Flexible List Colorings in Graphs with Special Degeneracy Conditions

Peter Bradshaw
Simon Fraser University, Burnaby, Canada
pabradsh@sfu.ca

Tomáš Masařík
Simon Fraser University, Burnaby, Canada
masarik@kam.mff.cuni.cz

Ladislav Stacho
Simon Fraser University, Burnaby, Canada
ladislav_stacho@sfu.ca

Abstract

For a given $\varepsilon > 0$, we say that a graph G is *ε-flexibly k-choosable* if the following holds: for any assignment L of lists of size k on $V(G)$, if a preferred color is requested at any set R of vertices, then at least $\varepsilon|R|$ of these requests are satisfied by some L-coloring. We consider flexible list colorings in several graph classes with certain degeneracy conditions. We characterize the graphs of maximum degree Δ that are ε-flexibly Δ-choosable for some $\varepsilon = \varepsilon(\Delta) > 0$, which answers a question of Dvořák, Norin, and Postle [List coloring with requests, JGT 2019]. We also show that graphs of treewidth 2 are $\frac{1}{3}$-flexibly 3-choosable, answering a question of Choi et al. [arXiv 2020], and we give conditions for list assignments by which graphs of treewidth k are $\frac{1}{k+1}$-flexibly $(k+1)$-choosable. We show furthermore that graphs of treedepth k are $\frac{1}{k}$-flexibly k-choosable. Finally, we introduce a notion of *flexible degeneracy*, which strengthens flexible choosability, and we show that apart from a well-understood class of exceptions, 3-connected non-regular graphs of maximum degree Δ are flexibly $(\Delta - 1)$-degenerate.

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1 Introduction

A *proper coloring* of a graph G is a function $\varphi : V(G) \to S$ by which each vertex of G receives a color from some color set S, such that no pair of adjacent vertices is assigned the same color. Proper graph coloring is one of the oldest concepts in graph theory. The *precoloring extension* problem is a related question which asks whether a graph can be properly colored using a given color set even when some vertices have preassigned colors. Surprisingly, the precoloring extension problem often has a negative answer, even for relatively simple graph classes and for a small number of precolored vertices [16, 18]. In particular, it is NP-complete to decide whether an interval graph can be properly colored when only two vertices are precolored by different colors [1].

In practical applications, proper graph coloring is often used to represent scheduling problems, in which case preassigned colors may be used to represent scheduling preferences or requests. Thus, the precoloring extension problem is not only an interesting concept in theory, but also has various practical applications, such as register allocation, scheduling, and many others (see [1] for an overview of basic precoloring extension applications). However, in many applications that arise from graph coloring with requests, it is not always necessary to satisfy all coloring requests. With this in mind, Dvořák, Norin, and Postle [11] recently introduced a relaxed notion of the precoloring extension problem in which it is not mandatory to satisfy every coloring request and it is sufficient to satisfy a positive fraction of all coloring requests. They named this new concept *flexibility*.

Dvořák, Norin, and Postle observed that for k-colorable graphs, the problem of finding a proper k-coloring that satisfies a positive fraction of some set of coloring requests is trivial, since by permuting the k colors of any proper k-coloring, one can satisfy at least a fraction of $\frac{1}{k}$ of any set of coloring requests. However, this approach does not work for list colorings, and thus the flexibility problem applied to list colorings becomes attractive and challenging. A graph where each vertex v has a list $L(v)$ of available colors is called *L-colorable* if there exists a proper coloring in which each vertex v receives a color from $L(v)$. We call such a coloring an *L-coloring*. A graph is *k-choosable* if every assignment L of at least k colors to each vertex guarantees an L-coloring. A graph class $\mathcal{G}$ is *k-choosable* if every $G \in \mathcal{G}$ is k-choosable. The *choosability* of a graph G, written $\mathrm{ch}(G)$, is the minimum k such that G is k-choosable.

We now formally introduce the concept of flexibility. A *weighted request* on a graph G with list assignment L is a function w that assigns a non-negative real number to each pair (v, c) where $v \in V(G)$ and $c \in L(v)$. For $\varepsilon > 0$, we say that w is *ε-satisfiable* if there exists an L-coloring φ of G such that

$$\sum_{v \in V(G)} w(v, \varphi(v)) \geq \varepsilon \cdot \sum_{\substack{v \in V(G) \\ c \in L(v)}} w(v, c).$$

Then, we say that G is *weighted ε-flexible* with respect to L if every weighted request on G is ε-satisfiable.

We also consider an unweighted version of flexibility. We say that a *request* on a graph G with list assignment L is a function r with $\mathrm{dom}(r) \subseteq V(G)$ such that $r(v) \in L(v)$ for all $v \in \mathrm{dom}(r)$.[1] Analogously, for $\varepsilon > 0$, we say that a request r is *ε-satisfiable* if there exists an L-coloring φ of G such that at least $\varepsilon|\mathrm{dom}(r)|$ vertices v in $\mathrm{dom}(r)$ receive the color $r(v)$. Then, we say that G is ε-flexible with respect to L if every request on G is ε-satisfiable.

An interesting special case of unweighted flexibility, which was brought up recently in [7], arises when each vertex of G requests exactly one color, i.e., $\mathrm{dom}(r) = V(G)$. We call such a request r *widespread*. Analagously, we say that a graph G with list assignment L is *weakly ε-flexible* if every widespread request is ε-satisfiable.

If G is ε-flexible for every list assignment with lists of size k, we say that G is *ε-flexible for lists of size k*. To simplify the terminology, we often say that G is *ε-flexibly k-choosable*. For a graph class $\mathcal{G}$, we may omit ε and say that $\mathcal{G}$ is *flexibly k-choosable* if there exists a universal constant $\varepsilon > 0$ such that every graph in $\mathcal{G}$ is ε-flexibly k-choosable. We will use this notation whenever we do not care about the precise value of the constant ε and only

[1] Note that if for all $v \in V(G)$ and for all $c \in L(v)$ $w(v, c) \in \{0, 1\}$ then weighted request w is actually a request.

care that a constant $\varepsilon > 0$ exists. A meta-question which is central in the study of flexibility asks whether a given graph class is flexibly k-choosable. We sometimes refer to this question simply as *flexibility*. Note that all the notation mentioned in this paragraph can be also stated for weak or weighted flexibility.

