On Rich Lenses in Planar Arrangements of Circles and Related Problems

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Abstract

We show that the maximum number of pairwise non-overlapping k-rich lenses (lenses formed by at least k circles) in an arrangement of n circles in the plane is $O(n^{3/2}\log(n/k^3)k^{-5/2} + n/k)$, and the sum of the degrees of the lenses of such a family (where the degree of a lens is the number of circles that form it) is $O(n^{3/2}\log(n/k^3)k^{-3/2} + n)$. Two independent proofs of these bounds are given, each interesting in its own right (so we believe). We then show that these bounds lead to the known bound of Agarwal et al. (JACM 2004) and Marcus and Tardos (JCTA 2006) on the number of point-circle incidences in the plane. Extensions to families of more general algebraic curves and some other related problems are also considered.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Lenses, Circles, Polynomial partitioning, Incidences

Digital Object Identifier 10.4230/LIPIcs.SoCG.2021.35

Related Version *Full Version*: `https://arxiv.org/abs/2012.04204`

Funding *Esther Ezra*: Work partially supported by NSF CAREER under grant CCF:AF-1553354 and by Grant 824/17 from the Israel Science Foundation.
Micha Sharir: Work partially supported by ISF Grant 260/18, by grant 1367/2016 from the German-Israeli Science Foundation (GIF), and by Blavatnik Research Fund in Computer Science at Tel Aviv University.
Joshua Zahl: Work supported by an NSERC Discovery Grant.

1 Introduction

Let C be a set of circles in the plane. A *lens* in the arrangement $\mathcal{A}(C)$ consists of a pair of distinct points p, q and a set of circles $C' \subset C$, each of which contain p and q. We will denote a lens by $\lambda_{p,q}(C')$. We say that two lenses $\lambda_{p,q}(C')$ and $\lambda_{s,t}(C'')$ are *overlapping* if there is a circle $c \in C' \cap C''$ so that the shorter arc of c containing p and q intersects the shorter arc of c containing s and t.[1] If two lenses are not overlapping, we call them *non-overlapping*. Finally, the *degree* of a lens $\lambda_{p,q}(C')$ is the cardinality of C', and we say a lens is *k-rich* if it has degree at least k.

[1] This definition requires a small modification if either p, q or s, t are antipodal points of c; if p, q are antipodal points of c, we say the corresponding lens overlaps every lens that contains c.

In this paper, we will be concerned with bounding the maximum size of a collection of pairwise non-overlapping k-rich lenses determined by a set of n circles in the plane. As we will see below, this question is closely related to the problem of *lens cutting*, which has a host of applications in combinatorial geometry; chief among these is the problem of obtaining incidence bounds for points and circles in the plane.

In [8] (sharpening a bound earlier obtained in [1, 3]) Marcus and Tardos proved that if C is a set of n circles, then any set of pairwise non-overlapping 2-rich lenses in C has cardinality $O(n^{3/2}\log n)$. Using standard random sampling techniques, this implies that any set of pairwise non-overlapping k-rich lenses has cardinality $O\big(\frac{n^{3/2}\log(n/k)}{k^{3/2}}\big)$. Our main result considerably improves this bound.

▶ **Theorem 1.** *Let C be a set of n circles in the plane, let $k \geq 2$, and let Λ be a set of pairwise non-overlapping k-rich lenses. Then $|\Lambda| = O\left(\frac{n^{3/2}\log(n/k^3)}{k^{5/2}} + \frac{n}{k}\right)$, and the sum of the degrees of the lenses in Λ is $O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}} + n\right)$.*

As mentioned, Theorem 1 was proved for the case $k = O(1)$ by Marcus and Tardos [8]. When k is large, we will show that Theorem 1 can be recast as an incidence problem between points and lines in $\mathbb{R}^3$. Crucially, we will show that only a few incidences of the type we analyze can occur inside any plane, and this will allow us to use a variant of Guth and Katz's point-line incidence bound from [7] to prove Theorem 1 when $k \geq n^{1/3}$. The details of this argument will be discussed in Section 2.

For intermediate values of k, we will give two proofs of Theorem 1. The first proof yields the bounds stated above, and the second proof gives a sightly weaker bound, in which the $\log(n/k^3)$ factor is weakened to $\operatorname{polylog} n$. Both of these proofs will use polynomial partitioning to divide the arrangement C of circles into smaller sub-arrangements. This breaks the problem of estimating $|\Lambda|$ into many smaller sub-problems, with smaller corresponding parameters n' (for the number of circles) and k' (for the richness). In the first proof, we will construct our partitioning so that in each sub-problem we have $k' = 2$, while in the second proof we will construct our partitioning so that in each sub-problem we have[2] $k' = (n')^{1/3}$.

▶ Remark 2. When $k \geq n^{1/3}\log^{2/3} n$, Theorem 1 states that $|\Lambda| = O(n/k)$. This bound is tight, since we can choose C to be a union of n/k sets of circles, where each set of circles has cardinality k and the circles in each set contain a common pair of points. For smaller values of k we conjecture that the bound in Theorem 1 is not tight.

▶ Remark 3. Theorem 1 implies that the circles in C can be cut into $O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}} + n\right)$ arcs, so that no pair of points is contained in k of the arcs, i.e., the resulting collection of arcs do not form any k-rich lens. Combining this observation with a variant of Székely's crossing-lemma technique [12], yields a new proof that the number of incidences between m points and n circles in the plane is (see Section 5):

$$O\left(m^{2/3}n^{2/3} + m^{6/11}n^{9/11}\log^{2/11} n + m + n\right).$$

This bound was first proved in [1, 8]. The point-circle incidence problem is among the most basic problems in incidence geometry, and has been studied intensively during the first half of the 2000's [1, 3, 8], culminating in the bound above. This bound is strongly suspected not to be tight for $n^{1/3} \leq m < n^{5/4}\log^{3/2} n$ (which is the range where the second term dominates),

[2] To simplify the presentation we ignore, throughout the paper, the issue of rounding non-integer values, and regard any such value as being rounded to the nearest integer.

but no improvement has been found in the last 15 years. While our result also does not yield an improvement, it provides a new proof (two proofs as a matter of fact), and we hope that this development will spur efforts to improve the above bound.

2 Preliminaries: The Case of Large or Small k

In this section we will prove Theorem 1 when k is small or $k \geq n^{1/3}$. As discussed above, when k is small (smaller than some constant) then the result immediately follows from [8].

▶ **Theorem 4** (Marcus and Tardos [8]). *Let C be a set of n circles in the plane, let $k \geq 2$, and let Λ be a set of pairwise non-overlapping k-rich lenses in C. Then Λ has cardinality $O(n^{3/2} \log n)$, and the sum of the degrees of the lenses in Λ is also $O(n^{3/2} \log n)$.*

When $k \geq n^{1/3}$, Theorem 1 will follow from a variant of Guth and Katz's point-line incidence bound [7]. Before stating this result we will need to introduce some additional notation. In what follows, C will be a set of circles in the plane, $k \geq 2$, and Λ will be a set of pairwise non-overlapping k-rich lenses in C. We define $\deg(\Lambda)$ to be the sum of the degrees of the lenses in Λ. We say that a circle $c \in C$ *participates* in a lens $\lambda_{p,q}(C') \in \Lambda$ if $c \in C'$.

