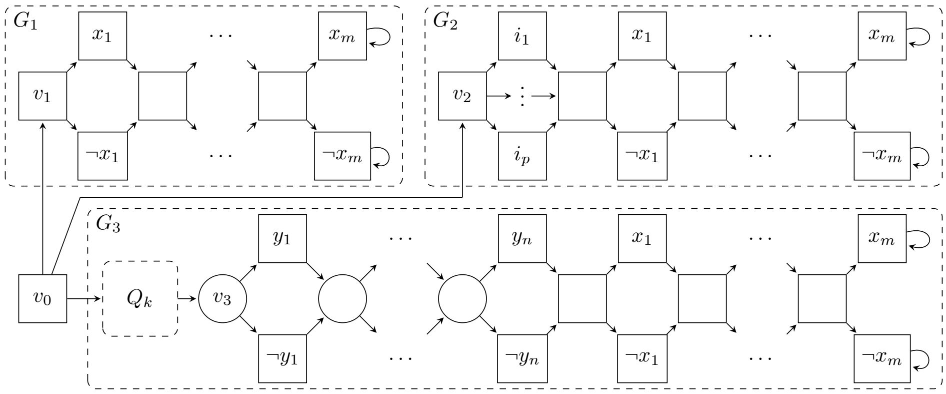

Figure 5 The arena G used in the reduction from the SSC problem.

from v_0 in G which is a solution to the SPS problem. The arena G, provided in Figure 5, can be viewed as three sub-arenas reachable from v_0. Sub-arenas G_1 and G_2 are completely controlled by Player 1. Plays entering these sub-arenas are therefore consistent with any strategy of Player 0. Sub-arena G_3 starts with a gadget Q_k whose vertices belong to Player 1 and which provides exactly k different paths from v_0 to v_3.

Objectives. The game is played between Player 0 with reachability objective Ω_0 and Player 1 with $t = 1 + 2 \cdot m + p + q$ reachability objectives. The payoff of a play therefore consists in a single Boolean for objective Ω_1, a vector of $2 \cdot m$ Booleans for objectives $\Omega_{x_1}, \Omega_{\neg x_1}, \dots, \Omega_{x_m}, \Omega_{\neg x_m}$, a vector of p Booleans for objectives $\Omega_{C_1}, \dots, \Omega_{C_p}$ and a vector of q Booleans for objectives $\Omega_{D_1}, \dots, \Omega_{D_q}$. The objectives are defined as follows.

- The target set for objective Ω_0 of Player 0 and objective Ω_1 of Player 1 is $\{v_2, v_3\}$.
- The target set for objective Ω_{x_i} (resp. $\Omega_{\neg x_i}$) with $i \in \{1, \dots, m\}$ is the set of vertices labeled x_i (resp. $\neg x_i$) in G_1, G_2 and G_3.
- The target set for objective Ω_{C_i} with $i \in \{1, \dots, p\}$ is the set of vertices in G_1 and G_3 corresponding to the literals of X which make up the clause C_i in ϕ. In addition, vertex i_j in G_2 belongs to the target set of objective Ω_{C_ℓ} for all $\ell \in \{1, \dots, p\}$ such that $\ell \neq j$.
- The target set of objective Ω_{D_i} with $i \in \{1, \dots, q\}$ is the set of vertices in G_3 corresponding to the literals of X and Y which make up the clause D_i in ψ. In addition, vertices v_1 and v_2 satisfy every objective Ω_{D_i} with $i \in \{1, \dots, q\}$.

Payoff of Plays in G_1. Plays in G_1 do not satisfy objective Ω_0 of Player 0 nor objective Ω_1 of Player 1. A play in G_1 is of the form $v_0\, v_1\, z_1\, \square \cdots \square\, (z_m)^\omega$ where z_i is either x_i or $\neg x_i$. It follows that a play satisfies the objective Ω_{x_i} or $\Omega_{\neg x_i}$ for each $x_i \in X$. The vector of payoffs for these objectives corresponds to a valuation of the variables in X, expressed as a vector of $2 \cdot m$ Booleans. In addition, due to the way the objectives are defined, objective Ω_{C_i} is satisfied in a play if and only if clause C_i of ϕ is satisfied by the valuation this play corresponds to. The objective Ω_{D_i} for $i \in \{1, \dots, q\}$ is satisfied in every play in G_1.