Typically, research in flexibility focuses on bounding the list size k needed for a graph class to be ε-flexibly k-choosable for some $\varepsilon > 0$, and the precise value of ε is not usually of concern. Apart from the original paper introducing flexibility [11], where some basic results in terms of maximum average degree were established, the main focus in flexibility research has been on planar graphs. In particular, for many subclasses $\mathcal{G}$ of planar graphs, there has been a vast effort to reduce the gap between the choosability of $\mathcal{G}$ and the list size needed for flexibility in $\mathcal{G}$. As of now, some tight bounds on list sizes are known: namely, triangle-free[2] planar graphs [10], $\{C_4, C_5\}$-free planar graphs [21], and $\{K_4, C_5, C_6, C_7, B_5\}$-free[3] planar graphs [17] are flexibly 4-choosable, and planar graphs of girth 6 [9] are flexibly 3-choosable. For other subclasses $\mathcal{G}$ of planar graphs, an upper bound is known for the list size k required for $\mathcal{G}$ to be flexibly k-choosable [7, 19]. However, these upper bounds are not known to be tight; see [7] for a comprehensive overview and a discussion of the related results. The main question in this direction, of determining whether planar graphs are flexibly 5-choosable, remains open.

1.1 Our Results

As discussed, the list size k needed for a graph to be ε-flexibly k-choosable for some $\varepsilon > 0$ has a basic lower bound equal to the graph's choosability. Similarly to list coloring, a graph's *degeneracy* d, which is the largest minimum degree over all induced subgraphs, plays a natural role in establishing upper bounds on the list size needed for flexibility in a graph. However, while a simple greedy argument gives an upper bound of $d + 1$ on the choosability (the list size needed just for a proper list-coloring), only the weaker upper bound of $d + 2$ is currently known to hold for the list size needed for flexibility, as shown in [11]. In the same paper, the authors ask whether an upper bound of $d + 1$ can always be achieved – that is, whether d-degenerate graphs are flexibly $(d + 1)$-choosable. However, answering this question seems to be out of reach with current knowledge. Even for 2-degenerate graphs, the question seems rather tough, as it would imply the non-trivial result that planar graphs of girth 6 are flexibly 3-choosable, proven in [9], as this class of graphs is 2-degenerate.

In this direction, Dvořák et al. [11] asked a more specific question about non-regular[4] graphs of bounded degree. A non-regular connected graph of maximum degree Δ is $(\Delta - 1)$-degenerate and therefore Δ-choosable. With this in mind, Dvořák et al. asked the following question:

▶ **Question 1** ([11]). *For each $\Delta \geq 2$, does there exist a value $\varepsilon = \varepsilon(\Delta) > 0$ such that any non-regular connected graph G of maximum degree Δ is ε-flexibly Δ-choosable?*

Later, in [7], Choi et al. asked another specific question regarding degeneracy and flexibility. Knowing that outer-planar graphs are 2-degenerate, the authors asked:

▶ **Question 2** ([7]). *Are outer-planar graphs flexibly 3-choosable?*

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[2] A graph G is $\mathcal{F}$-free if G does not contain any graph $F \in \mathcal{F}$ as a subgraph.
[3] B_5 denotes the *book* on 5 vertices, which is the graph consisting of 3 triangles sharing a common edge.
[4] A non-regular graph is a graph that contains two vertices of different degrees.

We will answer both questions in the affirmative.

We dedicate Section 2 to solving Question 1. We in fact show a stronger characterization for flexibility in connected graphs G of maximum degree Δ. When $\Delta = 2$, G is flexibly 2-choosable if and only if G is a path (Theorem 6). When $\Delta \geq 3$, G is flexibly Δ-choosable if and only if G is not a $(\Delta+1)$-clique (Theorem 8). In our proof, we use a seminal result by Erdős, Rubin, and Taylor [14], which characterizes graphs that can be list-colored whenever each vertex's color list has size equal to its degree. We prove a theorem (Theorem 9) of a similar flavour that describes a sufficient condition for flexibility based on the degrees of the vertices in G. Moreover, we provide an example (Figure 2) that hints at which situations need to be avoided while aiming for a characterization of graphs that are flexibly choosable with lists of size equal to their vertex degree.

We dedicate Section 3 to solving Question 2. In fact, we prove the stronger statement that graphs of treewidth 2 are weighted flexibly 3-choosable (Theorem 13); hence, this result encompasses not only outer-planar graphs, but also series-parallel graphs and other graphs of treewidth 2. Given a graph G of treewidth 2 with a 2-treewidth decomposition, our method finds a list coloring on G satisfying a fraction of $\frac{1}{3}$ of any weighted request in linear time. Furthermore, as a k-tree decomposition can be constructed in linear time for constant k [2], our method therefore also runs in linear time. At the end of the section, we give a sufficient condition for lists of size $k+1$ that allow every weighted request on a k-tree to be $\frac{1}{k+1}$-satisfiable with respect to these lists. We state our result using a new concept of Zhu [22].

Next, we append a short section (Section 4) dedicated to the more restrictive graph parameter of treedepth. We show that graphs of treedepth k are weighted $\frac{1}{k^2}$-flexibly k-choosable (Theorem 17).

In the last section (Section 5), we propose a study of a new property stronger than flexibility that is motivated by degeneracy ordering. We also explain relations between this property and a standard line of research concerning spanning trees with many leaves [4, 15]. First, we give a standard definition of a *k-degeneracy ordering*, which is an ordering of the vertices of a graph such that each vertex has at most k neighbours appearing previously in the order. Then, we say that a graph G is *ε-flexibly k-degenerate* if for any subset $R \subseteq V(G)$, there exists a k-degeneracy ordering D of $V(G)$ such that for at least $\varepsilon|R|$ vertices $r \in R$, each neighbor of r appears after r in the ordering D.

For technical reasons, we will actually consider the slightly weaker notion of *almost ε-flexible k-degeneracy*, which we define formally in Section 5, Definition 18. We will show that for each $\Delta \geq 3$, there exists a value $\varepsilon = \varepsilon(\Delta) > 0$ such that if a non-regular graph G of maximum degree Δ is 3-connected, then G is almost ε-flexibly $(\Delta-1)$-degenerate (Corollary 23).

