

# Revisiting Priority $k$ -Center: Fairness and Outliers

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## Abstract

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In the *Priority  $k$ -Center* problem, the input consists of a metric space  $(X, d)$ , an integer  $k$  and for each point  $v \in X$  a priority radius  $r(v)$ . The goal is to choose  $k$ -centers  $S \subseteq X$  to *minimize*  $\max_{v \in X} \frac{1}{r(v)} d(v, S)$ . If all  $r(v)$ 's were uniform, one obtains the classical  $k$ -center problem. Plesník [32] introduced this problem and gave a 2-approximation algorithm matching the best possible algorithm for vanilla  $k$ -center. We show how the Priority  $k$ -Center problem is related to two different notions of *fair* clustering [23, 28]. Motivated by these developments we revisit the problem and, in our main technical contribution, develop a framework that yields constant factor approximation algorithms for Priority  $k$ -Center with *outliers*. Our framework extends to generalizations of Priority  $k$ -Center to matroid and knapsack constraints, and as a corollary, also yields algorithms with fairness guarantees in the lottery model of Harris et al.

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## 1 Introduction

Clustering is a basic task in a variety of areas, and clustering problems are ubiquitous in practice, and are well-studied in algorithms and discrete optimization. Recently *fairness* has become an important concern as automated data analysis and decision making have become increasingly prevalent in society. This has motivated several problems in fair clustering and associated algorithmic challenges. In this paper, we show that two different fairness views are inherently connected with a previously studied clustering problem called the *Priority  $k$ -Center* problem.

The input to Priority  $k$ -Center is a metric space  $(X, d)$  and a priority radius  $r(v)$  for each  $v \in X$ . The objective is to choose  $k$ -centers  $S \subseteq X$  such that  $\max_{v \in X} \frac{d(v, S)}{r(v)}$  is minimized. If one imagines clients located at each point in  $X$ , and  $r(v)$  is the “speed” of a client at point  $v$ , then the objective is to open  $k$ -centers so that every client can reach an open center as quickly as possible. When all the  $r(v)$ 's are the same, then one obtains the classic



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$k$ -center problem [26]. Plesník [32] introduced this problem and showed how to generalize Hochbaum and Shmoys' [26] 2-approximation algorithm for the  $k$ -center problem, to obtain a 2-approximation for Priority  $k$ -Center. This approximation ratio is tight since  $(2 - \varepsilon)$ -factor approximation is ruled out even for the classic  $k$ -center problem under the assumption that  $P \neq NP$  [19, 25].

**Connections to Fair Clustering.** Our motivation to revisit Priority  $k$ -Center came from two recent papers that considered fair variants in clustering, without explicitly realizing the connection to Priority  $k$ -Center. One of them is the paper of Jung, Kannan and Lutz [28] who defined a version of fair clustering as follows. Given  $(X, d)$  representing clients/people in a geographic area, and an integer  $k$ , for each  $v \in X$  let  $r_\ell(v)$  denote the smallest radius  $r$  such that there are at least  $\ell$  points of  $X$  inside a ball of radius  $r$  around  $v$ . They suggested a notion of fair  $k$ -clustering as one in which each point  $v \in X$  should be served by a center not farther than  $r(v) = r_{n/k}(v)$  since the average size of a cluster in a  $k$ -clustering is  $n/k$ . [28] describe an algorithm that finds  $k$  centers such that each point  $v$  is served by a center at most distance  $\leq 2r_{n/k}(v)$  away from  $v$ . Once the radii are fixed for the points, then one obtains an instance of Priority  $k$ -Center, and the result essentially<sup>1</sup> follows from the algorithm in [32]; indeed, the algorithm in [28] is the same.

Another notion of fairness related to the Priority  $k$ -Center is the *lottery model* introduced by Harris et al. [22]. In this model, every client  $v \in X$  has a “probability demand”  $p(v)$  and a “distance demand”  $r(v)$ . The objective is to find a *distribution*  $\mathcal{S}$  over  $k$ -center locations such that for every client  $v \in X$ ,  $\Pr_{S \sim \mathcal{S}}[d(v, S) \leq r(v)] \geq p(v)$ . One needs to either prove such a solution is not possible, or provide a distribution where the distance to  $S$  can be relaxed to  $\alpha r(v)$ . Using a by now almost standard reduction via the ellipsoid method [6, 1], this boils down to the *outlier* version of Priority  $k$ -Center, where some points in  $X$  are allowed to be discarded. The outlier version of Priority  $k$ -Center had not been explicitly studied before.

**Our Contributions.** Motivated by these connections to fairness, we study the natural generalizations of Priority  $k$ -Center that have been studied for the classical  $k$ -center problem. The main generalization is the *outlier* version of Priority  $k$ -Center: the algorithm is allowed to discard a certain number of points when evaluating the quality of the centers chosen. First, the outlier version arises in the lottery model of fairness. Second, in many situations it is useful and important to discard outliers to obtain a better solution. Finally, it is also interesting from a technical point of view. We also consider the situation when the constraint on where centers can be opened is more general than the cardinality constraint. In particular, we study the matroid priority center problem where the set of centers must be an independent set of a given matroid, and the knapsack priority center problem where the total weight of centers opened is at most a certain amount. Our main contribution is an algorithmic framework to study the outlier problems in all these variations. Our results also imply interesting generalizations for fair clustering.

## 1.1 Statement of Results

We briefly describe some variants of Priority  $k$ -Center. In the supplier version, the metric space is partitioned into facilities  $F$  and clients  $C$ , and goal is to select  $k$  facilities  $S \subseteq F$  to minimize  $\max_{v \in C} d(v, S)/r(v)$ . In the Priority Matroid Supplier problem, the subset of

<sup>1</sup> One needs to observe that Plesník's analysis [32] can be made with respect to a natural LP which has a feasible solution with  $r(v) := r_{n/k}(v)$ .

facilities need to be an independent set of matroid on  $F$ . In the Priority Knapsack Supplier problem, the subset of facilities must have weight at most a certain amount. All these generalizations have a 3-approximation [26, 14] in the vanilla version where all  $r(v)$ 's are the same. Our first observation is that these extend to Priority  $k$ -Center in a simple fashion. This result also implicitly relates the approximation ratio to the integrality gap of the natural LP relaxation. This allows us to rederive and extend the algorithmic results in [28] we give details in Section 6.

► **Result 1.** *There is a 3-approximation for Priority  $k$ -Supplier and Priority Matroid Supplier and Priority Knapsack Supplier.*

Our second, and the main technical contribution, is a general framework to handle *outliers*. Given an instance of Priority  $k$ -Center and an integer  $m \leq n$ , the outlier version that we refer to as  $PkCO$ , is to find  $k$  centers  $S$  and a set  $C'$  of at least an  $m$  points from  $C$  such that  $\max_{v \in C'} \frac{1}{r(v)} d(v, S)$  is minimized. While the  $k$ -Center with outliers admits a clever, yet relatively simple, greedy 3-approximation due to Charikar et al. [10], a similar approach seems difficult to adapt for Priority  $k$ -Center. Instead, we take a more general and powerful LP-based approach from [7, 8] to develop a framework to handle  $PkCO$ , and also the outlier version of Matroid Center (PMCO), where the opened centers must be an independent set, and Knapsack Center (PKnapCO), where the total weight of the open centers must fit in a budget. We obtain the following results.

► **Result 2.** *There is a 9-approximation for  $PkCO$  and PMCO and a 14-approximation for PKnapCO. Moreover the approximation ratio for  $PkCO$  and PMCO are based on a natural LP relaxation.*

At this point we remark that a result in Harris et al. [22] (Theorem 2.8 in the arXiv version) also indirectly gives a 9-approximation for  $PkCO$ . We believe that our framework is more general and can handle PMCO and PKnapCO easily. The [22] paper do not consider these versions, and indeed for the PKnapCO problem their framework cannot give a constant factor approximation for they (in essence) use a weak LP relaxation.

Furthermore, our framework yields *better* approximation factors when either the number of distinct priorities are small, or they are in different scales. In practice, one indeed expects this to be the case. In particular, when there are only two distinct types of radii, then we get a 3-approximation which is tight; it is not too hard to show that it is NP-hard to obtain a better than 3-approximation for  $PkCO$  with two types<sup>2</sup> of priorities. We get improved factors (5 and 7) when the number of radii are three and four as well. On the other hand, if all the different priorities are powers of  $b$  (for some parameter  $b > 1$ ), then we get a  $\frac{3b-1}{b-1}$ -approximation. Thus, if all the priorities are in vastly different scales ( $b \rightarrow \infty$ ), then our approximation factor approaches 3.

