Beth Semantics and Labelled Deduction for Intuitionistic Sentential Calculus with Identity

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Abstract

In this paper we consider the intuitionistic sentential calculus with Suszko's identity (ISCI). After recalling the basic concepts of the logic and its associated Hilbert proof system, we introduce a new sound and complete class of models for ISCI which can be viewed as algebraic counterparts (and extensions) of sheaf-theoretic topological models of intuitionistic logic. We use this new class of models, called Beth semantics for ISCI, to derive a first labelled sequent calculus and show its adequacy w.r.t. the standard Hilbert axiomatization of ISCI. This labelled proof system, like all other current proof systems for ISCI that we know of, does not enjoy the subformula property, which is problematic for achieving termination. We therefore introduce a second labelled sequent calculus in which the standard rules for identity are replaced with new special rules and show that this second calculus admits cut-elimination. Finally, using a key regularity property of the forcing relation in Beth models, we show that the eigenvariable condition can be dropped, thus leading to the termination and decidability results.

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1 Introduction

In this paper we consider the intuitionistic sentential calculus with identity (ISCI) which extends intuitionistic logic with Suszko's identity operator $\approx$ introduced in [12] for non-Fregean logics, and studied in the context of classical logic in [9] and [1].

Under the usual Fregean interpretation, the question of the equivalence of two formulas reduces to the problem of asking whether or not they have the same logical value. In presence of the non-truth functional identity operator, the rejection of the Fregean axiom makes it possible for two logically equivalent formulas to be considered non-identical in Suszko's sense. The philosophical motivation behind the Sentential Calculus with Identity (SCI) is related to the ontology of situations. In classical logic, only two situations are possible: truth and falsity, and truth (resp. falsity) is described and witnessed by any true (resp. false) proposition. According to [1], this is unfortunate and could be remedied by allowing a new identity connective $\approx$ to describe and witness the fact that two propositions denote the same situation. From this point of view, SCI can be considered as a generalization of classical logic in which we assume that there are more than (and at least) two different situations [7, 9].

In this paper, our aim is to revisit the interpretation of the identity connective on the grounds of intuitionistic logic [3] and to propose a new labelled sequent calculus with good properties like termination from which we can obtain the decidability of the logic. Related works include sequent calculi for both the classical and intuitionistic variants of SCI [2]. Such sequent calculi are obtained following the strategy described in [10] and do not have the

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subformula property. They have been compared with other proof systems for SCI [6, 13] but cannot lead to a decidability procedure for SCI [2]. In the case of the intuitionistic version ISCI, there exists an initial algebraic semantics that combines the ideas of the matrix semantics for sentential calculi with the Kripke semantics of intuitionistic logic. An Hilbert proof system is provided in [9]. A Kripke semantics for ISCI is introduced in [3] along with a sequent calculus for which cut elimination holds. However, since the sequent calculus is not analytic, the cut elimination theorem does not provide a decidability argument.

In Section 2 we introduce ISCI and its standard Hilbert calculus $\mathsf{H}_{\mathsf{ISCI}}$. In Section 3, we propose a new class of models for ISCI, called Beth semantics, which can be viewed as algebraic counterparts (and extensions) of sheaf-theoretic topological models of intuitionistic logic. We first show that general Beth models are complete w.r.t. $\mathsf{H}_{\mathsf{ISCI}}$ (Th. 11). Then, we define the more specific class of regular Beth models and show that they are also complete w.r.t. $\mathsf{H}_{\mathsf{ISCI}}$ (Th. 14). In Section 4, we introduce a first labelled calculus $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ which is proved complete w.r.t. $\mathsf{H}_{\mathsf{ISCI}}$ (Th. 22) and also w.r.t. Beth models (Th. 23). In Section 5, we derive a second labelled calculus $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ with new rules for identity and show that $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ is also complete w.r.t. $\mathsf{H}_{\mathsf{ISCI}}$ (Th. 25) and w.r.t. Beth models (Th. 26), but more interestingly, we show that any $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$-proof can be translated into an $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$-proof (Th. 27). Moreover, we show that cut is admissible in $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$, leading to the cut-free labelled calculus $\mathsf{L}^{2e}_{\mathsf{ISCI}}$ (Th. 33). In Section 6, we derive $\mathsf{L}^{2}_{\mathsf{ISCI}}$, a liberalized variant of $\mathsf{L}^{2e}_{\mathsf{ISCI}}$ in which the eigenvariable condition can be dropped. We show the soundness of $\mathsf{L}^{2}_{\mathsf{ISCI}}$ w.r.t. regular Beth models (Th. 40), which implies the soundness of regular Beth models w.r.t. $\mathsf{H}_{\mathsf{ISCI}}$ and the soundness of all our labelled calculi w.r.t. regular Beth models, as depicted and summarized in the picture below

$$\begin{tikzcd}
\models \mathrm{A} \arrow[r, "\text{Th. 11}"] \arrow[dr] & \vdash_{\mathsf{H}_{\mathsf{ISCI}}} \mathrm{A} \arrow[r, "\text{Th. 22}"] & \vdash_{\mathsf{L}^{1ec}_{\mathsf{ISCI}}} \mathrm{A} \arrow[r, "\text{Th. 27}"] & \vdash_{\mathsf{L}^{2ec}_{\mathsf{ISCI}}} \mathrm{A} \arrow[r, "\text{Th. 33}"] & \vdash_{\mathsf{L}^{2e}_{\mathsf{ISCI}}} \mathrm{A} \arrow[r, "\text{Th. 34}"] & \vdash_{\mathsf{L}^{2}_{\mathsf{ISCI}}} \mathrm{A} \arrow[dllll, "\text{Th. 40}"] \\
& \models_{\mathrm{r}} \mathrm{A} \arrow[u, "\text{Th. 14}"'] & & & &
\end{tikzcd}$$

Finally, we discuss and give arguments for the termination of $\mathsf{L}^{2}_{\mathsf{ISCI}}$, from which we deduce the decidability of ISCI.

2 Intuitionistic Sentential Calculus with Identity

In this section, we recall the basic notions of intuitionistic sentential calculus with Suszko's identity (ISCI) [9, 12]. ISCI extends propositional intuitionistic logic by adding a set of axioms that formalizes the non-truth functional nature of the identity connective $\approx$. The Hilbert-style system for ISCI [3, 9] is introduced and illustrated with examples.

▶ **Definition 1.** *Let* $\mathbf{P} = \{\, \mathrm{p}, \mathrm{q}, \dots \}$ *be a countable set of propositional letters. The formulas of* ISCI*, the set of which is denoted* $\mathbf{F}$*, are given by the grammar:*

$$\mathrm{A} ::= \mathbf{P} \mid \bot \mid \mathrm{A} \wedge \mathrm{A} \mid \mathrm{A} \vee \mathrm{A} \mid \mathrm{A} \supset \mathrm{A} \mid \mathrm{A} \approx \mathrm{A}$$

Formulas of the form $\mathrm{A} \approx \mathrm{B}$ are called *equations*. We write $\mathbf{F}_{/\approx}$ for the restriction of $\mathbf{F}$ to equations. Negation $\neg\mathrm{A}$ and truth $\top$ are respectively defined as $\mathrm{A} \supset \bot$ and $\bot \supset \bot$. To reduce the amount of parentheses, we interpret connectives up to left associativity according to the following strictly decreasing order of precedence: $\neg, \approx, \wedge, \vee, \supset$. Therefore, $\mathrm{A} \wedge \mathrm{B} \wedge \mathrm{A} \vee \mathrm{C} \supset \neg\mathrm{A} \approx \mathrm{B} \supset \mathrm{C}$ means $((((\mathrm{A} \wedge \mathrm{B}) \wedge \mathrm{A}) \vee \mathrm{C}) \supset ((\neg\mathrm{A}) \approx \mathrm{B})) \supset \mathrm{C}$.

ISCI can be axiomatized by adding the four identity axioms described in Figure 1 to any axiom schemata for intuitionistic logic (IL). We call "$\mathsf{H}_{\mathsf{ISCI}}$" the Hilbert proof system consisting of the four axioms for identity, the ten axioms for IL and the rule of modus

$$
\begin{array}{llll}
(\approx_1) & \mathrm{A} \approx \mathrm{A} & & \\
(\approx_2) & (\mathrm{A} \approx \mathrm{B}) \supset (\neg \mathrm{A} \approx \neg \mathrm{B}) & & \\
(\approx_3) & (\mathrm{A} \approx \mathrm{B}) \supset (\mathrm{B} \supset \mathrm{A}) & & \\
(\approx_4) & \multicolumn{3}{l}{(\mathrm{A} \approx \mathrm{B}) \wedge (\mathrm{C} \approx \mathrm{D}) \supset (\mathrm{A} \otimes \mathrm{C}) \approx (\mathrm{B} \otimes \mathrm{D}) \text{ where } \otimes \in \{\wedge, \vee, \supset, \approx\}} \\
(\mathrm{IL}_1) & \mathrm{A} \supset (\mathrm{B} \supset \mathrm{A}) & (\mathrm{IL}_2) & (\mathrm{A} \supset \mathrm{B}) \supset ((\mathrm{A} \supset (\mathrm{B} \supset \mathrm{C})) \supset (\mathrm{A} \supset \mathrm{C})) \\
(\mathrm{IL}_3) & \mathrm{A} \supset (\mathrm{B} \supset (\mathrm{A} \wedge \mathrm{B})) & (\mathrm{IL}_4) & (\mathrm{A} \wedge \mathrm{B}) \supset \mathrm{A} \\
(\mathrm{IL}_5) & (\mathrm{A} \wedge \mathrm{B}) \supset \mathrm{B} & (\mathrm{IL}_6) & (\mathrm{A} \supset \mathrm{C}) \supset ((\mathrm{B} \supset \mathrm{C}) \supset ((\mathrm{A} \vee \mathrm{B}) \supset \mathrm{C})) \\
(\mathrm{IL}_7) & \mathrm{A} \supset (\mathrm{A} \vee \mathrm{B}) & (\mathrm{IL}_8) & \mathrm{B} \supset (\mathrm{A} \vee \mathrm{B}) \\
(\mathrm{IL}_9) & (\mathrm{A} \supset \mathrm{B}) \supset ((\mathrm{A} \supset \neg \mathrm{B}) \supset \neg \mathrm{A}) & (\mathrm{IL}_{10}) & \neg \mathrm{A} \supset (\mathrm{A} \supset \mathrm{B}) \\
(\mathrm{MP}) & \text{From A and } \mathrm{A} \supset \mathrm{B} \text{ deduce B.} & &
\end{array}
$$

Figure 1 Axioms for ISCI.

$$
\begin{array}{rrr}
(1) & \mathrm{A} \approx \mathrm{B} & \text{assumption} \\
(2) & \mathrm{B} \approx \mathrm{B} & \approx_1 \\
(3) & ((\mathrm{B} \approx \mathrm{B}) \wedge (\mathrm{A} \approx \mathrm{B})) \supset ((\mathrm{B} \approx \mathrm{A}) \approx (\mathrm{B} \approx \mathrm{B})) & \approx_4 \\
(4) & (\mathrm{B} \approx \mathrm{B}) \supset ((\mathrm{A} \approx \mathrm{B}) \supset ((\mathrm{B} \approx \mathrm{B}) \wedge (\mathrm{A} \approx \mathrm{B}))) & \mathrm{IL}_3 \\
(5) & (\mathrm{A} \approx \mathrm{B}) \supset ((\mathrm{B} \approx \mathrm{B}) \wedge (\mathrm{A} \approx \mathrm{B})) & \mathrm{MP}\ 2,4 \\
(6) & (\mathrm{B} \approx \mathrm{B}) \wedge (\mathrm{A} \approx \mathrm{B}) & \mathrm{MP}\ 1,5 \\
(7) & (\mathrm{B} \approx \mathrm{A}) \approx (\mathrm{B} \approx \mathrm{B}) & \mathrm{MP}\ 3,6 \\
(8) & ((\mathrm{B} \approx \mathrm{A}) \approx (\mathrm{B} \approx \mathrm{B})) \supset ((\mathrm{B} \approx \mathrm{B}) \supset (\mathrm{B} \approx \mathrm{A})) & \approx_3 \\
(9) & (\mathrm{B} \approx \mathrm{B}) \supset (\mathrm{B} \approx \mathrm{A}) & \mathrm{MP}\ 7,8 \\
(10) & \mathrm{B} \approx \mathrm{A} & \mathrm{MP}\ 2,9
\end{array}
$$

Figure 2 Proof of $\approx$-symmetry: $\mathrm{A} \approx \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B} \approx \mathrm{A}$.

ponens. We write $\mathrm{S} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ to mean that a formula B is derivable in $\mathsf{H_{ISCI}}$ from a finite set $\mathrm{S} = \{\mathrm{A}_1, \ldots, \mathrm{A}_n\}$ of assumptions. Whenever S is empty, B is called a *thesis* or a *theorem* of $\mathsf{H_{ISCI}}$ and we write $\vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ instead of $\emptyset \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$. Let us note that $\emptyset \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ iff $\top \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ and that the deduction theorem holds for $\mathsf{H_{ISCI}}$, i.e. $\mathrm{A}_1, \ldots, \mathrm{A}_n \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ iff $\vdash_{\mathsf{H_{ISCI}}} \mathrm{A}_1 \wedge \ldots \wedge \mathrm{A}_n \supset \mathrm{B}$.

Figure 2 and Figure 3 show that identity is a symmetric and transitive connective.

3 Beth Semantics for ISCI

In this section we propose a new class of models, which we call Beth semantics for ISCI. Let us recall that there already exists an algebraic semantics for ISCI [9]. A Kripke semantics has also been recently investigated in [3]. Kripke-style semantics are usually better suited to the construction of labelled proof systems than algebraic semantics since the forcing relation enables an easy interpretation of a labelled formula $\mathrm{A} : \mathrm{x}$ as $\rho(\mathrm{x}) \Vdash \mathrm{A}$, where $\rho(\mathrm{x})$ is the denotation of the label x in some suitable class of models. Kripke models have succeeded in becoming the most popular forcing semantics for intuitionistic logic. One reason for this success is their very natural interpretation of disjunction as $m \Vdash \mathrm{A} \vee \mathrm{B}$ iff $m \Vdash \mathrm{A}$ or $m \Vdash \mathrm{B}$, whereas Beth and topological models require more complex notions such as bars and covers.

The models we propose in this section interpret disjunctions in a way which is similar to their interpretation in (sheaf-theoretic) topogical models of intuitionistic logic, but in the more algebraic context of distributive bounded lattices (Heyting algebras). While we pay the price of losing the very natural Kripke interpretation of disjunction, we gain a regularity property that allows us to build a labelled proof system that does not require any eigenvariable conditions, thus opening the way for simpler termination arguments.

▶ **Definition 2.** *Let* $\mathbf{M}$ *be a set of elements, called* worlds, *such that* $\omega, \pi \in \mathbf{M}$ *and* $\omega \neq \pi$.

(1)	$A \approx B$	assumption
(2)	$B \approx C$	assumption
(3)	$(A \approx B) \supset (B \approx A)$	$\approx$ - symmetry
(4)	$B \approx A$	MP 1, 3
(5)	$(B \approx A) \supset ((B \approx C) \supset ((B \approx A) \wedge (B \approx C)))$	IL_3
(6)	$(B \approx C) \supset ((B \approx A) \wedge (B \approx C))$	MP 4, 5
(7)	$(B \approx A) \wedge (B \approx C)$	MP 3, 6
(8)	$((B \approx A) \wedge (B \approx C)) \supset ((B \approx B) \approx (A \approx C))$	$\approx_4$
(9)	$(B \approx B) \approx (A \approx C)$	MP 7, 8
(10)	$(B \approx B) \approx (A \approx C) \supset (A \approx C) \approx (B \approx B)$	$\approx$ - symmetry
(11)	$(A \approx C) \approx (B \approx B)$	MP 9, 10
(12)	$((A \approx C) \approx (B \approx B)) \supset ((B \approx B) \supset (A \approx C))$	$\approx_3$
(13)	$(B \approx B) \supset (A \approx C)$	MP 11, 12
(14)	$B \approx B$	$\approx_1$
(15)	$A \approx C$	MP 13, 14

Figure 3 Proof of $\approx$ - transitivity: $(A \approx B), (B \approx C) \vdash_{H_{ISCI}} A \approx C$.