We identify each circle c, with center (x, y) and radius r, with the point

$$c^* = (x, y, r^2 - x^2 - y^2)$$

in $\mathbb{R}^3$. We define $C^* = \{c^* \mid c \in C\}$, and identify each point $p = (p_x, p_y) \in \mathbb{R}^2$ with the plane

$$p^* = \{(x, y, z) \mid z = -2p_x x - 2p_y y + (p_x^2 + p_y^2)\}.$$

Observe that $(x, y, r^2 - x^2 - y^2) \in p^*$ if and only if $(x - p_x)^2 + (y - p_y)^2 = r^2$, i.e. the point p is contained in the circle centered at (x, y) of radius r. We identify each lens $\lambda = \lambda_{p,q}(C')$ with the line $\lambda^* = p^* \cap q^*$. Note that the lines $p^* \cap q^*$ and $s^* \cap t^*$ coincide if and only if $\{p, q\} = \{s, t\}$, and therefore our setting does not contain coinciding lines.[3] Define

$$\Lambda^* = \{\lambda^* \mid \lambda \in \Lambda\}.$$

For technical reasons, it will be convenient to require that no two distinct lenses in Λ share the same pair $\{p, q\}$ of endpoints, and thus the map $\lambda_{p,q}(C') \mapsto \lambda^*_{p,q}$ is injective on Λ, i.e., $|\Lambda^*| = |\Lambda|$. As we will see, this additional assumption is harmless, and we will discuss it briefly in Remark 10 below. We define ("nov" is an abbreviation for "non-overlapping")

$$I_{\text{nov}}(\Lambda) = \{(c^*, \lambda^*) \mid c \in C,\ \lambda \in \Lambda,\ c \text{ participates in } \lambda\}.$$

If the sets C and Λ are apparent from the context, we write I_{nov} in place of $I_{\text{nov}}(\Lambda)$. Note that $|I_{\text{nov}}| = \deg(\Lambda)$. If $(c^*, \lambda^*) \in I_{\text{nov}}$ then $c^* \in \lambda^*$. Thus $I_{\text{nov}} \subset I(C^*, \Lambda^*)$, where $I(C^*, \Lambda^*)$ denotes the set of all incidences between the points of C^* and the lines of Λ^*. Note that I_{nov} may be a proper subset of $I(C^*, \Lambda^*)$ due to the pairwise non-overlapping property of the lenses in Λ. That is, a circle c might pass through the vertices p, q of λ but it does not participate in λ because there is another lens $\lambda' \in \Lambda$, in which c does participate, so that the arc of c in λ' overlaps its arc between p and q; see Figure 1 for an illustration.

[3] This is because all planes of the form p^* are tangent to the paraboloid $\Pi :\ z = -x^2 - y^2$. A line $p^* \cap q^*$ that is disjoint from Π is contained in exactly two such tangent planes, which determine p and q.

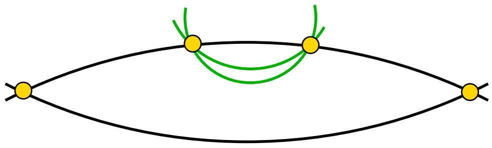

Figure 1 The non-overlapping property may exclude some arcs from participating in a lens.

As a first attempt to bound $|I_{\rm nov}|$, observe that the Szemerédi-Trotter theorem [13] implies that $|I_{\rm nov}| = O(n^{2/3}|\Lambda|^{2/3} + n + |\Lambda|)$, and thus, since each $\lambda \in \Lambda$ is k-rich, we have, when k is sufficiently large, $|\Lambda| = O(n^2/k^3 + n/k)$. Unfortunately this bound is too weak to prove Theorem 1. To strengthen the bound, we use a crucial property about non-overlapping lenses, which implies that few of the incidences in $I_{\rm nov}$ can concentrate in a plane.

▶ **Lemma 5.** *Let C be a set of circles and let Λ be a set of pairwise non-overlapping lenses in C. Let $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ be distinct lenses, and suppose that there is a circle $c \in C$ that participates in all three lenses, that is, $(c^*, \lambda_i^*) \in I_{\rm nov}$ for $i = 1, 2, 3$. Then $\lambda_1^*, \lambda_2^*, \lambda_3^*$ are not coplanar.*

Proof. Throughout the paper, $|ab|$ denotes the length of the segment ab (the Euclidean distance between a and b). The transformations $c \mapsto c^*$ and $\lambda \mapsto \lambda^*$ described above have the following property. For each nonvertical plane $h \subset \mathbb{R}^3$ there exists a point $w \in \mathbb{R}^2$ and a *power*[4] π such that

$$h = \{c^* \mid c \text{ is a circle, and } w \text{ has power } \pi \text{ with respect to } c\}.$$

Next, suppose that $\lambda_1^*, \lambda_2^*, \lambda_3^*$ lie in a common nonvertical plane h (this plane must necessarily contain c^*), and let w and π be the point and power associated with h. Let p_i and q_i be the vertices of λ_i, for $i = 1, 2, 3$. It is then easy to see that w must lie on each of the lines (in the xy-plane) through p_i and q_i, for $i = 1, 2, 3$, and the power π is $\pm|wp_i| \cdot |wq_i|$, where the sign is positive (resp. negative) if w lies outside (resp. inside) the segment $p_i q_i$; this is so because any other point cannot have a fixed power with respect to all the circles c whose dual points c^* lie on λ_i^*. In other words, the lines through p_1q_1, p_2q_2, and p_3q_3 are concurrent and meet at w. This however is impossible, because the circle c participates in all three lenses, which implies, as is easily verified (see Figure 2), that at least two of the lenses λ_1, λ_2, λ_3 must be overlapping, a contradiction that completes the proof for nonvertical planes.

The situation is similar when h is vertical. In this case all the circles c for which $c^* \in h$ are centered at points on the line ℓ of intersection of h with the xy-plane. For a lens $\lambda = \lambda_{p,q}$, the associated line λ^* is contained in h if and only if ℓ is the bisector of pq. Again, the fact that the lenses of Λ are pairwise non-overlapping is easily seen to imply that a circle c can contain at most two pairs p, q such that $\lambda_{p,q}$ is a lens in L, with $\lambda^* \subset h$, and c participates in $\lambda_{p,q}$. Hence, $\lambda_1^*, \lambda_2^*, \lambda_3^*$ cannot all lie in h. ◀

Note that a point c^* can be incident to arbitrarily many lines on a plane h, but Lemma 5 implies that at most two of them can contribute to $I_{\rm nov}$.

▶ **Corollary 6.** *Let C be a set of circles and let Λ be a set of pairwise non-overlapping lenses in C. Let $h \subset \mathbb{R}^3$ be a plane, and let $C' \subset C^*$ and $L' \subset \Lambda^*$ be the set of points and lines contained in h, respectively. Then*

$$|I_{\rm nov}(\Lambda) \cap I(C',\ L')| \le 2|C'|.$$

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[4] Recall that the power of a point w with respect to a circle c, centered at ξ and having radius r, is $|w\xi|^2 - r^2$.