► **Result 3.** *Suppose there are only two distinct priority radii among the clients. Then there is a 3-approximation for  $PkCO$ , PMCO and PKnapCO. With  $t$  distinct types of priorities, the approximation factor for  $PkCO$  and PMCO is  $2t - 1$ . If all distinct types are powers of  $b$ , the approximation factor for  $PkCO$  and PMCO becomes  $(3b - 1)/(b - 1)$ .*

It is possible that the  $PkCO$  problem has a 3-approximation in general, and even the natural LP-relaxation may suffice; we have not been able to obtain a worse than 3 integrality gap example. As we explain in Section 1.2 below, many approaches to the  $k$ -center type problems

<sup>2</sup> Interestingly, when there is a single priority, the vanilla  $k$ -center with outliers has a 2-approximation [7] showing a gap between the two problems.

begin with a Hochbaum-Shmoys [26] style partition of the points  $X$  to representatives. We could show examples where such an *approach* has a gap worse than 3, though not showing an integrality gap instance. Resolving the integrality gap of the natural LP-relaxation and/or obtaining improved approximation ratios are interesting open questions highlighted by our work.

## 1.2 Technical Discussion

Almost all clustering algorithms for the  $k$ -center objective proceeds via a partitioning subroutine due to Hochbaum and Shmoys [26] (HS, henceforth). This procedure returns a partition  $\Pi$  of  $X$  along with a representative for each part such that all vertices of a part “piggy-back” on the representative. More precisely, if the representative is assigned to a center  $f \in X$ , then so are all other vertices in that part. To ensure a good algorithm in vanilla  $k$ -center, it suffices to ensure the radius of each part is small.

For the Priority  $k$ -Center objective, one needs to be more careful : to use the above idea, one needs to make sure that if vertex  $v$  is piggybacking on vertex  $u$ , then  $r(v)$  better be more than  $r(u)$ . Indeed, this can be ensured by running the HS procedure in a particular order, namely by allowing vertices with smaller  $r(v)$  to form the parts first. This precisely gives Plesník’s algorithm [32]. In fact, this idea easily gives a 3-approximation for the matroid and supplier versions as well.

Outliers are challenging in the setting of Priority  $k$ -Center. We start with the approach of Chakrabarty et al. [7] for  $k$ -center. First, they construct an LP where  $\text{cov}(v)$  denotes the fractional coverage (amount to which one is *not* an outlier) of any point, and then write a natural LP for it. They show that if the HS algorithm is run according to the  $\text{cov}(v)$  order (higher coverage vertices first), then the resulting partition can be used to obtain a 2-approximation for the  $k$ -center with outliers problem.

When one moves to the priority  $k$ -center with outliers, one sees the obvious trouble: what if the  $r(v)$  order and the  $\text{cov}(v)$  order are at loggerheads? Our approach out of this is a simple bucketing idea. We first write a natural LP with fractional coverages  $\text{cov}(v)$  for every point. Then, we partition vertices into classes: all vertices  $v$  with  $r(v)$  between  $2^i$  and  $2^{i+1}$  are in the same class. We then use the HS partitioning algorithm in the decreasing  $\text{cov}(v)$  order separately on each class. The issue now is to handle the interaction across classes. To handle this, we define a directed acyclic graph across these various partitions where representative  $u$  has an edge to representative  $v$  iff  $d(u, v)$  is small ( $\leq r(u) + r(v)$ ). It is a DAG because we point edges from higher  $r(u)$  to the lower  $r(v)$ . Our main observation is that if we can peel out  $k$  paths with “large value” (each representative’s value is how many points piggyback on it), then we can get a 9-approximation for the priority  $k$ -center with outlier problem. We can show that a *fractional* solution of large value does exist using the fact that the DAG was constructed in a greedy fashion. Also, since the graph is a DAG, this LP is an integral min-cost max-flow LP. The factor 9 arises out of a geometric series and bucketing. Indeed, when the radii are exact powers of 2, we get a 5-approximation, and when there are only two type of radii, we get a 3 approximation which is tight.

The above framework can handle the outlier versions for the matroid and knapsack version. For the matroid version, the flow problem is no longer a min-cost max-flow problem, but rather it reduces to a *submodular flow* problem which is solvable in polynomial time. Modulo this, the above framework gives a 9-approximation. For the knapsack version, there are two issues. One is that the flow problem involves non-uniform numbers and is no longer integral and solving the underlying optimization problem is likely to be NP-hard (we did not attempt a formal proof). Nevertheless, our framework has sufficient flexibility that by

increasing the approximation factor from 9 to 14, the DAG can in fact be made into a rooted forest. In this rooted forest, we can employ dynamic programming to solve the problem of finding the desired paths. The second issue is that a fractional LP solution of the natural LP does not suffice when using the DP based algorithm on the forest; indeed the natural LP has an unbounded gap. Here we need to use the round-or-cut framework from [8]; either the DP on the rooted forest succeeds or we find a violated inequality for the large implicit LP that we use.

### 1.3 Other Related Works

There is a huge literature on clustering, and instead of summarizing the landscape, we mention a few works relevant to our paper. Gørtz and Wirth [20] study the priority  $k$ -center problem in the asymmetric metric case, and prove that it is NP-hard to obtain any non-trivial approximation. A related problem to priority  $k$ -clustering is the *non-uniform*  $k$ -center problem by Chakrabarty et al. [7] where instead of clients having radii bounds, the objective is to figure out centers of balls for different types of radii. Another related problem [21] is the local  $k$ -median problem where clients need to connect to facilities within a certain radius, but the objective is the sum instead of the max.

Fairness in clustering has also seen a lot of works recently. Apart from the two notions of fairness described above, which can be thought of as “individual fairness” guarantees, Chierichetti et al. [16] introduce the “group fairness” notion where points have color classes, and each cluster needs to contain similar proportion of colors as in the universe. Their results were generalized by a series of follow ups [33, 5, 4]. A similar concept for outliers led to the study of *fair colorful  $k$ -center*. In this problem, the objective is to find  $k$  centers which covers at least a prescribed number of points from each color class. This was introduced by Bandapadhyay et al. [3], and recently true approximation algorithms were concurrently obtained by Jia et al. [27] and Anegg et al. [1].

Another notion of fairness is introduced by Chen et al. [15] in which a solution is called fair if there is no facility and a group of at least  $n/k$  clients, such that opening that facility lowers the cost of all members of the group. They give a  $(1 + \sqrt{2})$ -approximation for  $L_1$ ,  $L_2$ , and  $L_\infty$  norm distances for the setting where facilities can be places anywhere in the real space. Recently Micha and Shah [31] showed that a modification of the same approach can give a close to 2-approximation for  $L_2$  case and proved  $(1 + \sqrt{2})$  factor is tight for  $L_1$  and  $L_\infty$ .

Coming back to the model of Jung et al. [28], the local notion of neighborhood radius is also present in the metric embedding works of [9, 11] and were recently used by Mahabadi and Vakilian [30] to extend the results in [28] to other objectives such as  $k$ -median and  $k$ -means. We leave the outlier versions of these problems as an open direction of study.

## 2 Preliminaries

We provide some formal definitions and describe a clustering routine from [26].

► **Definition 1** (Priority  $k$ -Center). *The input is a metric space  $(X, d)$  and radius function  $r : X \rightarrow \mathbb{R}^+$ , and integer  $k$ . The goal is to find  $S \subseteq X$  of size at most  $k$  to minimize  $\alpha$  such that for all  $v \in X$ ,  $d(v, S) \leq \alpha \cdot r(v)$*

► **Definition 2** (Priority  $\mathcal{F}$ -supplier). *(Generalization from [8]). The input is a metric space  $(X, d)$  where  $X = F \cup C$ ,  $C$  is the set of points, and  $F$  the set of facilities. We are also given a radius function  $r : C \rightarrow \mathbb{R}^+$ . The goal is to find  $S \subseteq F$  to minimize  $\alpha$  such that for all*

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$v \in C$ ,  $d(v, S) \leq \alpha \cdot r(v)$ . The constraint on  $F$  is that it must be selected from a down-ward closed family  $\mathcal{F}$ . Different families lead to different problems. We get the priority  $k$ -supplier problem if  $\mathcal{F} = \{F : |F| \leq k\}$ . We get the priority matroid supplier problem when  $(F, \mathcal{F})$  is a matroid. We get the priority knapsack supplier problem when there is a weight function  $w : F \rightarrow \mathbb{R}_{\geq 0}$  and  $\mathcal{F} = \{F : w(F) \leq B\}$  for some budget  $B$ .

For the remainder of this manuscript, we focus on the *feasibility* version of the problem. More precisely, given an instance of the problem, we either want to show there is no solution with  $\alpha = 1$ , or find a solution with  $\alpha \leq \rho$ . If we succeed, then via binary search we get a  $\rho$ -approximation.

Plesnik [32] obtained a 2-approximation for Priority  $k$ -Center. Algorithm 1 is a slight generalization of his algorithm; in addition to the radius function and the metric, we take as input a function  $\phi : X \rightarrow \mathbb{R}_{\geq 0}$  which encodes an ordering over the points (we can think of the points as being ordered from largest to smallest  $\phi$  values). The algorithm is a similar procedure to that of Hochbaum and Shmoys from [26], but while [26] picks points arbitrarily, points get picked in the order mandated by  $\phi$ .

