# Cake Cutting: An Envy-Free and Truthful Mechanism with a Small Number of Cuts

**Takao Asano**
Chuo University, Tokyo, Japan
asano@ise.chuo-u.ac.jp

**Hiroyuki Umeda**
Chuo University, Tokyo, Japan
wordmindblack@gmail.com

## Abstract

The mechanism for the cake-cutting problem based on the expansion process with unlocking proposed by Alijani, Farhadi, Ghodsi, Seddighin, and Tajik [1, 18] uses a small number of cuts, but is not actually envy-free and truthful, although they claimed that it is envy-free and truthful. In this paper, we consider the same cake-cutting problem and give a new envy-free and truthful mechanism with a small number of cuts, which is not based on their expansion process with unlocking.

**2012 ACM Subject Classification** Theory of computation → Algorithmic game theory and mechanism design

**Keywords and phrases** cake-cutting problem, envy-freeness, fairness, truthfulness, mechanism design

**Digital Object Identifier** 10.4230/LIPIcs.ISAAC.2020.15

**Acknowledgements** The first author would like to thank Professor Shigeo Tsujii of Chuo University.

## 1 Introduction

The problem of dividing a cake among players in a fair manner has attracted the attention of mathematicians, economists, political scientists and computer scientists [4, 3, 9, 10, 11, 16, 17] since it was first considered by Steinhaus [19]. The cake-cutting problem is often used as a metaphor for prominent real-world problems that involve the division of a heterogeneous divisible good [6]. Formally, the cake-cutting problem is stated as follows: Given a divisible heterogeneous cake $C$ and $n$ strategic players $N = \{1, 2, \ldots, n\}$, where each player $i \in N$ has a valuation function $v_i$ over $C$, find an allocation of $C$ to the players $N$ that satisfies one or several fairness criteria. In the cake cutting literature, one of the most important criteria is *envy-freeness* [4]. In an envy-free allocation, each player considers his/her own allocation at least as good as any other player's allocation.

Stromquist [21] showed that there is no finite envy-free cake cutting algorithm that outputs a contiguous allocation to each player for any $n \geq 3$, although an envy-free allocation with a contiguous allocation to each player is guaranteed to exist [20, 22]. Note that any cake cutting algorithm that outputs a contiguous allocation to each player uses $n-1$ cuts on cake $C$. Deng, Qi and Saberi [10] showed that finding an envy-free allocation using $n-1$ cuts on cake $C$ is PPAD-complete when valuation functions are given explicitly by polynomial-time algorithms, although their result requires very general (e.g., non-additive, non monotone) valuation functions [12].

In recent papers, some restricted classes of valuation functions have been studied [4, 6, 8, 9, 15]. Piecewise constant and piecewise uniform valuation functions are two special classes of valuation functions which are very important in practice [1, 4, 9, 18]. For a valuation function $v$ on cake $C$, let $D(v) = \{x \in C \mid v(x) > 0\}$ (thus, $D(v)$ consists of several disjoint maximal contiguous intervals). Then the valuation function $v$ is called *piecewise constant* if, for each contiguous interval $I$ in $D(v)$, $v(x') = v(x'')$ holds for all $x', x'' \in I$. Note that, in

© Takao Asano and Hiroyuki Umeda;
licensed under Creative Commons License CC-BY

a piecewise constant valuation $v$, $v(x) \neq v(y)$ may hold for $x \in I$ and $y \in J$ when $I, J$ are two distinct contiguous intervals in $D(v)$. In a piecewise constant valuation $v$, if $v(x) = v(y)$ holds for all $x, y \in D(v)$, then $v$ is called a *piecewise uniform* function. Kurokawa, Lai, and Procaccia [14] proved that finding an envy-free allocation when the valuation functions are piecewise uniform is as hard as solving the problem without any restriction on the valuation functions.

The cake-cutting problem has been studied not only from the viewpoint of computational complexity but also from the game theoretical point of view [1, 4, 5, 9, 15, 18]. Chen, Lai, Parkes, and Procaccia [9] considered a strong notion of truthfulness, in which the players' dominant strategies are to reveal their true valuations over the cake. They presented an envy-free and truthful mechanism (i.e., polynomial-time algorithm) for the cake-cutting problem when the valuation functions are piecewise uniform. Aziz and Ye [4] considered the problem when valuation functions are piecewise constant and piecewise uniform. They designed three algorithms called CCEA, MEA and CDA for piecewise constant valuations. They showed that CCEA becomes essentially the same as the envy-free and truthful mechanism proposed by Chen, et al. [9], if it is restricted for piecewise uniform valuations. However, CCEA and the mechanism in [9] uses $\Omega(n^2 m)$ cuts, where $m$ is the largest number of maximal contiguous subintervals in $D(v_i) = \{x \in C \mid v_i(x) > 0\}$ in piecewise uniform valuations $v_i$.

Alijani, Farhadi, Ghodsi, Seddighin, and Tajik [1, 18] considered that the number of cuts is important, noting that, in some cases, each cut might have additional cost: if the cake models a processing time that must be fairly allocated among a set of tasks, then every task-switch imposes an overhead and minimizing the total amount of overhead would be equivalent to minimizing the number of cuts on the cake. Therefore, from the viewpoint of a small number of cuts, they considered the following cake-cutting problem by restricting each piecewise uniform valuation $v_i$ to satisfy that $D(v_i) = \{x \in C \mid v_i(x) > 0\}$ is a single contiguous interval $C_i$ in cake $C$: Given a divisible heterogeneous cake $C$, $n$ strategic players $N = \{1, 2, \ldots, n\}$ with valuation interval $C_i \subseteq C$ of each player $i \in N$, find a mechanism for dividing $C$ into pieces and allocating pieces of $C$ to $n$ players $N$ to meet the following conditions: (i) the mechanism is envy-free; (ii) the mechanism is truthful; and (iii) the number of cuts made on cake $C$ is small. And they gave an envy-free and truthful mechanism with at most $2n - 2$ cuts based on the expansion process with unlocking, the main result in the paper [1, 18]. However, the mechanism is not actually envy-free and truthful [2].