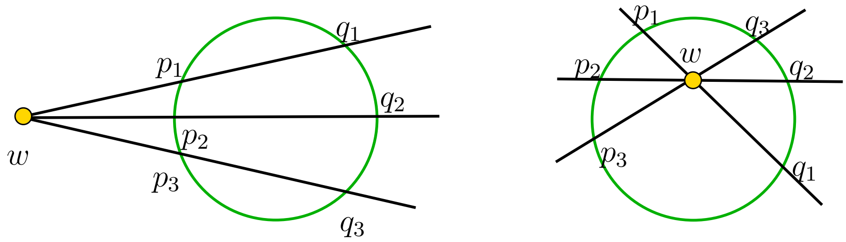

Figure 2 The lines in $\mathbb{R}^3$ corresponding to three pairwise non-overlapping lenses that share a common circle, which participates in all three of them, cannot be coplanar. (They are not the lines in the xy-plane drawn in the figure.)

Corollary 6 suggests that lines contained in a plane contribute few incidences to I_{nov}. Later in our arguments, we will need to study the contribution to I_{nov} coming from lines contained in an algebraic surface. The following lemma says that after removing a small number of ill-behaved lines, the contribution to I_{nov} is still small.

▶ **Lemma 7.** *Let C be a set of circles and let Λ be a set of pairwise non-overlapping lenses in C. Let $P \in \mathbb{R}[x, y, z]$ be a polynomial of degree D, and let $L' \subset \Lambda^*$ be a set of lines contained in the zero set $Z(P)$ of P. Then there is a set $L'' \subset L'$ of cardinality at most $11D^2$, so that*

$$|I_{\text{nov}}(\Lambda) \cap I(C^* \cap Z(P), L' \backslash L'')| \leq 2|C^* \cap Z(P)| + D|L' \backslash L''|.$$

Proof. This result follows immediately from the statements in Guth and Katz [7, Section 3], so we just briefly sketch the proof. Write $Z(P) = Z^{(1)} \cup Z^{(2)} \cup Z^{(3)} \cup Z^{(4)}$, where $Z^{(1)}$ is a union of planes, $Z^{(2)}$ is a union of reguli, $Z^{(3)}$ is a union of irreducible surfaces that are singly ruled by lines and are not planes or reguli, and $Z^{(4)}$ is the union of all irreducible components of $Z(P)$ that are not ruled. Let L'' be the set of lines contained in $Z^{(4)}$. By [7, Corollary 3.3] we have $|L''| \leq 11D^2$.

Let $Z_1, \ldots, Z_h$ be the irreducible components of $Z(P)$. For each index $i = 1, \ldots, h$, let C_i^* be the set of points $c^* \in C^*$ that are contained in Z_i and are not contained in any Z_j with $j < i$. Similarly, let L_i be the set of lines $\ell \in L' \backslash L''$ that are contained in Z_i and are not contained in any Z_j with $j < i$ (note that if the component Z_i is not ruled, then by definition L_i is empty).

First, we count the number of incidences $(c^*, \ell) \in I(C^* \cap Z(P), L' \backslash L'')$ for which $c^* \in C_i$ and $\ell \in L_j$ with $j \neq i$. For such an incidence, we must have that ℓ properly intersects Z_i. Thus there are at most $(D-1)|L' \backslash L''|$ incidences of this form.

Next we count the number of incidences $(c^*, \ell) \in I(C^* \cap Z(P), L' \backslash L'')$ for which $c^* \in C_i$ and $\ell \in L_i$. If Z_i is a plane, then by Corollary 6, there are $\leq 2|C_i^*|$ incidences of this type. If Z_i is a regulus, and hence doubly ruled, then it immediately follows that there are at most $2|C_i^*|$ incidences of this type. Finally, if Z_i is singly ruled, then Z_i has at most one exceptional point (incident to infinitely many lines contained in Z_i), and at most two exceptional (non-generator) lines, in the terminology of [7], which then implies that there are at most $2|C_i^*| + |L_i|$ incidences of this type. Summing the above contributions, we conclude

$$\begin{aligned} |I_{\text{nov}} \cap I(C^* \cap Z(P), L' \backslash L'')| &\leq (D-1)|L' \backslash L''| + \sum_i (2|C_i^*| + |L_i|) \\ &= 2|C^* \cap Z(P)| + D|L' \backslash L''|. \end{aligned}$$ ◀

▶ **Proposition 8.** *Let C be a set of circles and let Λ be a set of pairwise non-overlapping lenses in C. Then there is an absolute constant A so that*

$$|I_{\text{nov}}(\Lambda)| \leq A\big(|C|^{1/2}|\Lambda|^{3/4} + |C| + |\Lambda|\big). \tag{1}$$

Proof. The proposition is a slight variant of Guth and Katz's point-line incidence bound from [7], so we just briefly sketch the proof. We prove the result by induction on $|\Lambda|$. Let $M = |C|$, let $L = \Lambda^*$, and let $N = |L|$. First we can suppose that $N \leq M^2$. If not, then Proposition 8 follows immediately from the Kővári-Sós-Turán theorem (see [2, Theorem 9.5]), because the incidence graph of the points and the lines does not contain $K_{2,2}$ as a subgraph.

Let $D = \lfloor \min\{M^{1/2}N^{-1/4},\ N^{1/2}/10\}\rfloor$. We can suppose that $N^{1/2}/10$ (and thus D) is at least one, since otherwise $|I_{\text{nov}}(\Lambda)| \leq 100|C|$ and we are done. Using the polynomial partitioning for varieties established by Guth [6], we can find a polynomial $P \in \mathbb{R}[x, y, z]$ of degree $\leq D$ so that $\mathbb{R}^3\backslash Z(P)$ is a union of $O(D^3)$ open connected sets (such sets are often called *cells*), so that each cell contains $O(M/D^3)$ points from C^*, and each cell is intersected by $O(N/D^2)$ lines from L. If $D = N^{1/2}/10$ then each cell intersects $O(1)$ lines from L. Since each point from C^* is contained in at most one cell, we have in this case

$$I(C^*\backslash Z(P), \Lambda) = O(M).$$

If $D = M^{1/2}N^{-1/4}$, then standard incidence estimates allow us to bound

$$I(C^*\backslash Z(P), \Lambda) = O(M^{1/2}N^{3/4}).$$

Similarly, standard incidence estimates allow us to bound

$$|\{(c^*, \ell) \in I(C^* \cap Z(P), \Lambda) \mid \ell \not\subset Z(P)\}| = O(ND) = O(M^{1/2}N^{3/4}).$$

Let $L' \subset L$ be the set of lines contained in $Z(P)$. Applying Lemma 7, we obtain a set $L'' \subset L'$ with $|L''| \leq 11D^2 \leq |L|/2$, and

$$|I_{\text{nov}} \cap I(C^* \cap Z(P), L'\backslash L'')| \leq 2|C^* \cap Z(P)| + D|L'\backslash L''| = O(M^{1/2}N^{3/4} + M).$$

Finally, we apply the induction hypothesis to bound

$$|I_{\text{nov}} \cap I(C^* \cap Z(P), L'')| \leq A\big(M^{1/2}|L''|^{3/4} + M + |L''|\big) \leq 2^{-3/4}AM^{1/2}N^{3/4} + A(M + N).$$

Combining these bounds, we conclude that

$$|I_{\text{nov}}| \leq 2^{-3/4}AM^{1/2}N^{3/4} + A(M + N) + O(M^{1/2}N^{3/4}),$$

with implicit constant independent of A. Selecting A sufficiently large closes the induction. ◀

If C is a set of n circles and Λ is a set of pairwise non-overlapping k-rich lenses in C, then $|I_{\text{nov}}(\Lambda)| \geq k|\Lambda|$. Substituting this inequality in (1), we see that if n is sufficiently large ($n > 8A^3$ will suffice), then $|\Lambda| = O\left(\frac{n^2}{k^4} + \frac{n}{k}\right)$. Thus for all values of $n = |C|$, we have

$$\deg(\Lambda) = |I_{\text{nov}}(\Lambda)| = O\left(\frac{n^2}{k^3} + n\right). \tag{2}$$

In particular, Theorem 1 is true when $k \geq n^{1/3}\log^{-2/3} n$.