► **Fact 1.** *The following is true for the output of HS: (a)  $\forall u, v \in S, d(u, v) > r_u + r_v$ , (b) The set  $\{D(u) : u \in S\}$  partitions  $X$ , (c)  $\forall u \in S, \forall v \in D(u), \phi(u) \geq \phi(v)$ , and (d)  $\forall u \in S, \forall v \in D(u), d(u, v) \leq r_u + r_v$ .*

### ■ Algorithm 1 HS.

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**Input:** Metric  $(X, d)$ , radius function  $r : X \rightarrow \mathbb{R}_{>0}$ , and ordering  $\phi : X \rightarrow \mathbb{R}_{\geq 0}$

- 1:  $U \leftarrow X$  ▷ The set of uncovered points
- 2:  $S \leftarrow \emptyset$  ▷ The set of “representatives”
- 3: **while**  $U \neq \emptyset$  **do**
- 4:  $u \leftarrow \arg \max_{v \in U} \phi(v)$  ▷ The first point in  $U$  in non-increasing  $\phi$  order
- 5:  $S \leftarrow S \cup u$
- 6:  $D(u) \leftarrow \{v \in U : d(u, v) \leq r_u + r_v\}$  ▷ Note:  $D(u)$  includes  $u$  itself
- 7:  $U \leftarrow U \setminus D(u)$
- 8: **end while**

**Output:**  $S, \{D(u) : u \in S\}$

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► **Theorem 3** ([32]). *There is a 2-approximation for Priority  $k$ -Center.*

**Proof.** (For completeness and later use.) We claim that  $S$ , the output of Algorithm 1 for  $\phi := 1/r$ , is a 2-approximate solution; this follows from the observations in Fact 1. For any  $v \in X$  there is some  $u \in S$  for which  $v \in D(u)$ . By our choice of  $\phi$ ,  $r_u \leq r_v$ . Since  $d(u, v) \leq r_u + r_v$ , we have  $d(u, v) \leq 2r_v$ . To see why  $|S| \leq k$ , recall that for any  $u, v \in S$ , by Fact 1,  $d(u, v) > r_u + r_v$  so no two points in  $S$  can be covered by the same center. Thus any feasible solution needs at least  $|S|$  many points to cover all of  $S$ . ◀

In fact, the algorithm almost immediately gives a 3-approximation for Priority  $\mathcal{F}$ -Supplier for many families via the framework in [8].

One needs to check if given any partition  $\Pi$  of  $F$ , whether the following *partition feasibility* problem is solvable: does there exist  $A \in \mathcal{F}$  such that  $|A \cap P| = 1$  for all  $P \in \Pi$ ? We ask this for the partition returned by Algorithm 1, that is,  $\Pi = \{\{f \in F : d(f, u) \leq r_u\} : u \in S\}$ . If no such  $A$  exists, then the instance is infeasible since the centers  $S$  of the parts cannot be covered. If such an  $A$  exists, then by construction every  $v \in X$  in part  $D(u)$  satisfies

$d(v, A) \leq d(u, v) + d(u, A) \leq 2r_u + r_v \leq 3r_v$  since  $r_u \leq r_v$ . It is easy to see for the supplier, knapsack, and matroid center versions, the partition feasibility problem is solvable in polynomial time. This leads to the following theorem.

► **Theorem 4.** *There is a 3-approximation for Priority  $k$ -Supplier, Priority Knapsack Center, and the Priority Matroid Center problem.*

### 3 Priority $k$ -Center with Outliers

In this section we describe our framework for handling priorities and outliers and give a 9-approximation algorithm for the following problem.

► **Definition 5** (Priority  $k$ -Center with Outliers (PkCO)). *The input is a metric space  $(X, d)$ , a radius function  $r : X \rightarrow \mathbb{R}_{>0}$ , and parameters  $k, m \in \mathbb{N}$ . The goal is to find  $S \subseteq X$  of size at most  $k$  to minimize  $\alpha$  such that for at least  $m$  points  $v \in X$ ,  $d(v, S) \leq \alpha \cdot r(v)$ .*

► **Theorem 6.** *There is a 9-approximation for PkCO.*

The following is the natural LP relaxation for the feasibility version of PkCO. For each point  $v \in X$ , there is a variable  $0 \leq x_v \leq 1$  that denotes the (fractional) amount by which  $v$  is opened as a center. We use  $\text{cov}(v)$  to indicate the amount by which  $v$  is covered by itself or other open facilities. To be precise,  $\text{cov}(v)$  is the sum of  $x_u$  over all  $u \in X$  at distance at most  $r_v$  from  $v$ . Note that  $\text{cov}(v)$  is an auxiliary variable. We want to ensure that at least  $m$  units of coverage are assigned using at most  $k$  centers (hence the first two constraints).

$$\begin{aligned} \sum_{v \in X} \text{cov}(v) &\geq m && \text{(PkCO LP)} \\ \sum_{v \in X} x_v &\leq k \\ \text{cov}(v) &:= \sum_{\substack{u \in X: \\ d(u, v) \leq r_v}} x_u \leq 1 && \forall v \in X \\ 0 \leq x_v &\leq 1 && \forall v \in X. \end{aligned}$$

Next, we define another problem called Weighted  $k$ -Path Packing (WkPP) on a DAG. Our approach is to do an LP-aware reduction from PkCO to WkPP. To be precise, we use a fractional solution of the PkCO LP to reduce to a WkPP instance  $\mathcal{J}$ . We show that a good integral solution for  $\mathcal{J}$  translates to a 9-approximate solution for the PkCO instance. We prove that  $\mathcal{J}$  has a good integral solution by constructing a feasible fractional solution for an LP relaxation of WkPP; this LP relaxation is integral. Henceforth,  $\mathcal{P}(G)$  denotes the set of all the paths in  $G$  where each path is an ordered subset of the edges in  $G$ .

► **Definition 7** (Weighted  $k$ -Path Packing (WkPP)). *The input is  $\mathcal{J} = (G = (V, E), \lambda, k)$  where  $G$  is a DAG,  $\lambda : V \rightarrow \{0, 1, \dots, n\}$  for some integer  $n$ . The goal is to find a set of  $k$  vertex disjoint paths  $P \subseteq \mathcal{P}(G)$  that maximizes:*

$$\text{val}(P) := \sum_{p \in P} \sum_{v \in p} \lambda(v).$$

Even though this problem is NP-hard on general graphs<sup>3</sup>, it can be easily solved if  $G$  is a DAG by reducing to Min-Cost Max-Flow (MCMF). To build the corresponding flow network,

<sup>3</sup>  $k = 1$  and unit  $\lambda$  is the longest path problem which is known to be NP-hard [18].

we augment  $G$  to a new DAG  $G' = (V', E')$  with source and sink nodes  $s, t$ .  $V' = V \cup \{s, t\}$ . Each node  $v \in V$  has unit capacity and cost equal to  $-\lambda(v)$ .  $s$  and  $t$  have zero cost with capacities  $\infty$  and  $k$  respectively. As for the arcs,  $E'$  includes the entirety of  $E$ , plus arcs  $(s, v)$  and  $(v, t)$  for all  $v \in V$ . All the arcs have unit capacity and zero cost. One can now write the MCMF LP for  $WkPP$  which is known to be integral. We use  $\delta^+(v)$  and  $\delta^-(v)$  to denote the set of outgoing and incoming edges of a vertex  $v$  respectively. The LP has a variable  $y_e$  for each arc  $e \in E'$  to denote the amount of (fractional) flow passing through it. Similarly, the amount of flow entering a vertex is denoted by  $\text{flow}(v) := \sum_{e \in \delta^-(v)} y_e$ . The objective is to minimize the cost of the flow which is equivalent to maximizing the negation of the costs.

$$\begin{aligned} \max \sum_{v \in V} \text{flow}(v) \lambda(v) & \quad (WkPP \text{ LP}) \\ \text{flow}(v) := \sum_{e \in \delta^-(v)} y_e = \sum_{e \in \delta^+(v)} y_e & \quad \forall v \in V \\ \text{flow}(t) \leq k & \\ \text{flow}(v) \leq 1 \quad \forall v \in V, \quad 0 \leq y_e \leq 1 \quad \forall e \in E'. & \end{aligned}$$

▷ **Claim 8.**  $WkPP$  is equivalent to solving MCMF on  $G'$ .

*Proof.* Observe that any solution  $P$  for the  $WkPP$  instance translates to a valid flow of cost  $-\text{val}(P)$  for the flow problem. For any path  $p \in P$  with start vertex  $u$  and sink vertex  $v$ , send one unit of flow from  $s$  to  $u$ , through  $p$  to  $v$  and then to  $t$ . Since the paths in  $P$  are vertex disjoint and there are at most  $k$  of them, the edge and vertex capacity constraints in the network are satisfied.