Thus, we give an alternative envy-free and truthful mechanism with at most $2n - 2$ cuts which is not based on the expansion process with unlocking. Furthermore, it runs in $O(n^3)$ time. Our approach uses properties in the structures of the valuation intervals.

## 2 Notation and Fundamental Notions

We are given a divisible heterogeneous cake $C = [0, 1) = \{x \mid 0 \leq x < 1\}$ [1], $n$ strategic players $N = \{1, 2, \ldots, n\}$ with valuation interval $C_i = [\alpha_i, \beta_i) = \{x \mid 0 \leq \alpha_i \leq x < \beta_i \leq 1\} \subseteq C$ of each player $i \in N$. We denote by $\mathcal{C}_N$ the (multi-)set of valuation intervals of all the players $N$, i.e., $\mathcal{C}_N = (C_1, C_2, \ldots, C_n)$. We also write $\mathcal{C}_N = (C_i : i \in N)$.

The valuation intervals $\mathcal{C}_N$ is called *solid*, if, for every point $x \in C$, there is a player $i \in N$ whose valuation interval $C_i \in \mathcal{C}_N$ contains $x$. As assumed in [1, 4, 18], we will also assume that the valuation intervals $\mathcal{C}_N$ is solid throughout this paper, i.e., $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$.

[1] To guarantee that the pieces allocated to the players by a mechanism are mutually disjoint, we represent a given cake $C$ to be $C = [0, 1) = \{x \mid 0 \leq x < 1\}$ in this paper and we assume that if a subinterval $X = [x', x'') = \{x \mid x' \leq x < x''\}$ of $C = [0, 1)$ is cut at $y \in X$ with $x' < y < x''$ then $X$ is divided into two subintervals $X' = [x', y)$ and $X'' = [y, x'')$.

A union $X$ of mutual disjoint sets $X_1, X_2, \ldots, X_k$ is denoted by $X = X_1 + X_2 + \cdots + X_k = \sum_{\ell=1}^{k} X_\ell$. A *piece* $A_i$ of cake $C$ is a union of mutually disjoint subintervals $A_{i_1}, A_{i_2}, \ldots, A_{i_{k_i}}$ of $C$. Thus, $A_i = A_{i_1} + A_{i_2} + \cdots + A_{i_{k_i}} = \sum_{\ell=1}^{k_i} A_{i_\ell}$. A partition $A_N = (A_1, A_2, \ldots, A_n)$ of cake $C$ into $n$ disjoint pieces $A_1, A_2, \ldots, A_n$ is called an *allocation* of $C$ to $n$ players $N$ if each piece $A_i = \sum_{\ell=1}^{k_i} A_{i_\ell}$ is allocated to player $i$. We also write $A_N = (A_i : i \in N)$. Thus, $\sum_{i \in N} A_i = C$ in allocation $A_N = (A_i : i \in N)$ of $C$ to $n$ players $N$, and $A_i = \sum_{\ell=1}^{k_i} A_{i_\ell}$ is called an *allocated piece* of $C$ to player $i$.

For an interval $X = [x', x'')$ of $C$, the *length* of $X$, denoted by $len(X)$, is defined by $x'' - x'$. For a piece $A = \sum_{\ell=1}^{k} X_\ell$ of cake $C$, the *length* of $A$, denoted by $len(A)$, is defined by the total sum of $len(X_\ell)$, i.e., $len(A) = \sum_{\ell=1}^{k} len(X_\ell)$. For each $i \in N$ and valuation interval $C_i$ of player $i$, the *value* of piece $A = \sum_{\ell=1}^{k} X_\ell$ for player $i$, denoted by $V_i(A)$, is the total sum of $len(X_\ell \cap C_i)$, i.e., $V_i(A) = \sum_{\ell=1}^{k} len(X_\ell \cap C_i)$.

For an allocation $A_N = (A_i : i \in N)$ of cake $C$ to $n$ players $N$, if $V_i(A_i) \geq V_i(A_j)$ for all $j \in N$, then the allocated piece $A_i$ to player $i$ is called *envy-free* for player $i$. If, for every player $i \in N$, the allocated piece $A_i$ to player $i$ is envy-free for player $i$, then the allocation $A_N = (A_i : i \in N)$ of cake $C$ to $n$ players $N$ is called *envy-free*.

Let $\mathcal{M}$ be a mechanism for the cake-cutting problem. Let $\mathcal{C}_N = (C_i : i \in N)$ be an arbitrary input to $\mathcal{M}$ and $A_N = (A_i : i \in N)$ be an allocation of cake $C$ to $n$ players $N$ obtained by $\mathcal{M}$. If $A_N = (A_i : i \in N)$ with $A_i = \sum_{\ell=1}^{k_i} A_{i_\ell}$ for every input $\mathcal{C}_N = (C_i : i \in N)$ to $\mathcal{M}$ is envy-free then $\mathcal{M}$ is called *envy-free*.

