Failure of Cut-Elimination in the Cyclic Proof System of Bunched Logic with Inductive Propositions

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Abstract

Cyclic proof systems are sequent-calculus style proof systems that allow circular structures representing induction, and they are considered suitable for automated inductive reasoning. However, Kimura et al. have shown that the cyclic proof system for the symbolic heap separation logic does not satisfy the cut-elimination property, one of the most fundamental properties of proof systems. This paper proves that the cyclic proof system for the bunched logic with only nullary inductive predicates does not satisfy the cut-elimination property. It is hard to adapt the existing proof technique chasing contradictory paths in cyclic proofs since the bunched logic contains the structural rules. This paper proposes a new proof technique called proof unrolling. This technique can be adapted to the symbolic heap separation logic, and it shows that the cut-elimination fails even if we restrict the inductive predicates to nullary ones.

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1 Introduction

Static verification of software often needs to check the validity of entailments, which are implications between logical formulas. One of the ways to check entailments is an automated proof search in some proof systems.

The *bunched logic* [9] was introduced to reason compositional properties of resources with some additional logical connectives such as the multiplicative conjunction. The *separation logic* [11], which is based on the bunched logic, is one of the most successful logical foundations for verification of heap-manipulating programs using pointers. For inductive reasoning in these logics, Brotherston et al. proposed some *cyclic proof systems* for the bunched logic [3] and the separation logic [4, 5]. The cyclic proof systems allow cycles in proofs, which correspond to induction. They offer an efficient way for automated validity checking of entailments with inductive definitions since they provide a proof search algorithm that does not require finding induction hypothesis formulas a priori.

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The *cut-elimination property* of proof systems means that the provability does not change with or without the cut rule:

$$\frac{A \vdash C \quad C \vdash B}{A \vdash B}\ (Cut).$$

From a theoretical viewpoint, the cut-elimination property means that applying lemma is admissible, and it implies significant properties such as the subformula property and consistency. The cut-elimination property is also important from a practical viewpoint: When the cut rule is included as a candidate of the next rules during an automated proof search, we have to find a suitable cut formula, namely the formula C in the cut rule above. In general, cut formulas are independent of formulas in the conclusion of cut rules, and we have to find them heuristically.

Hence, we expect proof systems to enjoy the cut-elimination property, and it holds in many proof systems such as Gentzen's LK for the first-order logic and the (non-cyclic) proof system LBI for the bunched logic [10]. Furthermore, it has been shown that the cut-elimination property holds in some infinitary proof systems [6, 7, 2]. The cut-elimination processes in the existing proofs are not closed under the regularity of infinitary proof trees, and that suggests that the cut-elimination does not hold in the cyclic proof systems since cyclic proofs are regular infinitary proofs.

Kimura et al. [8] showed that the cut-elimination property fails for Brotherston's cyclic proof system [4] for the symbolic heaps, which are restricted forms of the separation logic formulas. They gave a counterexample entailment $\mathrm{ls}(x, y) \vdash \mathrm{sl}(x, y)$, where both $\mathrm{ls}(x, y)$ and $\mathrm{sl}(x, y)$ are inductive predicates that represent the semantically same data structure, namely singly-linked list from x to y, but are defined in the different ways. They assumed the existence of a cut-free cyclic proof of this counterexample and showed that a unique infinite path in the cyclic proof is a contradictory path, namely, an infinite path in which the sizes of sequents are strictly increasing. The contradictory path leads to a contradiction since it breaks the finiteness of the cyclic proof.

In [8], they guessed that the cut-elimination would not hold for the bunched logic either, but suggested that their proof technique needs some modification to handle the structural rules, the left weakening and the left contraction rules, in the bunched logic. The structural rules cause much more possibilities of paths than the symbolic heap separation logic, and we have to find a contradictory path from them. For example, we can assume a segment of a cyclic proof of the sequent $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ in the bunched logic as in Figure 1, where P_{AB} and P_{BA} are inductively defined as

$$\mathrm{P}_{AB} := \mathrm{P}_B \mid \mathrm{P}_{AB} * A \qquad \mathrm{P}_A := I \mid \mathrm{P}_A * A$$
$$\mathrm{P}_{BA} := \mathrm{P}_A \mid \mathrm{P}_{BA} * B \qquad \mathrm{P}_B := I \mid \mathrm{P}_B * B.$$

Here, the separators "," and ";" on the left-hand sides of sequents correspond to the multiplicative conjunction ($*$) and the additive conjunction ($\wedge$), respectively. The proposition constants I and $\top$ are the units for $*$ and $\wedge$, respectively. The rule (UL) unfolds predicates on the left-hand side from bottom to top. The rule (E) replaces the left-hand side with an equivalent one. The rules (W) and (C) are the left weakening and the left contraction rules, respectively. The rule $(\top)$ is admissible using the left weakening rule, and a link between two sequents marked with $(\dagger)$ forms a cycle, which satisfies the soundness condition for the cyclic proofs, the global trace condition [6]. Therefore, the rightmost path contains no contradiction. Furthermore, the part $(\star)$ is easily proved. This means that, to find a contradictory path, we

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ (\star)}{\mathrm{P}_B \vdash \mathrm{P}_{BA}}\ (W)}{\mathrm{P}_{AB}; \mathrm{P}_B \vdash \mathrm{P}_{BA}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots\ (\#)}{\mathrm{P}_{AB}; (\mathrm{P}_B, \top) \vdash \mathrm{P}_{BA}} \quad \dfrac{\dfrac{\dfrac{\mathrm{P}_{AB}; (\mathrm{P}_{AB}, \top) \vdash \mathrm{P}_{BA}\ (\dagger)}{\mathrm{P}_{AB}; (\mathrm{P}_{AB}, (A, \top)) \vdash \mathrm{P}_{BA}}\ (\top)}{\mathrm{P}_{AB}; ((\mathrm{P}_{AB}, A), \top) \vdash \mathrm{P}_{BA}}\ (E)}{\mathrm{P}_{AB}; (\mathrm{P}_{AB} * A, \top) \vdash \mathrm{P}_{BA}}\ (*L)}{\mathrm{P}_{AB}; (\mathrm{P}_{AB}, \top) \vdash \mathrm{P}_{BA} (\dagger)}\ (UL)^2}{\mathrm{P}_{AB}; (\mathrm{P}_{AB}, A) \vdash \mathrm{P}_{BA}}\ (\top)}{\mathrm{P}_{AB}; \mathrm{P}_{AB} * A \vdash \mathrm{P}_{BA}}\ (*L)}{\mathrm{P}_{AB}; \mathrm{P}_{AB} \vdash \mathrm{P}_{BA}}\ (UL)^1}{\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}}\ (C)
$$

Figure 1 A proof segment in the cyclic proof system of the bunched logic.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ (\#')}{I * A^m; (I, \top) \vdash \mathrm{P}_{BA}\ (\dagger)}}{I * A^m; (I, (A, \top)) \vdash \mathrm{P}_{BA}}\ (\top)}{I * A^m; ((I, A), \top) \vdash \mathrm{P}_{BA}}\ (E)}{\vdots}}{I * A^m; (I * A^{m-2}, \top) \vdash \mathrm{P}_{BA}}}{I * A^m; (I * A^{m-2}, (A, \top)) \vdash \mathrm{P}_{BA}}\ (\top)}{I * A^m; ((I * A^{m-2}, A), \top) \vdash \mathrm{P}_{BA}}\ (E)}{\dfrac{I * A^m; (I * A^{m-1}, \top) \vdash \mathrm{P}_{BA}\ (\dagger)}{\dfrac{I * A^m; (I * A^{m-1}, A) \vdash \mathrm{P}_{BA}}{\dfrac{I * A^m; I * A^m \vdash \mathrm{P}_{BA}}{I * A^m \vdash \mathrm{P}_{BA}}\ (C)}\ (*L)}\ (\top)}\ (*L)
$$

Figure 2 Proof unrolling.

have to chase it in the part (#), and hence we sometimes have to choose the right assumption (at $(UL)^1$), and also have to choose the left assumption (at $(UL)^2$). Therefore, it is hard to find such a contradictory path in cyclic proofs.