▶ **Lemma 12.** *Plays in G_1 are consistent with any strategy of Player 0. Their payoff are of the form $(0, val, sat(\phi, val), 1, \dots, 1)$ where val is a valuation of the variables in X expressed as a vector of payoffs for objectives Ω_{x_1} to $\Omega_{\neg x_m}$ and $sat(\phi, val)$ is the vector of payoffs for objectives Ω_{C_1} to Ω_{C_p} corresponding to that valuation. All plays in G_1 are lost by Player 0.*

Payoff of Plays in G_2. Plays in G_2 satisfy the objectives Ω_0 of Player 0 and Ω_1 of Player 1. A play in G_2 is of the form $v_0\, v_2\, i_j \,\square\, z_1 \,\square \cdots \square\, (z_m)^\omega$ where z_ℓ is either x_ℓ or $\neg x_\ell$. It follows that a play satisfies either the objective Ω_x or $\Omega_{\neg x}$ for each $x \in X$ which again corresponds to a valuation of the variables in X. The objective Ω_{D_i} for $i \in \{1, \dots, q\}$ is satisfied in every play in G_2. Compared to the plays in G_1, the difference lies in the objectives corresponding to clauses of ϕ which are satisfied. In any play in G_2, a vertex i_j with $j \in \{1, \dots, p\}$ is first visited, satisfying all the objectives Ω_{C_ℓ} with $\ell \in \{1, \dots, p\}$ and $\ell \neq j$. All but one objective corresponding to the clauses of ϕ are therefore satisfied.

▶ **Lemma 13.** *Plays in G_2 are consistent with any strategy of Player 0. Their payoff are of the form $(1, val, vec, 1, \dots, 1)$ where val is a valuation of the variables in X expressed as a vector of payoffs for objectives Ω_{x_1} to $\Omega_{\neg x_m}$ and vec is a vector of payoffs for objectives Ω_{C_1} to Ω_{C_p} in which all of them except one are satisfied. All plays in G_2 are won by Player 0.*

Plays in G_2 are such that their payoff is strictly larger than the payoff of plays in G_1 whose valuation of X does not satisfy ϕ. It is easy to see that, when considering G_1 and G_2, the only plays in G_1 with a Pareto-optimal payoff are exactly those whose valuation satisfies all clauses of ϕ. The following lemma therefore holds.

▶ **Lemma 14.** *The set of payoffs of plays in G_1 that are σ_0-fixed Pareto-optimal when considering $G_1 \cup G_2$ for any strategy σ_0 of Player 0 is equal to the set of payoffs of plays in G_1 whose valuation of X satisfy ϕ.*

Problematic Payoffs in G_1. The plays described in the previous lemma correspond exactly to the valuations of X which satisfy ϕ and therefore to the elements we aim to cover in the SSC problem. They are σ_0-fixed Pareto-optimal when considering $G_1 \cup G_2$ and are lost by Player 0. All other σ_0-fixed Pareto-optimal payoffs in $G_1 \cup G_2$ are only realized by plays in G_2 which are all won by Player 0. It follows that in order for Player 0 to find a strategy σ_0 from v_0 that is solution to the SPS problem, it must hold that those payoffs are not σ_0-fixed Pareto-optimal when considering $G_1 \cup G_2 \cup G_3$. Otherwise, a play consistent with σ_0 with a σ_0-fixed Pareto-optimal payoff is lost by Player 0. We call those payoffs *problematic payoffs*.

In order for Player 0 to find a strategy σ_0 which is a solution to the SPS problem, this strategy must be such that for each problematic payoff in G_1, there is a play in G_3 consistent with σ_0 and with a strictly larger payoff. Since the plays in G_3 are all won by Player 0, this would ensure that the strategy σ_0 is a solution to the problem. This corresponds in the SSC problem to selecting a series of sets in order to cover the valuations of X which satisfy ϕ.

Payoff of Plays in G_3. Plays in G_3 satisfy the objectives Ω_0 of Player 0 and Ω_1 of Player 1. A play in G_3 consistent with a strategy σ_0 is of the form $v_0 \square \cdots \square v_3\, r_1 \bigcirc \cdots \bigcirc r_n \square z_1 \square \cdots \square (z_m)^\omega$ where r_i is either y_i or $\neg y_i$ and z_i is either x_i or $\neg x_i$. Since only the vertices leading to y or $\neg y$ for $y \in Y$ belong to Player 0, it holds that $v_3\, r_1 \bigcirc \cdots \bigcirc r_n$ is the only part of any play in G_3 which is directly influenced by σ_0. That part of a play comes after a history from v_0 to v_3 of which there are k, provided by gadget Q_k. By definition of a strategy, this can be interpreted as Player 0 making a choice of valuation of the variables in Y after each of those k histories. After this, the play satisfies either the objective Ω_x or $\Omega_{\neg x}$ for each $x \in X$ which corresponds to a valuation of X. Due to the way the objectives are defined, the objective Ω_{C_i} (resp. Ω_{D_i}) is satisfied if and only if clause C_i of ϕ (resp. D_i of ψ) is satisfied by the valuation of the variables in X (resp. X and Y) the play corresponds to.