Now, assume that only player $i$ gives a false valuation interval $C'_i$ and let $\mathcal{C}'_N(i) = (C'_j : j \in N)$ (all the other players $j \neq i$ give true valuation intervals $C_j$ and thus $C'_j = C_j$ for each $j \neq i$) be an input to $\mathcal{M}$ and let an allocation of cake $C$ to $n$ players $N$ obtained by $\mathcal{M}$ be $A'_N(i) = (A'_j : j \in N)$ with $A'_j = \sum_{\ell=1}^{k'_j} A'_{j_\ell}$ for each $j \in N$. The values of $A_i = \sum_{\ell=1}^{k_i} A_{i_\ell}$ and $A'_i = \sum_{\ell=1}^{k'_i} A'_{i_\ell}$ for player $i$ are $V_i(A_i) = \sum_{\ell=1}^{k_i} len(A_{i_\ell} \cap C_i)$ and $V_i(A'_i) = \sum_{\ell=1}^{k'_i} len(A'_{i_\ell} \cap C_i)$ (note that $V_i(A'_i) \neq \sum_{\ell=1}^{k'_i} len(A'_{i_\ell} \cap C'_i)$). If $V_i(A_i) \geq V_i(A'_i)$, then player $i$ does not want to give false valuation interval $C'_i$ and player $i$ will report true valuation interval $C_i$ to $\mathcal{M}$ (thus, to report true valuation interval $C_i$ is a *dominant strategy* of player $i$). For each player $i \in N$, if this holds, then $\mathcal{M}$ is called *truthful* (allocation $A_N = (A_i : i \in N)$ obtained by $\mathcal{M}$ is also called *truthful*).

For valuation intervals $\mathcal{C}_N = (C_i : i \in N)$ and an interval $X = [x', x'')$ of cake $C$, let $N(X)$ be the set of players $i$ in $N$ whose valuation intervals $C_i$ are entirely contained in $X$ and let $\mathcal{C}_{N(X)}$ be the (multi-)set of valuation intervals in $\mathcal{C}_N$ which are entirely contained in $X$. Let $n_X$ be the cardinality of $N(X)$. Thus,

$$N(X) = \{i \in N \mid C_i \subseteq X, C_i \in \mathcal{C}_N\},\ \mathcal{C}_{N(X)} = (C_i \in \mathcal{C}_N : i \in N(X)),\ n_X = |N(X)|. \quad (1)$$

As we defined the solidness of the valuation intervals $\mathcal{C}_N$ in cake $C$, the valuation intervals $\mathcal{C}_{N(X)}$ for interval $X = [x', x'')$ of $C$ is called *solid*, if for every point $x \in X$, there is a valuation interval $C_i \in \mathcal{C}_{N(X)}$ containing $x$. Thus, the valuation intervals $\mathcal{C}_{N(X)}$ is solid if and only if $\bigcup_{C_i \in \mathcal{C}_{N(X)}} C_i = X$. Solidness will play a central role in this paper. Similarly, the notion of density defined below will also play the central role in this paper.

The *density* $\rho(X)$ of interval $X = [x', x'')$ of $C$ is defined by $\rho(X) = \frac{len(X)}{|N(X)|} = \frac{x''-x'}{n_X}$. The density $\rho(X)$ is the average length of pieces of the players in $N(X)$ when the part $X$ of cake $C$ is divided among the players in $N(X)$. Note that, if $X \neq \emptyset$ (i.e., $len(X) \neq 0$) and $n_X = 0$ then $\rho(X) = \infty$. Let $\mathcal{X}$ be the set of all nonempty intervals in $C$. Let $\rho_{\min}$ be the minimum density among the densities of all nonempty intervals in $C$, i.e., $\rho_{\min} = \min_{X \in \mathcal{X}} \rho(X)$. Let $\mathcal{X}_{\min} = \{X \in \mathcal{X} \mid \rho(X) = \rho_{\min}\}$. Thus, $\mathcal{X}_{\min}$ is the set of all intervals of minimum density

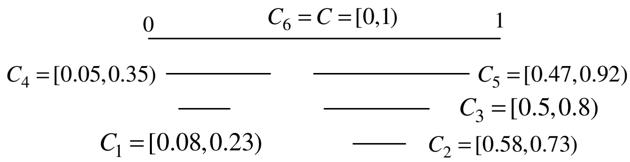


**Figure 1** Example of the valuation intervals $\mathcal{C}_N = (C_1, C_2, \ldots, C_6)$ where $N = \{1, 2, \ldots, 6\}$. The minimum density is $\rho_{\min} = 0.15$ and the set of all intervals of minimum density is $\mathcal{X}_{\min} = \{[0.08, 0.23) = C_1,\ [0.58, 0.73) = C_2,\ [0.5, 0.8) = C_3,\ [0.05, 0.35) = C_4,\ [0.47, 0.92) = C_5\}$. Among them, $C_1$ and $C_2$ are the minimal intervals of minimum density and $C_4$ and $C_5$ are the maximal intervals of minimum density. Interval $[0, 1) = C$ is of density $\rho(C) = \frac{1}{6}$.

in $C$. An interval $X \in \mathcal{X}_{\min}$ is called a *minimal interval of minimum density* if $X$ contains no other interval of $\mathcal{X}_{\min}$ properly. Similarly, $X \in \mathcal{X}_{\min}$ is called a *maximal interval of minimum density* if no other interval of $\mathcal{X}_{\min}$ contains $X$ properly (Figure 1).

Interval $X = [x', x'')$ of cake $C$ is called a *minimal interval with respect to valuations*, if there are valuation intervals $C_i = [\alpha_i, \beta_i)$ and $C_j = [\alpha_j, \beta_j)$ in $\mathcal{C}_{N(X)} = (C_k \in \mathcal{C}_N : k \in N(X))$ such that $x' = \alpha_i$ and $x'' = \beta_j$. The following lemmas and corollaries can be obtained by almost the same arguments. We will give only a proof of the first lemma.