Kimura et al. also mentioned a possibility to recover the cut-elimination property by restricting the number of arities (to unary or nullary) for inductive predicates. Restricting arities of inductive predicates may drastically change the situation as the result of Tatsuta et al [12]. They showed the decidability of the entailment checking problem for the symbolic heap separation logic with only unary inductive predicates whereas the problem for that with general inductive predicates is known to be undecidable [1].

In this paper, we show that the cut-elimination property fails for the cyclic proof system of the bunched logic [3] by a counterexample only with nullary inductive predicates. We develop a proof technique called *proof unrolling*. For a cut-free cyclic proof of $\Gamma \vdash \phi$, by using proof unrolling, we can construct a cut-free non-cyclic proof of $\Delta \vdash \phi$ for any Δ obtained by unfolding inductive predicates in Γ. For the example in Figure 1 and the formula $I * A^m = ((I * A) * \cdots * A) * A$ (m copies of A's) obtained by unfolding P_{AB}, we can construct the non-cyclic proof of $I * A^m \vdash \mathrm{P}_{BA}$ in Figure 2 by proof unrolling. During the proof unrolling, we unroll the cycle (at (†)), and choose cases at the rule (UL) depending on the unfolding tree of P_{AB} to obtain $I * A^m$. We will show that, for any cyclic proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$, if m is sufficiently large, any path in the non-cyclic proof by proof unrolling corresponds to a contradictory path in the original cyclic proof. The remaining path in the part (#′) of Figure 2 corresponds to a contradictory path in the part (#) of Figure 1. Hence, the existence of a cyclic proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ derives a contradiction.

The proof unrolling is a general technique almost independent of a choice of logic. We can straightforwardly adapt our proof to any cyclic proof system of a logic that contains a connective representing resource composition such as the separation logic and the multiplicative linear logic. Hence, the cut-elimination fails for the cyclic proof system of the separation logic even if we restrict inductive predicates to nullary ones.

The structure of the paper is as follows. Section 2 introduces a simple fragment of the propositional bunched logic BI_{ID0} with inductive definitions, and its cyclic proof system $CLBI^{\omega}_{ID0}$, which is a subsystem of $CLBI^{\omega}_{ID}$ given by Brotherston [3]. Section 3 presents our proof unrolling technique. Section 4 proves the main result of this paper, which shows that the cut-elimination property does not hold in $CLBI^{\omega}_{ID0}$ using the proof unrolling technique. It also discusses that our proof technique can be adapted to other systems including $CLBI^{\omega}_{ID}$. Section 5 concludes.

2 Bunched Logic with Inductive Propositions

In this section, we define the syntax and semantics of a core of the bunched logic BI_{ID0}, which is based on the logic in [3]. In BI_{ID0}, atomic and inductive predicates are restricted to nullary ones, which we call atomic propositions and inductive propositions, respectively. We also define proof systems for BI_{ID0}: one is the ordinary proof system LBI_{ID0}, and the other is the cyclic proof system $CLBI^{\omega}_{ID0}$.

In the following sections, we will prove that cuts cannot be eliminated in $CLBI^{\omega}_{ID0}$, and this result can be easily extended to the system in [3].

2.1 Syntax of BI_{ID0}

We use metavariables A, B,... for atomic propositions and P, Q,... for inductive propositions. We implicitly fix a language Σ consisting of atomic and inductive propositions. Note that in BI_{ID0}, we have neither terms, variables, nor function symbols.

▶ **Definition 1** (Formulas of BI_{ID0}). *Let I and $\top$ be propositional constants. The* formulas *of BI_{ID0}, denoted by ϕ, ψ,..., are defined as*

$$\phi ::= I \mid \top \mid A \mid P \mid \phi * \phi \mid \phi \wedge \phi.$$

In this paper, $*$ and $\wedge$ are treated as left-associative operators, that is, we write $\phi_1 * \phi_2 * \phi_3$ for $(\phi_1 * \phi_2) * \phi_3$. The notation A^n denotes $A * \cdots * A$ where the number of A's is n. We also use the notation $P * A^n$ for $P * A * \cdots * A$, namely $(\cdots((P * A) * A)\cdots) * A)$.

▶ **Definition 2** (Bunch). *The* bunches*, denoted by $\Gamma, \Delta, \ldots$, are defined as*

$$\Gamma, \Delta ::= \phi \mid \Gamma, \Gamma \mid \Gamma; \Gamma.$$

We sometimes use terminologies of trees to bunches by identifying a bunch as a tree whose internal nodes are labeled by "," or ";", and whose leaves are labeled by a formula. We write $\Gamma(\Delta)$ to mean that Γ of which Δ is a subtree. For a bunch $\Gamma(\Delta)$, $\Gamma(\Delta')$ is a bunch obtained by replacing the subtree Δ of Γ by Δ'.

The labels "," and ";" intuitively mean $*$ and $\wedge$, respectively. For a bunch Γ, we define the bunch formula ϕ_Γ as the formula defined as:

$$\begin{aligned}
\phi_\Gamma &= \Gamma, && (\Gamma \text{ is a formula});\\
\phi_{\Gamma_1,\Gamma_2} &= \phi_{\Gamma_1} * \phi_{\Gamma_2};\\
\phi_{\Gamma_1;\Gamma_2} &= \phi_{\Gamma_1} \wedge \phi_{\Gamma_2}.
\end{aligned}$$

▶ **Definition 3** (Equivalence of bunches). *Define the bunch equivalence $\equiv$ as the least equivalence relation satisfying:*

- *commutative monoid equations for ',' and I;*
- *commutative monoid equations for ';' and $\top$;*
- *congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$.*

▶ **Definition 4** (Size of formulas and bunches). *Let ϕ be a formula and Γ be a bunch. The size of ϕ (denoted by $|\phi|$) is as*

$$\begin{aligned}
|\phi| &= 1 && (\phi = I \text{ or } \top \text{ or } A \text{ or } P);\\
|\phi| &= |\psi| + |\psi'| + 1 && (\phi = \psi * \psi' \text{ or } \psi \wedge \psi').
\end{aligned}$$