Creating Strictly Larger Payoffs in G_3. In order to create a play with a payoff r' that is strictly larger than a problematic payoff r, σ_0 must choose a valuation of Y such that there exists a valuation of the remaining variables X which together with this valuation of Y satisfies ψ and ϕ (since in r every objective Ω_{C_i} for $i \in \{1, \dots, p\}$ and Ω_{D_i} for $i \in \{1, \dots, q\}$ is satisfied). Since the plays in G_3 also satisfy the objective Ω_1 and plays in G_1 do not, this ensures that $r < r'$.

We finally briefly explain why the proposed reduction is correct. In case of a positive instance of the SSC problem, by carefully selecting k valuations of Y, Player 0 ensures that for each valuation val_X satisfying ϕ, there is a play in G_3 with a valuation val_Y such that $val_X \in [\![\psi[val_Y]]\!]$. Therefore, when considering the whole arena, no play in G_1 is Pareto-optimal and every Pareto-optimal play is won by Player 0. In case of a negative instance, Player 0 is not able to do so and some play in G_1 thus has a Pareto-optimal payoff and is lost by Player 0.

5 Conclusion

We have introduced in this paper the class of two-player SP games and the SPS problem in those games. We provided a reduction from SP games to a two-player zero-sum game called the C-P game, which we used to obtain FPT results on solving this problem. We showed how the arena and the generic objective of this C-P game can be adapted to specifically handle reachability and parity SP games. This allowed us to prove that reachability (resp. parity) SP games are in FPT when the number t of objectives of Player 1 (resp. when t and the maximal priority according to each priority function in the game) is a parameter. We then turned to the complexity class of the SPS problem and sketched the main arguments used in our proof of its NEXPTIME-membership, which relied on showing that any solution to the SPS problem in a reachability or parity SP game can be transformed into a solution with an exponential memory. We showed the NP-completeness of the problem in the simple setting of reachability SP games played on tree arenas. We then came back to regular game arenas and established the NEXPTIME-hardness of the SPS problem in reachability and parity SP games. This result relied on a reduction from the SSC problem which we proved to be NEXPTIME-complete, a result of potential independent interest.

In future work, we want to study other ω-regular objectives as well as quantitative objectives such as mean-payoff in the framework of SP games and the SPS problem. It would also be interesting to study whether other works, such as rational synthesis, could benefit from the approaches used in this paper.

References

1 Mrudula Balachander, Shibashis Guha, and Jean-François Raskin. Fragility and robustness in mean-payoff adversarial Stackelberg games. *CoRR*, abs/2007.07209, 2020. arXiv:2007.07209.

2 Dietmar Berwanger. Admissibility in infinite games. In Wolfgang Thomas and Pascal Weil, editors, *STACS 2007, 24th Annual Symposium on Theoretical Aspects of Computer Science, Aachen, Germany, February 22-24, 2007, Proceedings*, volume 4393 of *Lecture Notes in Computer Science*, pages 188–199. Springer, 2007. doi:10.1007/978-3-540-70918-3_17.

3 Romain Brenguier, Lorenzo Clemente, Paul Hunter, Guillermo A. Pérez, Mickael Randour, Jean-François Raskin, Ocan Sankur, and Mathieu Sassolas. Non-zero sum games for reactive synthesis. In Adrian-Horia Dediu, Jan Janousek, Carlos Martín-Vide, and Bianca Truthe, editors, *Language and Automata Theory and Applications – 10th International Conference, LATA 2016, Prague, Czech Republic, March 14-18, 2016, Proceedings*, volume 9618 of *Lecture Notes in Computer Science*, pages 3–23. Springer, 2016. doi:10.1007/978-3-319-30000-9_1.

4 Romain Brenguier, Jean-François Raskin, and Ocan Sankur. Assume-admissible synthesis. In Luca Aceto and David de Frutos-Escrig, editors, *26th International Conference on Concurrency Theory, CONCUR 2015, Madrid, Spain, September 1.4, 2015*, volume 42 of *LIPIcs*, pages 100–113. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2015. doi:10.4230/LIPIcs.CONCUR.2015.100.