A Beth frame *is a bounded distributive lattice* $\mathcal{F} = (\mathbf{M}, \leqslant, \sqcup, \omega, \sqcap, \pi)$ *with* ω *and* π *as least and greatest elements respectively.*

▶ **Definition 3.** *A* Beth pre-model *is a triple* $\mathcal{M} = (\mathcal{F}, [\cdot], \Vdash)$*, where* $\mathcal{F}$ *is a Beth frame, and* $[\cdot]$ *is a* valuation function *from* $\mathbf{M}$ *to* $\wp(\mathbf{P} \cup \mathbf{F}_{/\approx})$*, such that for all worlds* m *and* n*:*
$(\mathcal{M}_\pi)$ $[\pi] = \mathbf{P} \cup \mathbf{F}_{/\approx}$,
$(\mathcal{M}_K)$ *if* $m \leqslant n$ *then* $[m] \subseteq [n]$,
$(\mathcal{M}_{\approx_1})$ $A \approx A \in [m]$,
$(\mathcal{M}_{\approx_2})$ *if* $A \approx B \in [m]$ *then* $\neg A \approx \neg B \in [m]$,
$(\mathcal{M}_{\approx_4})$ *for all* $\otimes \in \{\wedge, \vee, \supset, \approx\}$*, if* $A \approx B, C \approx D \in [m]$ *then* $A \otimes C \approx B \otimes D \in [m]$.
The forcing relation $\Vdash$ *is inductively defined as the smallest relation on* $\mathbf{M} \times \mathbf{F}$ *such that:*

- $m \Vdash p$ *iff* $p \in [m]$,
- $m \Vdash A \approx B$ *iff* $A \approx B \in [m]$,
- $m \Vdash \bot$ *iff* $\pi \leqslant m$,
- $m \Vdash A \wedge B$ *iff* $m \Vdash A$ *and* $m \Vdash B$,
- $m \Vdash A \supset B$ *iff for all worlds* n*, if* $n \Vdash A$ *then* $m \sqcup n \Vdash B$,
- $m \Vdash A \vee B$ *iff there exist two worlds* n_1, n_2 *such that* $n_1 \sqcap n_2 \leqslant m$*,* $n_1 \Vdash A$ *and* $n_2 \Vdash B$.

A Beth-model *is a Beth pre-model in which* $\Vdash$ *satisfies the following admissibility condition:*
$(\mathcal{M}_{\approx_3})$ *if* $m \Vdash A \approx B$ *then* $m \Vdash B \supset A$.

As usual, a formula A is *true* (or *satisfied*) in a Beth model $\mathcal{M}$, written $\mathcal{M} \vDash A$, iff $m \Vdash A$ for all worlds m in $\mathcal{M}$ (or equivalently, iff $\omega \Vdash A$) and *valid*, written $\vDash A$, iff it is true in all Beth models. It is routine to show that $\mathcal{M}_\pi$ and $\mathcal{M}_K$ extend from propositional letters and equations to all formulas. $\mathcal{M}_K$ is the well-known Kripke monotonicity condition, which applies to equations in our setting (see [3] for a discussion on alternative choices). Let us remark that $\mathcal{M}_\pi$ implies that all Beth models have a world π that forces all formulas, including inconsistency ($\bot$).

3.1 Completeness of Beth models

A standard way of proving the completeness of a given semantics is to build a canonical model that relates the denotation of formulas to a derivability relation that syntactically defines the logic under consideration (often an Hilbert proof system). Algebraic semantics are usually obtained through Lindenbaum-Tarski constructions that mostly rely on equivalence classes of formulas w.r.t. the underlying derivability relation (for ISCI, we would consider classes such as $\dot{\mathrm{A}} = \{\, \mathrm{B} \mid \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A} \text{ and } \mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B} \,\}$). Following an idea of Beth, we replace equivalence classes with theories of formulas to build a canonical model for ISCI in which the forcing relation faithfully mimics the derivability relation in $\mathsf{H_{ISCI}}$.

▶ **Definition 4.** *The* theory A^{t} *associated with a formula* A *is the set* $\{\, \mathrm{B} \mid \mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B} \,\}$.

Let $\mathbf{T}$ denote the set $\{\, \mathrm{A}^{\mathrm{t}} \mid \mathrm{A} \in \mathbf{F} \,\}$ of theories generated by all formulas of ISCI. Reading $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ as "$\mathrm{A} \leqslant \mathrm{B}$", all sets of formulas can be preordered by derivability in $\mathsf{H_{ISCI}}$. We define $min(X)$ as the set $\{\, \mathrm{A} \in X \mid \forall \mathrm{B} \in X, \mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B} \,\}$ of all formulas that are minimal in X w.r.t. $\vdash_{\mathsf{H_{ISCI}}}$. It follows that for all theories $X \in \mathbf{T}$, $X = \mathrm{A}^{\mathrm{t}}$ for all $\mathrm{A} \in min(X)$. Moreover, for all formulas $\mathrm{A}, \mathrm{B} \in \mathbf{F}$, if $X = \mathrm{A}^{\mathrm{t}} = \mathrm{B}^{\mathrm{t}}$ then both $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ and $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$.

▶ **Definition 5.** *The* canonical Beth frame for ISCI *is the structure* $\mathcal{T} = (\mathbf{T}, \subseteq, \sqcup, \top^{\mathrm{t}}, \sqcap, \bot^{\mathrm{t}})$, *where for all theories* $X, Y \in \mathbf{T}$*:*

$$X \sqcap Y = X \cap Y \ \textit{and}\ X \sqcup Y = \bigcup \{\, (\mathrm{A} \wedge \mathrm{B})^{\mathrm{t}} \mid \mathrm{A} \in min(X), \mathrm{B} \in min(Y) \,\}.$$

▶ **Lemma 6.** *For all theories* $X, Y \in \mathbf{T}$ *and all formulas* $\mathrm{A} \in min(X), \mathrm{B} \in min(Y)$*, the canonical Beth frame for* ISCI *satisfies the following properties:*

$$(a)\ X \sqcap Y = (\mathrm{A} \vee \mathrm{B})^{\mathrm{t}}, \quad (b)\ X \sqcup Y = (\mathrm{A} \wedge \mathrm{B})^{\mathrm{t}}, \quad (c)\ X \subseteq Y \ \textit{iff}\ \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}.$$

Proof. Since $\mathrm{A} \in min(X)$ and $\mathrm{B} \in min(Y)$ we have both $X = \mathrm{A}^{\mathrm{t}}$ and $Y = \mathrm{B}^{\mathrm{t}}$.

For (a), by definition $\mathrm{A}^{\mathrm{t}} \sqcap \mathrm{B}^{\mathrm{t}} = \mathrm{A}^{\mathrm{t}} \cap \mathrm{B}^{\mathrm{t}}$. Firstly, we show $\mathrm{A}^{\mathrm{t}} \cap \mathrm{B}^{\mathrm{t}} \subseteq (\mathrm{A} \vee \mathrm{B})^{\mathrm{t}}$. If $\mathrm{C} \in \mathrm{A}^{\mathrm{t}} \cap \mathrm{B}^{\mathrm{t}}$ then $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$ and $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$, which implies $\mathrm{A} \vee \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$ (by axiom IL_6). Thus, $\mathrm{C} \in (\mathrm{A} \vee \mathrm{B})^{\mathrm{t}}$. Secondly, we show $(\mathrm{A} \vee \mathrm{B})^{\mathrm{t}} \subseteq \mathrm{A}^{\mathrm{t}} \cap \mathrm{B}^{\mathrm{t}}$. If $\mathrm{C} \in (\mathrm{A} \vee \mathrm{B})^{\mathrm{t}}$, then $\mathrm{A} \vee \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$. Since axioms IL_7 and IL_8 imply $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A} \vee \mathrm{B}$ and $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A} \vee \mathrm{B}$, we have $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$ and $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$. Thus, $\mathrm{C} \in \mathrm{A}^{\mathrm{t}} \cap \mathrm{B}^{\mathrm{t}}$.

For (b), by definition, $(\mathrm{A} \wedge \mathrm{B})^{\mathrm{t}} \subseteq X \sqcup Y$. We show $X \sqcup Y \subseteq (\mathrm{A} \wedge \mathrm{B})^{\mathrm{t}}$. If $\mathrm{C} \in X \sqcup Y$ then $\mathrm{C} \in (\mathrm{F} \wedge \mathrm{G})^{\mathrm{t}}$ for some $\mathrm{F} \in min(X)$ and some $\mathrm{G} \in min(Y)$. Since $X = \mathrm{A}^{\mathrm{t}} = \mathrm{F}^{\mathrm{t}}$ and $Y = \mathrm{B}^{\mathrm{t}} = \mathrm{G}^{\mathrm{t}}$, we have $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{F}$ and $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{G}$, which implies $\mathrm{A} \wedge \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{F} \wedge \mathrm{G}$. By definition, $\mathrm{C} \in (\mathrm{F} \wedge \mathrm{G})^{\mathrm{t}}$ implies $\mathrm{F} \wedge \mathrm{G} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$. Thus, $\mathrm{A} \wedge \mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$ implies $\mathrm{C} \in (\mathrm{A} \wedge \mathrm{B})^{\mathrm{t}}$.

For (c), we show that $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$ iff $\mathrm{A}^{\mathrm{t}} \subseteq \mathrm{B}^{\mathrm{t}}$. If $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$ then for all $\mathrm{C} \in \mathrm{A}^{\mathrm{t}}$, we have $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$, from which it follows that $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}$, i.e. $\mathrm{C} \in \mathrm{B}^{\mathrm{t}}$. Conversely, since $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$ implies $\mathrm{A} \in \mathrm{A}^{\mathrm{t}}$, if $\mathrm{A}^{\mathrm{t}} \subseteq \mathrm{B}^{\mathrm{t}}$ then $\mathrm{A} \in \mathrm{B}^{\mathrm{t}}$, i.e. $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$. ◀

Lemma 6 shows that, in the canonical Beth frame $\mathcal{T}$, the partial order defined as set inclusion mimics derivability in $\mathsf{H_{ISCI}}$. Moreover, the lattice meet $\sqcap$ and join $\sqcup$ respectively correspond to disjunction and conjunction in the logic. It then easily follows that $\mathcal{T}$ is a bounded distributive lattice since $\wedge$ and $\vee$ distribute over one another in the logic. Let us note that while the meet of two theories coincides with intersection, their join does not coincide with union since for any two distinct propositional letters p and q, we have $\mathrm{p} \wedge \mathrm{q} \in (\mathrm{p} \wedge \mathrm{q})^{\mathrm{t}}$, but neither $\mathrm{p} \wedge \mathrm{q} \in \mathrm{p}^{\mathrm{t}}$, nor $\mathrm{p} \wedge \mathrm{q} \in \mathrm{q}^{\mathrm{t}}$ (since neither $\mathrm{p} \vdash_{\mathsf{H_{ISCI}}} \mathrm{p} \wedge \mathrm{q}$, nor $\mathrm{q} \vdash_{\mathsf{H_{ISCI}}} \mathrm{p} \wedge \mathrm{q}$).

▶ **Definition 7.** *The* canonical Beth model for ISCI *is the triple* $\mathcal{M}^{\mathrm{t}} = (\mathcal{T}, [\cdot], \Vdash)$, *where the* canonical valuation *is defined as* $[X] = \bigcup \{\, \mathrm{A}^{\mathrm{t}} \mid \mathrm{A} \in min(X) \,\} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx})$ *for all* $X \in \mathbf{T}$.

▶ **Lemma 8.** *The canonical valuation satisfies the conditions of Definition 3 and for all theories* $X \in \mathbf{T}$ *and all formulas* $\mathrm{A} \in min(X)$, $[X] = \mathrm{A}^{\mathrm{t}} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx})$.

Proof. $[X] = \mathrm{A}^{\mathrm{t}} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx})$ for all $\mathrm{A} \in min(X)$ follows from the fact that $\mathrm{C}^{\mathrm{t}} = \mathrm{D}^{\mathrm{t}}$ for all $\mathrm{C}, \mathrm{D} \in min(X)$, which implies $\bigcup\{\, \mathrm{B}^{\mathrm{t}} \mid \mathrm{B} \in min(X) \,\} = \mathrm{A}^{\mathrm{t}}$ for all $\mathrm{A} \in min(X)$.

Case $\mathcal{M}_{\pi}$: By definition, $[\bot^{\mathrm{t}}] = \{\, \mathrm{B} \mid \mathrm{B} \in \bot^{\mathrm{t}} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx}) \,\}$. Since $\bot \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$ for all formulas B, we have $\bot^{\mathrm{t}} = \mathbf{F}$, which implies $\bot^{\mathrm{t}} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx}) = (\mathbf{P} \cup \mathbf{F}_{/\approx}) = [\bot^{\mathrm{t}}]$.

Case $\mathcal{M}_{\mathrm{K}}$: Suppose we have $X, Y \in \mathbf{T}$ such that $X \subseteq Y$, then $X = \mathrm{A}^{\mathrm{t}}$ and $Y = \mathrm{B}^{\mathrm{t}}$ for some $\mathrm{A} \in min(X)$ and some $\mathrm{B} \in min(Y)$. Since $X \subseteq Y$ implies $\mathrm{A}^{\mathrm{t}} \subseteq \mathrm{B}^{\mathrm{t}}$, if $\mathrm{C} \in [X] = \mathrm{A}^{\mathrm{t}} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx})$, then $\mathrm{C} \in \mathrm{B}^{\mathrm{t}} \cap (\mathbf{P} \cup \mathbf{F}_{/\approx}) = [Y]$. Thus, $[X] \subseteq [Y]$.

The other cases $\mathcal{M}_{\approx_{i \in \{1,2,4\}}}$ easily follow from the $\mathsf{H_{ISCI}}$ axioms $\approx_{i \in \{1,2,4\}}$. ◀

▶ **Lemma 9.** *For all $X \in \mathbf{T}$, for all $\mathrm{A} \in min(X)$, $X \Vdash \mathrm{B}$ iff $\mathrm{A}^{\mathrm{t}} \Vdash \mathrm{B}$ iff $\mathrm{B} \in \mathrm{A}^{\mathrm{t}}$ iff $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$.*

Proof. By definition of a theory we have $\mathrm{B} \in \mathrm{A}^{\mathrm{t}}$ iff $\mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}$. Moreover, since $X = \mathrm{A}^{\mathrm{t}}$ for all $\mathrm{A} \in min(X)$, we only need to prove that $\mathrm{A}^{\mathrm{t}} \Vdash \mathrm{B}$ iff $\mathrm{B} \in \mathrm{A}^{\mathrm{t}}$ by structural induction on B.

Base case: $\mathrm{B} \in (\mathbf{P} \cup \mathbf{F}_{/\approx})$. Lemma 8 implies $\mathrm{B} \in [\mathrm{A}^{\mathrm{t}}]$ iff $\mathrm{B} \in \mathrm{A}^{\mathrm{t}}$. Since $\mathrm{A}^{\mathrm{t}} \Vdash \mathrm{B}$ iff $\mathrm{B} \in [\mathrm{A}^{\mathrm{t}}]$ by Definition 3, $\mathrm{A}^{\mathrm{t}} \Vdash \mathrm{B}$ iff $\mathrm{B} \in \mathrm{A}^{\mathrm{t}}$.