The size of Γ (denoted by $|\Gamma|$) is as

$$\begin{aligned}
|\Gamma| &= |\phi| && (\Gamma = \phi);\\
|\Gamma| &= |\Delta| + |\Delta'| + 1 && (\Gamma = \Delta, \Delta' \text{ or } \Delta; \Delta').
\end{aligned}$$

▶ **Definition 5** (Inductive definition). *An* inductive definition clause *of P is of the form $P := \phi$. For a set Φ of inductive definition clauses of inductive propositions, we define $\Phi_P = \{\phi \mid P := \phi \in \Phi\}$. We say that P is defined by $P := \phi_1 \mid \cdots \mid \phi_k$ in Φ if and only if $\Phi_P = \{\phi_1, \cdots, \phi_k\}$.*

▶ **Definition 6** (BI_{ID0} sequent). *Let Γ be a bunch and ϕ be a formula. $\Gamma \vdash \phi$ is called a* BI_{ID0} sequent. *Γ is called the* antecedent *of $\Gamma \vdash \phi$ and ϕ is called the* succedent *of $\Gamma \vdash \phi$. We define $L(\Gamma \vdash \phi) = \Gamma$ and $R(\Gamma \vdash \phi) = \phi$.*

2.2 Semantics of BI_{ID0}

We recall a *standard model* [3] as the semantics of BI_{ID0}. In the following, we fix a set Φ of inductive definition clauses.

▶ **Definition 7** (BI_{ID0} standard model). A BI_{ID0} standard model *is a tuple $M = (\langle R, \circ, e\rangle, \mathbf{A}^M)$ satisfying the following:*

- *$\langle R, \circ, e\rangle$ is a partial commutative monoid with the unit e;*
- *$\mathbf{A}^M$ is a set consisting of $A^M \subseteq R$ for each atomic proposition A.*

Let M be a BI_{ID0} standard model and let $r \in R$. We define the satisfaction relation $M, r \models \phi$ by

$$\begin{aligned}
M, r \models \top &\iff true \\
M, r \models I &\iff r = e \\
M, r \models A &\iff r \in A^M \text{ (for atomic proposition } A) \\
M, r \models P^{(0)} &\quad \text{never holds} \\
M, r \models P^{(m+1)} &\iff M, r \models \phi[P_1^{(m)}, \ldots, P_k^{(m)}/P_1, \ldots, P_k] \\
&\quad\ \ \text{for some } \phi \in \Phi_P \text{ containing inductive propositions } P_1, \ldots, P_k \\
M, r \models P &\iff M, r \models P^{(m)} \text{ for some } m \\
M, r \models \phi_1 \wedge \phi_2 &\iff M, r \models \phi_1 \text{ and } M, r \models \phi_2 \\
M, r \models \phi_1 * \phi_2 &\iff r = r_1 \circ r_2 \text{ and } M, r_1 \models \phi_1 \text{ and } M, r_2 \models \phi_2 \text{ for some } r_1, r_2 \in R\ ,
\end{aligned}$$

where $P^{(m)}$ are auxiliary proposition symbols, and $\phi[P_1^{(m)}, \ldots, P_k^{(m)}/P_1, \ldots, P_k]$ is the formula obtained by replacing each P_i by $P_i^{(m)}$. We define $M, r \models \Gamma$ as $M, r \models \phi_\Gamma$.

By defining in this way, the satisfaction relation for inductive propositions is the same as that in the standard model of [3].

▶ **Definition 8** (Validity). *Let M be a standard model. A sequent $\Gamma \vdash \phi$ is true in M, denoted by $\Gamma \models_M \phi$, if and only if, $M, r \models \Gamma$ implies $M, r \models \phi$ for any r. A sequent $\Gamma \vdash \phi$ is valid, denoted by $\Gamma \models \phi$, if and only if, it is true for any standard models. $\Gamma \models_M \Delta$ and $\Gamma \models \Delta$ are similarly defined.*

▶ **Example 9.** An example of the standard models is the *multiset model*. Let the set of atomic propositions Σ be $\{A, B\}$. The multiset model M_{multi} for Σ is the tuple $(\langle R_{\text{multi}}, \uplus, \emptyset\rangle, \mathbf{A}^{M_{\text{multi}}})$ such that

- R_{multi} is the set of multisets consisting of a and b;
- $\uplus$ is the merging operation of two multisets;
- A^M and B^M are $\{\{\mathsf{a}\}\}$ and $\{\{\mathsf{b}\}\}$, respectively.

For example, $M_{\text{multi}}, \{\mathsf{a}\} \models A$, $M_{\text{multi}}, \{\mathsf{a}, \mathsf{b}\} \models A * B$, and $M_{\text{multi}}, \{\mathsf{a}, \mathsf{a}\} \models A * A * I$ are true, and $M_{\text{multi}}, \{\mathsf{a}\} \models B$ and $M_{\text{multi}}, \{\mathsf{a}\} \models A * A$ are false.

2.3 Inference rules of LBI_{ID0} and $CLBI^{\omega}_{ID0}$

This and the next subsection define two proof systems LBI_{ID0} and $CLBI^{\omega}_{ID0}$. The system LBI_{ID0} is a non-cyclic proof system and the system $CLBI^{\omega}_{ID0}$is a cyclic proof system. The common inference rules of them are given as follows.

▶ **Definition 10.** *The common inference rules of the proof systems LBI_{ID0} and $CLBI^{\omega}_{ID0}$ are the following.*

$$\frac{}{\phi \vdash \phi}\ (Ax) \qquad \frac{\Gamma \vdash \phi \quad \Delta(\phi) \vdash \psi}{\Delta(\Gamma) \vdash \psi}\ (Cut),$$

$$\frac{\Gamma(\Delta) \vdash \phi}{\Gamma(\Delta; \Delta') \vdash \phi}\ (W) \qquad \frac{\Gamma(\Delta; \Delta) \vdash \phi}{\Gamma(\Delta) \vdash \phi}\ (C) \qquad \frac{\Gamma \vdash \phi}{\Delta \vdash \phi}\ (E) \quad (\Delta \equiv \Gamma),$$

$$\frac{\Gamma(\phi, \psi) \vdash \chi}{\Gamma(\phi * \psi) \vdash \chi}\ (*L) \qquad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi * \psi}\ (*R) \qquad \frac{\Gamma(\phi; \psi) \vdash \chi}{\Gamma(\phi \wedge \psi) \vdash \chi}\ (\wedge L) \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ (\wedge R).$$

$$\frac{\Gamma(\phi_1) \vdash \phi \quad \cdots \quad \Gamma(\phi_n) \vdash \phi}{\Gamma(P) \vdash \phi} (UL) \qquad \frac{\Gamma \vdash \phi_i}{\Gamma \vdash P} (UR) \quad (1 \leq i \leq n),$$

where the inductive predicate P is defined by $P := \phi_1 \mid \ldots \mid \phi_n$. (UL) and (UR) are called unfolding rules. The formula ϕ in (Cut) is called its cut formula.

2.4 Proofs in LBI_{ID0} and $CLBI^{\omega}_{ID0}$

Let Seq be the set of the BI_{ID0} sequents, Rules be the set of the common inference rules of LBI_{ID0} and $CLBI^{\omega}_{ID0}$, and Rules^{+} be the set $\mathrm{Rules} \cup \{(Bud)\}$.