► **Lemma 1.** *Let $X = [x', x'')$ be a minimal interval with respect to valuations in cake $C$. Suppose that $\rho(Y) \geq \rho(X)$ holds for each minimal interval $Y = [y', y'')$ with respect to valuations which is properly contained in $X$. Then the valuation intervals $\mathcal{C}_{N(X)}$ is solid.*

**Proof.** Suppose that no valuation interval in $\mathcal{C}_{N(X)}$ contains a point $x \in X = [x', x'')$. Thus, each valuation interval $C_i = [\alpha_i, \beta_i) \in \mathcal{C}_{N(X)}$ satisfies $\beta_i \leq x$ or $x < \alpha_i$. Since $X = [x', x'')$ is a minimal interval with respect to valuations, there are valuation intervals $C_j = [\alpha_j, \beta_j)$ and $C_k = [\alpha_k, \beta_k)$ in $\mathcal{C}_{N(X)}$ with $\alpha_j = x' < x$ and $\beta_k = x'' > x$. Thus, we have $\beta_j \leq x$ and $x < \alpha_k$. Let $y$ be the largest right endpoint among valuation intervals in $\mathcal{C}_{N(X)}$ whose right endpoints are smaller than or equal to $x$. Similarly, let $z$ be the smallest left endpoint among valuation intervals in $\mathcal{C}_{N(X)}$ whose left endpoints are larger than $x$. Thus $\epsilon = x - y \geq 0$ and $\delta = z - x > 0$. Let $Y = [x', y)$ and $Z = [z, x'')$. Then both $Y = [x', y)$ and $Z = [z, x'')$ are minimal intervals with respect to valuations. Furthermore, $C_j = [\alpha_j, \beta_j) \subseteq Y$, $C_k = [\alpha_k, \beta_k) \subseteq Z$, $Y \cap Z = \emptyset$ and each valuation interval $C_i = [\alpha_i, \beta_i) \in \mathcal{C}_{N(X)}$ satisfies $\beta_i \leq y \leq x$ or $x < z \leq \alpha_i$. Thus, each valuation interval $C_i = [\alpha_i, \beta_i) \in \mathcal{C}_{N(X)}$ is either in $\mathcal{C}_{N(Y)}$ or in $\mathcal{C}_{N(Z)}$ and we have $\mathcal{C}_{N(X)} = \mathcal{C}_{N(Y)} + \mathcal{C}_{N(Z)}$ and $n_X = n_Y + n_Z$. Since $len(Y) = \rho(Y) n_Y$, $\rho(Y) \geq \rho(X)$, $len(Z) = \rho(Z) n_Z$, and $\rho(Z) \geq \rho(X)$, we have

$$\begin{aligned}
\rho(X) &= \frac{len(X)}{n_X} = \frac{x'' - x'}{n_X} = \frac{x'' - z + z - x + x - y + y - x'}{n_Z + n_Y} \\
&= \frac{len(Z) + \delta + \epsilon + len(Y)}{n_Z + n_Y} = \frac{\rho(Z) n_Z + \rho(Y) n_Y + \delta + \epsilon}{n_Z + n_Y} \\
&> \frac{\rho(Z) n_Z + \rho(Y) n_Y}{n_Z + n_Y} \geq \frac{\rho(X) n_Z + \rho(X) n_Y}{n_Z + n_Y} = \rho(X),
\end{aligned}$$

a contradiction. Thus, we have $\mathcal{C}_{N(X)}$ is solid (i.e., $\bigcup_{C_i \in \mathcal{C}_{N(X)}} C_i = X$). ◄

► **Corollary 2.** *An interval $X = [x', x'')$ of minimum density $\rho_{\min}$ in cake $C$ is a minimal interval with respect to valuations and the valuation intervals $\mathcal{C}_{N(X)}$ is solid.*

▶ **Lemma 3.** *For two distinct minimal intervals $X_i = [x'_i, x''_i)$ and $X_j = [x'_j, x''_j)$ with respect to valuations in cake $C$ such that $X_i \cap X_j \neq \emptyset$, if $\rho(X_i) \geq \rho(X_j)$ and $\rho(X_i \cap X_j) \geq \rho(X_j)$, then $\rho(X_i \cup X_j) \leq \rho(X_i)$.*

Now we discuss structures of intervals of minimum density which play a central role in our mechanism. By Lemmas 1 and 3, we have the following corollaries.

▶ **Corollary 4.** *Let $X_i = [x'_i, x''_i)$ and $X_j = [x'_j, x''_j)$ be two distinct intervals of minimum density $\rho_{\min}$ in cake $C$. If $X_i \cap X_j \neq \emptyset$ then both $Y = X_i \cap X_j$ and $Z = X_i \cup X_j$ are intervals of minimum density $\rho_{\min}$.*

▶ **Corollary 5.** *If $X_i = [x'_i, x''_i)$ and $X_j = [x'_j, x''_j)$ are two distinct minimal intervals of minimum density $\rho_{\min}$ in cake $C$, then $X_i \cap X_j = \emptyset$. Furthermore, if $X_i = [x'_i, x''_i)$ lies to the left of $X_j = [x'_j, x''_j)$ then $x''_i \leq x'_j$. In this case, if $x''_i = x'_j$ then $Z = X_i + X_j = [x'_i, x''_j)$ is an interval of minimum density and there is no valuation interval $C_k = [x'_k, x''_k) \in \mathcal{C}_N$ such that $x'_i \leq x'_k < x''_i = x'_j < x''_k \leq x''_j$. Similarly, for two distinct maximal intervals $X_i = [x'_i, x''_i)$ and $X_j = [x'_j, x''_j)$ of minimum density $\rho_{\min}$ in cake $C$, we have $X_i \cap X_j = \emptyset$, and if $X_i = [x'_i, x''_i)$ lies to the left of $X_j = [x'_j, x''_j)$ then $x''_i < x'_j$.*

## 3 Outline of Our Mechanism

For a given input of cake $C = [0, 1)$, $n$ players $N = \{1, 2, \ldots, n\}$, and solid valuation intervals $\mathcal{C}_N = (C_i : i \in N)$ with valuation interval $C_i = [\alpha_i, \beta_i)$ of each player $i \in N$, we will give a mechanism $\mathcal{M}$ which finds an allocation $A_N = (A_i : i \in N)$ to players $N$ satisfying the following properties: (a) $\mathcal{M}$ is envy-free; (b) $\mathcal{M}$ is truthful; (c) $A_i \subseteq C_i$ for each $i \in N$; and (d) $\sum_{i \in N} A_i = C$. We first give a brief outline of our mechanism.