We remark that in addition to Theorem 13, Theorem 9 can also be straightforwardly turned into a polynomial-time algorithm that finds a list coloring satisfying a positive fraction of coloring requests. These algorithmic results can be compared with previous tools, which only give non-constructive proofs for the existence of an ε-satisifable coloring for an ε-flexible graph (as discussed in [6]).

1.2 Preliminaries

Let $G = (V, E)$ be a graph. For an edge $e = \{uv\} \in E$, we say that a vertex $w \in V$ is adjacent to e if $\{wu\} \in E$ or $\{wv\} \in E$. We will always use L to denote an assignment of color lists to each vertex of a graph G, and for a vertex $v \in V(G)$, we will use $L(v)$ to denote the list of colors assigned to v.

We assume throughout the entire paper that all graphs are connected, as questions about colorings of a disconnected graph may be answered by analyzing each component separately. We also define a slightly weaker version of weighted requests which we call *uniquely weighted request*, where $w(v,c)$ can be nonzero only for at most one color $c \in L(v)$ for each vertex $v \in V(G)$. This notion will be important only when we aim to calculate a specific (tight) value of ε, as for a general weighted request, one can disregard all but the largest weight request at each vertex while only losing an $|L(v)|$ factor. We formalize this idea in the following observation.

▶ **Observation 3.** *Let G be a graph with list assignment L. If every uniquely weighted request on G is ε-satisfiable, then G is weighted $\frac{\varepsilon}{\max_{v\in V(G)}|L(v)|}$-flexible with respect to L.*

We will make use of a lemma from Dvořák, Norin, and Postle [11] that serves as a useful tool and is easy to prove. This lemma tells us that in order to show weighted ε-flexibility in a graph G, we do not need to consider every possible request, and it is enough to find a distribution on colorings such that each individual vertex $v \in V(G)$ is colored by a given color $c \in L(v)$ with probability at least ε.

▶ **Lemma 4** (Lemma 3 in [11]). *Let G be a graph with a list assignment L. Suppose there exists a probability distribution on L-colorings φ of G such that for every $v \in V(G)$ and $c \in L(v)$,*

$$\Pr[\varphi(v) = c] \geq \varepsilon.$$

Then G is weighted ε-flexible with respect to L.

2 Graphs of Bounded Degree

In this section, we will investigate which graphs of maximum degree Δ are flexibly Δ-choosable, for all integers $\Delta \geq 2$. While every non-regular graph of maximum degree Δ is Δ-choosable, the complete graph $K_{\Delta+1}$ shows that not every Δ-regular graph is Δ-choosable. In [14], Erdős, Rubin, and Taylor give a complete characterization of Δ-regular graphs G with $\text{ch}(G) = \Delta$. Furthermore, the authors give a characterization of the more general notion of *degree choosability*, which is defined as follows. We say that a graph G is *degree choosable* if G can be list colored for any assignment L of lists such that $|L(v)| \geq \deg(v)$ for all $v \in V(G)$. Erdős, Rubin, and Taylor's characterization of degree choosable graphs is given in terms of the *blocks* of a graph, where a block of a graph G is defined as a maximal connected subgraph of G with no cut-vertex. By this definition, a block of a graph G is either a cut-edge or a 2-connected subgraph. The characterization of degree choosable graphs is as follows. Recall that in this paper we only consider connected graphs.

▶ **Theorem 5** ([14]). *A graph G is degree choosable if and only if G contains some block that is not a clique and is not an odd cycle.*

The question of whether there exists a characterization of graphs that are flexibly degree choosable is open. However, we may straightforwardly show that Theorem 5 does not give a characterization of flexible degree choosability. Indeed, Theorem 5 implies that a 2-regular graph G is 2-choosable if and only if G is an even cycle. However, the following theorem, which characterizes flexible 2-choosability in graphs of maximum degree 2, shows that in general, cycles are not flexibly 2-choosable.

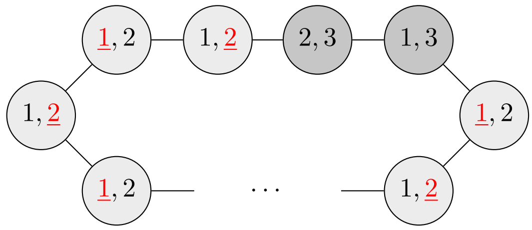

Figure 1 The graph G in the figure is an even cycle with color lists of size two. However, not even a single one of the red underlined requests can be satisfied. Therefore, the class of even cycles is not weakly flexibly 2-choosable.

▶ **Theorem 6.** *Let G be a graph of maximum degree 2. Then G is weakly flexibly 2-choosable if and only if G is a path. Furthermore, if G is a path, G is weighted $\frac{1}{2}$-flexibly 2-choosable.*

Proof. If G is a path, then it is straightforward to show that there exist two list colorings φ_1, φ_2 on G such that for each $v \in V(G)$, $L(v) = \{\varphi_1(v), \varphi_2(v)\}$. Then G is weighted $\frac{1}{2}$-flexibly 2-choosable by Lemma 4. If G is not a path, then G is a cycle. If G is an odd cycle, then G is not 2-choosable. If G is an even cycle, then the color list assignment in Figure 1 shows that no positive fraction of coloring requests on $V(G)$ can be satisfied, even when a color is requested at every vertex of G. ◀

Thus, we see that Theorem 5 does not characterize graphs that are flexibly degree-choosable, and the question of which graphs are flexibly degree-choosable is open. However, we will give a complete characterization of graphs of maximum degree $\Delta \geq 3$ that are flexibly Δ-choosable, which is a step toward characterizing flexible degree-choosability. We will show that for a graph G of maximum degree $\Delta \geq 3$, if $G \ncong K_{\Delta+1}$, then G is flexibly Δ-choosable. As $K_{\Delta+1}$ is not Δ-choosable, this gives a complete characterization of the graphs of maximum degree Δ that are flexibly Δ-choosable. With our characterization, we answer Question 1. First, it will be convenient to establish a corollary of Theorem 5.