▶ **Definition 11** (LBI_{ID0} Proof)**.** *An LBI_{ID0}* proof *is a tuple $\mathrm{Pr} = (N, l, r)$ satisfying the following:*

- *N is the set of nodes for a finite tree. The elements of N are strings of positive integers, the root is the empty string ε, and children of v are $v1$, $v2, \ldots$, where vi is a concatenation of the string v and the integer i.*
- *$l : N \to \mathrm{Seq}$ is a label function.*
- *$r : N \to \mathrm{Rules}$ is a rule function.*
- *If $r(v) \in \mathrm{Rules}$ is a rule with n premises, then v has exactly n children, and $\dfrac{l(v1) \quad \ldots \quad l(vn)}{l(v)}\ r(v)$ is a correct rule instance of LBI_{ID0}.*

An LBI_{ID0} proof $\mathrm{Pr} = (N, l, r)$ is called an LBI_{ID0} proof of $l(\varepsilon)$. When $r(v)$ is not (Cut) for any $v \in N$, Pr is called a cut-free LBI_{ID0} proof.

▶ **Definition 12** ($CLBI^{\omega}_{ID0}$ pre-proof)**.** *A $CLBI^{\omega}_{ID0}$ pre-proof is a tuple $\mathrm{Pr} = (N, l, r, \rho)$ satisfying the following:*

- *N and l are defined similarly as those of the LBI_{ID0} proofs.*
- *$r : N \to \mathrm{Rules}^{+}$ is a rule function.*
- *$\rho : \{v \in N \mid r(v) = (Bud)\} \to N$ is a bud-companion function.*
- *If $r(v) \in \mathrm{Rules}$ is a rule with n premises, then v has exactly n children, and $\dfrac{l(v1) \quad \ldots \quad l(vn)}{l(v)}\ r(v)$ is a correct rule instance.*
- *If $r(v) = (Bud)$, then v has no child and we have $l(v) = l(\rho(v))$.*

When $r(v) = (Bud)$, v is called a bud, *and $\rho(v)$ is called the* companion *of v.*

▶ **Definition 13** (Path)**.** *Let $\mathrm{Pr} = (N, l, r, \rho)$ be a $CLBI^{\omega}_{ID0}$ pre-proof. The proof graph $G(\mathrm{Pr})$ is a directed graph whose set of the nodes are N, and which has an edge from v to v' if and only if either v' is a child of v or v' is the companion of v. A* path *in Pr is a path in $G(\mathrm{Pr})$.*

The path of LBI_{ID0} is defined in the same way except for the bud-companion edges. We consider both finite and infinite paths in proofs. We use α for either a natural number or the ordinal ω, and we denote a path by $(v_i)_{i<\alpha}$.

▶ **Definition 14** (Trace)**.** *Let $(v_i)_{i<\alpha}$ be a path in a $CLBI^{\omega}_{ID0}$ pre-proof Pr. A* trace *along $(v_i)_{i<\alpha}$ is a sequence of occurrences of inductive predicates $(P_i)_{i<\alpha}$ such that each P_i occurs in $L(l(v_i))$, and satisfies the following conditions:*

- *If $r(v_i) = (UL)$ and P_i is unfolded by this rule instance, P_{i+1} appears as a subformula in the unfolding result of P_i in $L(l(v_{i+1}))$. In this case, i is called a* progressing point *of the trace $(P_i)_{i<\alpha}$.*
- *Otherwise, P_{i+1} is the subformula occurrence in $L(l(v_{i+1}))$ corresponding to P_i in $L(l(v_i))$.*

If a trace contains infinitely many progressing points, it is called an infinitely progressing trace.

▶ **Definition 15** ($CLBI^{\omega}_{ID0}$ Proof). *A $CLBI^{\omega}_{ID0}$ pre-proof* $\mathrm{Pr} = (N, l, r, \rho)$ *is called a* $CLBI^{\omega}_{ID0}$ proof *when it satisfies the* global trace condition, *that is, for every infinite path* $(v_i)_{i<\omega}$ *in* Pr, *there is an infinitely progressing trace following some tail of the path* $(v_i)_{n\le i<\omega}$. *A $CLBI^{\omega}_{ID0}$ proof* $\mathrm{Pr} = (N, l, r, \rho)$ *is called a $CLBI^{\omega}_{ID0}$ proof of* $l(\varepsilon)$. *When* $r(v)$ *is not* (Cut) *for any* $v \in N$, Pr *is called a* cut-free $CLBI^{\omega}_{ID0}$ proof.

Both the proof systems LBI_{ID0} and $CLBI^{\omega}_{ID0}$ are subsystems of $CLBI^{\omega}_{ID}$ in [3], and hence their soundness follows from the soundness of $CLBI^{\omega}_{ID}$.

▶ **Theorem 16** (Soundness of LBI_{ID0} and $CLBI^{\omega}_{ID0}$). *If* $\Gamma \vdash \phi$ *is provable in either* LBI_{ID0} *or* $CLBI^{\omega}_{ID0}$, *then* $\Gamma \vdash \phi$ *is valid.*

3 Proof Unrolling

In this section, we introduce a new technique, called *proof unrolling*, for constructing a non-cyclic proof from a given cyclic proof: we first define a non-cyclic proof system that is a variant of LBI_{ID0} (say LBI'_{ID0}), and then, for a cyclic proof of $\Gamma \vdash \phi$ in $CLBI^{\omega}_{ID0}$ and Γ' obtained from Γ by unfolding inductive propositions, construct a non-cyclic proof of $\Gamma' \vdash \phi$ in LBI'_{ID0}.

▶ **Definition 17** (Unfolded formula and unfolded bunch). *The set* $\mathrm{Unf}(\phi)$ *of* unfolded formulas *of ϕ is defined with auxiliary sets* $\mathrm{Unf}^{m}(\phi)$, *which is the set of formulas without inductive propositions obtained by at most m-time unfoldings of inductive predicates in ϕ, as follows:*

$$
\begin{aligned}
\mathrm{Unf}(\phi) &= \bigcup_{m} \mathrm{Unf}^{(m)}(\phi);\\
\mathrm{Unf}^{(m)}(\phi) &= \{\phi\} \qquad \textit{(when ϕ is I, $\top$, or an atomic proposition)};\\
\mathrm{Unf}^{(m)}(\phi_1 * \phi_2) &= \{\phi'_1 * \phi'_2 \mid \phi'_1 \in \mathrm{Unf}^{(m)}(\phi_1) \textit{ and } \phi'_2 \in \mathrm{Unf}^{(m)}(\phi_2)\};\\
\mathrm{Unf}^{(m)}(\phi_1 \wedge \phi_2) &= \{\phi'_1 \wedge \phi'_2 \mid \phi'_1 \in \mathrm{Unf}^{(m)}(\phi_1) \textit{ and } \phi'_2 \in \mathrm{Unf}^{(m)}(\phi_2)\};\\
\mathrm{Unf}^{(0)}(P) &= \emptyset;\\
\mathrm{Unf}^{(m+1)}(P) &= \bigcup_{\phi \in \Phi_P} \mathrm{Unf}^{(m)}(\phi).
\end{aligned}
$$

The set $\mathrm{Unf}(\Gamma)$ *of* unfolded bunches *of* Γ *is defined as follows:*

$$
\begin{aligned}
\mathrm{Unf}(\Gamma) &= \mathrm{Unf}(\phi) \qquad \textit{(when $\Gamma = \phi$)}\\
\mathrm{Unf}(\Gamma, \Gamma') &= \{\Delta, \Delta' \mid \Delta \in \mathrm{Unf}(\Gamma) \textit{ and } \Delta' \in \mathrm{Unf}(\Gamma')\}\\
\mathrm{Unf}(\Gamma; \Gamma') &= \{\Delta; \Delta' \mid \Delta \in \mathrm{Unf}(\Gamma) \textit{ and } \Delta' \in \mathrm{Unf}(\Gamma')\}.
\end{aligned}
$$

Before discussing the proof unrolling technique, we define an weakened variant of the rule (Ax) in LBI_{ID0}.