Let $H_1 = [h'_1, h''_1)$, $H_2 = [h'_2, h''_2), \ldots,$ $H_L = [h'_L, h''_L)$ be the maximal intervals of minimum density $\rho_{\min}$ in cake $C = [0, 1)$. We first cut $C = [0, 1)$ at both endpoints of each $H_\ell$ ($\ell = 1, 2, \ldots, L$). By Corollary 5, two distinct maximal intervals of minimum density are disjoint and we can cut the cake at both endpoints of each maximal interval of minimum density, independently. By these cuts, we can reduce the original cake-cutting problem into two types of cake-cutting subproblems of type (i) and type (ii) as follows (Figure 2):

**(i)** the cake-cutting problem within each maximal interval $H_\ell = [h'_\ell, h''_\ell)$ ($\ell = 1, 2, \ldots, L$) of minimum density (which consists of cake $H_\ell$, players $N(H_\ell)$ whose valuation intervals are in $H_\ell$ and valuations $\mathcal{C}_{N(H_\ell)}$ with density $\rho$); and

**(ii)** the cake-cutting problem obtained by deleting all $H_\ell = [h'_\ell, h''_\ell)$ ($\ell = 1, 2, \ldots, L$), i.e., the cake-cutting problem for cake $C' = C \setminus \sum_{\ell=1}^{L} H_\ell$, players $N' = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ and valuations $\mathcal{C}'_{N'}$ (which consists of valuations $C'_k = C_k \setminus \sum_{\ell=1}^{L} H_\ell \neq \emptyset$ for all $k \in N'$) with density $\rho'$ and $\bigcup_{C'_k \in \mathcal{C}'_{N'}} C'_k = C'$.

Note that the cake-cutting problem of type (i) is almost the same as the original cake-cutting problem, since cake $H_\ell$ is a single interval, each valuation $C_k \in \mathcal{C}_{N(H_\ell)}$ is also a single interval, and the valuation intervals $\mathcal{C}_{N(H_\ell)}$ is solid by Corollary 2.

On the other hand, the cake-cutting problem of type (ii) is different from the original cake-cutting problem, because the resulting cake $C' = C \setminus \sum_{\ell=1}^{L} H_\ell$ may become a set of two or more disjoint intervals and each remaining valuation $C'_k = C_k \setminus \sum_{\ell=1}^{L} H_\ell \neq \emptyset$ may also become a set of two or more disjoint intervals. However, the cake-cutting problem of type (ii) can be solved in almost the same way by using an idea proposed by Alijani et al. [1, 18]: for each $\ell = 1, 2, \ldots, L$, perform *shrinking* of $H_\ell$. That is, we virtually shrink each *hollow interval* $H_\ell = [h'_\ell, h''_\ell)$ (since $H_\ell$ was already deleted) and virtually consider $h'_\ell = h''_\ell$. Let $H_\ell^{(S)}$ be the shrunken interval obtained by shrinking the corresponding hollow interval $H_\ell$.

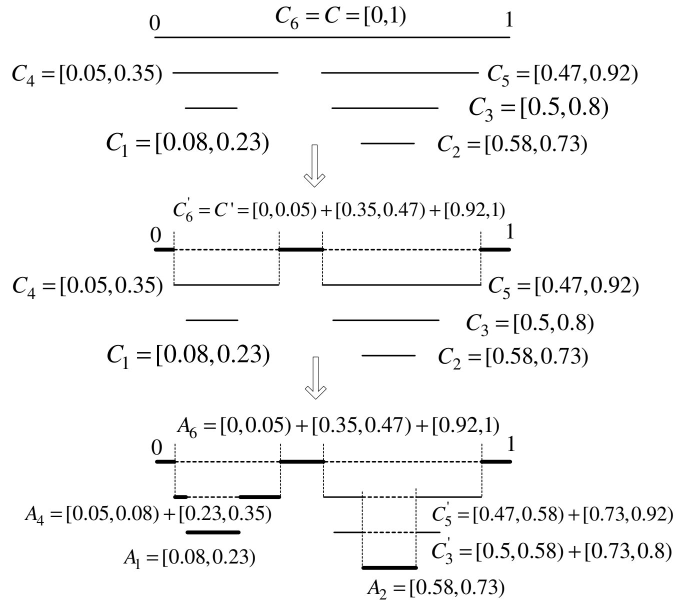


**Figure 2** The cake-cutting problem can be reduced into two types of cake-cutting subproblems by cutting cake $C = [0, 1)$ at both endpoints of each maximal interval of minimum density: (i) one within each maximal interval of minimum density (cake $[0.05, 0.35)$ with players $R_1 = \{1, 4\}$ and cake $[0.47, 0.92)$ with players $R_2 = \{2, 3, 5\}$), and (ii) one for cake $[0, 0.05) + [0.35, 0.47) + [0.92, 1)$ with the remaining players whose valuations are obtained by deleting all valuation intervals contained in maximal intervals of minimum density (players $P = \{6\}$). The maximal interval $[0.05, 0.35)$ of minimum density for cake $[0.05, 0.35)$ with players $R_1 = \{1, 4\}$ is further divided and $A_1 = [0.08, 0.23)$ is allocated to player 1 and $A_4 = [0.05, 0.08) + [0.23, 0.35)$ is allocated to player 4. Since the maximal interval $[0.47, 0.92)$ of minimum density contains the minimal interval $C_2 = [0.58, 0.73)$ of minimum density, $C_5 = [0.47, 0.92)$ is cut at both endpoints of $C_2$ and $A_2 = C_2$ is allocated to player 2. The remaining cake $C'_5 = C_5 \setminus C_2 = [0.47, 0.58) + [0.73, 0.92)$ is further divided and $A_3 = [0.5, 0.58) + [0.73, 0.8)$ is allocated to player 3 and $A_5 = [0.47, 0.5) + [0.8, 0.92)$ is to player 5.