Case $\mathrm{B} = \mathrm{B}_1 \vee \mathrm{B}_2$:

$$\begin{array}{rcll}
\mathrm{A}^{\mathrm{t}} \Vdash \mathrm{B}_1 \vee \mathrm{B}_2 & \Leftrightarrow & \exists\, {\mathrm{C}_1}^{\mathrm{t}}, {\mathrm{C}_2}^{\mathrm{t}}.\ {\mathrm{C}_1}^{\mathrm{t}} \sqcap {\mathrm{C}_2}^{\mathrm{t}} \subseteq \mathrm{A}^{\mathrm{t}}, {\mathrm{C}_1}^{\mathrm{t}} \Vdash \mathrm{B}_1, {\mathrm{C}_2}^{\mathrm{t}} \Vdash \mathrm{B}_2 & \\
& \Leftrightarrow & \exists\, {\mathrm{C}_1}^{\mathrm{t}}, {\mathrm{C}_2}^{\mathrm{t}}.\ (\mathrm{C}_1 \vee \mathrm{C}_2)^{\mathrm{t}} \subseteq \mathrm{A}^{\mathrm{t}}, \mathrm{B}_1 \in {\mathrm{C}_1}^{\mathrm{t}}, \mathrm{B}_2 \in {\mathrm{C}_2}^{\mathrm{t}} & \text{Lem. 6, I.H.} \\
& \Leftrightarrow & \exists\, \mathrm{C}_1, \mathrm{C}_2.\ \mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{C}_1 \vee \mathrm{C}_2, \mathrm{C}_1 \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}_1, \mathrm{C}_2 \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}_2 & \text{Lem. 6, Def. 4} \\
& \Leftrightarrow & \mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}_1 \vee \mathrm{B}_2 & \text{Logic} \\
& \Leftrightarrow & \mathrm{B}_1 \vee \mathrm{B}_2 \in \mathrm{A}^{\mathrm{t}} & \text{Def. 4}
\end{array}$$

Case $\mathrm{B} = \mathrm{B}_1 \supset \mathrm{B}_2$:

$$\begin{array}{rcll}
\mathrm{A}^{\mathrm{t}} \Vdash \mathrm{B}_1 \supset \mathrm{B}_2 & \Leftrightarrow & \forall\, \mathrm{C}^{\mathrm{t}}.\ \text{if } \mathrm{C}^{\mathrm{t}} \Vdash \mathrm{B}_1 \text{ then } \mathrm{A}^{\mathrm{t}} \sqcup \mathrm{C}^{\mathrm{t}} \Vdash \mathrm{B}_2 & \\
& \Leftrightarrow & \forall\, \mathrm{C}^{\mathrm{t}}.\ \text{if } \mathrm{B}_1 \in \mathrm{C}^{\mathrm{t}} \text{ then } \mathrm{B}_2 \in (\mathrm{A} \wedge \mathrm{C})^{\mathrm{t}} & \text{Lem. 6, I.H.} \\
& \Leftrightarrow & \forall\, \mathrm{C}.\ \text{if } \mathrm{C} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}_1 \text{ then } \mathrm{A} \wedge \mathrm{C} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}_2 & \text{Def. 4} \\
& \Leftrightarrow & \mathrm{A} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B}_1 \supset \mathrm{B}_2 & \text{Logic} \\
& \Leftrightarrow & \mathrm{B}_1 \supset \mathrm{B}_2 \in \mathrm{A}^{\mathrm{t}} & \text{Def. 4}
\end{array}$$

The other cases are similar. ◀

▶ **Lemma 10.** *The canonical Beth model $\mathcal{M}^{\mathrm{t}}$ satisfies the admissibility condition $\mathcal{M}_{\approx_3}$.*

Proof. Any $X \in \mathbf{T}$ such that $X \Vdash \mathrm{A} \approx \mathrm{B}$ entails $\mathrm{C} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A} \approx \mathrm{B}$ for all $\mathrm{C} \in min(X)$ by Lemma 9, which implies $\mathrm{C} \vdash_{\mathsf{H_{ISCI}}} \mathrm{B} \supset \mathrm{A}$ by axiom ($\approx_3$). Thus, $X \Vdash \mathrm{B} \supset \mathrm{A}$ by Lemma 9. ◀

▶ **Theorem 11.** *Beth models for* ISCI *are complete, i.e., if $\vDash \mathrm{A}$ then $\vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$.*

Proof. We show that $\nvdash_{\mathsf{H_{ISCI}}} \mathrm{A}$ implies $\nvDash \mathrm{A}$. Suppose that $\nvdash_{\mathsf{H_{ISCI}}} \mathrm{A}$, then $\top \nvdash_{\mathsf{H_{ISCI}}} \mathrm{A}$ which implies $\mathrm{A} \notin \top^{\mathrm{t}}$. By Lemma 9 we get $\top^{\mathrm{t}} \nVdash \mathrm{A}$ in $\mathcal{M}^{\mathrm{t}}$, which by definition implies $\nvDash \mathrm{A}$. ◀

3.2 Regular Beth Models

We now show that the canonical Beth model for ISCI satisfies a regularity property that is essential for the termination arguments in Section 6.3.

▶ **Definition 12.** *Let $\mathcal{M} = (\mathcal{F}, [\cdot], \Vdash)$ be a Beth model. $\mathcal{M}$ is* regular *iff for all formulas* A*, if $m \Vdash \mathrm{A}$ for some world m, then there exists a world m_{A}, called* A-minimal*, such that $m_{\mathrm{A}} \Vdash \mathrm{A}$ and for all worlds n, $n \Vdash \mathrm{A}$ implies $m_{\mathrm{A}} \leqslant n$. We write $\vDash_{\mathrm{r}}$ (instead of $\vDash$) for the restriction of validity to the class of* regular *Beth models.*

▶ **Lemma 13.** *The canonical model $\mathcal{M}^{\mathrm{t}}$ is regular: for all formulas* A*, A^{t} is* A*-minimal.*

Proof. Suppose that $\mathrm{B}^{\mathrm{t}} \Vdash \mathrm{A}$ for an arbitrary theory B^{t}. Then, $\mathrm{B} \vdash_{\mathsf{H_{ISCI}}} \mathrm{A}$ by Lemma 9, which implies $\mathrm{A}^{\mathrm{t}} \subseteq \mathrm{B}^{\mathrm{t}}$ by Lemma 6. ◀

▶ **Theorem 14.** *Regular Beth models for* ISCI *are complete: if* $\models_r A$ *then* $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$.

Proof. The result is an immediate consequence of Lemma 13. ◀

▶ **Theorem 15.** *Regular Beth models for* ISCI *are sound: if* $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$ *then* $\models_r A$.

Proof. The result follows from Theorems 22, 27, 33, 34 and 40. ◀

Let us remark that non-regular Beth models are neither sound for ISCI, nor for IL. Indeed, $p \vee p \supset p$ is a theorem of IL, but $\omega \nVdash p \vee p \supset p$ in the Beth model $((\mathbf{M}, \leqslant, \sqcup, \omega, \sqcap, \pi), [\cdot], \Vdash)$, where $\mathbf{M} = \{\omega, m_1, m_2, \pi\}$, $m \leqslant n$ iff $m = \omega$ or $n = \pi$, $[\omega] = \{A \approx A \mid A \in \mathbf{F}\}$, $[m_1] = [m_2] = [\omega] \cup \{p\}$, and $[\pi] = \mathbf{P} \cup \mathbf{F}_{/\approx}$.

▶ **Theorem 16.** *In a regular Beth model* $\mathcal{M}$*, if* $m \Vdash A$ *and* $n \Vdash A$ *then* $m \sqcap n \Vdash A$.

Proof. Since $\mathcal{M}$ is regular, $m \Vdash A$ and $n \Vdash A$ imply the existence of an A-minimal world m_A. Since $m_A \leqslant m$ and $m_A \leqslant n$ imply $m_A \leqslant m \sqcap n$, $m \sqcap n \Vdash A$ by Kripke monotonicity. ◀

4 Labelled Deduction for ISCI

In this section we propose a new labelled sequent calculus, called $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$, which is derived from the Beth models described in Section 3. The methodology is inspired by and in the spirit of our works on labelled deduction in BI and bi-intuitionistic logic [4, 5]. Let us note that there exists a label-free sequent calculus for ISCI [3], built following the strategy described in [10, 11], which like $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$ does not enjoy the subformula property.

4.1 A Labelling Algebra

Let $\mathbf{L}^n$ be the set $\{S \mid S \subset \mathbb{N} \text{ and } |S| = n\}$ of all subsets of $\mathbb{N}$ of size (cardinal) n. The set $\mathbf{L}^*$ of *label letters* is defined as $\bigcup_{n \in \mathbb{N}} \mathbf{L}^n$. Let $\mathbf{L}^u = \{\emptyset, \mathbb{N}\}$ be the set of *label units*, the set $\mathbf{L}$ of *labels* is then defined as $\mathbf{L}^* \cup \mathbf{L}^u$. We use the (possibly subscripted or primed) letters a, b, c to denote labels which are singletons (i.e., elements of $\mathbf{L}^1$) and save the letters x, y, z to denote arbitrary labels. A label x is a *sublabel* of a label y if $x \subseteq y$.

We work with a labelling algebra $\mathcal{L}$ defined as the lattice $(\mathbf{L}, \subseteq, \cup, \emptyset, \cap, \mathbb{N})$, where join $\cup$ and meet $\cap$ are standard set union and intersection. We consider that $\cup$ binds stronger than $\cap$ and we shall frequently write xy instead of $x \cup y$ ($xx' \cap yy'$ should therefore be read as $(x \cup x') \cap (y \cup y')$). In this paper, we shall only use examples with label letters built from the subset $\{1, \ldots, 9\}$. Therefore, we shall use the more concise notation 13 to unambiguously refer to $\{1, 3\}$ (and not to the label letter $\{13\}$).

4.2 The Labelled Sequent Calculus $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$

▶ **Definition 17.** *A* labelled formula *is a pair* (C, x)*, written* $C : x$*, where* C *is a formula and* x *is a label. A* labelled sequent *is a pair* (Γ, Δ)*, written* $\Gamma \vdash \Delta$*, where* Γ, Δ *are multi-sets of labelled formulas.*

We use the generic notation $O(T)$ to mean that the object T is a subobject of an object O (for some well defined notion of object inclusion). For example, when S is a set, $S(e_1, \ldots, e_n)$ means that $\{e_1, \ldots, e_n\}$ is a subset of S. Similarly, if F and G are formulas, F(G) means that G is a subformula of F and if x is a label, x(y) means that y is a sublabel of x. If Δ is a set or multi-set of labelled formulas, we define $x \subseteq \Delta$ as $\exists A : y \in \Delta$ such that $x \subseteq y$, which is more shortly written $\Delta(x)$. The notation $x \subseteq A : y$ is a shorthand for $x \subseteq \{A : y\}$. A labelled sequent $\Gamma \vdash \Delta$ is *connected* iff $x \subseteq \Delta$ for all $A : x \in \Gamma$.

$$\dfrac{}{\Gamma(A:x)\vdash\Delta(A:y)}\,\mathrm{id}(x\subseteq y)\qquad\dfrac{}{\Gamma(\bot:x)\vdash\Delta(A:y)}\,\bot_L(x\subseteq y)$$

$$\dfrac{\Gamma,A:x,B:x\vdash\Delta}{\Gamma(A\wedge B:x)\vdash\Delta}\,\wedge_L\qquad\dfrac{\Gamma\vdash\Delta,A:x\quad\Gamma\vdash\Delta,B:x}{\Gamma\vdash\Delta(A\wedge B:x)}\,\wedge_R$$

$$\dfrac{\Gamma\vdash\Delta,A:y\quad\Gamma,B:x\cup y\vdash\Delta}{\Gamma(A\supset B:x)\vdash\Delta}\,\supset_L(x\cup y\subseteq\Delta)\qquad\dfrac{\Gamma,A:a\vdash\Delta,B:x\cup a}{\Gamma\vdash\Delta(A\supset B:x)}\,\supset_R(a\not\subseteq\Gamma\cup\Delta)$$

$$\dfrac{\Gamma,A:x\cup a\vdash\Delta,C:y\cup a\quad\Gamma,B:x\cup b\vdash\Delta,C:y\cup b}{\Gamma(A\vee B:x)\vdash\Delta(C:y)}\,\vee_L(a\neq b\not\subseteq\Gamma\cup\Delta,x\subseteq y)$$

$$\dfrac{\Gamma\vdash\Delta,A:x,B:x}{\Gamma\vdash\Delta(A\vee B:x)}\,\vee_R\qquad\dfrac{\Gamma\vdash\Delta,C:x\quad C:x,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}\,\mathrm{cut}(x\subseteq\Delta)$$

$$\dfrac{\Gamma,A\approx A:x\vdash\Delta}{\Gamma\vdash\Delta}\,\approx_{L1}(x\subseteq\Delta)\qquad\dfrac{\Gamma,\neg A\approx\neg B:x\vdash\Delta}{\Gamma(A\approx B:x)\vdash\Delta}\,\approx_{L2}\qquad\dfrac{\Gamma,B\supset A:x\vdash\Delta}{\Gamma(A\approx B:x)\vdash\Delta}\,\approx_{L3}$$

$$\dfrac{\Gamma,A\otimes C\approx B\otimes D:x\vdash\Delta}{\Gamma(A\approx B:x,C\approx D:x)\vdash\Delta}\,\approx_{L4}\qquad\dfrac{\Gamma,A\otimes A\approx B\otimes B:x\vdash\Delta}{\Gamma(A\approx B:x)\vdash\Delta}\,\approx_{L4'}$$

Figure 4 Labelled Sequent Calculus $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B\approx B:\emptyset\vdash B\approx B:\emptyset}\,\mathrm{id}}{\vdash B\approx B:\emptyset}\,\approx_{L1}\quad\dfrac{}{B\approx A:1\vdash B\approx A:1}\,\mathrm{id}}{(B\approx B)\supset(B\approx A):1\vdash B\approx A:1}\,\supset_L}{(B\approx A)\approx(B\approx B):1\vdash B\approx A:1}\,\approx_{L3}}{B\approx B:1,A\approx B:1\vdash B\approx A:1}\,\approx_{L4}}{A\approx B:1\vdash B\approx A:1}\,\approx_{L1}}{\vdash(A\approx B)\supset(B\approx A):\emptyset}\,\supset_R$$

Figure 5 $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$-Proof of $\approx$-symmetry.

The labelled calculus $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ is given in Figure 4. The only structural rule in $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ is cut. All lattice properties of Beth models are implicity reflected in our labelling algebra by our choice of labels as subsets of $\mathbb{N}$. The rules $\supset_R$ and $\vee_L$ have *eigenvariable* (or *freshness*) conditions on the label letters a, b they introduce. Since connectedness plays a significant role in our forthcoming proof of cut elimination, the rules of $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ have been carefully designed so as to preserve this property from their conclusion to their premise. For instance, the cut rule has a side condition that requires the label of the cut formula to occur as a sublabel on the right-hand side of the conclusion.

▶ **Definition 18.** *A formula* A *is a* theorem of *(or* derivable in*)* $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$*, written* $\vdash_{\mathsf{L}^{1ec}_{\mathsf{ISCI}}} A$*, if the labelled sequent* $\vdash A:\emptyset$ *is derivable from the rules given in Figure 4.*

The proof rules in Figure 4 are formulated in a non-destructive way, i.e. they preserve (a copy of) their principal formulas in their premise. This is only a technical choice that makes the proofs of the forthcoming admissibility results shorter, but we shall use the more standard destructive versions of the rules in our examples to keep them more concise.