5 Léonard Brice, Jean-François Raskin, and Marie van den Bogaard. Subgame-perfect equilibria in mean-payoff games. *CoRR*, abs/2101.10685, 2021. arXiv:2101.10685.

6 Thomas Brihaye, Véronique Bruyère, Aline Goeminne, Jean-François Raskin, and Marie van den Bogaard. The complexity of subgame perfect equilibria in quantitative reachability games. *Log. Methods Comput. Sci.*, 16(4), 2020. URL: https://lmcs.episciences.org/6883.

7 Véronique Bruyère. Computer aided synthesis: A game-theoretic approach. In Émilie Charlier, Julien Leroy, and Michel Rigo, editors, *Developments in Language Theory – 21st International Conference, DLT 2017, Liège, Belgium, August 7-11, 2017, Proceedings*, volume 10396 of *Lecture Notes in Computer Science*, pages 3–35. Springer, 2017. doi:10.1007/978-3-319-62809-7_1.

8 Véronique Bruyère, Quentin Hautem, and Jean-François Raskin. Parameterized complexity of games with monotonically ordered omega-regular objectives. In Sven Schewe and Lijun Zhang, editors, *29th International Conference on Concurrency Theory, CONCUR 2018, September 4-7, 2018, Beijing, China*, volume 118 of *LIPIcs*, pages 29:1–29:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. doi:10.4230/LIPIcs.CONCUR.2018.29.

9 Véronique Bruyère, Jean-François Raskin, and Clément Tamines. Stackelberg-Pareto synthesis (full version). *CoRR*, abs/2102.08925, 2021. arXiv:2102.08925.

10 Krishnendu Chatterjee and Monika Henzinger. Efficient and dynamic algorithms for alternating Büchi games and maximal end-component decomposition. *J. ACM*, 61(3):15:1–15:40, 2014. doi:10.1145/2597631.

11 Krishnendu Chatterjee and Thomas A. Henzinger. Assume-guarantee synthesis. In Orna Grumberg and Michael Huth, editors, *Tools and Algorithms for the Construction and Analysis of Systems, 13th International Conference, TACAS 2007, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2007 Braga, Portugal, March 24 – April 1, 2007, Proceedings*, volume 4424 of *Lecture Notes in Computer Science*, pages 261–275. Springer, 2007. doi:10.1007/978-3-540-71209-1_21.

12 Krishnendu Chatterjee, Thomas A. Henzinger, and Marcin Jurdzinski. Games with secure equilibria. *Theor. Comput. Sci.*, 365(1-2):67–82, 2006. doi:10.1016/j.tcs.2006.07.032.

13 Rodica Condurache, Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. The complexity of rational synthesis. In Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi, editors, *43rd International Colloquium on Automata, Languages, and Programming, ICALP 2016, July 11-15, 2016, Rome, Italy*, volume 55 of *LIPIcs*, pages 121:1–121:15. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2016. doi:10.4230/LIPIcs.ICALP.2016.121.

14 Bireswar Das, Patrick Scharpfenecker, and Jacobo Torán. CNF and DNF succinct graph encodings. *Inf. Comput.*, 253:436–447, 2017. doi:10.1016/j.ic.2016.06.009.

15 R.G. Downey and M.R. Fellows. *Parameterized Complexity*. Monographs in Computer Science. Springer New York, 2012. URL: https://books.google.be/books?id=HyTjBwAAQBAJ.

16 Emmanuel Filiot, Raffaella Gentilini, and Jean-François Raskin. The adversarial Stackelberg value in quantitative games. In Artur Czumaj, Anuj Dawar, and Emanuela Merelli, editors, *47th International Colloquium on Automata, Languages, and Programming, ICALP 2020, July 8–11, 2020, Saarbrücken, Germany (Virtual Conference)*, volume 168 of *LIPIcs*, pages 127:1–127:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.ICALP.2020.127.

17 Dana Fisman, Orna Kupferman, and Yoad Lustig. Rational synthesis. In Javier Esparza and Rupak Majumdar, editors, *Tools and Algorithms for the Construction and Analysis of Systems, 16th International Conference, TACAS 2010, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2010, Paphos, Cyprus, March 20-28, 2010. Proceedings*, volume 6015 of *Lecture Notes in Computer Science*, pages 190–204. Springer, 2010. doi:10.1007/978-3-642-12002-2_16.

18 Erich Grädel, Wolfgang Thomas, and Thomas Wilke, editors. *Automata, Logics, and Infinite Games: A Guide to Current Research [outcome of a Dagstuhl seminar, February 2001]*, volume 2500 of *Lecture Notes in Computer Science*. Springer, 2002. doi:10.1007/3-540-36387-4.