In the next two sections, we will prove Theorem 1 when $2 < k < n^{1/3}$, and also give a second proof of a slightly weaker bound. Note that Theorem 1 consists of two statements: a bound on $|\Lambda|$ and a bound on $\deg(\Lambda)$. The second statement immediately implies the first, by dividing the resulting bound by k. The next lemma shows that the first statement also implies the second.

▶ **Lemma 9.** *Suppose that for every set C of circles in the plane and every $k \geq 2$, every set of pairwise disjoint k-rich lenses in C has cardinality at most $A\left(\frac{|C|^{3/2}\log(|C|/k^3)}{k^{5/2}} + \frac{|C|}{k}\right)$. Then for every set C of circles in the plane, every $k \geq 2$, and every set Λ of pairwise disjoint k-rich lenses in C, we have* $\deg(\Lambda) = O\left(\frac{|C|^{3/2}\log(|C|/k^3)}{k^{3/2}} + |C|\right)$*, where the implicit constant depends only on A.*

Proof. Let C be a set of n circles in the plane. Let $k \geq 2$ and let Λ be a set of pairwise disjoint k-rich lenses in C. If $k \geq n^{1/3}$, then by (2) we have $\deg(\Lambda) = O(n)$ and we are done.

Suppose now that $2 \leq k \leq n^{1/3}$. Let $\Lambda_0 \subset \Lambda$ be the set of lenses that are $n^{1/3}$-rich. Let j_0 be the smallest integer so that $2^{-j_0}n^{1/3} \leq k$, and for each $j = 1, \ldots, j_0$, let $\Lambda_j \subset \Lambda \setminus \bigcup_{i=0}^{j-1}\Lambda_j$ be the set of lenses that are $2^{-j}n^{1/3}$-rich. By construction, $\Lambda = \bigsqcup_{j=0}^{j_0}\Lambda_j$, and for each index $1 \leq j \leq j_0$, the lenses in Λ_j have degree between $2^{-j}n^{1/3}$ and $2^{-j+1}n^{1/3}$. Thus

$$\begin{aligned}
\deg(\Lambda) &= \deg(\Lambda_0) + \sum_{j=1}^{j_0}\deg(\Lambda_j) \\
&\leq \deg(\Lambda_0) + \sum_{j=1}^{j_0}(2^{-j+1}n^{1/3})|\Lambda_j| \\
&\leq O(n) + \sum_{j=1}^{j_0}(2^{-j+1}n^{1/3})\left(\frac{An^{3/2}\log 2^{3j}}{(2^{-j}n^{1/3})^{5/2}}\right) \\
&= O(n) + O\left(2^{\frac{3}{2}j_0}\cdot\frac{n^{3/2}\log 2^{3j_0}}{n^{1/2}}\right) \\
&= O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}} + n\right),
\end{aligned}$$

where the implicit constant depends on A. ◀

▶ Remark 10. Recall that at the beginning of this section, we added the assumption that no two distinct lenses in Λ share the same pair $\{p, q\}$ of endpoints. We can now explain why this assumption is harmless. Indeed, let C be a set of circles and let Λ be a set of pairwise non-overlapping k-rich lenses in C. Let Λ' be the set of lenses formed by "merging" all lenses in Λ that share common endpoints, i.e., if $\lambda_{p,q}(C')$ and $\lambda_{p,q}(C'')$ are k-rich lenses in Λ, then $\lambda_{p,q}(C' \sqcup C'')$ will replace these two lenses in Λ'. While $|\Lambda'|$ might be smaller than $|\Lambda|$, we have $\deg(\Lambda') = \deg(\Lambda)$, because Λ consists of pairwise non-overlapping lenses. To summarize: if we can prove that every set of k-rich lenses with distinct pairs of endpoints has cardinality $O\left(\frac{|C|^{3/2}\log(|C|/k^3)}{k^{5/2}} + \frac{|C|}{k}\right)$, then this implies that every set Λ' of k-rich lenses with distinct endpoints has degree $\deg(\Lambda') = O\left(\frac{|C|^{3/2}\log(|C|/k^3)}{k^{3/2}} + |C|\right)$. This implies the same bound for any set Λ of k-rich lenses (i.e., the distinct endpoint requirement can be dropped).

3 First Proof of Theorem 1: Reduction to Small k

Let C be a set of n circles in the plane, and let Λ be a set of pairwise non-overlapping k-rich lenses in C. Let C^*, $L = \Lambda^*$, and I_{nov} be as defined in Section 2, and let $2 \leq k \leq n^{1/3}$. By Lemma 9, to prove Theorem 1 it suffices to show that $|\Lambda| = O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}}\right)$. Let $\alpha > 0$ be a small absolute constant that will be specified below. We will suppose that $2/\alpha \leq k \leq \frac{1}{10}n^{1/3}$, since otherwise Theorem 1 follows from (2).

We can assume that $|\Lambda| \geq 16n/k$, since otherwise we are done. Together with the inequality $k \leq n^{1/3}/10$, this implies that $|\Lambda| \geq 100k^2$ (the constant is actually larger, but 100 will suffice for our purpose). Let $D = \alpha k$. As in [7], we construct a partitioning polynomial f of degree $O(D)$, so that each of the $O(D^3)$ cells of $\mathbb{R}^3 \setminus Z(f)$ contains at most n/D^3 points of C^* (note that some points of C^* might lie on the zero set $Z(f)$).