▶ **Corollary 7.** *Let G be a graph, and let L be a list assignment such that for each $v \in V(G)$, $|L(v)| \geq \deg(v)$. Then G has an L-coloring if and only if either G contains some block that is not a clique and is not an odd cycle, or G contains a vertex v for which $|L(v)| > \deg(v)$.*

We are now ready to characterize graphs of maximum degree $\Delta \geq 3$ that are flexibly Δ-choosable.

▶ **Theorem 8.** *Let G be a graph of maximum degree $\Delta \geq 3$. If $G \ncong K_{\Delta+1}$, then G is weighted $\frac{1}{2\Delta^4}$-flexibly Δ-choosable.*

One can go even further in the direction of Theorem 5 and reduce the size of the lists of vertices that have smaller degree than $\Delta(G)$.

▶ **Theorem 9.** *Let G be a graph of maximum degree $\Delta(G) \geq 3$, and let $|L(v)| \geq \deg(v) + 1$ whenever $\deg(v) < \Delta(G)$ and $|L(v)| \geq \deg(v)$ whenever $\deg(v) = \Delta(G)$. If $G \ncong K_{\Delta+1}$, then G is $\frac{1}{2\Delta^3}$-flexibly L-choosable. In fact, the same holds even when uniquely weighted requests are considered. Therefore, G is weighted $\frac{1}{2\Delta^4}$-flexible for L.*

The weighted flexibility statement of Theorem 9 follows from Observation 3, and Theorem 8 follows as an immediate corollary of Theorem 9. Note that Theorem 9 may not necessarily be best possible. On the other hand, Figure 2 provides some evidence that graphs that are not flexibly degree choosable may not be easily characterized. In particular, Figure 2 shows

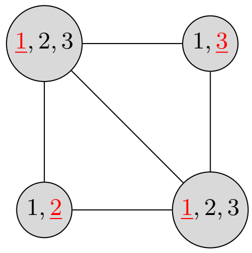

Figure 2 The graph shown here is not a clique or odd cycle and hence is list colorable with the given lists. However, not even a single vertex can be colored according to the widespread request shown by the red underlined colors. Therefore, this example distinguishes choosability and weak flexibility. Moreover, this example serves as an obstacle in further extensions of Theorem 9.

a diamond graph in which each vertex receives a color list of size equal to its degree along with a coloring request. However, in the figure, not a single coloring request can be satisfied. In contrast, Theorem 5 shows that diamond graphs themselves are degree choosable.

Proof of Theorem 9. Let G be a connected graph that is not isomorphic to $K_{\Delta+1}$. We assume that for each $v \in V(G)$, $L(v) \subseteq \mathbb{N}$. Let $R \subseteq V(G)$ be a set of vertices with coloring requests. As we consider only uniquely weighted requests, we can represent our coloring request with a function $f : R \to \mathbb{N}$. Let each vertex $r \in R$ have a weight $w(r)$ that corresponds to the nonzero weight of the request at r. For a subset $S \subseteq R$, let $w(S) = \sum_{r \in S} w(r)$. As $\chi(G^3) \leq \Delta^3$, we may choose a set $R' \subseteq R$ of weight at least $\frac{1}{\Delta^3} w(R)$ with no two vertices within distance three of each other. Note that R' can be constructed greedily. The next observation directly follows from our choice of R':

▷ Claim 10. Each edge of $G \setminus R'$ has at most one adjacent vertex in R'.

Now, suppose A is a component of some possibly disconnected graph with a color list assignment L' such that $|L'(v)| \geq \deg(v)$ for every $v \in V(A)$. We say that A is a *bad component* if $|L'(v)| = \deg(v)$ for every $v \in V(A)$ and every block of A is either an odd cycle or a clique. If A is not a *bad component*, then we say that A is a *good component*.

We consider the graph $G \setminus R'$. We give $G \setminus R'$ a color list assignment L' as follows. For a vertex $v \in V(G \setminus R')$, if $N_G(v) \cap R' = \emptyset$, then we let $L'(v) = L(v)$. If there exists a vertex $r \in N_G(v) \cap R'$, then we let $L'(v) = L(v) \setminus \{f(r)\}$. By Claim 10, every vertex of $G \setminus R'$ has at most one neighbor in R', and so L' is well-defined. By Corollary 7, if $G \setminus R'$ has no bad component, then G may be L-colored in a way that satisfies our request at all of R', in which case we satisfy a total weight of at least $\frac{1}{\Delta^3} w(R)$. Otherwise, let A be a bad component of $G \setminus R'$. We first observe that as A is a bad component, for every vertex $v \in V(A)$, $|L'(v)| = \deg_A(v)$, and hence $\deg_G(v) = \Delta$. By combining this fact with Claim 10, we obtain the following claim.

▷ Claim 11. If A is a bad component, then for every vertex $v \in V(A)$, $\Delta - 1 \leq \deg_A(v) = |L'(v)| \leq \Delta$.

We show that A is not a single block. Indeed, suppose A is a single block. Then, by Corollary 7, A must be a clique or odd cycle, and in particular, A is a regular graph. As G is not isomorphic to $K_{\Delta+1}$, $A \ncong K_{\Delta+1}$. Thus, by Claim 11, $\deg_A(v) = \Delta - 1$ for each $v \in V(A)$, and hence A must either be isomorphic to K_Δ or an odd cycle C_{2k+1}, $k \geq 2$ in the case that $\Delta = 3$. If $A \cong K_\Delta$, however, A must have exactly one neighbor $r \in R'$ by Claim 10, from which it follows that $\{r\} \cup A \cong K_{\Delta+1}$, a contradiction. If $A \cong C_{2k+1}$, $k \geq 2$ and

$\Delta = 3$, then again by Claim 10, A must have a single neighbor $r \in R'$ which is adjacent to every vertex of A. This is a contradiction, as this implies that $\deg_G(r) \geq 5 > \Delta$. Therefore, A has at least two blocks.