Figure 5 gives a labelled proof in $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ of the symmetry of the identity connective, from which one can easily derive a symmetry rule $\approx_{LS}$, as illustrated in Figure 6, which proves the transitivity of $\approx$. More examples are given in the proof of Lemma 20.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{B}\approx\mathrm{B}:\emptyset\vdash\mathrm{B}\approx\mathrm{B}:\emptyset}\,\mathrm{id}}{\vdash\mathrm{B}\approx\mathrm{B}:\emptyset}\approx_{\mathrm{L1}}\quad\dfrac{}{\mathrm{A}\approx\mathrm{C}:1\vdash\mathrm{A}\approx\mathrm{C}:1}\,\mathrm{id}}{(\mathrm{B}\approx\mathrm{B})\supset(\mathrm{A}\approx\mathrm{C}):1\vdash\mathrm{A}\approx\mathrm{C}:1}\supset_{\mathrm{L}}}{(\mathrm{A}\approx\mathrm{C})\approx(\mathrm{B}\approx\mathrm{B}):1\vdash\mathrm{A}\approx\mathrm{C}:1}\approx_{\mathrm{L3}}}{(\mathrm{B}\approx\mathrm{B})\approx(\mathrm{A}\approx\mathrm{C}):1\vdash\mathrm{A}\approx\mathrm{C}:1}\approx_{\mathrm{LS}}}{\mathrm{B}\approx\mathrm{A}:1,\mathrm{B}\approx\mathrm{C}:1\vdash\mathrm{A}\approx\mathrm{C}:1}\approx_{\mathrm{L4}}}{\mathrm{A}\approx\mathrm{B}:1,\mathrm{B}\approx\mathrm{C}:1\vdash\mathrm{A}\approx\mathrm{C}:1}\approx_{\mathrm{LS}}}{(\mathrm{A}\approx\mathrm{B})\wedge(\mathrm{B}\approx\mathrm{C}):1\vdash\mathrm{A}\approx\mathrm{C}:1}\wedge_{\mathrm{L}}}{\vdash(\mathrm{A}\approx\mathrm{B})\wedge(\mathrm{B}\approx\mathrm{C})\supset(\mathrm{A}\approx\mathrm{C}):\emptyset}\supset_{\mathrm{R}}$$

Figure 6 $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$-Proof of $\approx$-Transitivity.

4.3 Soundness and Completeness of $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$

▶ **Theorem 19** (Soundness). *If* $\vdash_{\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}} \mathrm{A}$ *then* $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} \mathrm{A}$.

Proof. A corollary of Theorems 27, 33, 34 and 40. ◀

▶ **Lemma 20.** *All of the axioms for* $\approx$ *given in Figure 1 are derivable in* $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$.

Proof. Axiom $\approx_1$:

$$\dfrac{\dfrac{}{\mathrm{A}\approx\mathrm{A}:\emptyset\vdash\mathrm{A}\approx\mathrm{A}:\emptyset}\,\mathrm{id}}{\vdash\mathrm{A}\approx\mathrm{A}:\emptyset}\approx_{\mathrm{L1}}$$

Axioms $\approx_2$, $\approx_3$:

$$\dfrac{\dfrac{\dfrac{}{\neg\mathrm{A}\approx\neg\mathrm{B}:1\vdash\neg\mathrm{A}\approx\neg\mathrm{B}:1}\,\mathrm{id}}{\mathrm{A}\approx\mathrm{B}:1\vdash\neg\mathrm{A}\approx\neg\mathrm{B}:1}\approx_{\mathrm{L2}}}{\vdash(\mathrm{A}\approx\mathrm{B})\supset(\neg\mathrm{A}\approx\neg\mathrm{B}):\emptyset}\supset_{\mathrm{R}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{B}:2\vdash\mathrm{B}:2}\,\mathrm{id}\quad\dfrac{}{\mathrm{A}:12\vdash\mathrm{A}:12}\,\mathrm{id}}{\mathrm{B}\supset\mathrm{A}:1,\mathrm{B}:2\vdash\mathrm{A}:12}\supset_{\mathrm{L}}}{\mathrm{A}\approx\mathrm{B}:1,\mathrm{B}:2\vdash\mathrm{A}:12}\approx_{\mathrm{L3}}}{\mathrm{A}\approx\mathrm{B}:1\vdash\mathrm{B}\supset\mathrm{A}:1}\supset_{\mathrm{R}}}{\vdash(\mathrm{A}\approx\mathrm{B})\supset(\mathrm{B}\supset\mathrm{A}):\emptyset}\supset_{\mathrm{R}}$$

Axiom $\approx_4$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(\mathrm{A}\otimes\mathrm{C})\approx(\mathrm{B}\otimes\mathrm{D}):1\vdash(\mathrm{A}\otimes\mathrm{C})\approx(\mathrm{B}\otimes\mathrm{D}):1}\,\mathrm{id}}{\mathrm{A}\approx\mathrm{B}:1,\mathrm{C}\approx\mathrm{D}:1\vdash(\mathrm{A}\otimes\mathrm{C})\approx(\mathrm{B}\otimes\mathrm{D}):1}\approx_{\mathrm{L4}}}{(\mathrm{A}\approx\mathrm{B})\wedge(\mathrm{C}\approx\mathrm{D}):1\vdash(\mathrm{A}\otimes\mathrm{C})\approx(\mathrm{B}\otimes\mathrm{D}):1}\wedge_{\mathrm{L}}}{\vdash(\mathrm{A}\approx\mathrm{B})\wedge(\mathrm{C}\approx\mathrm{D})\supset(\mathrm{A}\otimes\mathrm{C})\approx(\mathrm{B}\otimes\mathrm{D}):\emptyset}\supset_{\mathrm{R}}$$

◀

▶ **Lemma 21.** *All of the axioms for* IL *given in Figure 1 are derivable in* $\mathsf{L}^{\mathsf{1ec}}_{\mathsf{ISCI}}$.

Proof. Axiom IL_6:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{A}:34\vdash\mathrm{A}:234}\,\mathrm{id}\quad\dfrac{}{\mathrm{C}:1234\vdash\mathrm{C}:1234}\,\mathrm{id}}{\mathrm{A}\supset\mathrm{C}:1,\mathrm{A}:34\vdash\mathrm{C}:1234}\supset_{\mathrm{L}}\quad\dfrac{\dfrac{}{\mathrm{B}:35\vdash\mathrm{B}:135}\,\mathrm{id}\quad\dfrac{}{\mathrm{C}:1235\vdash\mathrm{C}:1235}\,\mathrm{id}}{\mathrm{B}\supset\mathrm{C}:2,\mathrm{B}:35\vdash\mathrm{C}:1235}\supset_{\mathrm{L}}}{\mathrm{A}\supset\mathrm{C}:1,\mathrm{B}\supset\mathrm{C}:2,\mathrm{A}\vee\mathrm{B}:3\vdash\mathrm{C}:123}\vee_{\mathrm{L}}}{\mathrm{A}\supset\mathrm{C}:1,\mathrm{B}\supset\mathrm{C}:2\vdash(\mathrm{A}\vee\mathrm{B})\supset\mathrm{C}:12}\supset_{\mathrm{R}}}{\mathrm{A}\supset\mathrm{C}:1\vdash(\mathrm{B}\supset\mathrm{C})\supset((\mathrm{A}\vee\mathrm{B})\supset\mathrm{C}):1}\supset_{\mathrm{R}}}{\vdash(\mathrm{A}\supset\mathrm{C})\supset((\mathrm{B}\supset\mathrm{C})\supset((\mathrm{A}\vee\mathrm{B})\supset\mathrm{C})):\emptyset}\supset_{\mathrm{R}}$$

Axioms IL_7, IL_8, IL_{10}:

$$\dfrac{\dfrac{\dfrac{}{\mathrm{A}:1\vdash\mathrm{A}:1,\mathrm{B}:1}\,\mathrm{id}}{\mathrm{A}:1\vdash\mathrm{A}\vee\mathrm{B}:1}\vee_{\mathrm{R}}}{\vdash\mathrm{A}\supset(\mathrm{A}\vee\mathrm{B}):\emptyset}\supset_{\mathrm{R}} \qquad \dfrac{\dfrac{\dfrac{}{\mathrm{B}:1\vdash\mathrm{A}:1,\mathrm{B}:1}\,\mathrm{id}}{\mathrm{B}:1\vdash\mathrm{A}\vee\mathrm{B}:1}\vee_{\mathrm{R}}}{\vdash\mathrm{B}\supset(\mathrm{A}\vee\mathrm{B}):\emptyset}\supset_{\mathrm{R}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{A}:2\vdash\mathrm{A}:2}\,\mathrm{id}\quad\dfrac{}{\bot:12\vdash\mathrm{B}:12}\,\bot_{\mathrm{L}}}{\neg\mathrm{A}:1,\mathrm{A}:2\vdash\mathrm{B}:12}\supset_{\mathrm{L}}}{\neg\mathrm{A}:1\vdash\mathrm{A}\supset\mathrm{B}:1}\supset_{\mathrm{R}}}{\vdash\neg\mathrm{A}\supset(\mathrm{A}\supset\mathrm{B}):\emptyset}\supset_{\mathrm{R}}$$

$$\frac{\Gamma \vdash \Delta, C\sigma : y}{\Gamma(A \approx B : x) \vdash \Delta(C : y)} \approx_{LR} (x \subseteq y, A \neq B) \qquad \frac{}{\Gamma \vdash \Delta(A \approx A : x)} \approx_R$$

$$\sigma = [B \mapsto A] \text{ if } |A| \leqslant |B| \text{ and } [A \mapsto B] \text{ otherwise.}$$

Figure 7 $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ "Special" Identity Rules.

Rule MP: We use admissibility of weakening, which is stated and proved in the paper for $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ in Lemma 29, but which also holds for $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ with a similar proof.

$$\dfrac{\dfrac{\vdash A:\emptyset}{\vdash B:\emptyset, A:\emptyset} \qquad \dfrac{\dfrac{\vdash A \supset B:\emptyset}{A:\emptyset \vdash B:\emptyset, A \supset B:\emptyset} \qquad \dfrac{\dfrac{}{A:\emptyset \vdash A:\emptyset}\,\text{id} \qquad \dfrac{}{B:\emptyset \vdash B:\emptyset}\,\text{id}}{A \supset B:\emptyset, A:\emptyset \vdash B:\emptyset}\supset_L}{A:\emptyset \vdash B:\emptyset}\,\text{cut}}{\vdash B:\emptyset}\,\text{cut}$$

The other cases are similar. ◀

▶ **Theorem 22** ($\mathsf{H}_{\mathsf{ISCI}}$ completeness). *If* $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$ *then* $\vdash_{\mathsf{L}^{1ec}_{\mathsf{ISCI}}} A$.

Proof. A direct consequence of Lemma 20 and Lemma 21. ◀

▶ **Theorem 23** (Beth completeness). *If* $\vDash A$ *then* $\vdash_{\mathsf{L}^{1ec}_{\mathsf{ISCI}}} A$.

Proof. If $\vDash A$ then Theorem 11 yields $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$, which by Theorem 22 implies $\vdash_{\mathsf{L}^{1ec}_{\mathsf{ISCI}}} A$. ◀

5 The Labelled Calculus $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$

$\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ is not very interesting from the point of view of termination as it lacks the subformula property. Indeed, even if we eliminate the cut rule from $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$, we can still introduce infinitely many subformulas using the identity rule $\approx_{L1}$. Moreover, defining the size $|A|$ of a formula A as the number of its connectives, it is easy to see that the identity rules $\approx_{L4}$ and $\approx_{L4'}$ introduce in their single premiss an active formula the size of which is greater than the size of the principal formula in their conclusion.

As a first step toward termination we define $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ as the variant of $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ in which all of the identity rules of Figure 4 are replaced with the identity rules of Figure 7. Depending on the size of A and B, the rule $\approx_{LR}$ simultaneously replaces all occurrences of the formula B in C with the formula A whenever $|A| \leqslant |B|$ and A is not syntactically equal to B.

5.1 Soundness and Completeness

▶ **Theorem 24** (Soundness). *If* $\vdash_{\mathsf{L}^{2ec}_{\mathsf{ISCI}}} A$ *then* $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$.

Proof. A corollary of Theorems 33, 34 and 40. ◀

▶ **Theorem 25** ($\mathsf{H}_{\mathsf{ISCI}}$ completeness). *If* $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$ *then* $\vdash_{\mathsf{L}^{2ec}_{\mathsf{ISCI}}} A$.

Proof. Similar to the proof of Theorem 22. ◀

▶ **Theorem 26** (Beth completeness). *If* $\vDash A$ *then* $\vdash_{\mathsf{L}^{2ec}_{\mathsf{ISCI}}} A$.

Proof. If $\vDash A$ then Theorem 11 yields $\vdash_{\mathsf{H}_{\mathsf{ISCI}}} A$, which by Theorem 25 implies $\vdash_{\mathsf{L}^{2ec}_{\mathsf{ISCI}}} A$. ◀

▶ **Theorem 27** ($\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ to $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$). *If* Π *is an* $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ *proof of* A*, then there exists a translation* $\mathrm{t}(\Pi)$ *of* Π *which is an* $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ *proof of* A*.*

Proof. The proof is by induction on the height of $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ proofs. Since $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ only differs from $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ on the identity rules, the base cases for axioms are immediate and we only need to show that $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$ can simulate $\mathsf{L}^{1ec}_{\mathsf{ISCI}}$ identity rules. We assume without loss of generality that $|\mathrm{A}| \leqslant |\mathrm{B}|$ and $|\mathrm{C}| \leqslant |\mathrm{D}|$. Moreover, in the translated proofs below, the occurrences of $\approx_{\mathrm{LR}}$ only actually exist when the formulas on both sides of the principal identity connective are not syntactically equal.