Let $L' \subset L$ be the set of lines contained in $Z(f)$. By Lemma 7, there is a set $L'' \subset L'$ with $|L''| \leq 11 \deg(f)^2 = O(\alpha^2 k^2)$ so that

$$|I_{\text{nov}} \cap I(C^* \cap Z(P), L' \backslash L'')| \leq 2|C^* \cap Z(P)| + D|L' \backslash L''| \leq 2n + \alpha k |L' \backslash L''|.$$

Recall that $|L| = |\Lambda| \geq 100k^2$, and thus if $\alpha > 0$ is chosen sufficiently small then $|L''| \leq |L|/4$. Since each line in $L' \backslash L''$ participates in at least k incidences in I_{nov}, we have

$$|L' \backslash L''| \leq \frac{1}{k}(2n + \alpha k |L' \backslash L''|) \leq \frac{2n}{k} + \alpha |L| \leq \frac{|L|}{4},$$

where the final inequality follows by choosing $\alpha < 1/8$ and by using the assumption that $|\Lambda| \geq 16n/k$. We conclude that $|L \backslash L'| \geq |L|/2$. Next, each $\ell \in L \backslash L'$ participates in at least k incidences in I_{nov}, at least $k - \deg(f) \geq (1 - O(\alpha))k$ of which must be inside the cells of $\mathbb{R}^3 \setminus Z(f)$. We say an incidence $(c^*, \ell) \in I_{\text{nov}}$ is *lonely* if c^* is inside a cell of $\mathbb{R}^3 \setminus Z(f)$, and (c^*, ℓ) is the only incidence in I_{nov} involving ℓ that occurs inside that cell (i.e., there are no other points of C^* on ℓ inside that cell, so in the primal plane this implies that this configuration does not form a lens). Since each $\ell \in L \backslash L'$ intersects at most $\deg(f) + 1 \leq \alpha k + 1$ cells, each $\ell \in L \backslash L'$ participates in at least $(1 - \alpha)k - 1$ incidences in I_{nov} that are not lonely. Let I'_{nov} be the set of incidences $(c^*, \ell) \in I_{\text{nov}}$ where c^* is inside a cell of $\mathbb{R}^3 \setminus Z(f)$, and the incidence is not lonely. Then if $\alpha > 0$ is selected sufficiently small, we have

$$|I'_{\text{nov}}| \geq |L \backslash L'|(k/2) \geq \frac{1}{4} k |L|. \tag{3}$$

On the other hand, Theorem 4 says that there are $O\big((n/D^3)^{3/2} \log(n/D^3)\big)$ non-lonely incidences inside each cell. Thus

$$|I'_{\text{nov}}| = O\left(D^3 \left(\frac{n}{D^3}\right)^{3/2} \log\left(\frac{n}{D^3}\right)\right) = O\left(\frac{n^{3/2} \log(n/k^3)}{k^{3/2}}\right), \tag{4}$$

where the implicit constant depends on α. Combining (3) and (4), we conclude that

$$|L| = O\left(\frac{n^{3/2} \log(n/k^3)}{k^{5/2}}\right).$$

This completes the first proof of Theorem 1.

▶ Remark 11. It is an interesting challenge to extend the analysis in this section from circles to more general families of algebraic curves. This topic will be discussed in Section 6.

4 Second Proof of Theorem 1: Reduction to Large k

In this section we prove a slightly weaker version of Theorem 1 using a different proof technique. We feel that each of the techniques is interesting in its own right, and that each has the potential of being extended into different and more general contexts.

Most of the analysis in this section extends to more general algebraic curves, except for one (significant) step. We will discuss possible generalizations in Section 6.

Sharpening our notation from the previous sections, we define $F(n,k)$ to be the smallest integer with the following property: Let C be a set of at most n circles in the plane; let Λ be a set of pairwise disjoint k-rich lenses in C. Then $\deg(\Lambda) \le F(n,k)$. Note that $F(n,1) = \infty$ (i.e., it is undefined for $k = 1$), and, trivially, $F(n,k) = O(n^2)$ for all $k \ge 2$. Furthermore, $F(n,k)$ is monotone increasing in n and monotone decreasing in k. Abusing notation slightly, we extend our definition of $F(n,k)$ to all real numbers $n \ge 1$ and $k \ge 2$ by defining $F(n,k) = F(\lfloor n \rfloor, \lceil k \rceil)$. Finally, note that if A and n are integers, then $F(An,k) \ge AF(n,k)$, since we may take A disjoint copies of a configuration of n circles that achieves the $F(n,k)$ bound.

In Section 2 we proved that $F(n,k) = O(n^2/k^3 + n)$, and in particular there is an absolute constant A_0 so that, for any $z > 1$ and any k,

$$F(k^3 z, k) \le A_0 k^3 z^2. \tag{5}$$

In this section we will establish the following recurrence relation for $F(n,k)$.

▶ **Lemma 12.** *There is a constant A so that for any $D, n, k \ge 1$ we have*

$$F(n,k) \le AD^3 F(n/D^2, k/4) + F(AD^2, k/4) + AD^2 n. \tag{6}$$

Before proving Lemma 12, we show that it implies

$$F(n,k) = O\left(\frac{n^{3/2}\log^b(n/k^3)}{k^{3/2}}\right), \tag{7}$$

for some constant b and for all $n > k^3$. To show this, we solve the recurrence in the lemma in several steps. First, given n and k, we construct a sequence of real numbers $n_0, n_1, \ldots, n_s = n$, where $n_0 = k^3 z$, for a suitable value of $z > 1$ (the actual value will be between $\sqrt{2}$ and 2, and its concrete choice will be given towards the end of the forthcoming analysis), and $n_{j+1} = n_j^2/k^3$, for $j \ge 0$. That is, $n_j = k^3 z^{2^j}$ for $j \ge 0$, as is easily verified by induction on j. Since we want n_s to be equal to n, we have $z^{2^s} = n/k^3$. We also define, for each j, $D_j := n_j^{1/2}/k^{3/2} = z^{2^{j-1}}$, and note that $n_{j+1} = D_j^2 n_j$. The rationale for choosing these sequences will become clear as the solution of the recurrence unfolds. We have

$$\frac{n_{j+1}^{3/2}}{k^{3/2}} = \frac{D_j^3 n_j^{3/2}}{k^{3/2}} = \frac{D_j^2 n_j^2}{k^3} = D_j^2 n_{j+1}. \tag{8}$$

Note that $AD_j^2 = An_j/k^3$. For simplicity, we shall suppose that $k \ge A^{1/3}$ and thus $AD_j^2 \le n_j$ (if this inequality failed then Theorem 1 follows from Theorem 4, since k becomes a constant).

We next prove that for each $j \ge 0$ we have

$$F(n_j, 4^j k) \le A_0 z^{1/2} (4A)^j n_j^{3/2}/k^{3/2}, \tag{9}$$

where A_0 is the constant from (5). The case $j = 0$ is precisely (5). For the induction step, we compute, using Lemma 12:

$$\begin{aligned}
F(n_{j+1}, 4^{j+1}k) &\le AD_j^3 F(n_{j+1}/D_j^2, 4^j k) + F(AD_j^2, 4^j k) + AD_j^2 n_{j+1} \\
&\le 2AD_j^3 F(n_j, 4^j k) + AD_j^2 n_{j+1} \\
&\le 2AD_j^3 (A_0 z^{1/2})(4A)^j n_j^{3/2}/k^{3/2} + AD_j^2 n_{j+1} \\
&\le 2 \cdot 4^j (A_0 z^{1/2}) A^{j+1} n_{j+1}^{3/2}/k^{3/2} + AD_j^2 n_{j+1} \\
&\le A_0 z^{1/2} (4A)^{j+1} n_{j+1}^{3/2}/k^{3/2},
\end{aligned}$$

where in the last inequality we used the equality (8), namely $D_j^2 n_{j+1} = n_{j+1}^{3/2}/k^{3/2}$.