Now, we consider a *terminal block* B which is a leaf in the *block-cut tree* of A (c.f. [8, Chapter 3.1]) – that is, B is a block that only shares a vertex with one other block of A. We claim that $B \cong K_\Delta$. To show this, we consider a vertex $v \in V(B)$ that is not a cut-vertex in A. If $\deg_A(v) = \Delta$, then $B \cong K_{\Delta+1}$, which is a contradiction. Hence, by Claim 11, $\deg_A(v) = \Delta - 1$, and B has a neighbor in R'. As A is a bad component, Claim 10 implies that B has exactly one neighbor $r \in R'$, which must be adjacent to every non cut-vertex of B. This implies that $|V(B)| \leq \Delta + 1$, which rules out the possibility that $B \cong C_{2k+1}$ for some $k \geq 2$ when $\Delta = 3$. Then, as A is a bad component, B is $(\Delta - 1)$-regular, and it follows for all values $\Delta \geq 3$ that $B \cong K_\Delta$ and that r is adjacent to $\Delta - 1$ vertices of B. We note that r must then be adjacent to exactly one terminal block of a bad component, namely B. Furthermore, as the block-cut tree of A has at least two leaves, A has at least two terminal blocks B, B' and hence two vertices $r, r' \in R'$ adjacent to B, B' and no other terminal blocks of any bad component.

Now, we will construct a set $R^+ \subseteq R'$. As we construct R^+, we will define $R'' = R' \setminus R^+$. To construct R^+, for each bad component A of $G \setminus R'$, we will choose a vertex $r \in R'$ of least weight adjacent to a terminal block of A, and we will add r to R^+. Note that such a vertex r has at least two neighbors $u, v \in V(A)$, and as u, v belong to a terminal block of A, uv must be an edge in a triangle uvw of A for which $w \not\sim r$. Therefore, $A \cup \{r\}$ contains an induced diamond subgraph, and by Theorem 5, $A \cup \{r\}$ is not contained in a bad component with respect to any color list assignment.

We also construct a color list assignment $L'' : V(G \setminus R'') \to \mathbb{N}$ such that $G \setminus R''$ has no bad component with respect to L''. For a vertex $v \in V(G \setminus R'')$, if $N_G(v) \cap R'' = \emptyset$, then we let $L''(v) = L(v)$. If there exists a vertex $r \in N_G(v) \cap R''$, then we let $L''(v) = L(v) \setminus \{f(r)\}$. Again, by Claim 10, L'' is well-defined. Any bad component of $G \setminus R''$ with respect to L'' must also be a bad component of $G \setminus R'$ with respect to L', and hence by our choice of R^+, $G \setminus R''$ has no bad component with respect to L''. Therefore, by first coloring each vertex $r \in R''$ with $f(r)$ and then giving G an L''-coloring by Theorem 5, we find an L-coloring on G that satisfies a total request weight of at least $w(R'')$. As $w(R'') \geq \frac{1}{2} w(R') \geq \frac{1}{2\Delta^3} w(R)$, the proof for uniquely weighted requests is complete. The general weighted flexibility statement then follows from Observation 3. ◀

3 Graphs of Bounded Treewidth

In this section, we consider graphs of bounded treewidth. We characterize *treewidth*[5] in terms of *k-trees*, which is defined as follows. Given a nonnegative integer k, a k-tree is a graph that may be constructed by starting with a k-clique and then repeatedly adding a vertex of degree k whose neighbors induce a k-clique. The treewidth of a graph G is then the smallest integer k for which G is a subgraph of a k-tree. For technical reasons, we define a 0-tree to be an independent set, which is an exception to our overall connectivity assumption. The class of connected graphs of treewidth 1 is simply the class of trees. The class of connected graphs of treewidth at most 2 includes connected outer-planar graphs and series-parallel graphs, among other graphs. Graphs of bounded treewidth are of particular interest in the study of graph algorithms, as problems that are intractable in general are often tractable on

[5] For an equivalent definition of treewidth using a tree decomposition, refer to e.g. [3].

graphs of bounded treewidth; for a survey on algorithmic aspects of treewidth, see [3]. As k-trees are k-degenerate, it follows that graphs of treewidth k are $(k+1)$-choosable. The following result, shown implicitly in [11], shows furthermore that graphs of treewidth 1 (i.e. trees) are $\frac{1}{2}$-flexibly 2-choosable.

▶ **Proposition 12** ([11]). *Let G be a 1-tree with lists of size 2. Then there exists a set $\Phi = \{\varphi_1, \varphi_2\}$ of two proper colorings on G such that for each vertex $v \in V(G)$, $\{\varphi_1(v), \varphi_2(v)\} = L(v)$. In particular, G is weighted $\frac{1}{2}$-flexibly 2-choosable.*

In this section, we will show that graphs of treewidth 2 are $\frac{1}{3}$-flexibly 3-choosable (Theorem 13). We will show furthermore that for any positive integer k, if a graph G of treewidth k has a list assignment L of size $k+1$ that obeys certain restrictions, then G is $\frac{1}{k+1}$-flexibly L-choosable (Theorem 15). By considering a $(k+1)$-clique whose vertices all have the same color lists in which the same color is requested at every vertex, we see that a $\frac{1}{k+1}$ flexibility constant is best possible.

When we prove a result for graphs of treewidth k, we will only consider k-trees, as a proper coloring on a graph must also give a proper coloring for every subgraph.

▶ **Theorem 13.** *Let G be a 2-tree with lists of size 3. Then there exists a set $\Phi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\}$ of six colorings on G such that for each vertex $v \in V(G)$, $\{\varphi_1(v), \varphi_2(v), \varphi_3(v), \varphi_4(v), \varphi_5(v), \varphi_6(v)\}$ is a multiset in which each color from $L(v)$ appears exactly twice. In particular, G is weighted $\frac{1}{3}$-flexibly 3-choosable.*

It is interesting to note that the result of Theorem 13 (as well as Proposition 12) can be easily turned into linear time algorithm that provides a 1/3-satisfiable (resp. 1/2-satisfiable) coloring, as the construction of the set Φ is algorithmic.