▶ **Definition 18.** *We consider the following inference rule.*

$$
\frac{}{\phi \vdash \psi}\ (Ax') \qquad \phi \in \mathrm{Unf}(\psi)
$$

We define LBI'_{ID0} *as* LBI_{ID0} *in which* (Ax) *is replaced by* (Ax').

▶ **Lemma 19.** *If a sequent is cut-free provable in* LBI'_{ID0}, *then it is cut-free provable in* LBI_{ID0}, *and hence* LBI'_{ID0} *is sound.*

Proof. It is sufficient to prove $\phi \vdash \psi$ is cut-free provable in LBI_{ID0} for any n and $\phi \in \mathrm{Unf}^{(n)}(\psi)$, and it is proved by induction on (n, ψ). The only nontrivial case is the case where $n > 1$, $\psi = P$, and $\phi \in \mathrm{Unf}^{(n)}(P)$. In this case, for some definition clause ψ' of P, we have $\phi \in \mathrm{Unf}^{(n-1)}(\psi')$. By the induction hypothesis, we have $\phi \vdash \psi'$, and hence we have $\phi \vdash P$ by the rule (UR). ◀

▶ **Lemma 20.** *If* $\Delta \in \mathrm{Unf}(\Gamma)$*, then* $\Delta \models \Gamma$ *holds.*

Proof. It is proved by induction on Γ and the soundness of the rule (Ax') by Lemma 19. ◀

▶ **Lemma 21.** *If an* LBI'_{ID0} *proof contains a finite path* $(v_i)_{i\le n}$ *such that* $l(v_0) = \Gamma \vdash \phi$*,* $l(v_n) = \Gamma' \vdash \phi$*, and* $r(v_i)$ *is either* (W)*,* (C)*,* (E)*, or* $(*L)$ *for* $0 \le i < n$*, then we have* $\Gamma \models \Gamma'$*.*

Proof. It is sufficient to show that $\Gamma \models \Gamma'$ holds for any rule instance

$$\frac{\Gamma' \vdash \phi}{\Gamma \vdash \phi}\,(R),$$

where (R) is either (W), (C), (E), or $(*L)$. It is easily proved. ◀

▶ **Lemma 22.** *Let* (R) *be a rule of* $CLBI^{\omega}_{ID0}$ *except for* (Cut)*. If* $\Gamma \vdash \phi$ *is inferred by* (R) *from the premises* $\Gamma_1 \vdash \phi_1, \ldots, \Gamma_n \vdash \phi_n$*, and* $\Delta \in \mathrm{Unf}(\Gamma)$*, we have the following.*

1. *If* $(R) = (Ax)$*,* $\Delta \vdash \phi$ *is inferred by* (Ax')*.*
2. *If* $(R) = (UL)$*,* $\Delta \in \mathrm{Unf}(\Gamma_i)$ *and* $\phi = \phi_i$ *hold for some* i*.*
3. *Otherwise,* $\Delta \vdash \phi$ *is inferred by* (R) *from* $\Delta_1 \vdash \phi_1, \ldots, \Delta_n \vdash \phi_n$ *for some* $\Delta_i \in \mathrm{Unf}(\Gamma_i)$ $(1 \le i \le n)$*.*

Proof.

1. By the definition of (Ax').
2. In the definition of $\Delta \in \mathrm{Unf}(\Gamma)$, we choose an inductive definition clause of P, which is unfolded by the rule (UL). If the clause is i-th one, we can choose a premise $\Gamma_i \vdash \phi$ such that $\Delta \in \mathrm{Unf}(\Gamma_i)$ holds.
3. If (R) is a left rule, by the definition of the unfolded bunches, $\Delta \vdash \phi$ contains the corresponding connectives of the principal formula in $\Gamma \vdash \phi$ for (R). Otherwise, it is easily proved. ◀

▶ **Definition 23** (UL path). *A finite path* $(v_i)_{i\le m}$ *in a cyclic proof* (N, l, r, ρ) *is called a UL path when* $r(v_i)$ *is either* (UL) *or* (Bud) *for any* i *such that* $0 \le i < m$*.*

▶ **Lemma 24** (Proof unrolling). *Let* $\mathrm{Pr}_1 = (N_1, l_1, r_1, \rho_1)$ *be a cut-free* $CLBI^{\omega}_{ID0}$ *proof of* $\Gamma_1 \vdash \phi$ *and* $\Gamma_2 \in \mathrm{Unf}(\Gamma_1)$*. We can construct a cut-free* LBI'_{ID0} *proof* $\mathrm{Pr}_2 = (N_2, l_2, r_2)$ *of* $\Gamma_2 \vdash \phi$ *accompanied with a mapping* $f : N_2 \to N_1$ *such that the following hold:*

- $f(\varepsilon) = \varepsilon$*.*
- *For any* $v \in N_2$*,* $L(l_2(v)) \in \mathrm{Unf}(L(l_1(f(v))))$ *and* $R(l_2(v)) = R(l_1(f(v)))$*.*
- *For any* $v \in N_2$*, there is a UL path* $(v_i)_{0\le i\le m}$ *in* Pr_1 *such that* $v_0 = f(v)$*,* $r_1(v_m) = r_2(v)$*, and* $f(vn) = v_m n$*.*

Proof. (Sketch) We can construct Pr_2 from Pr_1 by unrolling the cyclic structures and choosing the premises of (UL) depending on the definition of the unfolded bunch Γ_2. Lemma 22 guarantees that this construction works well and the global trace condition guarantees that the construction eventually terminates for the unfolded bunch Γ_2 since any infinite path in Pr_1 has an infinitely progressing trace. ◀

$$\dfrac{\dfrac{\dfrac{}{I \vdash I(2)}(Ax)}{I \vdash \mathrm{P}_A}(UR) \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathrm{P}_{AA} \vdash \mathrm{P}_A(8)(\dagger) \quad \dfrac{}{A \vdash A(9)}(Ax)}{\mathrm{P}_{AA}, A \vdash \mathrm{P}_A * A(7)}(*R)}{\mathrm{P}_{AA}, A \vdash \mathrm{P}_A(6)}(UR) \quad \dfrac{}{A \vdash A(10)}(Ax)}{(\mathrm{P}_{AA}, A), A \vdash \mathrm{P}_A * A(5)}(*R)}{(\mathrm{P}_{AA}, A), A \vdash \mathrm{P}_A(4)}(UR)}{\mathrm{P}_{AA} * A, A \vdash \mathrm{P}_A(3)}(*L)}{\mathrm{P}_{AA} * A * A \vdash \mathrm{P}_A}(*L)}{\mathrm{P}_{AA} \vdash \mathrm{P}_A(1)(\dagger)}(UL)$$