Case $\approx_{\mathrm{L1}}$:

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A}\approx \mathrm{A} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma\vdash\Delta}\approx_{\mathrm{L1}} \quad\rightsquigarrow\quad \dfrac{\dfrac{}{\Gamma\vdash\Delta, \mathrm{A}\approx \mathrm{A}:\mathrm{x}}\approx_{\mathrm{R}} \qquad \begin{array}{c} \mathrm{t}(\Pi_1)\text{ from I.H.}\\ \hdashline \Gamma, \mathrm{A}\approx \mathrm{A}:\mathrm{x}\vdash\Delta\end{array}}{\Gamma\vdash\Delta}\text{cut}$$

Case $\approx_{\mathrm{L2}}$:

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \neg\mathrm{A}\approx \neg\mathrm{B} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta}\approx_{\mathrm{L2}}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta, \neg\mathrm{A}\approx \neg\mathrm{A}:\mathrm{x}}\approx_{\mathrm{R}}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta, \neg\mathrm{A}\approx \neg\mathrm{B}:\mathrm{x}}\approx_{\mathrm{LR}}(\mathrm{A}\neq\mathrm{B}) \qquad \begin{array}{c} \mathrm{t}(\Pi_1)\text{ from I.H.}\\ \hdashline \Gamma, \neg\mathrm{A}\approx \neg\mathrm{B}:\mathrm{x}\vdash\Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta}\text{cut}$$

Case $\approx_{\mathrm{L3}}$:

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{B}\supset \mathrm{A} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta}\approx_{\mathrm{L3}} \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x}), \mathrm{A}:\mathrm{a}\vdash\Delta, \mathrm{A}:\mathrm{xa}}\text{id}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta, \mathrm{A}\supset \mathrm{A}:\mathrm{x}}\supset_{\mathrm{R}}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta, \mathrm{B}\supset \mathrm{A}:\mathrm{x}}\approx_{\mathrm{LR}}(\mathrm{A}\neq\mathrm{B}) \qquad \begin{array}{c} \mathrm{t}(\Pi_1)\text{ from I.H.}\\ \hdashline \Gamma, \mathrm{B}\supset \mathrm{A}:\mathrm{x}\vdash\Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta}\text{cut}$$

Case $\approx_{\mathrm{L4}}$:

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A}\otimes\mathrm{C}\approx \mathrm{B}\otimes\mathrm{D} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x}, \mathrm{C}\approx\mathrm{D}:\mathrm{x})\vdash\Delta}\approx_{\mathrm{L4}}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x}, \mathrm{C}\approx\mathrm{D}:\mathrm{x})\vdash\Delta, \mathrm{A}\otimes\mathrm{C}\approx \mathrm{A}\otimes\mathrm{C}:\mathrm{x}}\approx_{\mathrm{R}}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x}, \mathrm{C}\approx\mathrm{D}:\mathrm{x})\vdash\Delta, \mathrm{A}\otimes\mathrm{C}\approx \mathrm{A}\otimes\mathrm{D}:\mathrm{x}}\approx_{\mathrm{LR}}(\mathrm{C}\neq\mathrm{D})}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x}, \mathrm{C}\approx\mathrm{D}:\mathrm{x})\vdash\Delta, \mathrm{A}\otimes\mathrm{C}\approx \mathrm{B}\otimes\mathrm{D}:\mathrm{x}}\approx_{\mathrm{LR}}(\mathrm{A}\neq\mathrm{B}) \qquad \begin{array}{c} \mathrm{t}(\Pi_1)\text{ from I.H.}\\ \hdashline \Gamma, \mathrm{A}\otimes\mathrm{C}\approx \mathrm{B}\otimes\mathrm{D}:\mathrm{x}\vdash\Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x}, \mathrm{C}\approx\mathrm{D}:\mathrm{x})\vdash\Delta}\text{cut}$$

Case $\approx_{\mathrm{L4'}}$:

$$\dfrac{\begin{array}{c}\Pi_1\\ \hdashline \Gamma, \mathrm{A}\otimes\mathrm{A}\approx \mathrm{B}\otimes\mathrm{B} : \mathrm{x} \vdash \Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta}\approx_{\mathrm{L4'}}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta, \mathrm{A}\otimes\mathrm{A}\approx \mathrm{A}\otimes\mathrm{A}:\mathrm{x}}\approx_{\mathrm{R}}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta, \mathrm{A}\otimes\mathrm{A}\approx \mathrm{B}\otimes\mathrm{B}:\mathrm{x}}\approx_{\mathrm{LR}}(\mathrm{A}\neq\mathrm{B}) \qquad \begin{array}{c} \mathrm{t}(\Pi_1)\text{ from I.H.}\\ \hdashline \Gamma, \mathrm{A}\otimes\mathrm{A}\approx \mathrm{B}\otimes\mathrm{B}:\mathrm{x}\vdash\Delta\end{array}}{\Gamma(\mathrm{A}\approx\mathrm{B}:\mathrm{x})\vdash\Delta}\text{cut}$$

◀

5.2 Cut Elimination in $\mathsf{L}^{\mathsf{2ec}}_{\mathsf{ISCI}}$

We now eliminate the cut rule from $\mathsf{L}^{\mathsf{2ec}}_{\mathsf{ISCI}}$. The cut free version of $\mathsf{L}^{\mathsf{2ec}}_{\mathsf{ISCI}}$ is denoted $\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}$ (the c superscript is dropped). Let us write $h(\Pi)$ for the height of a proof Π defined as the length of its longest branch. For a proof system S and a formula or labelled sequent s, the notation $\vDash^{\mathrm{n}}_{\mathsf{S}} s$ means that s is derivable in S with a proof Π such that $h(\Pi) \leqslant n$ ($n \in \mathbb{N}$).

Label substitution is defined as follows: if $\mathrm{y} \subseteq \mathrm{x}$ then $\mathrm{x}[\mathrm{u}/\mathrm{y}] = (\mathrm{x} - \mathrm{y}) \cup \mathrm{u}$, otherwise $\mathrm{x}[\mathrm{u}/\mathrm{y}] = \mathrm{x}$. For instance, $374[\emptyset/7] = \{3,7,4\} - \{7\} \cup \emptyset = \{3,4\} = 34$. Label substitutions straightforwardly extend to labelled formulas and labelled sequents.

▶ **Lemma 28.** *Let $s = \Gamma \vdash \Delta$. If $\vDash^{\mathrm{n}}_{\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}} s$ then $\vDash^{\mathrm{n}}_{\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}} s[\mathrm{u}/\mathrm{c}]$, where $\mathrm{c} \in \mathbf{L}^1$ or $\mathrm{c} = \emptyset$.*

Proof. By induction on the height h of the proof of $\Gamma \vdash \Delta$. The base case $h = 0$ is when s is the conclusion of an axiom.

Case id: s is of the form $\Gamma(\mathrm{A} : \mathrm{x}) \vdash \Delta(\mathrm{A} : \mathrm{y})$ with $\mathrm{x} \subseteq \mathrm{y}$. If $\mathrm{c} \not\subseteq \mathrm{y}$ then $s[\mathrm{u}/\mathrm{c}] = s$ and the result is immediate. Otherwise, $\mathrm{c} \subseteq \mathrm{y}$ and $\mathrm{y} = (\mathrm{y} - \mathrm{c}) \cup \mathrm{c}$. Since $\mathrm{x} \subseteq \mathrm{y}$, $\mathrm{y} = (\mathrm{y} - \mathrm{x}) \cup \mathrm{x}$ implies $\mathrm{y} = (\mathrm{y} - (\mathrm{x} \cup \mathrm{c})) \cup (\mathrm{x} - \mathrm{c}) \cup \mathrm{c}$. Hence, $\mathrm{y}[\mathrm{u}/\mathrm{c}] = (\mathrm{y} - (\mathrm{x} \cup \mathrm{c})) \cup (\mathrm{x} - \mathrm{c}) \cup \mathrm{u}$. We then show that $s[\mathrm{u}/\mathrm{c}]$ remains an axiom for A by showing that $\mathrm{x}[\mathrm{u}/\mathrm{c}] \subseteq \mathrm{y}[\mathrm{u}/\mathrm{c}]$. If $\mathrm{c} \not\subseteq \mathrm{x}$ then $\mathrm{x}[\mathrm{u}/\mathrm{c}] = \mathrm{x} = \mathrm{x} - \mathrm{c}$ and $\mathrm{x} - \mathrm{c} \subseteq \mathrm{y}[\mathrm{u}/\mathrm{c}]$. If $\mathrm{c} \subseteq \mathrm{x}$ then $\mathrm{x}[\mathrm{u}/\mathrm{c}] = ((\mathrm{x} - \mathrm{c}) \cup \mathrm{c})[\mathrm{u}/\mathrm{c}] = (\mathrm{x} - \mathrm{c}) \cup \mathrm{u}$ and $(\mathrm{x} - \mathrm{c}) \cup \mathrm{u} \subseteq \mathrm{y}[\mathrm{u}/\mathrm{c}]$.

Case $\perp_{\mathrm{L}}$: Similar to Case id.

For the inductive case $h = n + 1$, let r be the last rule applied (which has s as a conclusion). If r requires the introduction of eigenvariables we proceed as follows.

Case $\vee_{\mathrm{L}}$: s is of the form $\Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{x} \vdash \Delta(\mathrm{C} : \mathrm{y})$ and is obtained by the rule $\vee_{\mathrm{L}}$ from the premise $s_1 = \Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{x}, \mathrm{A} : \mathrm{xa} \vdash \Delta, \mathrm{C} : \mathrm{ya}$ and $s_2 = \Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{x}, \mathrm{B} : \mathrm{xb} \vdash \Delta, \mathrm{C} : \mathrm{yb}$, where $\mathrm{a}, \mathrm{b} \not\subseteq \Gamma \cup \Delta$, which have proofs Π_1, Π_2 such that $h(\Pi_1), h(\Pi_2) \leqslant n$. We choose two labels $\mathrm{a}' \neq \mathrm{b}'$ such that $\mathrm{a}', \mathrm{b}' \not\subseteq \Gamma \cup \Delta$ and $\mathrm{a}', \mathrm{b}' \not\subseteq \mathrm{xyuabc}$. By I.H. on Π_1 and Π_2 with substitutions $[\mathrm{a}'/\mathrm{a}]$ and $[\mathrm{b}'/\mathrm{b}]$ we get proofs Π_1' and Π_2' of $\Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{x}, \mathrm{A} : \mathrm{xa}' \vdash \Delta, \mathrm{C} : \mathrm{ya}'$ and $\Gamma, \mathrm{A} \vee \mathrm{B} : \mathrm{x}, \mathrm{B} : \mathrm{xb}' \vdash \Delta, \mathrm{C} : \mathrm{yb}'$. Then, by I.H. on Π_1' and Π_2' with substitution $[\mathrm{u}/\mathrm{c}]$, we get proofs Π_1'' and Π_2'' of $\Gamma[\mathrm{u}/\mathrm{c}], \mathrm{A} \vee \mathrm{B} : \mathrm{x}[\mathrm{u}/\mathrm{c}], \mathrm{A} : \mathrm{x}[\mathrm{u}/\mathrm{c}]\mathrm{a}' \vdash \Delta[\mathrm{u}/\mathrm{c}], \mathrm{C} : \mathrm{y}[\mathrm{u}/\mathrm{c}]\mathrm{a}'$ and $\Gamma[\mathrm{u}/\mathrm{c}], \mathrm{A} \vee \mathrm{B} : \mathrm{x}[\mathrm{u}/\mathrm{c}], \mathrm{B} : \mathrm{x}[\mathrm{u}/\mathrm{c}]\mathrm{b}' \vdash \Delta[\mathrm{u}/\mathrm{c}], \mathrm{C} : \mathrm{y}[\mathrm{u}/\mathrm{c}]\mathrm{b}'$ from which we infer the conclusion $\Gamma[\mathrm{u}/\mathrm{c}], \mathrm{A} \vee \mathrm{B} : \mathrm{x}[\mathrm{u}/\mathrm{c}] \vdash \Delta[\mathrm{u}/\mathrm{c}]$ by the rule $\vee_{\mathrm{L}}$.

Case $\supset_{\mathrm{R}}$: Similar to Case $\vee_{\mathrm{L}}$.

If r does not require eigenvariables, we apply the I.H. on all of the premise of r since they have proofs of height strictly less than $n + 1$ and we conclude $s[\mathrm{u}/\mathrm{c}]$ by reapplying r. ◀

▶ **Lemma 29.** *If $\vDash^{\mathrm{n}}_{\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}} \Gamma \vdash \Delta$ then $\vDash^{\mathrm{n}}_{\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}} \Gamma, \Gamma' \vdash \Delta, \Delta'$.*

Proof. By induction on the height h of a proof Π of $\Gamma \vdash \Delta$. For $h = 0$, it is immediate that when $\Gamma \vdash \Delta$ is an axiom, then so is $\Gamma, \Gamma' \vdash \Delta, \Delta'$. For $h = n + 1$, let r be the last rule applied in Π. If r is not $\supset_{\mathrm{R}}$ or $\vee_{\mathrm{L}}$, we apply the I.H. on the premise of r and conclude by reapplying r. Otherwise, we first use Lemma 28 to replace the eigenvariables in all of the premise of r with variables not occurring in $\Gamma \cup \Gamma' \cup \Delta \cup \Delta'$ and then apply the I.H. to the modified premise before concluding with a new instance of r. ◀

▶ **Lemma 30.** *All $\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}$ rules are height preserving invertible.*

Proof. A k-ary proof rule r with premise $s_1 \ldots s_k$ and conclusion s is height preserving invertible if $\vDash^{\mathrm{n}}_{\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}} s$ implies $\vDash^{\mathrm{n}}_{\mathsf{L}^{\mathsf{2e}}_{\mathsf{ISCI}}} s_i$ for all $1 \leqslant i \leqslant k$. Let $s = \Gamma \vdash \Delta$. Since proof rules are non-destructive, each premiss s_i can be represented as $\Gamma, \Gamma_i \vdash \Delta, \Delta_i$, where Γ_i, Δ_i are the active parts of r. If $k = 0$ (for axioms), the result is immediate. Otherwise, if we have a proof Π of s, then by Lemma 29, we have a proof Π_i of s_i such that $h(\Pi_i) \leqslant h(\Pi)$. ◀

▶ **Lemma 31.** *If* $\vdash^{n}_{\mathsf{L}^{2e}_{\mathsf{ISCI}}} \Gamma(\mathrm{A}:\mathrm{x},\mathrm{A}:\mathrm{y}) \vdash \Delta$ *and* $\mathrm{x} \subseteq \mathrm{y}$ *then* $\vdash^{n}_{\mathsf{L}^{2e}_{\mathsf{ISCI}}} \Gamma(\mathrm{A}:\mathrm{x}) \vdash \Delta$*. Similarly, if* $\vdash^{n}_{\mathsf{L}^{2e}_{\mathsf{ISCI}}} \Gamma \vdash \Delta(\mathrm{A}:\mathrm{x},\mathrm{A}:\mathrm{y})$ *and* $\mathrm{y} \subseteq \mathrm{x}$ *then* $\vdash^{n}_{\mathsf{L}^{2e}_{\mathsf{ISCI}}} \Gamma \vdash \Delta(\mathrm{A}:\mathrm{x})$*.*

Proof. By induction on the height of the proofs, using Lemma 30. ◀

▶ **Lemma 32.** *If Π is a proof of either $\Gamma, \mathrm{A}:\mathrm{x} \vdash \Delta$, or $\Gamma \vdash \Delta, \mathrm{A}:\mathrm{x}$, in which $\mathrm{A}:\mathrm{x}$ is never principal for any sequent in Π, then there exists a proof Π' of $\Gamma \vdash \Delta$ such that $h(\Pi') \leqslant h(\Pi)$.*

Proof. By induction on the height of the proof Π deleting all occurrences of $\mathrm{A}:\mathrm{x}$. ◀

▶ **Theorem 33** (Cut elimination)**.** *The* cut *rule is admissible in* $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$.

Proof. Our proof follows the pattern given in [10] or in [8] for Boolean BI. We define the cut rank of (an instance) of the cut rule as the pair $(|\mathrm{C}|, h(\Pi_1) + h(\Pi_2))$, where C is the cut formula and $\Pi_{i\in\{1,2\}}$ is the proof whose conclusion is the sequent s_i corresponding to the i-th premiss above the cut. For the base case we consider that one of the premiss of the cut has a proof of height 0. For the inductive step, we distinguish three cases: $\mathrm{C}:\mathrm{z}$ is not principal in s_1, $\mathrm{C}:\mathrm{z}$ is principal only in s_1, $\mathrm{C}:\mathrm{z}$ is principal in both s_1 and s_2. We only do a few illustrative or difficult cases. More cases are given in the appendix (see Theorem 43).