Proof of Theorem 13. We will construct a set of six L-colorings $\Phi := \{\varphi_1, \dots, \varphi_6\}$ on G. Given an edge $uv \in E(G)$, we say that Φ is *admissible* at uv if the following conditions are satisfied:

- For each $i = 1, \dots, 6$, $\varphi_i(u) \neq \varphi_i(v)$.
- If $\varphi_i(u) = \varphi_j(u)$ and $\varphi_i(v) = \varphi_j(v)$, then $i = j$.
- For each color $c \in L(u)$, c appears exactly twice in the multiset $\{\varphi_1(u), \dots, \varphi_6(u)\}$.
- For each color $c' \in L(v)$, c' appears exactly twice in the multiset $\{\varphi_1(v), \dots, \varphi_6(v)\}$.

We establish the following claim, which will be the main tool of our proof. The proof of the claim is in the full version of the paper.

▷ Claim 14. Let G be a 2-tree, and let $uv \in E(G)$. Let Φ be a set of six L-colorings on G that is admissible at every edge of G. Suppose a vertex w is added to G with neighbors u, v. Then Φ may be extended to $G + w$ so that Φ is also admissible at uw and vw.

Now, as G is a 2-tree, G may be constructed by starting with a single edge and repeatedly adding a vertex of degree two whose two neighbors induce an edge. Suppose we start with an edge e. It is straightforward to construct an admissible set Φ of six colorings of e. Then, suppose that we have a partially constructed 2-tree G' and that Φ is an admissible coloring set at each edge of G'. We may add a vertex w to G' with adjacent neighbors u, v, and by Claim 14, we may extend Φ to w while still letting Φ be admissible at every edge of the new graph. By this process, we may construct a set Φ of six colorings that is admissible at every edge of G. We conclude the proof by a simple application of Lemma 4 on Φ with $\varepsilon = 1/3$. ◀

For $k \geq 3$, the question of whether or not k-trees are flexible with lists of size $k+1$ is still open. However, after adding some restrictions to our color lists, we can guarantee the existence of a flexible list coloring of any k-tree with lists of size $k+1$. Our method will be an application of the algorithm of Theorem 13. In order to state our result precisely, we will need a definition.

Given a partition $\lambda = \{\lambda_1, \dots, \lambda_t\}$ of $(k+1)$ – that is, an integer multiset for which $\lambda_1 + \cdots + \lambda_t = k+1$ – a λ-assignment L on a graph G is a list assignment for which $\bigcup_{v \in V(G)} L(v)$ may be partitioned into parts $C_1, \dots, C_t$ such that for each $v \in V(G)$ and each value $1 \leq i \leq t$, $|L(v) \cap C_i| = \lambda_i$. A graph G is λ-choosable if there exists a list coloring on G for any λ-assignment L. Zhu introduces λ-assignments in [22] and notes that for any integer k, $\{k\}$-choosability is equivalent to k-choosability, and $\underbrace{\{1, 1, \dots, 1\}}_{k \text{ times}}$-choosability is equivalent to k-colorability. With this definition, λ-choosability gives a notion of colorings that lie between traditional colorings and traditional list colorings. Zhu shows, for example, that while tripartite planar graphs are not 4-choosable in general, these graphs are always $\{1,3\}$-choosable. Choi and Kwon remark furthermore that while general planar graphs are not $\{1,3\}$-choosable, the question of whether all planar graphs are $\{1,1,2\}$-choosable is still open [6]. We may extend the concept of λ-choosability to flexible list colorings by saying that a graph G is ε-flexibly λ-choosable if, given any λ-assignment L and request r on G, r is ε-satisfiable with respect to L. Then we have the following theorem.

▶ **Theorem 15.** *Let G be a k-tree, and let $\lambda = \{\lambda_1, \dots, \lambda_t\}$ be a partition of $k+1$ with parts of size at most 3. Then G is $\frac{1}{k+1}$-flexibly λ-choosable.*

The proof is in the full version of the paper. We conclude the section by noting that by improving Theorem 13, one can also improve Theorem 15. In particular, suppose we could prove for some $k_0 \geq 3$ and all $k \leq k_0$ that for a k-tree G and a list assignment L of $k+1$ colors at each $v \in V(G)$, that there exists a set Φ of $(k+1)!$ colorings on G such that each color of each list $L(v)$ appears at v exactly $k!$ times in Φ. (This statement for $k=2$ is exactly the statement of Theorem 13.) Then, we could relax the requirement of Theorem 15 to allow parts of λ of size at most k_0. However, even proving this statement for $k=3$ seems like a difficult problem.

4 Graphs of Bounded Treedepth

In this section, we will consider graphs of bounded treedepth. For a rooted tree T, we define the *height* of T as the number of vertices in the longest path from the root of T to a leaf of T. Then, the *treedepth* $\mathtt{td}(G)$ of a graph G is defined as the minimum height of a rooted tree T for which $G \subseteq \text{Closure}(T)$, where the *closure* of a rooted tree T, written $\text{Closure}(T)$, is a graph on $V(T)$ in which each vertex is adjacent to all of its ancestors in T and all of its descendants in T.

If $\mathtt{td}(G) = k$, then G is $(k-1)$-degenerate, as each leaf of the corresponding tree has at most $k-1$ ancestors and no descendants. It follows that such a graph is k-choosable, and the complete graph on k vertices shows us that this is best possible. We will show that graphs of treedepth k are not only k-choosable, but weighted ε-flexibly k-choosable as well, with $\varepsilon = \frac{1}{k}$ in the unweighted case, and $\varepsilon = \frac{1}{k^2}$ in the weighted case. Note that K_k also shows that our value of $\varepsilon = \frac{1}{k}$ that we obtain for the unweighted case is the best possible.

Before we show our main proof, we develop a variant of Lemma 4 that is suited to uniquely weighted requests and allows us to build a different distribution for each request. With this new lemma, we can weaken the assumption on φ where we only require $\Pr[\varphi(v) = c] \geq \varepsilon$

for c such that $w(c, v) \neq 0$. As we are working with uniquely weighted requests, for each $v \in V(G)$ there is at most one color c that we need to use at v with positive probability. We often refer to this color c as *the requested color* at v. The proof of Lemma 16 is very similar to the original proof of Lemma 4. We include the proof in the full version of the paper.