Now we argue that any solution to the MCMF instance with cost  $-m$  translates to a solution  $P$  for the original  $WkPP$  instance with  $\text{val}(P) = m$ . To see this, note that the MCMF solution consists of at most  $k$  many  $s, t$  paths that are vertex disjoint with respect to  $V$ . This is because of our choice of vertex capacities. Let  $P$  be those paths modulo vertices  $s$  and  $t$ . For a  $v \in V$ ,  $-\lambda(v)$  is counted towards the MCMF cost iff  $v$  has a flow passing through it which means  $v$  is included in some path in  $P$ . Thus  $\text{val}(P) = m$ . ◁

### 3.1 Reduction to $WkPP$

Using a fractional solution of the  $PkCO$  LP we construct a  $WkPP$  instance. In particular, we use the cov assignment generated by the LP solution. Without loss of generality, by scaling the distances, we assume that the smallest neighborhood radius is 1. Let  $t := \lceil \log_2 r_{max} \rceil$ , where  $r_{max}$  is the largest value of  $r$  (after scaling). We use  $[t]$  to denote  $\{1, 2, \dots, t\}$ . Partition  $X$  according to each point's radius into  $C_1, \dots, C_t$ , where  $C_i := \{v \in X : 2^{i-1} \leq r_v < 2^i\}$  for  $i \in [t]$ . Note that some sets may be empty if no radius falls within its range.

Algorithm 2 shows the  $PkCO$  to  $WkPP$  reduction. The algorithm constructs a DAG called contact DAG (see Definition 9) as a part of the  $WkPP$  instance definition. We first run Algorithm 1 on each  $C_i$  to produce a set of representatives  $R_i$  and their respective clusters  $\{D(u) : u \in R_i\}$ . The  $\lambda$  function is constructed using the  $D(v)$ 's. Each  $R_i$  defines a row of the contact DAG starting with  $R_t$  at the top. Arcs in the contact DAG exist only between points in different rows, and only when they share a point in  $X$  that can cover them both within their desired radii. We always have arcs pointing downwards, that is, from points in  $R_i$  to points in  $R_j$  where  $i > j$ . See Figure 1 for an example on how a contact DAG looks like.



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**Algorithm 2** Reduction to WkPP.
 

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**Input:** PkCO instance  $\mathcal{I} = ((X, d), r, k, m)$  and assignment  $\{\text{cov}(v) \in \mathbb{R}_{\geq 0} : v \in X\}$

1:  $R_i, \{D(u) : u \in R_i\} \leftarrow \text{HS}((C_i, d), r, \text{cov})$  for all  $i \in [t]$

2: Construct contact DAG  $G = (V, E)$  per Definition 9

3:  $\lambda(v) \leftarrow |D(v)|$  for all  $v \in V$

**Output:** WkPP instance  $\mathcal{J} = (G = (V, E), \lambda, k)$

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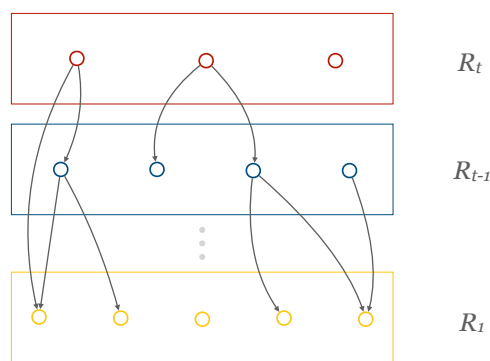


Figure 1 A contact DAG.

► **Definition 9** (contact DAG). Let  $R_i \subseteq C_i$ ,  $i \in [t]$  be the set of representatives acquired after running HS on  $C_i$  according to Line 1 of Algorithm 2. contact DAG  $G = (V, E)$  is a DAG on vertex set  $V = \bigcup_i R_i$  where the arcs are constructed by the following rule:

$$\begin{aligned} \text{For } u \in R_i \text{ and } v \in R_j \text{ where } i > j, (u, v) \in E \\ \iff \exists f \in X : d(u, f) \leq r_u \text{ and } d(v, f) \leq r_v. \end{aligned}$$

Our first observation is that the WkPP instance has a good fractional solution and since the LP is integral, it also has a good integral solution.

► **Lemma 10.** *There is a valid solution to WkPP LP of value  $\geq m$  for the WkPP instance  $\mathcal{J}$ . Since WkPP LP is integral, this implies  $\mathcal{J}$  has an integral solution of value  $\geq m$ .*

The proof of this lemma can be found in [2]. Theorem 6 now follows from the following lemma.

► **Lemma 11.** *Any solution with value at least  $m$  for the WkPP instance  $\mathcal{J}$  given by Algorithm 2 translates to a 9-approximation for the PkCO instance  $\mathcal{I}$ .*

**Proof.** We begin with a few observations. Per definition of contact DAG we have the following property. Note that the converse is not necessarily true.

► **Fact 2.** *If  $u \in R_i$ ,  $v \in R_j$ , and  $(u, v)$  is an arc in contact DAG,  $d(u, v) \leq r_u + r_v$ .*

► **Fact 3.**  *$\{D(v), v \in V\}$  as constructed in Algorithm 2 partitions  $X$ .*

**Proof.**  $\{C_i\}_{i \in [t]}$  partitions  $X$  and HS further partitions each  $C_i$  according to Fact 1. ◀

▷ **Claim 12.** For any  $u \in R_i$ ,  $v \in R_j$  reachable from  $u$  in a contact DAG,  $d(u, v) < 3 \cdot 2^i$ .

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Proof. Observe that by definition of contact DAG,  $i > j$ . A path from  $u$  to  $v$  may contain a vertex from any level of the DAG between  $i$  and  $j$ . In the worst case, the path has a vertex  $w_k$  from every level  $R_k$  for  $j < k < i$ :

$$\begin{aligned}
d(u, v) &\leq d(u, w_{i-1}) + d(w_{i-1}, w_{i-2}) + \dots + d(w_{j+1}, v) \\
&\leq (r_u + r_{w_{i-1}}) + (r_{w_{i-1}} + r_{w_{i-2}}) + \dots + (r_{w_{j+1}} + r_v) && \text{(by Fact 2)} \\
&= r_u + 2 \sum_{k=j+1}^{i-1} r_{w_k} + r_v < r_u + 2 \sum_{k=1}^{i-1} 2^k \\
&= r_u + 2 \cdot (2^i - 2) < 3 \cdot 2^i. && (r_u < 2^i) \quad \triangleleft
\end{aligned}$$

Now we are armed with all the facts we need to prove Lemma 11. We are assuming the constructed  $WkPP$  instance has a solution of value at least  $m$ , which means there exists a set of  $k$  disjoint paths  $P \subseteq \mathcal{P}(G)$  in the contact DAG such that  $\text{val}(P) \geq m$ . For any path  $p \in P$ , let  $\text{sink}(p)$  denote the last node in this path (i.e.  $\text{sink}(p) = \arg \min_{u \in p} r_u$ ). Our final solution would be  $S := \{\text{sink}(p) : p \in P\}$ . We argue that this  $S$  is a 9-approximate solution for the initial  $PkCO$  instance. Since  $P$  has at most  $k$  many paths,  $|S| \leq k$ .

Now we show any  $w \in D(u)$  where  $u \in p \in P$ , can be covered by  $v = \text{sink}(p)$  with dilation at most 9. Assume  $u \in R_i$  for some  $i \in [t]$ .

$$\begin{aligned}
d(w, v) &\leq d(w, u) + d(u, v) < r_w + r_u + 3 \cdot 2^i && \text{(by Fact 1 and above claim)} \\
&< r_w + 4 \cdot 2^i \leq 9r_w. && (r_u < 2^i \text{ and } 2^{i-1} \leq r_w)
\end{aligned}$$

The last piece is to argue at least  $m$  points will be covered by  $S$ . The set of points that are covered by  $S$  within 9 times their radius is precisely the set  $D_{\text{total}} := \bigcup_{p \in P} \bigcup_{u \in p} D(v)$ . So we need to show  $|D_{\text{total}}| \geq m$ . By Fact 3 we have:

$$|D_{\text{total}}| = \left| \bigcup_{p \in P} \bigcup_{v \in p} D(v) \right| = \sum_{p \in P} \sum_{v \in p} |D(v)| = \text{val}(P),$$

where the last equality is by the definition of  $\lambda(v)$ ,  $v \in V$  (in Line 3 of Algorithm 2) and definition of  $\text{val}(P)$ . By assumption  $\text{val}(P) \geq m$  thus  $D_{\text{total}}$  contains at least  $m$  points.  $\blacktriangleleft$

In the special case where there are 2 types of radii we can slightly modify our approach to get a 3-approximation algorithm. This result is tight. To see this consider  $PkCO$  instances where clients having priority radii in  $\{0, 1\}$  with  $n_0$  of the former type and  $n_1$  of the latter, and the number of outliers allowed is  $n_0 - k$ . Clients with priority radii 0 either need to have a facility opened at that same point, or need to be an outlier. Since only  $n_0 - k$  outliers and  $k$  centers are allowed, all the outliers and centers are on these  $n_0$  points. Thus, the  $n_0$  points act as facilities in the  $k$ -supplier problem which is hard to approximate with a factor better than 3. This shows a gap with the vanilla  $k$ -center with outliers has a 2-approximation [7].

In general, our framework yields improved approximation factors when the number of distinct priorities are less than 5 (see Theorem 13). In the special case when all radii are powers of 2, our algorithm is actually a 5-approximation. This factor improves if the radii are powers of some  $b > 2$  and approaches 3 as  $b$  goes to infinity (see Theorem 14).