Cases $\mathbf{n_1.\bot_L}$, $\mathbf{p_1.\bot_L}$: s_1 is the conclusion of $\bot_\mathrm{L}$. If $\mathrm{C}:\mathrm{z}$ is not principal in s_1 (Case $\mathrm{n_1.\bot_L}$), then let $\mathrm{A}:\mathrm{y}$ denote the principal formula of $\bot_\mathrm{L}$ in s_1. By the side condition of $\bot_\mathrm{L}$, we have $\mathrm{x} \subseteq \mathrm{y}$. If $\mathrm{C}:\mathrm{z}$ is principal in s_1 (Case $\mathrm{p_1.\bot_L}$), then by the connectedness property, $\bot:\mathrm{x} \in \Gamma$ implies $\mathrm{A}:\mathrm{y} \in \Delta$ for some A and y such that $\mathrm{x} \subseteq \mathrm{y}$. In both cases, we eliminate the cut rule as follows:

$$\dfrac{\dfrac{}{\Gamma(\bot:\mathrm{x}) \vdash \Delta(\mathrm{A}:\mathrm{y}), \mathrm{C}:\mathrm{z}}\,\bot_\mathrm{L} \qquad \dfrac{\Pi_2}{\mathrm{C}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{A}:\mathrm{y})}}{\Gamma(\bot:\mathrm{x}) \vdash \Delta(\mathrm{A}:\mathrm{y})}\,\text{cut} \quad \rightsquigarrow \quad \dfrac{}{\Gamma(\bot:\mathrm{x}) \vdash \Delta(\mathrm{A}:\mathrm{y})}\,\bot_\mathrm{L}$$

Case $\mathbf{p_1.\vee_R p_2.\vee_L}$: $\mathrm{C}:\mathrm{z}$ is principal in both s_1 and s_2, C has the form $\mathrm{A} \vee \mathrm{B}$, $\mathrm{z} \subseteq \mathrm{y}$.

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}}}{\Gamma \vdash \Delta, \mathrm{A}\vee\mathrm{B}:\mathrm{z}}\,\vee_\mathrm{R} \qquad \dfrac{\dfrac{\Pi_2^1}{\mathrm{A}:\mathrm{za}, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{D}:\mathrm{ya}} \quad \dfrac{\Pi_2^2}{\mathrm{B}:\mathrm{zb}, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{D}:\mathrm{yb}}}{\mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y})}\,\vee_\mathrm{L}}{\Gamma \vdash \Delta}\,\text{cut}$$

We first use a cut on $\mathrm{A} \vee \mathrm{B}:\mathrm{z}$ of strictly lower cut height to get the following proof:

$$\Pi_3 \left\{ \dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}} \qquad \dfrac{\Pi_2^{1'} \text{ from Lemma 29} \quad \Pi_2^{2'} \text{ from Lemma 29}}{\mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}}\,\vee_\mathrm{L}}{\Gamma \vdash \Delta, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}}\,\text{cut} \right.$$

We apply Lemma 28 on Π_2^1 with $[\emptyset/\mathrm{a}]$ and on Π_2^2 with $[\emptyset/\mathrm{b}]$ to get:

$$\Pi_4 \left\{ \dfrac{\Pi_2^1[\emptyset/\mathrm{a}]}{\mathrm{A}:\mathrm{z}, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y}), \mathrm{D}:\mathrm{y}} \right. \qquad \Pi_5 \left\{ \dfrac{\Pi_2^2[\emptyset/\mathrm{b}]}{\mathrm{B}:\mathrm{z}, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y}), \mathrm{D}:\mathrm{y}} \right.$$

We apply Lemma 31 on Π_4 and Π_5 to get:

$$\dfrac{\Pi_4'}{\mathrm{A}:\mathrm{z}, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y})} \qquad \dfrac{\Pi_5'}{\mathrm{B}:\mathrm{z}, \mathrm{A}\vee\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y})}$$

We use two cuts on $\mathrm{A} \vee \mathrm{B}:\mathrm{z}$ of strictly lower cut height to get Π_6, Π_7, which are finally combined with Π_3 to obtain a proof with two cuts on strictly smaller formulas.

$$
\Pi_6 \left\{ \dfrac{\dfrac{\dfrac{\Pi_1' \text{ from Lemma 29}}{\mathrm{A}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}}}{\mathrm{A}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B}:\mathrm{z}} \vee_{\mathrm{R}} \qquad \dfrac{\Pi_4'}{\mathrm{A}:\mathrm{z}, \mathrm{A} \vee \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y})}}{\mathrm{A}:\mathrm{z}, \Gamma \vdash \Delta} \text{cut} \right.
$$

$$
\Pi_7 \left\{ \dfrac{\dfrac{\dfrac{\Pi_1'' \text{ from Lemma 29}}{\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}}}{\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A} \vee \mathrm{B}:\mathrm{z}} \vee_{\mathrm{R}} \qquad \dfrac{\Pi_5'}{\mathrm{B}:\mathrm{z}, \mathrm{A} \vee \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta(\mathrm{D}:\mathrm{y})}}{\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta} \text{cut} \right.
$$

$$
\dfrac{\dfrac{\dfrac{\Pi_3}{\Gamma \vdash \Delta, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}} \qquad \dfrac{\Pi_7' \text{ from Lemma 29}}{\mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A}:\mathrm{z}}}{\Gamma \vdash \Delta, \mathrm{A}:\mathrm{z}} \text{cut} \qquad \dfrac{\Pi_6}{\mathrm{A}:\mathrm{z}, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta} \text{cut}
$$

◀

6 Liberalizing $\mathsf{L}^{2e}_{\mathsf{ISCI}}$ and Decidability

Even restricted to the simple case of intuitionistic logic, the termination of a labelled proof system is not straightforward. A problem is the rules $\supset_{\mathrm{L}}$ and $\vee_{\mathrm{R}}$ (called *β-rules*) can be used several times as long as there are yet untried labels satisfying their requirements. Combined with the fact that the rules $\supset_{\mathrm{R}}$ and $\vee_{\mathrm{L}}$ (called *α-rules*) require the systematic introduction of fresh singleton labels, the proof-search process might degenerate into the construction of infinite branches when there are α-formulas in the scope of β-formulas.

Let us assume a globally fixed[1] total injective *indexing function* $\mathrm{i} : \mathbf{F} \times \mathbb{N} \to \mathbf{L}^1$ that given a formula A and an index n maps the pair (A, n) to the singleton label denoted $\mathrm{i}^{\mathrm{n}}_{\mathrm{A}}$. We define $\mathsf{L}^{2}_{\mathsf{ISCI}}$ as the labelled proof system obtained from $\mathsf{L}^{2e}_{\mathsf{ISCI}}$ in which the eigenvariable requirements are dropped by replacing the α-rules of Figure 4 with the following ones:

$$
\dfrac{\Gamma, \mathrm{A}:\mathrm{i}^1_{\mathrm{A}\supset\mathrm{B}} \vdash \Delta, \mathrm{B}:\mathrm{x} \cup \mathrm{i}^1_{\mathrm{A}\supset\mathrm{B}}}{\Gamma \vdash \Delta(\mathrm{A} \supset \mathrm{B}:\mathrm{x})} \supset_{\mathrm{R}}
$$

$$
\dfrac{\Gamma, \mathrm{A}:\mathrm{x} \cup \mathrm{i}^1_{\mathrm{A}\vee\mathrm{B}} \vdash \Delta, \mathrm{C}:\mathrm{y} \cup \mathrm{i}^1_{\mathrm{A}\vee\mathrm{B}} \qquad \Gamma, \mathrm{B}:\mathrm{x} \cup \mathrm{i}^2_{\mathrm{A}\vee\mathrm{B}} \vdash \Delta, \mathrm{C}:\mathrm{y} \cup \mathrm{i}^2_{\mathrm{A}\vee\mathrm{B}}}{\Gamma(\mathrm{A} \vee \mathrm{B}:\mathrm{x}) \vdash \Delta(\mathrm{C}:\mathrm{y})} \vee_{\mathrm{L}}(\mathrm{x} \subseteq \mathrm{y})
$$

▶ **Theorem 34.** *If* $\Vdash^{\mathrm{n}}_{\mathsf{L}^{2e}_{\mathsf{ISCI}}} \mathrm{A}$ *then* $\Vdash^{\mathrm{n}}_{\mathsf{L}^{2}_{\mathsf{ISCI}}} \mathrm{A}$.

Proof. By induction on the height of the $\mathsf{L}^{2e}_{\mathsf{ISCI}}$ proof of A. ◀

6.1 Validity of the Replacement Law for ISCI

▶ **Lemma 35.** *Let* $\mathrm{A}, \mathrm{B}, \mathrm{C}$ *be formulas and let* $\mathrm{C}[\mathrm{A} \mapsto \mathrm{B}]$ *be the formula obtained from* C *by simultaneously replacing all occurrences of* A *in* C *with* B*. Then, the formula* $(\mathrm{A} \approx \mathrm{B}) \supset (\mathrm{C} \approx \mathrm{C}[\mathrm{A} \mapsto \mathrm{B}])$ *is valid in Beth semantics.*

Proof. Let $\mathcal{M}$ be a Beth model and m be a world such that $m \Vdash \mathrm{A} \approx \mathrm{B}$. If C does not contain any occurrence of A then $\mathrm{C}[\mathrm{A} \mapsto \mathrm{B}] = \mathrm{C}$ and condition $\mathcal{M}_{\approx_1}$ of Definition 3 then implies $m \Vdash \mathrm{C} \approx \mathrm{C}$. If C contains at least one occurrence of A, let $d(\mathrm{F}, \mathrm{C})$ denote the depth at which a subformula F is nested in C. We proceed by induction on the depth

[1] The use of a globally fixed indexing function is just for technical convenience. One could also associate each derivation with a partial indexing function defined only on the formulas occurring in that derivation.

$d = min\{\, d(\mathrm{A},\mathrm{C}) \mid \mathrm{A} \in \mathrm{C}\,\}$ of the least deeply nested occurrence(s) of A in C (e.g., if $\mathrm{C} = (\mathrm{A} \supset \mathrm{D}) \wedge ((\mathrm{A} \vee \mathrm{B}) \approx \mathrm{A})$ then $d = 2$). The base case is when $d = 0$, i.e., when $\mathrm{C} = \mathrm{A}$. Thus, $\mathrm{C}[\mathrm{A} \mapsto \mathrm{B}] = \mathrm{B}$ and we have $m \Vdash \mathrm{A} \approx \mathrm{B}$ by assumption. For the inductive case, C is of the form $\mathrm{C}_1 \otimes \mathrm{C}_2$, where $\otimes$ is a binary connective. We assume as an I.H. that the property holds for all formulas C and all d' such that $0 \leq d' < d$. By definition of a substitution, $(\mathrm{C}_1 \otimes \mathrm{C}_2)[\mathrm{A} \mapsto \mathrm{B}] = \mathrm{C}_1[\mathrm{A} \mapsto \mathrm{B}] \otimes \mathrm{C}_2[\mathrm{A} \mapsto \mathrm{B}]$. For $\mathrm{C_i} \in \{\,\mathrm{C}_1, \mathrm{C}_2\,\}$, if A does not occur in $\mathrm{C_i}$ then $\mathrm{C_i}[\mathrm{A} \mapsto \mathrm{B}] = \mathrm{C_i}$. Thus, $m \Vdash \mathrm{C_i} \approx \mathrm{C_i}[\mathrm{A} \mapsto \mathrm{B}]$ by condition $\mathcal{M}_{\approx_1}$ of Definition 3. Otherwise, if A occurs in $\mathrm{C_i}$ then $m \Vdash \mathrm{C_i} \approx \mathrm{C_i}[\mathrm{A} \mapsto \mathrm{B}]$ by I.H. Hence, by condition $\mathcal{M}_{\approx_4}$ of Definition 3, we get $(\mathrm{C}_1 \otimes \mathrm{C}_2) \approx (\mathrm{C}_1[\mathrm{A} \mapsto \mathrm{B}] \otimes \mathrm{C}_2[\mathrm{A} \mapsto \mathrm{B}])$. ◀

▶ **Lemma 36.** *If $m \Vdash \mathrm{A} \approx \mathrm{B}$ then for all formulas C, $m \Vdash \mathrm{C}$ iff $m \Vdash \mathrm{C}[\mathrm{A} \mapsto \mathrm{B}]$.*

Proof. By Lemma 35, if $m \Vdash \mathrm{A} \approx \mathrm{B}$ then $m \Vdash \mathrm{C} \approx \mathrm{D}$, where $\mathrm{D} = \mathrm{C}[\mathrm{A} \mapsto \mathrm{B}]$. By symmetry of $\approx$, $m \Vdash \mathrm{C} \approx \mathrm{D}$ implies $m \Vdash \mathrm{D} \approx \mathrm{C}$. Therefore, by condition $\mathcal{M}_{\approx_3}$ of Definition 3, we get both $m \Vdash \mathrm{D} \supset \mathrm{C}$ and $m \Vdash \mathrm{C} \supset \mathrm{D}$. Consequently, if $m \Vdash \mathrm{C}$, then $m \Vdash \mathrm{C} \supset \mathrm{D}$ implies $m \Vdash \mathrm{D}$. Conversely, if $m \Vdash \mathrm{D}$, then $m \Vdash \mathrm{D} \supset \mathrm{C}$ implies $m \Vdash \mathrm{C}$. ◀

6.2 Liberalized Soundness

To show that $\mathsf{L}^2_{\mathsf{ISCI}}$ is sound even in the absence of the eigenvariable condition, we take advantage of the completeness of ISCI w.r.t. regular Beth models (Theorem 14) by semantically interpreting (realizing) the unique index i_{A} of a formula A by an A-minimal world.

▶ **Definition 37** (Realization). *Let $\mathcal{M}$ be a regular Beth model. Let $s = \Gamma \vdash \Delta$ be a labelled sequent. A* realization of *s* in $\mathcal{M}$ *is a partial function $\rho : \mathbf{L} \to \mathbf{M}$ such that:*

- $\rho(\emptyset) = \omega$, $\rho(\mathbb{N}) = \pi$, $\rho(\mathrm{i}^{\mathrm{n}\in\{1,2\}}_{\mathrm{A}_1\otimes\mathrm{A}_2}) = m_{\mathrm{A}_\mathrm{n}}$ *for all* $\mathrm{i}^{\mathrm{n}\in\{1,2\}}_{\mathrm{A}_1\otimes\mathrm{A}_2} \subseteq s$ *and* $\rho(\mathrm{x} \cup \mathrm{y}) = \rho(\mathrm{x}) \sqcup \rho(\mathrm{y})$,
- *for all* $\mathrm{x}, \mathrm{y} \subseteq \Gamma$, *if* $\mathrm{x} \subseteq \mathrm{y}$ *then* $\rho(\mathrm{x}) \leqslant \rho(\mathrm{y})$ *holds in* $\mathcal{M}$,
- *for all* $\mathrm{A} : \mathrm{x}$ *in* Γ, $\rho(\mathrm{x}) \Vdash \mathrm{A}$ *and for all* $\mathrm{A} : \mathrm{x}$ *in* Δ, $\rho(\mathrm{x}) \nVdash \mathrm{A}$.

A sequent s is realizable in $\mathcal{M}$ *if there exists a realization of s in $\mathcal{M}$, and* realizable *if it is realizable in some regular Beth model $\mathcal{M}$.*

▶ **Lemma 38.** *If the sequent $s = \Gamma \vdash \Delta$ is an initial sequent in an $\mathsf{L}^2_{\mathsf{ISCI}}$-proof, i.e., a leaf sequent that is the conclusion of a zero-premiss rule, then s is not realizable.*

Proof. If s is realizable, then we have a realization ρ of s in some regular Beth model $\mathcal{M}$. We proceed by case analysis on the zero-premiss rule of which s is the conclusion.

Case id: $s = \Gamma, \mathrm{A} : \mathrm{x} \vdash \Delta, \mathrm{A} : \mathrm{y}$ with $\mathrm{x} \subseteq \mathrm{y}$, which implies the contradiction $\rho(\mathrm{y}) \nVdash \mathrm{A}$ since $\rho(\mathrm{x}) \leqslant \rho(\mathrm{y})$ and $\rho(\mathrm{x}) \Vdash \mathrm{A}$ imply $\rho(\mathrm{y}) \Vdash \mathrm{A}$ by Kripke monotonicity.