By shrinking of all $H_\ell = [h'_\ell, h''_\ell)$, cake $C' = C \setminus \sum_{\ell=1}^{L} H_\ell$ becomes a single interval $C'^{(S)}$, players $N' = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$ remains the same, each valuation $C'_k \in \mathcal{C}'_{N'}$ becomes a single interval $C'^{(S)}_k$ of $C'^{(S)}$, and the valuation intervals $\mathcal{C}'^{(S)}_{N'} = (C'^{(S)}_k : k \in N')$ becomes solid (i.e., $\bigcup_{k \in N'} C'^{(S)}_k = C'^{(S)}$). Thus, by shrinking of all $H_\ell$, the cake-cutting problem of type (ii) above can be reduced to the cake-cutting problem of type (i) for cake $C'^{(S)}$, players $N' = N \setminus \sum_{\ell=1}^{L} N(H_\ell)$, solid valuation intervals $\mathcal{C}'^{(S)}_{N'} = (C'^{(S)}_k : C'_k \in \mathcal{C}'_{N'})$ with $\bigcup_{k \in N'} C'^{(S)}_k = C'^{(S)}$ and the same density $\rho'^{(S)} = \rho'$, which can be solved recursively.

From an allocation $A'^{(S)}_{N'} = (A'^{(S)}_k : k \in N')$ to players $N'$ where $A'^{(S)}_k$ is the allocated piece of cake $C'^{(S)}$ to player $k \in N'$ with $A'^{(S)}_k \subseteq C'^{(S)}_k$ and $\sum_{i \in N'} A'^{(S)}_k = C'^{(S)}$, we obtain an allocation $A'_{N'} = (A'_k : k \in N')$ to players $N'$ where $A'_k$ is the allocated piece of cake $C'$ to player $k$ with $A'_k \subseteq C'_k$ and $\sum_{i \in N'} A'_k = C'$ as follows: if $A'^{(S)}_k$ contains a shrunken interval $H^{(S)}_\ell$ of hollow interval $H_\ell$, then let $A'_k$ be the set of disjoint intervals obtained from $A'^{(S)}_k$ by restoring each shrunken interval $H^{(S)}_\ell$ in $A'^{(S)}_k$ as original hollow interval $H_\ell = [h'_\ell, h''_\ell)$; otherwise, let $A'_k = A'^{(S)}_k$. We will call this *inverse shrinking* of all $H_\ell$ ($\ell = 1, 2, \ldots, L$).

In summary, we have the following lemma.

▶ **Lemma 6.** *For the two cake-cutting subproblems of type (i) and type (ii) above, the minimum density* $\rho_{\ell\,\min}$ *of intervals in the cake-cutting problem within each* $H_\ell = [\,h'_\ell, h''_\ell)$ *(*$\ell = 1, 2, \ldots, L$*) of type (i) satisfies* $\rho_{\ell\,\min} = \rho_{\min}$*, and if* $\rho(C) > \rho_{\min}$ *then the minimum density* $\rho'_{\min}$ *of intervals in the cake-cutting problem of type (ii) satisfies* $\rho'_{\min} > \rho_{\min}$*.*

The cake-cutting problem of type (i) within each maximal interval $H_\ell = [\,h'_\ell, h''_\ell)$ of minimum density $\rho_{\min}$ can be solved similarly. Let $X_1 = [\,x'_1, x''_1)$, $X_2 = [\,x'_2, x''_2)$, $\ldots$, $X_K = [\,x'_K, x''_K)$ be all the minimal intervals of minimum density $\rho_{\min}$ in $H_\ell$. Then by cutting cake $H_\ell$ at both endpoints of each $X_k = [\,x'_k, x''_k)$ we can reduce the original cake-cutting problem into two types of cake-cutting subproblems of type (i) and type (ii) as follows (Figure 2):

**(i)** the cake-cutting problem within each minimal interval $X_k = [\,x'_k, x''_k)$ ($k = 1, 2, \ldots, K$) of minimum density $\rho_{\min}$ (which consists of cake $X_k$, players $N(X_k)$ whose valuation intervals are in $X_k$ and solid valuation intervals $\mathcal{C}_{N(X_k)}$ with density $\rho$); and

**(ii)** the cake-cutting problem obtained by deleting all $X_k = [\,x'_k, x''_k)$ ($k = 1, 2, \ldots, K$), i.e., the cake-cutting problem for cake $D = H_\ell \backslash \sum_{k=1}^{K} X_k$, players $R = N(H_\ell) \backslash \sum_{k=1}^{K} N(X_k)$ and valuations $\mathcal{D}_R$ (which consists of valuations $D_i = C_i \setminus \sum_{k=1}^{K} X_k \neq \emptyset$ for all $i \in R$) with density $\rho'$ and $\bigcup_{D_i \in \mathcal{D}_R} D_i = D$.

For the same reason as above, we can solve the cake-cutting problem of type (ii) recursively by shrinking of all $X_k = [\,x'_k, x''_k)$. Thus, in summary, we have the following lemma.

▶ **Lemma 7.** *For the two cake-cutting subproblems of type (i) and type (ii) within each maximal interval* $H_\ell = [\,h'_\ell, h''_\ell)$ *above, the minimum density of intervals in the cake-cutting problem within each* $X_k = [\,x'_k, x''_k)$ *(*$k = 1, 2, \ldots, K$*) of type (i) is* $\rho_{\min}$*, and the minimum density* $\rho'_{\min}$ *of intervals in the cake-cutting problem of type (ii) also satisfies* $\rho'_{\min} = \rho_{\min}$*.*

Thus, the core of our mechanism is to solve the cake-cutting problem for cake $X_k$ which is a minimal interval of minimum density $\rho_{\min}$, players $N(X_k)$ and solid valuation intervals $\mathcal{C}_{N(X_k)}$. We call this as Procedure CutMinInterval$(N(X_k), X_k, \mathcal{C}_{N(X_k)})$ and will use it later.