**► Theorem 13.** *There is a  $(2t - 1)$ -approximation for  $PkCO$  instances where there are only  $t$  types of radii.*

**Proof.** Given  $PkCO$  instance  $\mathcal{I}$  obtain fractional solution  $x$  by solving the  $PkCO$  LP. Partition  $X$  according to each point's radius into  $C_1, \dots, C_t$ , where  $C_i$  is points of radius type  $i$  for  $i \in [t]$ . Run Algorithm 2 with input  $\text{cov}$  corresponding to  $x$  and take resulting  $WkPP$  instance

$\mathcal{J}$ . Assuming the  $WkPP$  instance has a solution with value at least  $m$ , we can show how to obtain a  $(2t - 1)$ -approximate solution as follows. Let  $P$  be the  $WkPP$  solution. Take any  $p \in P$ . If  $p$  is a single vertex, simply add it to solution  $S$ . Otherwise, instead of adding  $v' = \text{sink}(p)$  to  $S$ , if  $v$  is the vertex before  $v'$  in  $p$ , add a point  $f \in X$  that covers both the endpoints  $v$  and  $v'$  ( $f$  exists by Definition 9).

Take any  $w \in D(u)$  where  $u \in p$  and assume  $u \in R_i$  for some  $i \in [t]$ . Similar to the proof of Claim 12 one can show  $d(u, v) < 2(i - 2)r_u$  by bounding the radius of any vertex in between them by  $r_u$  and noting that  $v$  is in level 2 or higher (remember  $p$  ends at  $v'$  and  $(v, v')$  is an edge). Since  $d(w, f) \leq d(w, u) + d(u, v) + d(v, f)$  and  $d(w, u) \leq r_w + r_u$  (Fact 1), plus  $d(v, f) < r_v \leq r_u$  we have  $d(w, f) < r_w + 2(i - 1)r_u$ . But  $r_w = r_u$  by definition of  $C_i$  so  $w$  is covered by  $f$  with dilation at most  $2i - 1 \leq 2t - 1$ . The part to argue at least  $m$  points will be covered by  $S$ , is done similar to the proof of Lemma 11.

The remainder of this proof, i.e. showing that  $\mathcal{J}$  does indeed have a solution of value at least  $m$  that can be determined in polynomial time using an MCMF algorithm, is identical to the proof of Theorem 6.  $\blacktriangleleft$

► **Theorem 14.** *There is a  $((3b - 1)/(b - 1))$ -approximation for  $PkCO$  instances where the radii are powers of  $b \geq 2$ .*

**Proof.** Given  $PkCO$  instance  $\mathcal{I}$  obtain fractional solution  $x$  by solving the  $PkCO$  LP. Partition  $X$  according to each point's radius into  $C_1, \dots, C_t$ , where  $t := \lceil \log_b r_{max} \rceil$  and  $C_i := \{v \in X : r_v = b^{i-1}\}$  for  $i \in [t]$ . Run Algorithm 2 with input  $\text{cov}$  corresponding to  $x$  and take resulting  $WkPP$  instance  $\mathcal{J}$ . Assume the  $WkPP$  instance has a solution  $P$  with value at least  $m$ . For any  $p \in P$  add  $v = \text{sink}(p)$  to solution  $S$ . Consider arbitrary  $w \in D(u)$  where  $u \in p \in P$  and assume  $u \in R_i$  for some  $i \in [t]$ . Similar to the proof of Claim 12 one can show  $d(u, v) < ((b + 1)/(b - 1)) \times b^{i-1}$ . By Fact 1  $d(w, u) \leq r_w + r_u = 2b^{i-1}$ . Thus any  $w$  is covered by dilation  $(3b - 1)/(b - 1)$  as  $d(w, v) \leq d(w, u) + d(u, v) < 2b^{i-1} + ((b + 1)/(b - 1)) \times b^{i-1} = (3b - 1)/(b - 1)r_w$ . To argue at least  $m$  points will be covered by  $S$ , see the proof of Lemma 11. Showing that  $\mathcal{J}$  does indeed have a solution of value at least  $m$  that can be determined in polynomial time using an MCMF algorithm, is identical to the proof of Theorem 6 as well.  $\blacktriangleleft$

## 4 Priority Matroid-Center with Outliers

In this section, we show how to generalize the results from the previous section for the case of Priority Matroid-Center with Outliers (PMCO).

► **Definition 15** (Priority Matroid-Center with Outliers (PMCO)). *The input is a metric space  $(X, d)$ , parameter  $m \in \mathbb{N}$ , radius function  $r : X \rightarrow \mathbb{R}_{>0}$ , and  $\mathcal{F} \subseteq 2^X$  a family of independent sets of a matroid. The goal is to find  $S \in \mathcal{F}$  to minimize  $\alpha$  such that for at least  $m$  points  $v \in X$ ,  $d(v, S) \leq \alpha \cdot r(v)$ .*

► **Theorem 16.** *There is a 9-approximation for PMCO.*

As in the previous section, we assume  $\alpha = 1$  and consider the feasibility version of the problem. For any  $S \subseteq V$ , let  $\text{rank}_{\mathcal{F}}(S)$  be the rank of  $S$  in the given matroid. The natural LP relaxation for this problem is very similar to that of  $PkCO$  LP except that we replace the cardinality constraints with *rank constraints*  $x(S) \leq \text{rank}_{\mathcal{F}}(S)$  for all  $S \subseteq V$ . This is because for any  $S \in \mathcal{F}$ ,  $|S| = \text{rank}_{\mathcal{F}}(S)$ .

$$\begin{aligned}
 \sum_{v \in X} \text{cov}(v) &\geq m && \text{(PMCO LP)} \\
 \sum_{v \in S} x_v &\leq \text{rank}_{\mathcal{F}}(S) && \forall S \subseteq V \\
 \text{cov}(v) := \sum_{\substack{u \in X: \\ d(u,v) \leq r_v}} x_u &\leq 1 && \forall v \in X \\
 0 &\leq x_v \leq 1 && \forall v \in X.
 \end{aligned}$$

Similar to  $WkPP$ , we have the path packing version of PMCO defined below. Recall from the last section, that after reducing from  $PkCO$  to  $WkPP$  we returned a set of  $k$  vertices in DAG  $G$  as our final solution. Now that we have matroid constraints, we must instead return a set  $S$  of vertices such that  $S \in \mathcal{F}$ . Doing so is not as straightforward, since our reduction does not guarantee that such a subset of vertices actually exists and covers enough points in their corresponding vertex disjoint paths. Instead, we show there is an  $S \in \mathcal{F}$  such that each member of this  $S$  is *close* to some vertex of  $G$ . These close points in  $G$  will correspond to a set of vertex disjoint paths that will cover enough points.

► **Definition 17** (Weighted  $\mathcal{F}$ -Path Packing (WMatPP)). *The input is  $G = (V, E)$  and  $\lambda$  same as in  $WkPP$ , plus a finite set  $X$ ,  $\mathcal{Y} = \{Y_v \subseteq X : v \in V\}$ , and  $\mathcal{F} \subseteq 2^X$  a family of independent sets of a matroid. The goal is to find a set of disjoint paths  $P \in \mathcal{P}(G)$  with maximum  $\text{val}(P)$  for which there exists  $S \in \mathcal{F}$  such that  $\forall p \subseteq P, S \cap Y_{\text{sink}(p)} \neq \emptyset$ .*

Observe that the reduction procedure in Algorithm 2 and all of our subsequent observations in Section 3.1 do not rely on how we define a feasible set of centers. Hence, the main obstacle in proving Theorem 16 lies in our reduction to MCMF. Luckily, the result of [13] helps us address this by giving LP integrality results similar to MCMF using the following formulation on directed *polymatroidal flows* [17, 24, 29]: For a network  $G' = (V', E')$ , for all  $v \in V'$ , we are given polymatroids<sup>4</sup>  $\rho_v^-$  and  $\rho_v^+$  on  $\delta^-(v)$  and  $\delta^+(v)$  respectively. For every arc  $e \in E'$  there is a variable  $0 \leq y_e \leq 1$ . The capacity constraints for each  $v \in V'$  are defined as:

$$\begin{aligned}
 \sum_{e \in U} y_e &\leq \rho_v^-(U) && \forall U \subseteq \delta^-(v) \\
 \sum_{e \in U} y_e &\leq \rho_v^+(U) && \forall U \subseteq \delta^+(v).
 \end{aligned}$$

We augment the DAG  $G$  given in WMatPP to construct a polymatroidal flow network  $G'$ . In this new network,  $V' = V \cup X \cup \{s, t\}$  where each node  $v \in V$  has cost  $-\lambda(v)$ . **Note:** Even though a vertex  $v \in V$  might correspond to a point in  $X$ , in  $V'$  we make a distinction between the two copies.  $E'$  includes all of  $E$ , plus arcs  $(s, v)$  for all  $v \in V$ . Finally, instead of adding arcs  $(v, t)$ , we add arcs  $(v, f)$  and  $(f, t)$  for all  $f \in Y_v$ .