▶ **Lemma 16.** *Let G be a graph with list assignment L. Let $R \subseteq V(G)$ be a set of vertices with uniquely weighted requests given by a weight function w. Suppose that for any weighted request w on G, there exists a probability distribution on L-colorings φ of G such that for every $v \in V(G)$, $c \in L(v)$ such that $w(v, c) \neq 0$,*

$$\Pr[\varphi(v) = c] \geq \varepsilon.$$

Then G is ε-flexibly L-choosable.

▶ **Theorem 17.** *Let G be a graph of treedepth k. Then G is $\frac{1}{k}$-flexibly k-choosable. In fact, the same holds even when uniquely weighted requsts are considered. Therefore, G is weighted $\frac{1}{k^2}$-flexibly k-choosable.*

Proof. We will prove that G is weighted $\frac{1}{k}$-flexibly k-choosable with respect to some arbitrary assignment of uniquely weighted requests. We will inductively construct a coloring distribution on G and then apply Lemma 16. We induct on k. When $k = 1$, the statement is trivial.

Suppose that $k > 1$. Let G be a subgraph of the closure of a tree T of height k and root v. First, we color v with a color $c \in L(v)$ uniformly at random. Then, we delete c from all other lists in G. For any vertex $u \in V(G)$ whose list still has k colors, we arbitrarily delete another color from $L(u)$, taking care not to delete a requested color, if one exists, at u. Now, we obtain a coloring distribution on the remaining vertices of G by the induction hypothesis, which is possible, as each component of $G \setminus v$ is a graph of treedepth $k-1$ with lists of size $k-1$. For a vertex $w \neq v$ at which a color $a \in L(w)$ is requested, the probability that a is assigned to w is at least the probability that a is not deleted from $L(w)$ multiplied by the probability that w is assigned the color a by the induction hypothesis. The probability that a is not deleted from $L(w)$ is at least $\frac{k-1}{k}$, as the requested color a can only be deleted from $L(w)$ by using a at v. The probability that a is used at $L(w)$ by the induction hypothesis is $\frac{1}{k-1}$. Therefore, a is used at w with probability at least $\frac{k-1}{k} \cdot \frac{1}{k-1} = \frac{1}{k}$. Furthermore, the probability that the requested color $c \in L(v)$ is used at v, if such a request exists, is exactly $\frac{1}{k}$. Hence, by Lemma 16, the proof is complete. The weighted flexibility statement then follows from Observation 3. ◀

5 Flexible Degeneracy Orderings

Recall that a graph G with a k-degeneracy ordering is called k-degenerate. One of the earliest appearances of graph degeneracy is in a paper by Erdős and Hajnal [13], in which the authors define the *coloring number* $\text{col}(G)$ of a graph G as one more than the minimum k for which G is k-degenerate. The coloring number of G satisfies $\text{ch}(G) \leq \text{col}(G)$, and hence an upper bound on a graph's coloring number implies an upper bound on a graph's choosability. Similarly, the concept of *flexible degeneracy*, defined in the introduction, extends the notion of coloring number to the setting of flexibility.

Suppose we wish to determine if a k-degenerate graph is ε-flexibly k-degenerate for some $\varepsilon > 0$. If G has a single vertex w of degree at most k, then in any k-degenerate ordering of $V(G)$, w must appear as the last vertex. Therefore, if we allow our request R to contain the single vertex w, then we see that G is not ε-flexibly k-degenerate for any value $\varepsilon > 0$. Thus, in order to avoid this small problem, we give the following definition:

▶ **Definition 18.** *Let G be a graph. For a constant $\varepsilon > 0$ and an integer $k \geq 1$, we say that G is* almost ε-flexibly k-degenerate *if the following holds. Let $R \subseteq V(G)$ be a vertex subset, and if G contains a single vertex w of degree at most k, let $R \neq \{w\}$. Then there exists an ordering D on $V(G)$ such that*
- *For each vertex $v \in V(G)$, at most k neighbors of v appear before v in D.*
- *There exist at least $\varepsilon|R|$ vertices $r \in R$ for which no neighbor of r appears before r in D.*

In this section, we will investigate flexible degeneracy in non-regular graphs of maximum degree Δ. We will show that for each $\Delta \geq 3$, there exists a value $\varepsilon = \varepsilon(\Delta) > 0$ such that if a non-regular graph G of maximum degree Δ is 3-connected, then G is almost ε-flexibly $(\Delta - 1)$-degenerate.

Given a non-regular graph G of maximum degree Δ and a set R of requested vertices, the problem of finding a flexible degeneracy ordering on G with respect to R is closely related to the problem of finding a spanning tree on G whose leaves intersect a positive fraction of the vertices in R. We formalize this observation with the following lemma.

▶ **Lemma 19.** *Let G be a non-regular graph of maximum degree Δ with a vertex $w \in V(G)$ of degree less than Δ. Let T be a spanning tree of G, and let $\mathcal{L}$ be the set of leaves in T with w excluded if w is a leaf of T. Then there exists a $(\Delta - 1)$-degeneracy order D on $V(G)$ in which each vertex $v \in \mathcal{L}$ appears in D before all neighbors of v.*

Lemma 19 tells us that given a non-regular graph G of bounded degree, one way to find a flexible degeneracy order on G is to find a spanning tree T on G in which many requested vertices are leaves in T. The following definition describes essentially how readily a graph G may accommodate an arbitrary set of requested vertices as leaves of a spanning tree on G.

▶ **Definition 20.** *Let G be a graph. We define the game connectivity $\kappa_g(G)$ of G as follows. Given a spanning tree T of G, we let $\mathcal{L}(T)$ represent the set of leaves in T. Then, given a vertex subset $R \subseteq V(G)$, we define $l(R)$ to be the maximum value of $\frac{|R \cap \mathcal{L}(T)|}{|R|}$, maximized over all spanning trees T of G. Finally, we define*

$$\kappa_g(G) = \min_{R \subseteq V(G)} l(R).$$

Previous research considers the problem of finding a spanning tree in a graph with a large fraction of vertices as leaves [4] [15], as well as the problem of finding a spanning tree with leaves at prescribed vertices [12]. Game connectivity hence is a natural combination of these two ideas. Furthermore, similarly to Lemma 19, a degeneracy order on a graph may be obtained from a spanning tree of bounded degree. Hence, we observe that a tree of bounded degree whose leaves make up a large fraction of its vertices gives a notion of weakly flexible degeneracy. Note that game connectivity can also be expressed in the language of connected dominating sets. The next lemma shows that if the game connectivity of a non-regular G of maximum degree Δ is bounded below, then G is almost flexibly $(\Delta - 1)$-degenerate.