19 Anshul Gupta and Sven Schewe. Quantitative verification in rational environments. In Amedeo Cesta, Carlo Combi, and François Laroussinie, editors, *21st International Symposium on Temporal Representation and Reasoning, TIME 2014, Verona, Italy, September 8-10, 2014*, pages 123–131. IEEE Computer Society, 2014. doi:10.1109/TIME.2014.9.

20 Anshul Gupta, Sven Schewe, and Dominik Wojtczak. Making the best of limited memory in multi-player discounted sum games. In Javier Esparza and Enrico Tronci, editors, *Proceedings Sixth International Symposium on Games, Automata, Logics and Formal Verification, GandALF 2015, Genoa, Italy, 21-22nd September 2015*, volume 193 of *EPTCS*, pages 16–30, 2015. doi:10.4204/EPTCS.193.2.

21 Julian Gutierrez, Muhammad Najib, Giuseppe Perelli, and Michael J. Wooldridge. On computational tractability for rational verification. In Sarit Kraus, editor, *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI 2019, Macao, China, August 10-16, 2019*, pages 329–335. ijcai.org, 2019. doi:10.24963/ijcai.2019/47.

22 Julian Gutierrez, Muhammad Najib, Giuseppe Perelli, and Michael J. Wooldridge. Automated temporal equilibrium analysis: Verification and synthesis of multi-player games. *Artif. Intell.*, 287:103353, 2020. doi:10.1016/j.artint.2020.103353.

23 Richard M. Karp. Reducibility among combinatorial problems. In Raymond E. Miller and James W. Thatcher, editors, *Proceedings of a symposium on the Complexity of Computer Computations, held March 20-22, 1972, at the IBM Thomas J. Watson Research Center, Yorktown Heights, New York, USA*, The IBM Research Symposia Series, pages 85–103. Plenum Press, New York, 1972. doi:10.1007/978-1-4684-2001-2_9.

24 Orna Kupferman, Giuseppe Perelli, and Moshe Y. Vardi. Synthesis with rational environments. *Ann. Math. Artif. Intell.*, 78(1):3–20, 2016. doi:10.1007/s10472-016-9508-8.

25 John F. Nash. Equilibrium points in n-person games. In *PNAS*, volume 36, pages 48–49. National Academy of Sciences, 1950.

26 Amir Pnueli and Roni Rosner. On the synthesis of a reactive module. In *Conference Record of the Sixteenth Annual ACM Symposium on Principles of Programming Languages, Austin, Texas, USA, January 11-13, 1989*, pages 179–190. ACM Press, 1989. doi:10.1145/75277.75293.

27 Reinhard Selten. Spieltheoretische Behandlung eines Oligopolmodells mit Nachfrageträgheit. *Zeitschrift für die gesamte Staatswissenschaft*, 121:301–324 and 667–689, 1965.

28 Michael Ummels. Rational behaviour and strategy construction in infinite multiplayer games. In S. Arun-Kumar and Naveen Garg, editors, *FSTTCS 2006: Foundations of Software Technology and Theoretical Computer Science, 26th International Conference, Kolkata, India, December 13-15, 2006, Proceedings*, volume 4337 of *Lecture Notes in Computer Science*, pages 212–223. Springer, 2006. doi:10.1007/11944836_21.

29 Michael Ummels. The complexity of Nash equilibria in infinite multiplayer games. In Roberto M. Amadio, editor, *Foundations of Software Science and Computational Structures, 11th International Conference, FOSSACS 2008, Held as Part of the Joint European Conferences on Theory and Practice of Software, ETAPS 2008, Budapest, Hungary, March 29 – April 6, 2008. Proceedings*, volume 4962 of *Lecture Notes in Computer Science*, pages 20–34. Springer, 2008. doi:10.1007/978-3-540-78499-9_3.

30 Michael Ummels and Dominik Wojtczak. The complexity of Nash equilibria in limit-average games. In Joost-Pieter Katoen and Barbara König, editors, *CONCUR 2011 – Concurrency Theory – 22nd International Conference, CONCUR 2011, Aachen, Germany, September 6-9, 2011. Proceedings*, volume 6901 of *Lecture Notes in Computer Science*, pages 482–496. Springer, 2011. doi:10.1007/978-3-642-23217-6_32.

31 Heinrich Freiherr von Stackelberg. *Marktform und Gleichgewicht.* Wien und Berlin, J. Springer, Cambridge, MA, 1937.