The polymatroids for this instance are constructed as follows: for any  $v \in V \cup X$ ,  $\rho_v^-(U) = 1$  for all non-empty  $U \subseteq \delta^-(v)$  and  $\rho_v^+$  is defined similarly on  $\delta^+(v)$ . For  $s$ , we only have outgoing edges where  $\rho_s^+(U) = |U|$  for all  $U \subseteq \delta^+(s)$ . Finally, we enforce the matroid constraints of  $\mathcal{F}$  on  $t$ . For any  $U \subseteq \delta^-(t)$ , let  $T \subseteq X$  be the set of starting nodes in  $U$ . That is,  $U = \{(f, t) : f \in T\}$ . Set  $\rho_t^-(U) = \text{rank}_{\mathcal{F}}(T)$ . Since  $\delta^-(t) \subseteq X$ , these capacity constraints on  $t$  are equivalent to the following set of constraints:

$$\sum_{f \in T} y_{(f,t)} \leq \text{rank}_{\mathcal{F}}(T) \quad \forall T \subseteq X : \{(f, t) : f \in T\} \subseteq \delta^-(t).$$

<sup>4</sup> Monotone integer-valued submodular functions.

Now, we prove a claim analogous to that of Claim 8.

▷ **Claim 18.** WMatPP is equivalent to solving the polymatroidal flow on network  $G'$ .

*Proof.* Any solution  $P$  for the WMatPP instance translates to a valid flow of cost  $-\text{val}(P)$  for the flow problem. Let  $S \in \mathcal{F}$  be the independent set that intersects  $Y_{\text{sink}(p)}$  for all  $p \in P$ . For any path  $p \in P$  with start vertex  $u$  and sink vertex  $v$ , take arbitrary  $f \in S \cap Y_v$ . Send one unit of flow from  $s$  to  $u$ , through  $p$  to  $v$  and then to  $f$  and  $t$ . All the polymatroidal constraints in WMatPP LP are satisfied.

Now we argue that any solution to the flow instance with cost  $-m$  translates to a solution  $P$  for WMatPP with  $\text{val}(P) = m$ . To see this, note that the flow solution consists of  $s, t$  paths that are vertex disjoint with respect to  $V \cup X$ . This is due to our choice of  $V \cup X$  polymatroids. Each path passes through one  $v \in V$ , then immediately to  $f \in Y_v$  and then ends in  $t$ . By polymatroidal constraints on  $t$ , the subset of  $X$  that has a flow going through it will be an independent set of  $\mathcal{F}$ .

Let  $P$  be the described paths induced on  $V$ . For a  $v \in V$ ,  $-\lambda(v)$  is counted towards the MCMF cost iff  $v$  has a flow passing through it. This means  $v$  is included in some path in  $P$ . Thus  $\text{val}(P) = m$ .  $\triangleleft$

The polymatroidal LP for this particular construction is as follows (recall  $\text{flow}(v) := \sum_{e \in \delta^-(v)} y_e$ ):

$$\begin{aligned}
 & \max \sum_{v \in V} \text{flow}(v) \lambda(v) && \text{(WMatPP LP)} \\
 & \text{flow}(v) := \sum_{e \in \delta^-(v)} y_e = \sum_{e \in \delta^+(v)} y_e && \forall v \in V \cup X \\
 & \sum_{f \in T} y_{(f,t)} \leq \text{rank}_{\mathcal{F}}(T) && \forall T \subseteq X : \{(f,t) : f \in T\} \subseteq \delta^-(t) \\
 & \text{flow}(v) \leq 1 && \forall v \in V \cup X \\
 & 0 \leq y_e \leq 1 && \forall e \in E'
 \end{aligned}$$

By [13], WMatPP LP is integral and there are polynomial time algorithms to solve it.

As for reducing PMCO to WMatPP, most of the notation and results can be recycled from Section 3.1. Specially, the reduction itself (Algorithm 3) is just Algorithm 2 with Line 4 added. **Note:** By definition of an arc in contact DAG, for two nodes  $u, v \in V$ ,  $(u, v)$  is an arc iff  $Y_v$  intersects  $Y_u$ .

■ **Algorithm 3** Reduction to WMatPP.

---

**Input:** PMCO instance  $\mathcal{I} = ((X, d), r, \mathcal{F}, m)$  and assignment  $\{\text{cov}(v) \in \mathbb{R}_{\geq 0} : v \in X\}$

- 1:  $R_i, \{D(u) : u \in R_i\} \leftarrow \text{HS}((C_i, d), r, \text{cov})$  for all  $i \in [t]$
- 2: Construct contact DAG  $G = (V, E)$  per Definition 9
- 3:  $\lambda(v) \leftarrow |D(v)|$  for all  $v \in V$
- 4:  $Y_v \leftarrow \{u \in X : d(u, v) \leq r_v\}$  for all  $v \in V$

**Output:** WMatPP instance  $(G = (V, E), \lambda, X, \mathcal{Y}, \mathcal{F})$

---

Before we start to prove our 9-approximation result for PMCO, we need to slightly modify Claim 12 to account for the fact that a vertex covered by  $v$  (the sink of some path) has to travel slightly farther than  $v$  to reach an  $f \in Y_v$ . Fortunately, the proof of Claim 12 has a slight slack that allows us to derive the same distance guarantees even with this extra step.

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▷ **Claim 19.** For any  $u \in R_i$  and  $v \in R_j$  reachable from  $u$  in contact DAG  $G$ , and any  $f \in Y_v$ ,  $d(u, f) < 3 \cdot 2^i$ .

*Proof.* By definition of  $G$  it must be the case that  $i > j$ . Also for all  $f \in Y_v$ ,  $d(f, v) \leq r_v$ . If  $v$  is reachable from  $u$ , a path between  $u$  and  $v$  may contain a vertex  $w_k$  from every level  $R_k$  for  $j < k < i$ :

$$\begin{aligned}
 d(u, f) &\leq d(u, v) + d(v, f) \leq d(u, v) + r_v \\
 &\leq (r_u + r_{w_{i-1}}) + (r_{w_{i-1}} + r_{w_{i-2}}) + \dots + (r_{w_{j+1}} + r_v) + r_v && \text{(by Fact 2)} \\
 &\leq r_u + 2 \sum_{k=1}^{i-1} 2^k && \text{(by definition of } C_k) \\
 &= r_u + 2 \cdot (2^i - 2) < 3 \cdot 2^i. && (u \in C_i, r_u < 2^i)
 \end{aligned}$$

◀

Since the previous claim has the same guarantee as Claim 12, Lemma 11 easily translates to the following:

► **Lemma 20.** Any solution with value at least  $m$  for the output of Algorithm 3 translates to a 9-approximation for the input  $\mathcal{I}$ .

*Proof.* Let  $P \in \mathcal{P}(G)$  be the promised WMatPP solution. Let  $S \in \mathcal{F}$  be the independent set that intersects  $Y_{\text{sink}(p)}$  for all  $p \in P$ . By Claim 19,  $S$  covers all the vertices  $v \in V$  that are included in  $P$  by dilation 3. Proof of Lemma 11 shows that for any such  $v \in V$  covered by  $P$  and any  $w \in D(v)$ ,  $d(w, S) \leq 9r_w$ . This holds for at least  $m$  points. ◀

We can now prove our 9-approximation result for PMCO.

**Proof of Theorem 16.** The algorithm is very similar to that of Theorem 6: Given PMCO instance  $\mathcal{I} = ((X, d), r, \mathcal{F}, m)$ , solve the PMCO LP and use the solution in the procedure of Algorithm 3 to reduce to WMatPP instance  $\mathcal{J} = (G = (V, E), \lambda, X, \mathcal{Y}, \mathcal{F})$ . Let  $P \in \mathcal{P}(G)$  be the solution to this instance and  $S \in \mathcal{F}$  be the independent set that intersects  $Y_{\text{sink}(p)}$  for all  $p \in P$ . If  $\text{val}(P) \geq m$ ,  $S$  is a 9-approximate solution for  $\mathcal{I}$  via Lemma 20. So we prove such solution  $P$  exists by constructing a feasible (possibly fractional) WMatPP LP solution.