Thus if $n > k^3$, we can find $z > 1$ and s so that $n_s = k^3 z^{2^s} = n$ i.e., $2^s = \frac{\log(n/k^3)}{\log z}$ or $s = \log\log(n/k^3) - \log\log z$. Using (9), with $j = s$, $n_j = n$ and replacing k by $k/4^s$ we get

$$F(n,k) \le A_0 z^{1/2} (2^5 A)^s \frac{n^{3/2}}{k^{3/2}}.$$

Putting $B := 2^5 A$ and $b := \log B$, we get

$$(2^5 A)^s = B^s = (2^s)^b = \left(\frac{\log(n/k^3)}{\log z}\right)^b = \frac{\log^b(n/k^3)}{\log^b z},$$

and hence

$$F(n,k) \le A_0 z^{1/2} B^s \frac{n^{3/2}}{k^{3/2}} = \frac{A_0 z^{1/2}}{\log^b z} \cdot \frac{n^{3/2}\log^b(n/k^3)}{k^{3/2}}.$$

It remains to determine the value of z. Put $z_j = (n/k^3)^{1/2^j}$, for $j \ge 0$. This sequence converges to 1 and satisfies $z_j = \sqrt{z_{j-1}}$ for each j. We take s to be that (unique) value of j for which $\sqrt{2} < z_j \le 2$ (for such a z to exist we need to assume that $n > k^3\sqrt{2}$). We then have

$$\frac{A_0 z^{1/2}}{\log^b z} \le A_1 := \frac{A_0\sqrt{2}}{\log^b \sqrt{2}}, \qquad \text{and so} \qquad F(n,k) \le A_1 \frac{n^{3/2}\log^b(n/k^3)}{k^{3/2}}.$$

This establishes (7), and leaves us with the task of proving Lemma 12.

Proof of Lemma 12. Let C be a set of n circles in the plane and let Λ be a set of pairwise non-overlapping k-rich lenses in C. Following the technique of Ellenberg, Solymosi, and Zahl [5], for each circle $c \in C$ with defining polynomial g (i.e., $c = Z(g)$), consider the variety

$$\{(x,y,z) \in \mathbb{R}^3 \mid g(x,y) = 0,\ z\partial_y g(x,y) + \partial_x g(x,y) = 0\}.$$

As discussed in [5, Section 3.3], this variety is a union of three irreducible curves in $\mathbb{R}^3$, two of which are vertical lines (one above each of the points in c where the circle has infinite slope). Define $\gamma(c) \subset \mathbb{R}^3$ to be the irreducible component that is not a vertical line. If $(x,y) \in c$ is a point where c has finite slope, then $(x,y,z) \in \gamma(c)$ if and only if c has slope z at (x,y). In particular, if $\lambda_{p,q}(C')$ is a lens and if $c \in C'$, then the (shorter) arc $\beta \subset c$ with endpoints p and q lifts to a curve segment $\gamma(\beta) \subset \gamma(c)$. We will call this curve segment the *lifted arc* of c corresponding to the lens λ.

For a set of circles C, define $\gamma(C) = \{\gamma(c) \mid c \in C\}$. Let $\Lambda_{p,q}(C')$ be a lens in C, and suppose that none of the circles $c \in C'$ have infinite slope at the point p or q (this is a harmless assumption, since at most two circles containing p and q can have infinite slope at p or q). Let $\ell_p, \ell_q \subset \mathbb{R}^3$ be vertical lines passing through $(p,0)$ and $(q,0)$ respectively. Then each of the curves in $\gamma(C')$ intersect ℓ_p and ℓ_q. Furthermore, each of the intersection points $\{\gamma(c) \cap \ell_p \mid c \in C'\}$ are distinct, and similarly for ℓ_q. Define $z(\gamma(c) \cap \ell_p)$ to be the z-coordinate of $\gamma(c) \cap \ell_p$. The curves in $\gamma(C')$ have the following property:

Order Reversal Property. If we order the curves $c_1, \ldots, c_m \in C'$ so that

$$z(\gamma(c_1) \cap \ell_p) < z(\gamma(c_2) \cap \ell_p) < \cdots < z(\gamma(c_m) \cap \ell_p),$$

$$\text{then} \quad z(\gamma(c_1) \cap \ell_q) > z(\gamma(c_2) \cap \ell_q) > \cdots > z(\gamma(c_m) \cap \ell_q).$$

I.e., the order on C' given by the z-coordinates of $\gamma(c) \cap \ell_p$ is precisely the reverse of the order given by $\gamma(c) \cap \ell_q$. See Figure 3.

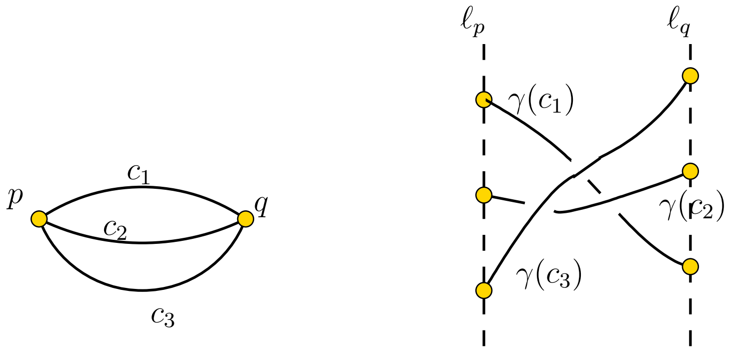

Figure 3 A lens is lifted to a multi-2-cycle in three dimensions.

We employ the approach of Aronov and Sharir [4], as detailed in Sharir and Zahl [10], with some modifications, as follows. We construct a partitioning polynomial f, of degree $O(D)$, so that we have $O(D^3)$ open connected cells of $\mathbb{R}^3 \setminus Z(f)$ (recall that $Z(f)$ is the zero set of f), and at most n/D^2 curves from $\gamma(C)$ intersect each cell. The existence of such a partitioning polynomial was established in Guth [6]. For each cell O of $\mathbb{R}^3 \setminus Z(f)$, define $C_O = \{c \in C \mid \gamma(c) \cap O \neq \emptyset\}$.

For a cell $O \subset \mathbb{R}^3 \setminus Z(f)$, we say that a lens $\lambda = \lambda_{p,q}(C') \in \Lambda$ is *preserved* within O if for at least $\deg(\lambda)/4 = |C'|/4$ circles $c \in C'$, the lifted arc of c corresponding to the lens λ is fully contained in O. In particular, if $\lambda_{p,q}(C')$ is preserved within O, then $|C' \cap C_O| \geq |C'|/4 \geq k/4$. Thus for each cell O, we have

$$\sum \deg(\lambda_{p,q}(C')) \leq 4 \sum |C' \cap C_O| \leq 4F(n/D^2, k/4),$$

where the sum is taken over all lenses $\lambda_{p,q}(C')$ that are preserved within O. Summing over all cells O, we conclude that

$$\sum_{\lambda \text{ preserved within a cell}} \deg(\lambda) = O(D^3) F(n/D^2, k/4). \tag{10}$$

If a lens is not preserved within any cell, we say that it is *disrupted* by $Z(f)$. It remains to bound the sum of the degrees of the disrupted lenses. The arguments here are very similar to those in [10], so we just sketch them briefly. First, for each $(x, y, z) \in \mathbb{R}^3$, define $h(x, y, z)$ to be the number of intersections between $Z(f)$ and the infinite ray $\{(x, y, t) \mid t > z\}$. This quantity is finite (indeed bounded by $\deg f$) unless the vertical line passing through (x, y, z) is contained in $Z(f)$. Following the arguments in [4, 10], there is a polynomial $g \in \mathbb{R}[x, y, z]$ of degree $O(D^2)$ with the following properties.