Figure 3 $CLBI^{\omega}_{ID0}$ proof of $\mathrm{P}_{AA} \vdash \mathrm{P}_A$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{I \vdash I(2)}(Ax')}{I \vdash \mathrm{P}_A(8)}(UR)}{I, A \vdash \mathrm{P}_A * A(7)}(*L)}{I, A \vdash \mathrm{P}_A(6)}(UR) \quad \dfrac{}{A \vdash A(10)}(Ax')}{(I, A), A \vdash \mathrm{P}_A * A(5)}(*L)}{(I, A), A \vdash \mathrm{P}_A(4)}(UR)}{I * A, A \vdash \mathrm{P}_A(3)}(*L)}{I * A * A \vdash \mathrm{P}_A(8)}(*L) \quad \dfrac{}{A \vdash A(9)}(Ax')}{I * A * A, A \vdash \mathrm{P}_A * A(7)}(*R)}{I * A * A, A \vdash \mathrm{P}_A(6)}(UR) \quad \dfrac{}{A \vdash A(10)}(Ax')}{(I * A * A, A), A \vdash \mathrm{P}_A * A(5)}(*R)}{(I * A * A, A), A \vdash \mathrm{P}_A(4)}(UR)}{I * A * A * A, A \vdash \mathrm{P}_A(3)}(*L)}{I * A * A * A * A \vdash \mathrm{P}_A(1)}(*L)$$

Figure 4 LBI'_{ID0} proof of $I * A * A * A * A \vdash \mathrm{P}_A$ constructed by proof unrolling.

Intuitively, a cyclic proof of $\Gamma \vdash \phi$ contains several (possibly infinite) cases according to the unfolding of inductive propositions in Γ. The proof unrolling technique takes one case among them by $\Gamma' \in \mathrm{Unf}(\Gamma)$ and extracts a non-cyclic proof of $\Gamma' \vdash \phi$ from the cyclic proof of $\Gamma \vdash \phi$.

▶ **Example 25.** We consider two inductive propositions P_A and P_{AA}, which are defined by

$$\mathrm{P}_A := I \mid \mathrm{P}_A * A \qquad\qquad \mathrm{P}_{AA} := I \mid \mathrm{P}_{AA} * A * A.$$

For these inductive propositions, the sequent $\mathrm{P}_{AA} \vdash \mathrm{P}_A$ is provable in $CLBI^{\omega}_{ID0}$ as Figure 3. The sequents marked (†) are corresponding bud and companion. The numbers (1), (2), ... are identifiers of sequents.

From this cyclic proof, we can construct an LBI'_{ID0} (non-cyclic) proof of $I * A * A * A * A \vdash P_A$ for $I * A * A * A * A \in \mathrm{Unf}(P_{AA})$ by the proof unrolling as Figure 4. The identifiers of sequents indicate the corresponding nodes in the cyclic proof, where we unroll the cycle at (†) twice, and for (UL) in the cyclic proof, we choose the right premise twice at (3) and the left premise at (2).

$$
\dfrac{\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}(@) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{P}_A \vdash \mathrm{P}_A}(Ax) \quad \dfrac{}{A \vdash A}(Ax)}{\mathrm{P}_A, A \vdash \mathrm{P}_A * A}(*R)}{\mathrm{P}_A, A \vdash \mathrm{P}_A}(UR)}{\mathrm{P}_A, A \vdash \mathrm{P}_{BA}}(UR) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathrm{P}_{BA}, A \vdash \mathrm{P}_{BA}(\#) \quad \dfrac{}{B \vdash B}(Ax)}{(\mathrm{P}_{BA}, A), B \vdash \mathrm{P}_{BA} * B}(*R)}{(\mathrm{P}_{BA}, B), A \vdash \mathrm{P}_{BA} * B}(E)}{(\mathrm{P}_{BA}, B), A \vdash \mathrm{P}_{BA}}(UR)}{\mathrm{P}_{BA} * B, A \vdash \mathrm{P}_{BA}}(*L)}{\mathrm{P}_{BA}, A \vdash \mathrm{P}_{BA}(\#)}(UL)}{\dfrac{\mathrm{P}_{AB}, A \vdash \mathrm{P}_{BA}}{\mathrm{P}_{AB} * A \vdash \mathrm{P}_{BA}\,(1)}(*L)}(Cut)
$$

is the subproof of the following proof figure:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{I \vdash I}(Ax)}{I \vdash \mathrm{P}_A}(UR)}{I \vdash \mathrm{P}_{BA}}(UR) \qquad \dfrac{\dfrac{\dfrac{\mathrm{P}_B \vdash \mathrm{P}_{BA}(\dagger) \quad \dfrac{}{B \vdash B}(Ax)}{\mathrm{P}_B, B \vdash \mathrm{P}_{BA} * B}(*R)}{\mathrm{P}_B, B \vdash \mathrm{P}_{BA}}(UR)}{\mathrm{P}_B * B \vdash \mathrm{P}_{BA}}(*L)}{\mathrm{P}_B \vdash \mathrm{P}_{BA}(\dagger)}(UL) \qquad \begin{array}{c} \vdots \text{ the above proof figure} \\ \mathrm{P}_{AB} * A \vdash \mathrm{P}_{BA}\,(1) \end{array}}{\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}(@)}(UL) \quad ,
$$

Each bud marked (†), (@), or (#) has its companion with the same mark.

Figure 5 $CLBI^{\omega}_{ID0}$ proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$.

4 Failure of Cut-Elimination

In this section, we give a counterexample of the cut-elimination property in $CLBI^{\omega}_{ID0}$. We fix the language Σ consisting of the atomic propositions A and B, and the inductive propositions P_{AB}, P_{BA}, P_A, and P_B. We also fix the set Φ of inductive definitions for P_{AB}, P_{BA}, P_A, and P_B defined by:

$$\mathrm{P}_{AB} := \mathrm{P}_B \mid \mathrm{P}_{AB} * A; \qquad \mathrm{P}_A := I \mid \mathrm{P}_A * A;$$
$$\mathrm{P}_{BA} := \mathrm{P}_A \mid \mathrm{P}_{BA} * B; \qquad \mathrm{P}_B := I \mid \mathrm{P}_B * B.$$

Intuitively, P_A and P_B mean $I * A^n$ and $I * B^m$ with arbitrary $n, m \geq 0$, respectively. P_{AB} and P_{BA} mean $(I * B^m) * A^n$ and $(I * A^m) * B^n$ with arbitrary $n, m \geq 0$, respectively. We note that P_{AB} and P_{BA} are logically equivalent in the standard models since the separating conjunction $*$ and the formula I are interpreted as a commutative monoid operator and the unit of it, respectively.