## 4 Details of Our Mechanism

In this section, we will give details of our mechanism based on the outline in the previous section. We denote, by Procedure CutCake$(P, D, \mathcal{D}_P)$, a method for solving the cake-cutting problem for cake $D$ which is a single interval, players $P$ and solid valuation intervals $\mathcal{D}_P$ (where each valuation $D_k \in \mathcal{D}_P$ for $k \in P$ is a single interval in $D$ and $\bigcup_{k \in P} D_k = D$). The original cake-cutting problem for cake $C$, players $N$ and solid valuation intervals $\mathcal{C}_N$ can be solved by calling CutCake$(N, C, \mathcal{C}_N)$. Thus, we can write our mechanism as follows.

**Mechanism 1 Our cake-cutting mechanism.**

---

**Input:** A cake $C = [\,0, 1)$, $n$ players $N = \{1, 2, \ldots, n\}$ and solid valuation intervals $\mathcal{C}_N$ with valuation interval $C_i = [\,\alpha_i, \beta_i)$ of each player $i \in N$ and $\bigcup_{C_i \in \mathcal{C}_N} C_i = C$.

**Output:** Allocation $A_N = (A_i : i \in N)$ to players $N$.

**Algorithm** { CutCake$(N, C, \mathcal{C}_N)$; }

---

We also denote, by Procedure CutMaxInterval$(R, H, \mathcal{D}_R)$ called in CutCake$(P, D, \mathcal{D}_P)$, a method for solving the cake-cutting problem of type (i) with cake $H = H_\ell$ which is a maximal interval of minimum density $\rho_{\min}$ in cake $D$, players $R = P(H_\ell) = \{i \in P \mid D_i \subseteq H_\ell, D_i \in \mathcal{D}_P\}$ and solid valuation intervals $\mathcal{D}_R = \mathcal{D}_{P(H_\ell)} = (D_i \in \mathcal{D}_P : i \in P(H_\ell))$ (thus, $\cup_{D_i \in \mathcal{D}_R} D_i = H_\ell$). Based on Lemma 6, we can write Procedure CutCake$(P, D, \mathcal{D}_P)$ as follows.

**Procedure** CutCake$(P, D, \mathcal{D}_P)$

---

Find all the maximal intervals of minimum density $\rho_{\min}$ in the cake-cutting problem with cake $D$, players $P$ and solid valuation intervals $\mathcal{D}_P$;
Let $H_1 = [h'_1, h''_1),\ H_2 = [h'_2, h''_2), \ldots,\ H_L = [h'_L, h''_L)$ be all the maximal intervals of minimum density $\rho_{\min}$; // $H_1, H_2, \ldots, H_L$ are mutually disjoint by Corollary 5
**for** $\ell = 1$ **to** $L$ **do**
  cut cake $D$ at both endpoints $h'_\ell, h''_\ell$ of $H_\ell$;
  $R_\ell = \{k \in P \mid D_k \subseteq H_\ell, D_k \in \mathcal{D}_P\}$; $\mathcal{D}_{R_\ell} = (D_k \in \mathcal{D}_P :\ k \in R_\ell)$;
  CutMaxInterval$(R_\ell, H_\ell, \mathcal{D}_{R_\ell})$;
$P' = P$; $D' = D$; **for** $\ell = 1$ **to** $L$ **do** $P' = P' \setminus R_\ell$; $D' = D' \setminus H_\ell$;
**if** $P' \neq \emptyset$ **then** // $P' = P \setminus \sum_{\ell=1}^{L} R_\ell$ and $D' = D \setminus \sum_{\ell=1}^{L} H_\ell$
  $\mathcal{D}'_{P'} = \emptyset$;
  **for** each $D_k \in \mathcal{D}_P$ with $k \in P'$ **do** $D'_k = D_k \setminus \sum_{\ell=1}^{L} H_\ell$; $\mathcal{D}'_{P'} = \mathcal{D}'_{P'} + \{D'_k\}$;
  Perform shrinking of all $H_1, H_2, \ldots, H_L$;
  Let $D^{(S)}$, $D_k^{(S)} \in \mathcal{D}_{P'}^{(S)}$, and $\mathcal{D}_{P'}^{(S)}$ be obtained from $D'$, $D'_k \in \mathcal{D}'_{P'}$, and $\mathcal{D}'_{P'}$ by shrinking of all $H_1, H_2, \ldots, H_L$, respectively;
  CutCake$(P', D^{(S)}, \mathcal{D}_{P'}^{(S)})$; Perform inverse shrinking of all $H_1, H_2, \ldots, H_L$;

---

Note that, if $P' \neq \emptyset$ after the deletion of $H_1, H_2, \ldots, H_L$ and CutCake$(P', D^{(S)}, \mathcal{D}_{P'}^{(S)})$ is recursively called, then the minimum density $\rho'_{\min}$ in CutCake$(P', D^{(S)}, \mathcal{D}_{P'}^{(S)})$ satisfies $\rho'_{\min} > \rho_{\min}$ by Lemma 6. Next, we give a detailed description of CutMaxInterval$(R, H, \mathcal{D}_R)$ based on Lemma 7 and Procedure CutMinInterval$(S, X, \mathcal{D}_S)$.