Case $\bot_{\mathrm{L}}$: $s = \Gamma, \bot : \mathrm{x} \vdash \Delta, \mathrm{A} : \mathrm{y}$ with $\mathrm{x} \subseteq \mathrm{y}$, which implies the contradiction $\rho(\mathrm{y}) \nVdash \mathrm{A}$ since $\rho(\mathrm{x}) = \rho(\mathrm{y}) = \pi$ and $\pi \Vdash \mathrm{A}$ for all A.

Case $\approx_{\mathrm{R}}$: $s = \Gamma \vdash \Delta, \mathrm{A} \approx \mathrm{A} : \mathrm{x}$, which implies the contradiction $\rho(\mathrm{x}) \nVdash \mathrm{A} \approx \mathrm{A}$. ◀

▶ **Lemma 39.** *Every proof rule in $\mathsf{L}^2_{\mathsf{ISCI}}$ preserves realizability in regular Beth models.*

Proof. By case analysis of the proof rules of $\mathsf{L}^2_{\mathsf{ISCI}}$. We show that whenever the conclusion of a rule is realizable in some regular model $\mathcal{M}$ for some realization ρ, then at least one of its premise is also realizable in $\mathcal{M}$ for some extension of ρ. We write $s = \Gamma \vdash \Delta$ for the sequent which is the conclusion of the rule and $s_i = \Gamma_i \vdash \Delta_i$ for the i-th premiss (for $i \in \{\,1, 2\,\}$). Since ρ realizes both Γ and Δ in s, ρ also realizes Γ_i and Δ_i in s_i since $\Gamma_i \subseteq \Gamma$ and $\Delta_i \subseteq \Delta$. Therefore, we only need to consider the principal and active parts of each rule.

Case $\vee_{\mathrm{L}}$**:** If ρ realizes $s = \Gamma(\mathrm{A} \vee \mathrm{B} : \mathrm{x}) \vdash \Delta(\mathrm{C} : \mathrm{y})$ in $\mathcal{M}$, then $\rho(\mathrm{x}) \Vdash \mathrm{A} \vee \mathrm{B}$ implies that there exist $n_1, n_2 \in \mathbf{M}$ such that $n_1 \sqcap n_2 \leqslant \rho(\mathrm{x})$, $n_1 \Vdash \mathrm{A}$ and $n_2 \Vdash \mathrm{B}$. Moreover, $\mathrm{a} = \mathrm{i}^1_{\mathrm{A}\vee\mathrm{B}}$ and $\mathrm{b} = \mathrm{i}^2_{\mathrm{A}\vee\mathrm{B}}$. If $\rho(\mathrm{a})$ is already defined then $\rho(\mathrm{a}) = m_{\mathrm{A}}$ by definition. Otherwise, we extend ρ by setting $\rho(\mathrm{a}) = m_{\mathrm{A}}$. We proceed similarly for $\rho(\mathrm{b})$ to get $\rho(\mathrm{b}) = m_{\mathrm{B}}$. Since m_{A} is A-minimal, we get $m_{\mathrm{A}} \leqslant n_1$ and $\rho(\mathrm{x}) \sqcup \rho(\mathrm{a}) \Vdash \mathrm{A}$. Similarly, since m_{B} is B-minimal, we get $m_{\mathrm{B}} \leqslant n_2$ and $\rho(\mathrm{x}) \sqcup \rho(\mathrm{b}) \Vdash \mathrm{B}$. Moreover, $m_{\mathrm{A}} \leqslant n_1$ and $m_{\mathrm{B}} \leqslant n_2$ imply $m_{\mathrm{A}} \sqcap m_{\mathrm{B}} \leqslant n_1 \sqcap n_2$. Thus, $\mathrm{xa} \cap \mathrm{xb} = \mathrm{x}$ implies $(\rho(\mathrm{x}) \sqcup \rho(\mathrm{a})) \sqcap (\rho(\mathrm{x}) \sqcup \rho(\mathrm{b})) = \rho(\mathrm{x})$. Now if both $\rho(\mathrm{y}) \sqcup \rho(\mathrm{a}) \Vdash \mathrm{C}$ and $\rho(\mathrm{y}) \sqcup \rho(\mathrm{b}) \Vdash \mathrm{C}$ then, since $\mathcal{M}$ is a regular Beth model, there exists a C-minimal world m_{C}. Thus, $m_{\mathrm{C}} \leqslant \rho(\mathrm{y}) \sqcup \rho(\mathrm{a})$ and $m_{\mathrm{C}} \leqslant \rho(\mathrm{y}) \sqcup \rho(\mathrm{b})$, which implies $m_{\mathrm{C}} \leqslant (\rho(\mathrm{y}) \sqcup \rho(\mathrm{a})) \sqcap (\rho(\mathrm{y}) \sqcup \rho(\mathrm{b})) = \rho(\mathrm{y})$. Hence, $\rho(\mathrm{y}) \Vdash \mathrm{C}$, which is a contradiction since $\rho(\mathrm{y}) \nVdash \mathrm{C}$ by definition. Therefore, either $\rho(\mathrm{y}) \sqcup \rho(\mathrm{a}) \nVdash \mathrm{C}$ and s_1 is realizable, or $\rho(\mathrm{y}) \sqcup \rho(\mathrm{b}) \nVdash \mathrm{C}$ and s_2 is realizable.

Case $\vee_{\mathrm{R}}$**:** If ρ realizes $s = \Gamma \vdash \Delta(\mathrm{A} \vee \mathrm{B} : \mathrm{x})$ then $\rho(\mathrm{x}) \nVdash \mathrm{A} \vee \mathrm{B}$. Suppose that $\rho(\mathrm{x}) \Vdash \mathrm{A}$. Since $\mathcal{M}$ is regular there exists an A-minimal world m_{A}. Since $m_{\mathrm{A}} \Vdash \mathrm{A}$ and $\pi \Vdash \mathrm{B}$ by definition, we have $m_{\mathrm{A}} \sqcap \pi = m_{\mathrm{A}} \leqslant \rho(\mathrm{x})$ which implies the contradiction $\rho(\mathrm{x}) \Vdash \mathrm{A} \vee \mathrm{B}$. Similarly, if $\rho(\mathrm{x}) \Vdash \mathrm{B}$ we also get the contradiction $\rho(\mathrm{x}) \Vdash \mathrm{A} \vee \mathrm{B}$. Hence, s_1 is realizable.

Case $\approx_{\mathrm{LR}}$**:** This case directly follows from Lemma 36.

The other cases are similar. ◀

▶ **Theorem 40** (Liberalized soundness)**.** *If* $\vdash_{\mathsf{L}^2_{\mathsf{ISCI}}} \mathrm{A}$ *then* $\vDash^{\mathrm{r}} \mathrm{A}$.

Proof. Suppose that $\vdash_{\mathsf{L}^2_{\mathsf{ISCI}}} \mathrm{A}$, then there exists an $\mathsf{L}^2_{\mathsf{ISCI}}$-proof Π of $\vdash \mathrm{A} : \emptyset$. If $\nvDash^{\mathrm{r}} \mathrm{A}$, then there is a regular Beth model $\mathcal{M}$ such that $\omega \nVdash \mathrm{A}$. Since $\vdash \mathrm{A} : \emptyset$ is trivially realizable, Lemma 39 implies that Π contains a branch the sequents of which are all realizable. Since Π is a proof, this branch ends with an initial sequent s that is the conclusion of an axiom rule. Lemma 38 then implies that s is not realizable, which is a contradiction. Therefore, $\vDash^{\mathrm{r}} \mathrm{A}$. ◀

6.3 Termination and Decidability

Giving a full-fledged proof that $\mathsf{L}^2_{\mathsf{ISCI}}$ is a terminating proof-system is out of the scope of this paper as it would require a detailed proof-search algorithm with a well defined proof strategy. Moreover, since $\mathsf{L}^2_{\mathsf{ISCI}}$ proof rules as formulated non-destructively, we would also need a suitable notion of (sequent) saturation to decide whether a labelled formula is fully analyzed or not. For instance, an occurrence of $\mathrm{A} \wedge \mathrm{B} : \mathrm{x}$ on the left-hand side of a sequent $\Gamma \vdash \Delta$ would be considered fully analyzed whenever $\mathrm{A} : \mathrm{y}$ and $\mathrm{B} : \mathrm{z}$ occur in Γ for some labels y, z such that $\mathrm{y}, \mathrm{z} \subseteq \mathrm{x}$. We now sketch the proof that $\mathsf{L}^2_{\mathsf{ISCI}}$ has a finite proof search space.

▶ **Theorem 41** (Termination)**.** $\mathsf{L}^2_{\mathsf{ISCI}}$ *is a terminating proof system.*

Proof sketch. Firstly, without any eigenvariable requirements, only finitely many singleton labels can occur in an $\mathsf{L}^2_{\mathsf{ISCI}}$ derivation of A. Since labels occurring in an $\mathsf{L}^2_{\mathsf{ISCI}}$ derivation of A are finite unions of singleton labels, there can only be finitely many of them. Secondly, let $n = |\mathrm{A}|$ and let $At(\mathrm{A})$ be the set of propositional letters occurring in A. It is easy to see that the active formula introduced by an instance of the rule $\approx_{\mathrm{LR}}$ has a size $m \leqslant n$ and is built using only atoms in $At(\mathrm{A})$ (this can be viewed as a weak form of subformula property). There can only be finitely many formulas of size $\leqslant n$ built from $At(\mathrm{A})$. Finally, with a finitely many subformulas and labels, one can only generate a finite number of labelled formulas. Therefore, only finitely many unsaturated labelled sequents can occur in a $\mathsf{L}^2_{\mathsf{ISCI}}$ derivation of A. Thus, the proof search space for $\vdash \mathrm{A} : \emptyset$ in $\mathsf{L}^2_{\mathsf{ISCI}}$ is finite. ◀

▶ **Corollary 42** (Decidability)**.** ISCI *is a decidable logic.*

References

1. S.L. Bloom and R. Suszko. Investigations into the Sentential Calculus with Identity. *Notre Dame Journal of Formal Logic*, 13(3):289–308, 1972.
2. S. Chlebowski. Sequent Calculi for SCI. *Studia Logica*, 106(3):541–563, 2018.
3. S. Chlebowski and D. Leszczyńska-Jasion. An Investigation into Intuitionistic Logic with Identity. *Bulletin of the Section of Logic*, 48(4):259–283, 2019.
4. D. Galmiche, D. Méry, and D. Pym. The semantics of BI and Resource Tableaux. *Mathematical Structures in Computer Science*, 15(6):1033–1088, 2005.
5. D. Galmiche and D. Méry. A Connection-based Characterization of Bi-intuitionistic Validity. *Journal of Automated Reasoning*, 51(1):3–26, 2013. doi:10.1007/s10817-013-9279-4.
6. J. Golińska-Pilarek. Rasiowa–Sikorski proof system for the non-Fregean sentential logic. *Journal of Applied Non-Classical Logics*, 17(4):511–519, 2007.
7. J. Golińska-Pilarek and T. Huuskonen. Non-Fregean Propositional Logic with Quantifiers. *Notre Dame Journal of Formal Logic*, 57(2):249–279, 2016. doi:10.1215/00294527-3470547.
8. Z. Hou, R. Goré, and A. Tiu. A labelled sequent calculus for BBI: Proof theory and proof search. *Journal of Logic and Computation*, 28(4):809–872, 2018. doi:10.1007/978-3-642-40537-2_16.
9. P. Lukowski. Intuitionistic sentential calculus with identity. *Bulletin of the Section of Logic*, 19(3):92–99, 1990.
10. S. Negri and J. von Plato. Cut elimination in the presence of axioms. *Bulletin of Symbolic Logic*, 4(4):418–435, 1998.
11. S. Negri and J. von Plato. *Proof Analysis - A Contribution to Hilbert's Last Problem*. Cambridge University Press, 2014. URL: http://www.cambridge.org/de/academic/subjects/philosophy/logic/proof-analysis-contribution-hilberts-last-problem?format=PB.
12. R. Suszko. Abolition of the Fregean axiom. In *Logic Colloquium*, pages 169–239, 1975. Springer.
13. A. Wasilewska. DFC-algorithms for Suszko logic and one-to-one Gentzen type formalizations. *Studia Logica*, 43(4):395–404, 1984.

A Appendix

A.1 Cut Elimination

▶ **Theorem 43** (Cut elimination). *The* cut *rule is admissible in* $\mathsf{L}^{2ec}_{\mathsf{ISCI}}$.

Proof. Our proof follows the pattern given in [10] or in [8] for Boolean BI. We define the cut rank of (an instance) of the cut rule as the pair $(|\mathrm{C}|, h(\Pi_1) + h(\Pi_2))$, where C is the cut formula and $\Pi_{i\in\{1,2\}}$ is the proof whose conclusion is the sequent s_i corresponding to the i-th premiss above the cut. For the base case we consider that one of the premiss of the cut has a proof of height 0. For the inductive step, we distinguish three cases: $\mathrm{C}:\mathrm{z}$ is not principal in s_1, $\mathrm{C}:\mathrm{z}$ is principal only in s_1, $\mathrm{C}:\mathrm{z}$ is principal in both s_1 and s_2.

Case $\mathrm{n_1}$**.id:** s_1 is the conclusion of id, $\mathrm{C}:\mathrm{z}$ is not principal in s_1, $\mathrm{x} \subseteq \mathrm{y}$.

$$\dfrac{\dfrac{}{\Gamma(\mathrm{A}:\mathrm{x}) \vdash \Delta(\mathrm{A}:\mathrm{y}), \mathrm{C}:\mathrm{z}}\,\mathrm{id} \qquad \begin{array}{c}\Pi_2\\ \hdashline \mathrm{C}:\mathrm{z}, \Gamma \vdash \Delta\end{array}}{\Gamma(\mathrm{A}:\mathrm{x}) \vdash \Delta(\mathrm{A}:\mathrm{y})}\,\mathrm{cut} \quad \rightsquigarrow \quad \dfrac{}{\Gamma(\mathrm{A}:\mathrm{x}) \vdash \Delta(\mathrm{A}:\mathrm{y})}\,\mathrm{id}$$

Case $\mathrm{p_1}$**.id:** s_1 is the conclusion of id, $\mathrm{C}:\mathrm{z}$ is principal in s_1, $\mathrm{x} \subseteq \mathrm{z}$.

$$\dfrac{\dfrac{}{\Gamma(\mathrm{C}:\mathrm{x}) \vdash \Delta, \mathrm{C}:\mathrm{z}}\,\mathrm{id} \qquad \begin{array}{c}\Pi_2\\ \hdashline \mathrm{C}:\mathrm{z}, \Gamma \vdash \Delta\end{array}}{\Gamma(\mathrm{C}:\mathrm{x}) \vdash \Delta}\,\mathrm{cut} \quad \rightsquigarrow \quad \begin{array}{c}\Pi_2' \text{ from Lemma 31}\\ \hdashline \Gamma(\mathrm{C}:\mathrm{x}) \vdash \Delta\end{array}$$

Case n_2.id: s_2 is the conclusion of id, $C:z$ is not principal in s_2. Similar to Case n_1.id.
Case p_2.id: s_2 is the conclusion of id, $C:z$ is principal in s_2. Similar to Case p_1.id.
Case $n_1.\bot_L$: s_1 is the conclusion of $\bot_L$, $C:z$ is not principal in s_1, $x \subseteq y$.

$$\dfrac{\dfrac{}{\Gamma(\bot:x) \vdash \Delta(A:y), C:z}\,\bot_L \qquad \dfrac{\Pi_2}{C:z, \Gamma \vdash \Delta}}{\Gamma(\bot:x) \vdash \Delta(A:y)}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{}{\Gamma(\bot:x) \vdash \Delta(A:y)}\,\bot_L$$

Case $p_1.\bot_L$: s_1 is the conclusion of $\bot_L$, $C:z$ is principal in s_1, $x \subseteq z$.

$$\dfrac{\dfrac{}{\Gamma(\bot:x) \vdash \Delta, C:z}\,\bot_L \qquad \dfrac{\Pi_2}{C:z, \Gamma \vdash \Delta}}{\Gamma(\bot:x) \vdash \Delta}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{\text{connectedness: } x \subseteq u}{\Gamma(\bot:x) \vdash \Delta(A:u)}\,\bot_L$$

Case $n_2.\bot_L$: s_2 is the conclusion of $\bot_L$, $C:z$ is not principal in s_2. Similar to Case $n_1.\bot_L$.
Case $p_2.\bot_L$: s_2 is the conclusion of $\bot_L$, $C:z$ is principal in s_2. Similar to Case $p_1.\bot_L$.
Case $n_1.\approx_R$: s_1 is the conclusion of $\approx_R$, $C:z$ is not principal in s_1.