- h is constant on each connected component of $\mathbb{R}^3 \setminus \big(Z(f) \cup Z(g)\big)$.
- h is constant on $Z(f) \setminus Z(g)$.
- $g(x, y, z)$ is independent of z, i.e., $g(x, y, z) = \tilde{g}(x, y)$ for some polynomial $\tilde{g}(x, y) \in \mathbb{R}[x, y]$.
- If $Q(x, y, z)$ is an irreducible component of f that is independent of z, then Q divides g, i.e., Q is also an irreducible component of g.

In brief, the polynomial $g(x, y, z) = \tilde{g}(x, y)$ is constructed by computing the resultant of f and $\partial_z f$; see [4, 10] for details.

We say that a lens $\lambda_{p,q}(C')$ is *preserved* by $Z(g)$ if at least $\deg(\lambda)/4 = |C'|/4$ of the curves from $\gamma(C')$ are contained in $Z(g)$. Recall that $g(x, y, z) = \tilde{g}(x, y)$, and thus if $\gamma(c) \subset Z(g)$, we must have $c \subset Z(\tilde{g})$. In particular, at most $\deg(g) = \deg(\tilde{g}) = O(D^2)$ circles from C can be contained in $Z(g)$. Arguing as before, we conclude that

$$\sum_{\lambda \text{ preserved by } Z(g)} \deg(\lambda) \leq 4F(O(D^2), k/4) = F(O(D^2), k/4). \tag{11}$$

It remains to bound the sum of the degrees of the lenses that are disrupted by $Z(f)$ and not preserved by $Z(g)$.

Claim. If $\lambda_{p,q}(C')$ is such a lens, then there are at least $|C'|/4 - 1$ circles $c \in C'$ so that the lifted arc of c corresponding to the lens λ properly intersects $Z(f)$ or $Z(g)$.

Once this claim has been established we are done, since the number of such proper intersections is at most $n(\deg f + \deg g) = O(D^2 n)$, and since the lenses are pairwise non-overlapping, each such intersection is counted towards at most one lens.

To verify this claim, let $\lambda_{p,q}(C')$ be a lens that is disrupted by $Z(f)$ and not preserved by $Z(g)$. We will divide our argument into the following two cases.

Case 1. At least half of the lifted circles in C' are contained in $Z(f) \cup Z(g)$.
Since the lens $\lambda_{p,q}(C')$ is not preserved by $Z(G)$, fewer than $|C'|/4$ circles from C' can be contained in $Z(g)$. Thus at least $|C'|/4$ circles from C' are contained in $Z(f)$. Enumerate these circles as $c_1, \ldots, c_w$, for some $w \geq |C'|/4$, so that $z(\gamma(c_1) \cap \ell_p) < \cdots < z(\gamma(c_w) \cap \ell_p)$. Since each circle c_i is contained in $Z(f)$ but not contained in $Z(g)$, we have

$$h(\gamma(c_1) \cap \ell_p) < \cdots < h(\gamma(c_w) \cap \ell_p).$$

However, by the Order Reversal Property, we have $z(\gamma(c_1) \cap \ell_q) > \cdots > z(\gamma(c_w) \cap \ell_q)$, so

$$h(\gamma(c_1) \cap \ell_q) > \cdots > h(\gamma(c_w) \cap \ell_q).$$

Since h is constant on $Z(f) \backslash Z(g)$, we conclude that for all but at most one index i, the lifted arc of c_i corresponding to the lens λ intersects $Z(g)$. Thus for at least $|C'|/4 - 1$ circles $c \in C'$, the lifted arc of c corresponding to the lens λ properly intersects $Z(g)$.

Case 2. At least half of the lifted circles in C' are not contained in $Z(f) \cup Z(g)$.
Let $c_1, \ldots, c_w$, for some $w \geq |C'|/2$, be circles in C' whose lifted curve is not contained in $Z(f) \cup Z(g)$. For each index $i = 1, \ldots, w$, let β_i be the (shorter) arc of c_i with endpoints p and q. Let v be the number of arcs $\gamma(\beta_i)$ that properly intersect $Z(f)$; if $v \geq |C'|/4$ then we are done. If not, then at least $w - v$ of the arcs $\gamma(\beta_i)$ are contained inside a cell of $Z(f)$ (though different arcs might be contained inside different cells). But an argument analogous to the one above shows that all of the arcs in all but one of the cells must properly intersect $Z(g)$. Since no cell contains more than $|C'|/4$ of the lifted arcs, at least $w - v - |C'|/4$ of the lifted arcs must properly intersect $Z(g)$. We conclude that v arcs properly intersect $Z(f)$ and at least $w - v - |C'|/4 \geq |C'|/4 - v$ arcs properly intersect $Z(g)$. Thus at least $|C'|/4$ arcs properly intersect either $Z(f)$ or $Z(g)$.

Combining the bounds in (10), (11), adding the overhead $O(D^2 n)$, and making the constants in the $O(\cdot)$ notation explicit, bounding all of them by the same constant A, we obtain the recurrence asserted in the lemma. ◀

5 Point-Circle and Lens-Circle Incidence Bounds

5.1 Point-circle incidence bounds

We can use Theorem 1 to bound the number of incidences between m points and n circles in the plane. As it turns out, the bound that we get is the same as the best known bound due to Agarwal et al. [1] (and to [8]). We describe the derivation nonetheless, as an illustration of the power of Theorem 1.

Let P be a set of m points and let C be a set of n circles. We fix a parameter k, to be determined below, and use a modified variant of Székely's technique [12]. We first construct a graph G whose vertices are the points of P, and whose edges connect pairs of consecutive points along each circle of C. Some edges of G form k-rich lenses, and we observe that these lenses are pairwise non-overlapping. Let Λ denote the set of these lenses. We split G into two subgraphs G_0 and G_1, where G_1 consists of all the edges in the lenses of Λ and G_0 consists of all the remaining edges.

By Theorem 1, the number of edges of G_1 is $O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}}+n\right)$.

The number $E_0(c)$ of edges of G_0 along a circle c is $|N_c|-E_1(c)$, where $N_c=P\cap c$ and $E_1(c)$ is the number of edges of G_1 along c. Note that the multiplicity of each edge of G_0 is smaller than k. An upper bound on the number of edges of G_0 then follows from a variant of Székely's technique (see Theorem 8 of [12]), which takes into account the maximum multiplicity of an edge in the graph (which is smaller than k). Concretely, denoting by $|G_0|$ (resp., $|G_1|$) the number of edges of G_0 (resp., G_1), we have

$$|G_0| = O\left(k^{1/3}m^{2/3}n^{2/3}+km\right), \qquad \text{and thus}$$

$$|G| = |G_0|+|G_1| = O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}}+k^{1/3}m^{2/3}n^{2/3}+km+n\right).$$

We balance the first two terms in the bound for $|G|$ by choosing $k=n^{5/11}(\log(n/k^3))^{6/11}/m^{4/11}$. This is meaningful when $k\ge 1$, which holds when $m\le n^{5/4}\log^{3/2} n$, which is indeed the interesting range. For larger values of m, we take $k=2$ and get the bound $O(m^{2/3}n^{2/3}+m+n^{3/2}\log n)$, which is dominated by $O(m^{2/3}n^{2/3}+m)$. The bound then becomes (see [1])

$$O\left(m^{2/3}n^{2/3}+m^{6/11}n^{9/11}\log^{2/11}(m^3/n)+m+n\right).$$

Note that the bound is meaningful only for $m>n^{1/3}$. For smaller values of m, the bound becomes $O(n)$. The logarithmic factor provides a "smooth" transition from the above bound to the linear bound as $m\searrow n^{1/3}$.