▶ **Lemma 21.** *Let G be a non-regular graph of maximum degree $\Delta \geq 3$. Then G is almost $\frac{1}{2(\Delta+1)}\kappa_g(G)$-flexibly $(\Delta - 1)$-degenerate.*

Lemma 21 shows that certain classes of graphs with robust game connectivity have flexible degeneracy orderings. However, calculating $\kappa_g(G)$ for an arbitrary graph G does not appear to be an easy problem. Therefore, it will be useful to find general lower bounds for $\kappa_g(G)$ in graphs of bounded degree. However, the example in Figure 3 shows that the game connectivity of a graph of bounded degree may be arbitrarily small, even when the graph is regular and of arbitrary degree. By also requiring 3-connectivity in addition to a bound on vertex degree, we will be able to obtain a lower bound on a graph's game connectivity.

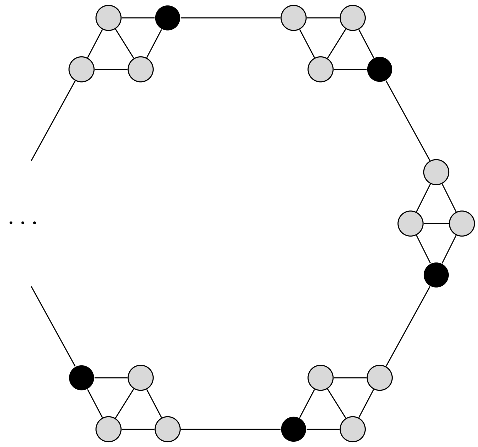

Figure 3 The graph G in the figure is an arbitrarily large two-connected 3-regular graph. If a set $R \subseteq V(G)$ is chosen as shown by the black vertices in the figure, then there does not exist a constant $\varepsilon > 0$ such that $\varepsilon|R|$ vertices of R may become leaves of some spanning tree of G. For any $k \geq 3$, a similar k-regular graph without flexible degeneracy may be constructed from a cycle C by replacing each vertex of C by a clique minus an edge.

We will show that for non-regular 3-connected graphs of maximum degree Δ, there exists a constant $\varepsilon = \varepsilon(\Delta) > 0$ for which such graphs are almost ε-flexibly $(\Delta - 1)$-degenerate. Our proof method may be applied to a more general hypergraph theorem, which we state and prove in the full version. Theorem 22, and hence Corollary 23, follow immediately.

▶ **Theorem 22.** *Let $\Delta \geq 3$ be an integer. Then there exists a value $\varepsilon = \varepsilon(\Delta) > 0$ for which the following holds. Let G be a 3-connected graph of maximum degree Δ. Then $\kappa_g(G) \geq \varepsilon$.*

▶ **Corollary 23.** *Let $\Delta \geq 3$ be an integer. Then there exists a value $\varepsilon(\Delta) > 0$ for which the following holds. Let G be a 3-connected non-regular graph of maximum degree Δ. Then G is almost ε-flexibly $(\Delta - 1)$-degenerate.*

The graph in Figure 3 shows that the 3-connectivity condition of Theorem 22 may not be replaced by 2-connectivity. Furthermore, the following example shows that the bounded degree condition of Theorem 22 may not be removed. Consider a graph G with an independent subset $R \subseteq V(G)$ satisfying the following properties. For each triplet $A \in \binom{R}{3}$, let there exist a vertex of $G \setminus R$ whose neighbors of R are exactly those vertices from A. Figure 4 shows such a construction with $|R| = 5$. It is straightforward to show that when $|R| \geq 4$, G is 3-connected. However, no more than two vertices of R may be removed from G without disconnecting G. Therefore, for any spanning tree T on G, the leaves of T include at most two vertices of R. As R may be arbitrarily large, this example shows that 3-connected graphs G do not satisfy $\kappa_g(G) \geq \varepsilon$ for any universal $\varepsilon > 0$.

6 Conclusion

In Section 2, we provide a characterization of flexibility in terms of the maximum degree of a graph. Moreover, we prove a more general theorem (Theorem 9), which somewhat resembles a famous theorem of Erdős, Rubin, and Taylor. It would be very interesting to discover whether a complete characterization of flexible degree choosability exists and, moreover, how closely such a characterization would resemble Erdős, Rubin, and Taylor's characterization of degree choosability. Perhaps, some more structural insight might be needed in order to find such a characterization, as hinted by Figure 2.

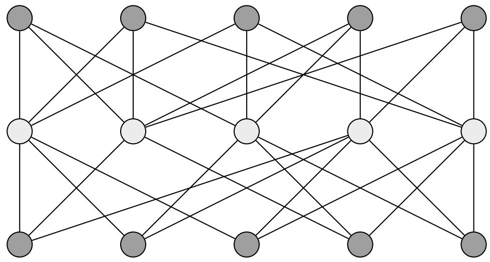

Figure 4 The figure shows a 3-connected graph G in which removing any three light vertices disconnects G. For each triplet u, v, w of light vertices in G, there exists a dark vertex whose neighbors are exactly u, v, w.

As we present tight bounds for list sizes needed for flexibility in graphs of bounded treedepth and graphs of treewidth 2, a natural question arises: Is it possible to show similar bounds for graphs of bounded pathwidth? More specifically, one can focus on the even more restricted class of (unit) interval graphs with bounded clique size. However, even for such a restricted graph class it seems to be challenging to show a tight bound on the list size needed for flexibility. In particular, the k-path seems to be a challenging example.

Another interesting direction could be a systematic search for graphs that are not flexibly choosable. In this paper we give quite simple examples (Theorem 6 and Figure 2) of graphs that are not flexible for the stronger reason that they do not allow a precoloring extension. These examples are not surprising, as many examples exist of graphs that do not allow a precoloring extension [1, 16, 18, 20]. It would be interesting to find some constructions that prohibit flexibility while allowing precoloring extension.

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