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta(A \approx A:x), C:z}\,\approx_R \qquad \dfrac{\Pi_2}{C:z, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta(A \approx A:x)}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{}{\Gamma \vdash \Delta(A \approx A:x)}\,\approx_R$$

Case $p_1.\approx_R$: s_1 is the conclusion of $\approx_R$, $C:z$ is principal in s_1. Then, C has the form $A \approx A$ for some A. Since $A \neq A$ is not satisfiable, $A \approx A:x$ can never be the principal formula of an occurrence of $\approx_{LR}$ in Π_2. Therefore, the only way for $A \approx A:x$ to be principal in Π_2 is if s_2 is the conclusion of an occurrence of id, which then implies that Δ contains an occurrence of $A \approx A:y$ for some $x \subseteq y$. In this case, we have

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta, A \approx A:x}\,\approx_R \qquad \dfrac{\Pi_2}{A \approx A:x, \Gamma \vdash \Delta(A \approx A:y)}}{\Gamma \vdash \Delta(A \approx A:y)}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{}{\Gamma \vdash \Delta(A \approx A:y)}\,\approx_R$$

Otherwise, $A \approx A:x$ is never principal in Π_2 and we apply Lemma 32 on Π_2 to get a proof Π_2' of $\Gamma \vdash \Delta$ as follows

$$\dfrac{\dfrac{}{\Gamma \vdash \Delta, A \approx A:x}\,\approx_R \qquad \dfrac{\Pi_2}{A \approx A:x, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{\Pi_2' \text{ from Lemma 32}}{\Gamma \vdash \Delta}$$

Case $n_2.\approx_R$: s_2 is the conclusion of $\approx_R$, $C:z$ is not principal in s_2. Similar to Case $n_1.\approx_R$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, C:z} \qquad \dfrac{}{C:z, \Gamma \vdash \Delta(A \approx A:x)}\,\approx_R}{\Gamma \vdash \Delta(A \approx A:x)}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{}{\Gamma \vdash \Delta(A \approx A:x)}\,\approx_R$$

Case $p_2.\approx_R$: cannot happen ($A \approx A:z$ on the left-hand side cannot be principal for $\approx_R$).

Case $n_1.r_1$: $C:z$ is not principal in s_1, r is a rule with one premiss and active parts Γ', Δ'.

$$\dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, \Gamma' \vdash \Delta, \Delta', C:z}}{\Gamma \vdash \Delta, C:z}\,r \qquad \dfrac{\Pi_2}{C:z, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta}\,\text{cut} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\Pi_1}{\Gamma, \Gamma' \vdash \Delta, \Delta', C:z} \qquad \dfrac{\Pi_2' \text{ from Lemma 29}}{C:z, \Gamma, \Gamma' \vdash \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}}{\Gamma \vdash \Delta}\,r$$

The rank of the new cut is $(|C|, h(\Pi_1) + h(\Pi_2'))$, which is strictly lower than the rank $(|C|, 1 + h(\Pi_1) + h(\Pi_2))$ of the original cut.

Case $\mathbf{p_1 n_2.r_1}$: $\mathrm{C}:\mathrm{z}$ is only principal in s_1, r is a rule with one premiss and active parts Γ', Δ'. Similar to Case $\mathrm{n_1.r_1}$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, \mathrm{C}:\mathrm{z}} \qquad \dfrac{\dfrac{\Pi_2}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma' \vdash \Delta, \Delta'}}{\mathrm{C}:\mathrm{z}, \Gamma \vdash \Delta}\,\mathrm{r}}{\Gamma \vdash \Delta}\,\mathrm{cut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Pi'_1 \text{ from Lemma 29}}{\Gamma, \Gamma' \vdash \Delta, \Delta', \mathrm{C}:\mathrm{z}} \qquad \dfrac{\Pi_2}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma' \vdash \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{cut}}{\Gamma \vdash \Delta}\,\mathrm{r}$$

The rank of the new cut is $(|\mathrm{C}|, h(\Pi'_1) + h(\Pi_2))$, which is strictly lower than the rank $(|\mathrm{C}|, h(\Pi_1) + h(\Pi_2) + 1)$ of the original cut.

Case $\mathbf{n_1.r_2}$: $\mathrm{C}:\mathrm{z}$ is not principal in s_1, r is a rule with two premise and active parts Γ', Δ' in the first premiss and Γ'', Δ'' in the second one. We apply Lemma 29 twice on Π_2 to get Π'_2 and Π''_2.

$$\dfrac{\dfrac{\dfrac{\Pi_1^1}{\Gamma, \Gamma' \vdash \Delta, \Delta', \mathrm{C}:\mathrm{z}} \qquad \dfrac{\Pi_1^2}{\Gamma, \Gamma'' \vdash \Delta, \Delta'', \mathrm{C}:\mathrm{z}}}{\Gamma \vdash \Delta, \mathrm{C}:\mathrm{z}}\,\mathrm{r} \qquad \dfrac{\Pi_2}{\mathrm{C}:\mathrm{z}, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta}\,\mathrm{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Pi_1^1}{\Gamma, \Gamma' \vdash \Delta, \Delta', \mathrm{C}:\mathrm{z}} \qquad \dfrac{\Pi'_2}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma' \vdash \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{cut} \qquad \dfrac{\dfrac{\Pi_1^2}{\Gamma, \Gamma'' \vdash \Delta, \Delta'', \mathrm{C}:\mathrm{z}} \qquad \dfrac{\Pi''_2}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma'' \vdash \Delta, \Delta''}}{\Gamma, \Gamma'' \vdash \Delta, \Delta''}\,\mathrm{cut}}{\Gamma \vdash \Delta}\,\mathrm{r}$$

The ranks $(|\mathrm{C}|, h(\Pi_1^1) + h(\Pi'_2))$ and $(|\mathrm{C}|, h(\Pi_1^2) + h(\Pi''_2))$ of the two new cuts are strictly lower than the rank $(|\mathrm{C}|, 1 + max(h(\Pi_1^1), h(\Pi_1^2)) + h(\Pi_2))$ of the original cut.

Case $\mathbf{p_1 n_2.r_2}$: $\mathrm{C}:\mathrm{z}$ is only principal in s_1, r is a rule with two premises and active parts Γ', Δ' in the first premiss and Γ'', Δ'' in the second one. We apply Lemma 29 twice on Π_1 to get Π'_1 and Π''_1. Similar to Case $\mathrm{n_1.r_2}$.

$$\dfrac{\dfrac{\Pi_1}{\Gamma \vdash \Delta, \mathrm{C}:\mathrm{z}} \qquad \dfrac{\dfrac{\Pi_2^1}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma' \vdash \Delta, \Delta'} \qquad \dfrac{\Pi_2^2}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma'' \vdash \Delta, \Delta''}}{\mathrm{C}:\mathrm{z}, \Gamma \vdash \Delta}\,\mathrm{r}}{\Gamma \vdash \Delta}\,\mathrm{cut} \quad \rightsquigarrow$$

$$\dfrac{\dfrac{\dfrac{\Pi'_1}{\Gamma, \Gamma' \vdash \Delta, \Delta', \mathrm{C}:\mathrm{z}} \qquad \dfrac{\Pi_2^1}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma' \vdash \Delta, \Delta'}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathrm{cut} \qquad \dfrac{\dfrac{\Pi''_1}{\Gamma, \Gamma'' \vdash \Delta, \Delta'', \mathrm{C}:\mathrm{z}} \qquad \dfrac{\Pi_2^2}{\mathrm{C}:\mathrm{z}, \Gamma, \Gamma'' \vdash \Delta, \Delta''}}{\Gamma, \Gamma'' \vdash \Delta, \Delta''}\,\mathrm{cut}}{\Gamma \vdash \Delta}\,\mathrm{r}$$

The ranks $(|\mathrm{C}|, h(\Pi'_1) + h(\Pi_2^1))$ and $(|\mathrm{C}|, h(\Pi''_1) + h(\Pi_2^2))$ of the two new cuts are strictly lower than the rank $(|\mathrm{C}|, h(\Pi_1) + max(h(\Pi_2^1), h(\Pi_2^2)) + 1)$ of the original cut.

Case $\mathbf{p_1.\wedge_R p_2.\wedge_L}$: $\mathrm{C}:\mathrm{z}$ is principal in both s_1 and s_2, C has the form $\mathrm{A} \wedge \mathrm{B}$.

$$\dfrac{\dfrac{\dfrac{\Pi_1^1}{\Gamma \vdash \Delta, \mathrm{A} \wedge \mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}} \qquad \dfrac{\Pi_1^2}{\Gamma \vdash \Delta, \mathrm{A} \wedge \mathrm{B}:\mathrm{z}, \mathrm{B}:\mathrm{z}}}{\Gamma \vdash \Delta, \mathrm{A} \wedge \mathrm{B}:\mathrm{z}}\,\wedge_{\mathrm{R}} \qquad \dfrac{\dfrac{\Pi_2}{\mathrm{A} \wedge \mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta}}{\mathrm{A} \wedge \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta}\,\wedge_{\mathrm{L}}}{\Gamma \vdash \Delta}\,\mathrm{cut}$$

We use three cuts on $\mathrm{A} \wedge \mathrm{B}:\mathrm{z}$ of strictly lower cut height to get the following proofs:

$$\Pi_3 \left\{ \dfrac{\dfrac{\Pi'_1 \text{ from Lemma 29}}{\mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta, \mathrm{A} \wedge \mathrm{B}:\mathrm{z}}\,\wedge_{\mathrm{R}} \qquad \dfrac{\Pi_2}{\mathrm{A} \wedge \mathrm{B}:\mathrm{z}, \mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta}}{\mathrm{A}:\mathrm{z}, \mathrm{B}:\mathrm{z}, \Gamma \vdash \Delta}\,\mathrm{cut} \right.$$

$$\Pi_4 \left\{ \dfrac{\dfrac{\Pi_1^2}{\Gamma \vdash \Delta, B:z, A \wedge B:z} \qquad \dfrac{\dfrac{\Pi_2' \text{ from Lemma 29}}{A \wedge B:z, A:z, B:z, \Gamma \vdash \Delta, B:z}}{A \wedge B:z, \Gamma \vdash \Delta, B:z} \wedge_L}{\Gamma \vdash \Delta, B:z} \text{cut} \right.$$

$$\Pi_5 \left\{ \dfrac{\dfrac{\Pi_1^1}{\Gamma \vdash \Delta, A:z, A \wedge B:z} \qquad \dfrac{\dfrac{\Pi_2'' \text{ from Lemma 29}}{A \wedge B:z, A:z, B:z, \Gamma \vdash \Delta, A:z}}{A \wedge B:z, \Gamma \vdash \Delta, A:z} \wedge_L}{\Gamma \vdash \Delta, A:z} \text{cut} \right.$$

We construct the following proof using two cuts on strictly smaller formulas:

$$\dfrac{\dfrac{\Pi_5}{\Gamma \vdash \Delta, A:z} \qquad \dfrac{\dfrac{\Pi_4' \text{ from Lemma 29}}{A:z, \Gamma \vdash \Delta, B:z} \qquad \dfrac{\Pi_3}{A:z, B:z, \Gamma \vdash \Delta}}{A:z, \Gamma \vdash \Delta} \text{cut}}{\Gamma \vdash \Delta} \text{cut}$$

Case $p_1.\supset_L p_2.\supset_R$: $C:z$ is principal in both s_1 and s_2, C has the form $A \supset B$.

$$\dfrac{\dfrac{\dfrac{\Pi_1}{A:a, \Gamma \vdash \Delta, B:z \cup a}}{\Gamma \vdash \Delta, A \supset B:z} \supset_R \qquad \dfrac{\dfrac{\Pi_2^1}{A \supset B:z, \Gamma \vdash \Delta, A:x} \qquad \dfrac{\Pi_2^2}{A \supset B:z, B:z \cup x, \Gamma \vdash \Delta}}{A \supset B:z, \Gamma \vdash \Delta} \supset_L}{\Gamma \vdash \Delta} \text{cut}$$

We first apply Lemma 28 on Π_1 to replace a with x.

$$\dfrac{\Pi_3}{A:x, \Gamma \vdash \Delta, B:z \cup x}$$

We then apply Lemma 29 on Π_1 to get Π_1':

$$\dfrac{\Pi_1'}{A:a, \Gamma \vdash \Delta, B:z \cup a, A:x}$$

We combine Π_1' and Π_2^1 to get the following proof with a cut of strictly lower cut height:

$$\Pi_4 \left\{ \dfrac{\dfrac{\dfrac{\Pi_1'}{A:a, \Gamma \vdash \Delta, B:z \cup a, A:x}}{\Gamma \vdash \Delta, A:x, A \supset B:z} \supset_R \qquad \dfrac{\Pi_2^1}{A \supset B:z, \Gamma \vdash \Delta, A:x}}{\Gamma \vdash \Delta, A:x} \text{cut} \right.$$

Applying Lemma 29 on Π_4 we get Π_4':

$$\dfrac{\Pi_4'}{\Gamma \vdash \Delta, B:z \cup x, A:x}$$

Since $B:z \cup x$ occurs on the left-hand side of the conclusion of Π_2^2, $z \cup x$ necessarily is a sublabel of some label in Δ. We can therefore apply Lemma 29 on Π_1 to get Π_1'':

$$\dfrac{\Pi_1''}{A:a, \Gamma, B:z \cup x \vdash \Delta, B:z \cup a}$$

We combine Π_1'' and Π_2^2 to get the following proof with a cut of strictly lower cut height:

$$\Pi_5 \left\{ \dfrac{\dfrac{\dfrac{\Pi_1''}{\mathrm{A:a}, \Gamma, \mathrm{B:z \cup x} \vdash \Delta, \mathrm{B:z \cup a}}}{\Gamma, \mathrm{B:z \cup x} \vdash \Delta, \mathrm{A \supset B:z}} \supset_\mathrm{R} \quad \dfrac{\Pi_2^2}{\mathrm{A \supset B:z}, \Gamma, \mathrm{B:z \cup x} \vdash \Delta}}{\Gamma, \mathrm{B:z \cup x} \vdash \Delta} \text{cut} \right.$$

We finally cut on strictly smaller formulas:

$$\dfrac{\dfrac{\dfrac{\Pi_4'}{\Gamma \vdash \Delta, \mathrm{B:z \cup x}, \mathrm{A:x}} \quad \dfrac{\Pi_3}{\mathrm{A:x}, \Gamma \vdash \Delta, \mathrm{B:z \cup x}}}{\Gamma \vdash \Delta, \mathrm{B:z \cup x}} \text{cut} \quad \dfrac{\Pi_5}{\mathrm{B:z \cup x}, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta} \text{cut}$$

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