The intention of the name P_{AB} is that, during the unfolding of P_{AB}, A's appear first, and then B's appear in the unfolding of P_B. P_{BA} is also named by a similar intention.

Our main result will be obtained by showing the entailment $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is a counter-example for the cut-elimination. We need to show two things: One is that $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is provable in $CLBI^{\omega}_{ID0}$ with (Cut), and the other is that $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is not cut-free provable in $CLBI^{\omega}_{ID0}$.

First, we show that $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is provable in $CLBI^{\omega}_{ID0}$ with (Cut).

▶ **Proposition 26.** $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ *is provable in* $CLBI^{\omega}_{ID0}$.

Proof. The proof figures in Figure 5 show this proposition. ◀

To show that $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is not cut-free provable in $CLBI^{\omega}_{ID0}$, we assume that it is cut-free provable to derive a contradiction. For this purpose, we will consider only the multiset model M_{multi} introduced in Example 9. We omit M_{multi} in the satisfaction relation, that is, $r \models \phi$ means $M_{\mathrm{multi}}, r \models \phi$. We write $\{\mathsf{a}^n\}$ for the multiset consisting of n a's.

We shall describe our proof approach before starting the formal discussion. We assume the existence of a cut-free cyclic proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$. By the proof unrolling, we can construct proofs of $\phi \vdash \mathrm{P}_{BA}$ in LBI'_{ID0} for any unfolded formula ϕ of P_{AB}. Hence we have proofs of $I * A^n \vdash \mathrm{P}_{BA}$ for arbitrary n. We consider parts of the proofs of $I * A^n \vdash \mathrm{P}_{BA}$ which contain the conclusion and do not contain the rule (UR). We call such parts the proof segments. In such a proof segment, $\{\mathsf{a}^n\} \in M_{\mathrm{multi}}$ satisfies every antecedent. Then, $\{\mathsf{a}^n\}$ also satisfies every antecedent in the corresponding part of the cyclic proof. Since the cyclic proof is finite, for a sufficiently large n, the antecedents cannot contain A^n, but they must contain either P_{AB} or $\top$, and then both $\{\mathsf{a}^n\}$ and $\{\mathsf{a}^n, \mathsf{b}\}$ satisfy the antecedents. On the other hand, since the proof segment does not contain (UR), every succedent is P_{BA}. When we unfold P_{BA}, we have to decide either P_A or $\mathrm{P}_{BA} * B$. However, neither of them can be satisfied by both $\{\mathsf{a}^n\}$ and $\{\mathsf{a}^n, \mathsf{b}\}$.

To achieve our plan, we prepare some definitions and theorems.

▶ **Definition 27** (P_{AB}-formula and P_{AB}-bunch)**.** *A P_{AB}-formula $\phi_{\mathrm{P}_{AB}}$ is defined as follows:*

$$\phi_{\mathrm{P}_{AB}} ::= I \mid \top \mid A \mid B \mid \mathrm{P}_{AB} \mid \mathrm{P}_B \mid \mathrm{P}_{AB} * A \mid \mathrm{P}_B * B.$$

A P_{AB}-bunch $\Gamma_{\mathrm{P}_{AB}}$ is a bunch all of whose leaves are P_{AB}-formulas.

▶ **Lemma 28.** *Let (N, l, r, ρ) be a cut-free $CLBI^{\omega}_{ID0}$ proof of $\mathrm{P}_{AB} \vdash \phi$. For any $v \in N$, $L(l(v))$ is a P_{AB}-bunch.*

Proof. This lemma is proved by induction on the size of N. ◀

▶ **Lemma 29.** *Let Γ be a P_{AB}-bunch. If we have $\{\mathsf{a}^i\} \models \Gamma$ for $i > 2^{|\Gamma|}$, then we also have $\{\mathsf{a}^i, \mathsf{b}\} \models \Gamma$.*

Proof. It is proved by induction on Γ. The only nontrivial case is the case of $\Gamma = \Delta, \Delta'$. In this case, we have $\{\mathsf{a}^j\} \models \Delta$ and $\{\mathsf{a}^{j'}\} \models \Delta'$ for some j and j' such that $j + j' = i$. By the assumption, we have $i > 2 \cdot 2^{|\Gamma|-1} > 2 \cdot 2^{\max(|\Delta|,|\Delta'|)}$. Hence either $j > 2^{|\Delta|}$ or $j' > 2^{|\Delta'|}$ holds. By the induction hypothesis, we have either $\{\mathsf{a}^j, \mathsf{b}\} \models \Delta$ or $\{\mathsf{a}^{j'}, \mathsf{b}\} \models \Delta'$ holds. Therefore we have $\{\mathsf{a}^i, \mathsf{b}\} \models \Gamma$. ◀

▶ **Definition 30** (Proof segment)**.** *Let $Pr_1 = (N_1, l_1, r_1)$ be a LBI'_{ID0} proof. $Pr = (N_2, l_2, r_2)$ is a proof segment of Pr_1 when it enjoys the following conditions:*

- *$N_2 \subseteq N_1$ holds, and $vi \in N_2$ implies $v \in N_2$.*
- *For any $v \in N_2$, $l_2(v) = l_1(v)$ and $r_2(v) = r_1(v)$ hold.*

Note that leaves of a proof segment are not necessarily assigned the rule (Ax').

▶ **Proposition 31.** $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ *is not cut-free provable in $CLBI^{\omega}_{ID0}$.*

Proof. This proposition is shown by contradiction. We assume that there is a cut-free $CLBI^{\omega}_{ID0}$ proof $\mathrm{Pr}_1 = (N_1, l_1, r_1, \rho_1)$ of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$. Let $n = max\{|L(l_1(v))| \mid v \in N_1\}$.

Since $I * A^{2^n+1} \in \mathrm{Unf}(\mathrm{P}_{AB})$, we can construct a cut-free LBI'_{ID0} proof $\mathrm{Pr}_2 = (N_2, l_2, r_2)$ of $I * A^{2^n+1} \vdash \mathrm{P}_{BA}$ and the mapping $f : N_2 \to N_1$ by Lemma 24.

Let $\mathrm{Pr}_2^{BA} = (N_2^{BA}, l_2^{BA}, r_2^{BA})$ be the biggest proof segment of Pr_2 such that $R(l_2^{BA}(v)) = \mathrm{P}_{BA}$ for any $v \in N_2^{BA}$. Note that Pr_2^{BA} is not empty since $R(l_2(\varepsilon)) = \mathrm{P}_{BA}$. For any $v \in N_2^{BA}$, $r_2^{BA}(v)$ is either (W), (C), $(*L)$, (E), (Ax'), or (UR). In particular, (Ax') and

(UR) are only applied to leaves of Pr_2^{BA}, and the other rules are not applied to leaves since these rules do not change the succedents. We have $\{\mathsf{a}^{2^n+1}\} \models I * A^{2^n+1}$ in the multiset model, and hence we have $\{\mathsf{a}^{2^n+1}\} \models L(l_2^{BA}(v))$ holds for any $v \in N_2^{BA}$ by Lemma 21.

Let v be a leaf node of Pr_2^{BA}. Then, $r_2^{BA}(v)$ is either (Ax') or (UR).