Take the contact DAG of Algorithm 3  $G = (V, E)$  and recall that each  $v \in V$  is also a point in  $X$ . For any  $f \in X$  let  $A_f := \{v \in V : d(f, v) \leq r_v\}$  be the set of points  $v \in V$  for which  $x_f$  contributes to  $\text{cov}(v)$ . By definition of an edge in contact DAG, for any  $u, v \in A_f$ , we have  $(u, v) \in E$ . Define  $p_f$  to be the  $s, t$  path that passes through  $A_f$  in the order of decreasing neighborhood radii. Formally, let  $(u_1, \dots, u_l)$  be  $A_f$  sorted in decreasing order of neighborhood radii. Then,  $p_f = ((s, u_1), (u_1, u_2), \dots, (u_l, f), (f, t))$ . Similar to the proof of Theorem 6 we define  $H_e := \{f \in X : e \in p_f\}$  and set  $y$  as follows:

$$y_e := \sum_{f \in H_e} x_f.$$

Now, we argue that  $y$  is a feasible solution for WMatPP LP with objective value at least  $m$ . The flow is conserved for each vertex  $v \in V \cup X$  since for any  $f \in X$ , we add the same amount  $x_f$  to  $y_e$  of all  $e \in p_f$ . Observe that  $\text{flow}(v) = \text{cov}(v)$  thus the constraint  $\text{cov}(v) \leq 1$  in PkCO LP implies  $\text{flow}(v) \leq 1$ . To see why the rank constraints are satisfied, the key observation is that any  $e \in \delta^-(t)$  must be of the form  $(f, t)$  for some  $f \in X$ , and by our construction  $y_e = x_f$ . So according to constraint  $\sum_{f \in T} x_f \leq \text{rank}_{\mathcal{F}}(T)$  in PMCO LP we have  $\sum_{f \in T} y_{(f, t)} \leq \text{rank}_{\mathcal{F}}(T)$ . Lastly, the WMatPP LP objective for this solution is at least  $m$ . The proof is identical to what we had for Theorem 6. ◀

## 5 Priority Knapsack-Center with Outliers

In this section, we show how to generalize the results from the previous section for the case of Priority Knapsack-Center with Outliers (PKnapCO).

► **Definition 21** (Priority Knapsack-Center with Outliers (PKnapCO)). *The input is a metric space  $(X, d)$ , a radius function  $r : X \rightarrow \mathbb{R}_{>0}$ , a weight function  $w : X \rightarrow \mathbb{R}_{\geq 0}$ , parameters  $B > 0$  and  $m \in \mathbb{N}$ . The goal is to find  $S \subseteq X$  with  $w(S) \leq B$  to minimize  $\alpha$  such that for at least  $m$  points  $v \in X$ ,  $d(v, S) \leq \alpha \cdot r(v)$ .*

► **Theorem 22.** *There is a 14-approximation for PKnapCO.*

As in PkCO and PMCO, we reduce to the following path packing problem.

► **Definition 23** (Weighted Knapsack-Path Packing (WnapPP)). *The input is  $G = (V, E)$  and  $\lambda$  same as in WkPP, plus  $X$  a finite set,  $w : X \rightarrow \mathbb{R}_{\geq 0}$ ,  $\mathcal{Y} = \{Y_v \subseteq X : v \in V\}$ , and parameter  $B > 0$ . The goal is to find a set of disjoint paths  $P \in \mathcal{P}(G)$  with maximum  $\text{val}(M)$  for which there exists  $S \subseteq X$  with  $w(S) \leq B$  such that  $\forall p \in P$ ,  $S \cap Y_{\text{sink}(p)} \neq \emptyset$ .*

There are two main issues in generalizing our techniques from Section 3 and Section 4. First, the WnapPP problem seems hard on a general DAG. To circumvent this, we make two changes to the LP-aware PKnapCO to WnapPP reduction (given in Algorithm 4). First, we modify Algorithm 1 so that a representative captures points at *larger* distances. To be precise, for a representative  $u$ , Algorithm 1 is modified to:  $D(u) \leftarrow \{v \in U : d(u, v) \leq r_u + 2r_v\}$ . Second, the partition induced by  $C_i$ 's in Algorithm 2 is done via powers of 4 instead of 2. This is what bumps our approximation factor from 9 to 14. However it helps, as the resulting contact DAG is in fact a directed out-forest. It is not too hard to solve WnapPP when  $G$  is a directed-out forest using dynamic programming.

### Algorithm 4 Reduction to WnapPP.

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**Input:** PKnapCO instance  $\mathcal{I} = ((X, d), r, w, B, m)$  and assignment  $\{\text{cov}(v) \in \mathbb{R}_{\geq 0} : v \in X\}$

- 1:  $R_i, \{D(u) : u \in R_i\} \leftarrow \text{ModHS}((C_i, d), r, \text{cov})$  for all  $i \in [t]$
- 2: Construct contact forest  $G = (V, E)$  per Definition 24
- 3:  $\lambda(v) \leftarrow |D(v)|$  for all  $v \in V$
- 4:  $Y_v \leftarrow \{u \in X : d(u, v) \leq r_v\}$  for all  $v \in V$

**Output:** WnapPP instance  $(G = (V, E), \lambda, X, \mathcal{Y}, w, B)$

---

► **Definition 24** (contact forest). *Let  $R_i \subseteq C_i$ ,  $i \in [t]$  be the set of representatives acquired after running ModHS procedure on  $C_i$  according to Line 2 of Algorithm 4. contact forest  $G = (V, E)$  is a directed forest on vertex set  $V = \bigcup_i R_i$  where the arcs are constructed by the as follows: For  $u \in R_i$  and  $v \in R_j$  where  $i > j$ , add the arc  $(u, v) \in E$  if there exists  $f \in X$  such that  $d(u, f) \leq r_u$  and  $d(v, f) \leq 2r_v$ . Next, remove all the forward edges<sup>5</sup>.*

The second issue is more difficult to handle. In PkCO and PMCO, we used the fact that WkPP and WMatPP LPs are integral to show that the instances constructed by the reduction have large value. This is not true any more as the WnapPP LP is not integral even when  $G$  is a forest. Indeed, the natural LP relaxation for PKnapCO has unbounded integrality gap even without priorities [14].

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<sup>5</sup> In a DAG, edge  $(u, v)$  is a forward edge if there is a path of length two or more in the graph that connects  $u$  to  $v$ .

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We circumvent this by using the round and cut framework of [8]. Instead of using the PKnapCO LP, we would use  $\text{cov}$  in the convex hull of the integral solutions (call it  $\mathcal{P}_{\text{cov}}$ ). Of course, we do not know the integral solutions and there may indeed be exponentially such solutions. So, we have to employ the ellipsoid algorithm. In each iteration of ellipsoid, we get some  $\text{cov}$  that may or may not be in  $\mathcal{P}_{\text{cov}}$ . In any case, if we manage to get a good path packing solution using this  $\text{cov}$ , we get an approximate PKnapCO solution and we are done. Otherwise, we are able to give ellipsoid a linear constraint that should be satisfied by any point in  $\mathcal{P}_{\text{cov}}$  but is violated by the current  $\text{cov}$ . Ultimately, either we find an approximate solution for PKnapCO along the way, or ellipsoid prompts that  $\mathcal{P}_{\text{cov}}$  is empty, indicating that the problem is infeasible.

From here on, let  $\mathcal{F}$  be the set of all possible centers that fit in the budget. That is,  $\mathcal{F} := \{S \subseteq X : w(S) \leq B\}$ . The following is the convex hull of the integral solutions for PKnapCO.

$$\mathcal{P}_{\text{cov}} = \{(\text{cov}(v) : v \in X) : \sum_{v \in X} \text{cov}(v) \geq m \quad (\mathcal{P}_{\text{cov}.1})$$

$$\forall v \in X, \quad \text{cov}(v) := \sum_{\substack{S \in \mathcal{F}: \\ d(v,S) \leq r_v}} z_S \quad (\mathcal{P}_{\text{cov}.2})$$

$$\sum_{S \in \mathcal{F}} z_S = 1 \quad (\mathcal{P}_{\text{cov}.3})$$

$$\forall S \in \mathcal{F}, \quad z_S \geq 0 \quad (\mathcal{P}_{\text{cov}.4})$$

We show if  $\text{cov}(v) \in \mathcal{P}_{\text{cov}}$ , then indeed the WNapPP instance obtained is “valuable”, that is, has value  $\geq m$ . More importantly, we show that if the instance is *not* valuable, then we can find a hyperplane separating  $\text{cov}$  from  $\mathcal{P}_{\text{cov}}$ . One can now use the ellipsoid method to get the 14-approximation: given  $\text{cov}$ , we either get a valuable WNapPP instance leading to a 14 approximation, or we find a separating hyperplane which can be fed to the ellipsoid method to obtain a new  $\text{cov}$  vector. The details are omitted in this version due to space restrictions and can be found in [2].

## 6 Connections to Fair Clustering

In this section, we show how our results imply results in the two fairness notions as defined by [28] and [22].

### 6.1 “A Center in your Neighborhood” notion of [28]

Jung et al. [28] argue that fairness in clustering should take into account population densities and geography. For every  $v \in X$ , they define a *neighborhood radius*  $\text{NR}(v)$  to be the distance to its  $(\lceil n/k \rceil - 1)$ th nearest neighbor. A solution is fair, they argue, if every  $v$  is served within their  $\text{NR}(v)$ . They also observe that this may not always be possible, and therefore they wish to find a placement  $S \subseteq X$  minimizing  $\max_v \frac{d(v,S)}{\text{NR}(v)}$ . As an optimization problem, the problem is precisely an instantiation of Priority  $k$ -Center. Thus, one can easily obtain a 2-approximation once  $r(v) = \text{NR}(v)$  is fixed.