5.2 Circle-lens incidence bounds

We can apply the bounds in Theorem 1 to obtain an upper bound on the number of incidences between m pairwise non-overlapping lenses and n circles, where a lens λ is said to be incident to a circle c if c participates in λ. To do so, let Λ be the given set of m lenses (which are not necessarily rich). Set $k:=\frac{n^{3/5}\log^{2/5}(n/k^3)}{m^{2/5}}$. (Note that $k=\Omega(1)$ since we always have $m=O(n^{3/2}\log n)$ [1, 8].) The non-k-rich lenses of Λ contribute at most $km=m^{3/5}n^{3/5}\log^{2/5}(n/k^3)$ incidences. The k-rich lenses contribute, by Theorem 1,

$$O\left(\frac{n^{3/2}\log(n/k^3)}{k^{3/2}}+n\right)=O\left(m^{3/5}n^{3/5}\log^{2/5}(n/k^3)+n\right)$$

incidences. Since $\log(n/k^3)=O(\log(m^3/n^2))$, we thus obtain:

▶ **Theorem 13.** *Let Λ be a family of m pairwise non-overlapping lenses in an arrangement of n circles in the plane. Then the number of incidences between the lenses of Λ and the circles of C is* $O\left(m^{3/5}n^{3/5}\log^{2/5}(m^3/n^2)+n\right)$.

▶ Remark 14. Aside from the log factor, this bound generalizes the recent result of Sharir and Zlydenko [11] (see also Sharir, Solomon, and Zlydenko [9]) on incidences between so-called directed points and circles. A directed point is a pair (p,u) where p is a point in the plane and u is a direction, and (p,u) is incident to a circle c if $p \in c$ and u is the direction of the tangent to c at p. The bound in [9, 11] is $O(m^{3/5}n^{3/5}+m+n)$ which is similar, albeit slightly sharper, than the bound in Theorem 13. The two setups are indeed related, as a directed point of degree at least two is a limiting case of a lens, and the resulting infinitesimal limit lenses are clearly pairwise non-overlapping. The novelty in Theorem 13 is that lenses are 4-parameterizable, that is, each lens is specified by four real parameters (the coordinates of its vertices p, q), whereas directed points are 3-parameterizable. This makes the analysis in [9, 11] inapplicable to the case of lenses, and yet the bound is more or less preserved.

6 Discussion

Each of the two proofs of the main result, given in Sections 3 and 4, can be extended to more general contexts, provided that certain key properties can be established, or alternatively are assumed. In this section we discuss such possible extensions, and then summarize the state of affairs developed in this paper.

First proof. We offer a few informal comments on a possible approach to extending Theorem 1 to more general plane curves. First, we need to assume that the curves in our family C are 3-parameterizable, so that we can represent them as points in a dual 3-space, and also that they are algebraic of some constant degree. Each point $p \in \mathbb{R}^2$ then becomes a two-dimensional surface p^*, consisting of the points in $\mathbb{R}^3$ whose corresponding curves contain p. Then a lens with endpoints p, q becomes the curve $\ell_{p,q} = p^* \cap q^*$ (we ignore in this informal discussion various issues involving degeneracies and various assumptions that one might need to impose).

We can then apply the same partitioning argument. Inside each cell, we use the (slightly weaker) bound $O(n^{3/2}\text{polylog}(n))$, due to Sharir and Zahl [10], on the number of lenses formed by a set of bounded-degree algebraic curves.

The main difference is in handling points and curves that lie on the zero set of the partitioning polynomial. The preceding analysis strongly relied on Lemma 5, which requires that the curves in C be circles. This in turn allowed us to control the number of incidences occurring on the zero-set $Z(f)$ of the partitioning polynomial. With an analogue of Lemma 5 for more general curves, it seems plausible that the rest of the argument will work with standard modifications.

Second proof. Lemma 12 holds with almost no modification if the circles in C are replaced by arbitrary (bounded degree, algebraic) curves. Indeed, the only important difference is that the Order Reversal Property might not be true, but the dichotomy that a lens must either be preserved within a cell or disrupted by $Z(f)$ remains true, and the bound on the number of lenses that are disrupted by $Z(f)$ and not preserved by $Z(g)$ also remains true. Thus the only obstruction to extending Theorem 1 to more general curves is that the estimate $F(n,n^{1/3}) = O(n)$ (or, more precisely, $F(nz,n^{1/3}) = O(nz^2)$ for $z > 1$), which serves as the base case of the induction, might not be true. We conjecture that for other classes of curves, an estimate of the form $F(n,n^b) = O(n)$ should hold (where $b > 0$ depends on the class of curves). As b becomes larger, the corresponding analogue of Theorem 12 becomes weaker.

References

1 P. Agarwal, E. Nevo, J. Pach, R. Pinchasi, M. Sharir, and S. Smorodinsky. Lenses in arrangements of pseudocircles and their applications. *J. ACM*, 51:139–186, 2004.

2 P. Agarwal and J. Pach. *Combinatorial Geometry.* Wiley Interscience, 1995.

3 B. Aronov and M. Sharir. Cutting circles into pseudo-segments and improved bounds for incidences. *Discrete Comput. Geom.*, 28:475–490, 2002.

4 B. Aronov and M. Sharir. Almost tight bounds for eliminating depth cycles in three dimensions. *Discrete Comput. Geom.*, 59:725–741, 2018.

5 J. Ellenberg, J. Solymosi, and J. Zahl. New bounds on curve tangencies and orthogonalities. *Discrete Anal.*, 18, 2016.

6 L. Guth. Polynomial partitioning for a set of varieties. *Math. Proc. Cambridge Philos. Soc.*, 159:459–469, 2015.

7 L. Guth and N.H. Katz. On the Erdős distinct distances problem in the plane. *Ann. of Math.*, 181:155–190, 2015.

8 A. Marcus and G. Tardos. Intersection reverse sequences and geometric applications,. *J. Combin. Theory Ser. A*, 113:675–691, 2006.

9 M. Sharir, N. Solomon, and O. Zlydenko. Incidences with curves with almost two degrees of freedom. *arXiv e-prints*, 2020. `arXiv:2003.02190v2`.

10 M. Sharir and J. Zahl. Cutting algebraic curves into pseudo-segments and applications. *J. Combin. Theory Ser. A*, 150:1–35, 2017.

11 M. Sharir and O. Zlydenko. Incidences between points and curves with almost two degrees of freedom. In *36th International Symposium on Computational Geometry (SoCG 2020)*, volume 164, pages 66:1–66:14, 2020.

12 L. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.*, 11:1–10, 1993.

13 E. Szemerédi and W.T. Trotter. Extremal problems in discrete geometry,. *Combinatorica*, 3:381–392, 1983.