In the case of (Ax'), by Lemma 24, there is a UL path from $f(v)$ to some v' in Pr_1 such that $r_1(v') = (Ax)$. By Lemma 28, $l_1(v') = \Gamma \vdash \mathrm{P}_{BA}$ for some P_{AB}-bunch Γ, and it contradicts $r_1(v') = (Ax)$ since P_{BA} is not a P_{AB}-bunch. Hence, (Ax') is not the case.

In the case of (UR), let v' be the premise of v in Pr_2. Since we have $l_2^{BA}(v) = l_2(v) = \Gamma \vdash \mathrm{P}_{BA}$ for some Γ, $l_2(v')$ is either $\Gamma \vdash \mathrm{P}_{BA} * B$ or $\Gamma \vdash \mathrm{P}_A$, but it is proved as follows that both of them are not the case.

For $l_2(v') = \Gamma \vdash \mathrm{P}_{BA} * B$, we have $\{\mathsf{a}^{2^n+1}\} \models \Gamma$ and $\{\mathsf{a}^{2^n+1}\} \not\models \mathrm{P}_{BA} * B$, and hence $\Gamma \vdash \mathrm{P}_{BA} * B$ is invalid. It contradicts the soundness of LBI'_{ID0}. Hence, this is not the case.

For $l_2(v') = \Gamma \vdash \mathrm{P}_A$, we have $l_1(f(v')) = \Gamma' \vdash \mathrm{P}_A$ for some P_{AB}-bunch Γ' such that $\Gamma \in \mathrm{Unf}(\Gamma')$. Then, we have $\{\mathsf{a}^{2^n+1}\} \models \Gamma'$ by Lemma 20, and $\{\mathsf{a}^{2^n+1}, \mathsf{b}\} \models \Gamma'$ by Lemma 29 and $2^n + 1 > 2^{|\Gamma'|}$. Since $\{\mathsf{a}^{2^n+1}, \mathsf{b}\} \not\models \mathrm{P}_A$, it contradicts the soundness of LBI'_{ID0}. Hence, this is not the case.

Therefore, there is no possible rule at the leaves of Pr_2^{BA}, and hence there is no cut-free $CLBI^{\omega}_{ID0}$ proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$. ◀

▶ **Theorem 32** (Failure of cut-elimination in $CLBI^{\omega}_{ID0}$). *$CLBI^{\omega}_{ID0}$ does not enjoy the cut-elimination property.*

Proof. By Proposition 26 and Proposition 31, $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is a counterexample. ◀

This result is easily extended to the original cyclic proof system $CLBI^{\omega}_{ID}$ in [3], which contains full logical connectives of the bunched logic and inductive predicates with arbitrary arity.

▶ **Corollary 33** (Failure of cut elimination in $CLBI^{\omega}_{ID}$). *$CLBI^{\omega}_{ID}$ does not enjoy cut-elimination property.*

Proof. $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is a counterexample. It is provable in $CLBI^{\omega}_{ID}$, since the proof in Figure 5 is also a $CLBI^{\omega}_{ID}$ proof with cuts. If there is a cut-free $CLBI^{\omega}_{ID}$ proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$, it is a cut-free $CLBI^{\omega}_{ID0}$ proof since neither logical connectives other than $*$, inductive predicates accompanied by some arguments, nor first-order terms can occur in the proof. ◀

5 Conclusion and Future Work

We have proved by the proof unrolling technique that the cut-elimination fails for the cyclic proof system of the bunched logic $CLBI^{\omega}_{ID}$ in [3] only with nullary inductive predicates.

For a logic with a connective representing resource composition such as the separation logic and the multiplicative linear logic, we can straightforwardly adapt our proof technique to the cyclic proof system for the logic.

For the separation logic, we allow arbitrary substitution in the definition of Unf for existentially quantified variables as

$$\mathrm{Unf}^{(m+1)}(P) = \bigcup_{\exists \vec{x}.\phi(\vec{x}) \in \Phi_P \text{ and } \vec{t}\,:\text{ arbitrary terms}} \mathrm{Unf}^{(m)}(\phi(\vec{t})),$$

and we reread the atomic propositions A and B in our proof as to the following nullary predicates, for example,

$$A = \exists x(x \mapsto x) \qquad\qquad B = \exists x(x \mapsto \mathrm{nil}),$$

and then we can prove that the cut-elimination fails for the cyclic proof system of the separation logic with only nullary predicates.

We can adapt the proof unrolling to cyclic proof system $CLKID^\omega$ [6] for the first-order logic when we consider a cut-free cyclic proof that contains only positive occurrences of inductive predicates. However, the proof in Section 4 depends on the multiset model, and it is an interesting question if we can apply our proof idea for the first-order logic. Another direction of future work is to find reasonable restrictions for the inductive predicates to recover the cut-elimination property in the cyclic proof systems. Our result shows that the restriction on the arity of predicates is not sufficient.

References

1. T. Antonopoulos, N. Gorogiannis, C. Haase, M. Kanovich, and J. Ouaknine. Foundations for decision problems in separation logic with general inductive predicates. In *Proceedings of FoSSaCS 2014*, volume 8412 of *LNCS*, pages 411–425, 2014.
2. D. Baelde, A. Doumane, and A. Saurin. Infinitary proof theory: the multiplicative additive case. In *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, volume 62 of *LIPIcs*, pages 42:1–42:17, 2016.
3. J. Brotherston. Formalised inductive reasoning in the logic of bunched implications. In *Proceedings of SAS '07*, volume 4634 of *LNCS*, pages 87–103, 2007.
4. J. Brotherston, D. Distefano, and R. L. Petersen. Automated cyclic entailment proofs in separation logic. In *Proceedings of CADE-23*, volume 6803 of *LNAI*, pages 131–146, 2011.
5. J. Brotherston, N. Gorogiannis, and R. L. Petersen. A generic cyclic theorem prover. In *Proceedings of APLAS 2012*, volume 7705 of *LNCS*, pages 350–367, 2012.
6. J. Brotherston and A. Simpson. Sequent calculi for induction and infinite descent. *Journal of Logic and Computation*, 21(6):1177–1216, 2011.
7. J. Fortier and L. Santocanale. Cuts for circular proofs: semantics and cut-elimination. In *Computer Science Logic 2013 (CSL 2013)*, volume 23 of *LIPIcs*, pages 248–262, 2013.
8. D. Kimura, K. Nakazawa, T. Terauchi, and H. Unno. Failure of cut-elimination in cyclic proofs of separation logic. *Comupter Software*, 37(1):39–52, 2020.
9. P. O'Hearn and D. Pym. The logic of bunched implications. *Bulletin of Symbolic Logic*, 5(2):215–244, 1999.
10. D. J. Pym. *The Semantics and Proof Theory of the Logic of Bunched Implications*. Springer, 2002.
11. J. C. Reynolds. Separation logic: A logic for shared mutable data structures. In *Proceedings of LICS 2002*, pages 55–74, 2002.
12. M. Tatsuta and D. Kimura. Separation logic with monadic inductive definitions and implicit existentials. In *Proceedings of APLAS 2015*, volume 9458 of *LNCS*, pages 69–89, 2015.