[28] in fact show that it is always possible to find  $S$  such that  $d(v,S) \leq 2\text{NR}(v)$ . They do so by looking at the centers obtained from running their algorithm which is the same as that of Plesník. Note that a 2-approximation to the instance of Priority  $k$ -Center defined by



$r(v) = \text{NR}(v)$  does not necessarily imply this additional property. Here we show why it is not a coincidence by considering the natural LP relaxation for Priority  $k$ -Center. Given an instance of Priority  $k$ -Center one can obtain a lower bound on the optimum value by finding the smallest  $\alpha$  such that the following LP is feasible.

$$\text{PkCFeasLP}(\alpha) := \{(y_u \geq 0 : u \in X) : \sum_{u \in X} y_u \leq k; \quad \forall v \in X : \sum_{u: d(u,v) \leq \alpha r(v)} y_u \geq 1\} \quad (1)$$

▷ **Claim 25.** Suppose  $\text{PkCFeasLP}(\alpha)$  has a feasible solution, then Algorithm 1 run with  $\phi(v) = \frac{1}{r(v)}$  finds at most  $k$  centers that cover each point  $v$  within distance  $2\alpha r(v)$ .

*Proof.* The proof is similar to that of Theorem 3. Without loss of generality we can assume  $\alpha = 1$ , otherwise we can scale all the radii by  $1/\alpha$ . We need to argue  $|S| \leq k$ . For any  $u \in S$ , we have  $\sum_{v \in D(u)} y_v \geq 1$  since  $B(u, r(u)) \subseteq D(u)$ . Since  $D(u)$ 's are disjoint and  $\sum_{u \in X} y_u \leq k$ , the claim follows. ◁

The preceding discussion and the claim show the utility of viewing the clustering problem of [28] as a special case of Priority  $k$ -Center. One can then bring to bear all the positive algorithmic results on Priority  $k$ -Center (such as Theorem 4) to fine-tune the fair clustering model. Below we list a few high-level speculative ideas on how the Priority  $k$ -Center view can help.

- The LP relaxation could be useful in obtaining better empirical solutions. For example, it has been shown that for  $k$ -center, the LP relaxation is integral under notions of stability [12].
- The model of [28] allows  $\text{NR}(v)$  to be very large for points  $v$  which may not be near many points. However, one may want to put an upper bound  $M$  on the radius that is independent of  $\text{NR}(v)$ . The same algorithm works to give a 2-approximation but one may no longer have the property that all points are covered within twice  $\text{NR}(v)$ .
- In many scenarios it makes sense to work with the supplier version since centers cannot necessarily be placed at all locations in  $X$ . Second, there could be several additional constraints on the set of centers that can be chosen. Theorem 4 shows that more general constraints than cardinality can be handled.
- Related to the first point above, far away points in less dense regions (outliers) can be harmed by setting  $\text{NR}(v)$  to be a large number. Alternatively, one can skew the choice of centers if one tries to set a small radius for these points. In this situation it is useful to have algorithms that can handle outliers such that one can find a good solution for vast majority of points and help the outliers via other techniques.

## 6.2 The Lottery Model of Harris et al. [22]

Harris et al. [22] define a lottery model of fairness where every client  $v \in X$  has a “distance demand”  $r(v)$  and a “probability demand”  $p(v)$ . They deem a lottery or a distribution over feasible solutions fair if every client is connected to a facility within their distance demand with probability at least the probability demand. The computational question is to figure out if this is (approximately) feasible. We show a connection to the outlier version of the priority  $k$ -center problem, and then generalize their results. We start with a definition.

► **Definition 26** (Lottery Priority  $\mathcal{F}$ -Center (LP $\mathcal{F}$ C)). *The input is a metric space  $(X, d)$  where each point  $v$  has a distance demand  $r(v) > 0$  and probability demand  $\text{prob}(v)$ . The input also (implicitly) specifies a family  $\mathcal{F} \subseteq 2^X$  of allowed locations where centers can be opened. A distribution over  $\mathcal{F}$  is  $\alpha$ -approximate if*

$$\forall v \in X : \Pr_{S \sim \mathcal{F}} [d(v, S) \leq \alpha \cdot r(v)] \geq \text{prob}(v).$$

*An  $\alpha$ -approximation algorithm in the lottery model either asserts the instance infeasible in that an 1-approximate distribution doesn't exist, or returns an  $\alpha$ -approximate distribution.*

Harris et al. [22] show that for the case when  $\mathcal{F}$  is simply  $\{S : |S| \leq k\}$ , there is a 9-approximate distribution. Using our results described before, and the by now standard framework using the ellipsoid method (as in [6, 1]), we can get the following results.

► **Theorem 27.** *There is a 9-approximation for LP $\mathcal{F}$ C where  $\mathcal{F}$  is the independent set of a matroid.*

► **Theorem 28.** *There is a 14-approximation for LP $\mathcal{F}$ C on points  $X$  where  $\mathcal{F} = \{S \subseteq X : w(S) \leq B\}$  for a poly-bounded weight function  $w : X \rightarrow \mathbb{R}_{\geq 0}$  and parameter  $B > 0$ .*

We first describe the reduction. For this, we need to define the Fractional Priority  $\mathcal{F}$ -Center where each point comes with a (possibly fractional) weight  $\mu_v$  and given  $m \geq 0$ , the goal is to find a set  $S \in \mathcal{F}$  that covers a total weight of more than  $m$  with minimum dilation of neighborhood radii.

► **Definition 29** (Fractional Priority  $\mathcal{F}$ -Center (FP $\mathcal{F}$ C)). *The input is a metric space  $(X, d)$  where each point  $v$  has a radius  $r_v > 0$  and a weight  $\mu_v \geq 0$ . Given parameter  $m \geq 0$  and a family of subsets of points  $\mathcal{F} \subseteq 2^X$ , the goal is to find  $S \in \mathcal{F}$  to minimize  $\alpha$  such that  $\mu(\{v \in X : d(v, S) \leq \alpha \cdot r_v\}) > m$ .*

An instance of FP $\mathcal{F}$ C is specified by the tuple  $((X, d), r, \mu, \mathcal{F}, m)$ . The following theorem states the reduction from LP $\mathcal{F}$ C to FP $\mathcal{F}$ C using the ellipsoid method. The proof of this theorem can be found in [2].

► **Theorem 30.** *Given LP $\mathcal{F}$ C instance  $\mathcal{I}$  and a black-box  $\alpha$ -approximate algorithm  $\mathcal{A}$  for FP $\mathcal{F}$ C that runs in time  $T(\mathcal{A})$ , one can get an  $\alpha$ -approximate solution for  $\mathcal{I}$  in time  $\text{poly}(|\mathcal{I}|)T(\mathcal{A})$ .*

Now we discuss how our results generalize to solve FP $\mathcal{F}$ C for matroid and knapsack constraints.

**Proof of Theorem 27.** According to Theorem 30 we only need to prove that we can find a 9-approximate solution for any given FP $\mathcal{F}$ C instance  $\mathcal{I} = ((X, d), r, \mu, \mathcal{F}, m)$ . First, observe that the LP for  $\mathcal{I}$  is the same as PMCO LP with a minor modification: The constraint  $\sum_{v \in X} \text{cov}(v) \geq m$  is changed to  $\sum_{v \in X} \mu_v \text{cov}(v) > m$ . Solve the LP for  $\mathcal{I}$  and use the obtained  $\text{cov}$  to run the reduction in Algorithm 3; but with a change in Line 3): instead of setting  $\lambda(v) \leftarrow |D(v)|$  for all  $v \in V$ , we will have  $\lambda(v) \leftarrow \mu(D(v))$ . This results in a WMatPP instance  $\mathcal{J}$  with fractional  $\lambda$ . The procedure in [13] can handle fractional  $\lambda$ 's so we can still compute the solution for  $\mathcal{J}$  in polynomial time. If this solution has value less than or equal to  $m$ , we know that  $\mathcal{I}$  is infeasible. Otherwise, Lemma 20 tells us that this solution for  $\mathcal{J}$  translates to a 9-approximation for  $\mathcal{I}$  and we are done. ◀

**Proof of Theorem 28.** We follow a procedure similar to the proof of Theorem 27. Per Theorem 30 we only need to prove there is a 14-approximation for the  $\text{FP-}\mathcal{F}\text{C}$  instance where  $\mathcal{F}$  is a set of feasible knapsack solutions with poly-bounded weights  $w : X \rightarrow \mathbb{R}_{\geq 0}$  and budget  $B > 0$ . Change the constraint  $\mathcal{P}_{\text{cov.1}}$  in  $\mathcal{P}_{\text{cov}}$  to  $\sum_{v \in X} \mu_v \text{cov}(v) > m$  and modify Line 3 of Algorithm 4 to  $\lambda(v) \leftarrow \mu(D(v))$  then follow the round-or-cut procedure in the proof of Theorem 22. The only challenge here is to prove the  $\text{WNapPP}$  problem can be solved in polynomial time. The dynamic program (which can be found in [2]) depends on the assumption that  $\lambda$ 's are poly-bounded. But here, our  $\lambda$ 's are real numbers so instead, we assume that our weights  $w : X \rightarrow \mathbb{R}_{\geq 0}$  are poly-bounded so we can still solve the problem via dynamic programming. ◀

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