**Procedure** CutMaxInterval$(R, H, \mathcal{D}_R)$

---

Let $X_1 = [x'_1, x''_1),\ X_2 = [x'_2, x''_2),\ \ldots,\ X_K = [x'_K, x''_K)$ be all the minimal intervals of minimum density $\rho_{\min}$ in $H$; // $X_1, X_2, \ldots, X_K$ are mutually disjoint by Corollary 5
**for** $k = 1$ **to** $K$ **do**
  cut cake $H$ at both endpoints $x'_k, x''_k$ of $X_k$;
  $S_k = \{i \in R \mid D_i \subseteq X_k, D_i \in \mathcal{D}_R\}$; $\mathcal{D}_{S_k} = (D_i \in \mathcal{D}_R :\ i \in S_k)$;
  CutMinInterval$(S_k, X_k, \mathcal{D}_{S_k})$;
$R' = R$; $H' = H$; **for** $k = 1$ **to** $K$ **do** $R' = R' \setminus S_k$; $H' = H' \setminus X_k$;
**if** $R' \neq \emptyset$ **then** // $R' = R \setminus \sum_{k=1}^{K} S_k$ and $H' = H \setminus \sum_{k=1}^{K} X_k$
  $\mathcal{D}'_{R'} = \emptyset$;
  **for** each $D_i \in \mathcal{D}_R$ with $i \in R'$ **do** $D'_i = D_i \setminus \sum_{k=1}^{K} X_k$; $\mathcal{D}'_{R'} = \mathcal{D}'_{R'} + \{D'_i\}$;
  Perform shrinking of all $X_1, X_2, \ldots, X_K$;
  Let $H^{(S)}$, $D_i^{(S)} \in \mathcal{D}_{R'}^{(S)}$, and $\mathcal{D}_{R'}^{(S)}$ be obtained from $H'$, $D'_i \in \mathcal{D}'_{R'}$, and $\mathcal{D}'_{R'}$ by shrinking of all $X_1, X_2, \ldots, X_K$, respectively;
  CutMaxInterval$(R', H^{(S)}, \mathcal{D}_{R'}^{(S)})$; Perform inverse shrinking of all $X_1, X_2, \ldots, X_K$;

---

Note that, if $R' \neq \emptyset$ after deletion of $X_1, X_2, \ldots, X_K$ and CutMaxInterval$(R', H^{(S)}, \mathcal{D}_{R'}^{(S)})$ is recursively called, then the minimum density $\rho'_{\min}$ in CutMaxInterval$(R', H^{(S)}, \mathcal{D}_{R'}^{(S)})$ satisfies $\rho'_{\min} = \rho_{\min}$ by Lemma 7. As mentioned before, Procedure CutMinInterval$(S, X, \mathcal{D}_S)$ is the core method for solving the cake-cutting problem where cake $X$ is a minimal interval of minimum density in maximal interval $H$ of minimum density $\rho_{\min}$, players $S = R(X) = \{i \in R \mid D_i \in \mathcal{D}_R,\ D_i \subseteq X\}$ and solid valuation intervals $\mathcal{D}_S = \mathcal{D}_{R(X)} = (D_i \in \mathcal{D}_R : i \in S)$.

## 4.1 Core Method: Cutting Minimal Interval of Minimum Density

We need some more definitions and notations to give the core method.

▶ **Definition 8.** *Let $X = [x', x'')$ be a minimal interval of minimum density $\rho_{\min}$ in cake $C$. A minimal interval $Y = [y', y'')$ with respect to valuations which is properly contained in $X$ (i.e., $Y \subset X$) is called a separable interval of $X$, if len$(Y)$ is less than $(n_Y + 1)\,\rho_{\min}$ where $n_Y$ is the number of players whose valuation intervals are entirely contained in $Y$. If there is no separable interval of $X = [x', x'')$, then $X$ is called nonseparable.*

We first consider the case when a minimal interval $X$ of minimum density $\rho_{\min}$ is nonseparable. This has a nice property which can be proved by Hall's Theorem [13].

▶ **Lemma 9.** *Let $X = [x', x'')$ be a nonseparable minimal interval of minimum density $\rho_{\min}$. For simplicity, we assume $N(X) = \{1, 2, \ldots, n_X\}$. Let $I_j = [x' + (j-1)\rho_{\min}, x' + j\rho_{\min})$ for each $j \in N(X)$, and let $\mathcal{I}_{N(X)} = \{I_1, I_2, \ldots, I_{n_X}\}$ (thus, $\sum_{j \in N(X)} I_j = X$). Let $G_{N(X)} = (\mathcal{C}_{N(X)}, \mathcal{I}_{N(X)}, E)$ be a bipartite graph with vertex set $\mathcal{C}_{N(X)} + \mathcal{I}_{N(X)}$ and edge set $E$ where $(C_i, I_j) \in E$ if and only if $I_j \subseteq C_i$. Then $G_{N(X)}$ has a perfect matching $M = \{(C_i, I_{\pi(i)}) \mid i \in N(X)\} \subseteq E$, where $\pi$ is a permutation on $N(X)$.*

*(Thus, we can allocate $A_i = I_{\pi(i)} \subseteq C_i$ of cake $X$ to player $i \in N(X)$ with $\sum_{i \in N(X)} A_i = X$ and we call this Procedure AllocateInterval($N(X), X, \mathcal{C}_{N(X)}$).)*

Next we consider the case when a minimal interval $X = [x', x'')$ of minimum density $\rho_{\min}$ has a separable interval (Figure 3). Let $\mathcal{Y}$ be the set of all separable intervals in $X$ and let

$$y^* = \max_{Y = [y',\, y'') \in \mathcal{Y}} y' \quad \text{and} \quad \mathcal{Y}_{y^*} = \{Y = [y',\, y'') \in \mathcal{Y} \mid y' = y^*\}. \tag{2}$$

That is, $y^*$ is the largest left endpoint of the separable intervals in $X$ and $\mathcal{Y}_{y^*}$ is the set of all separable intervals with left endpoint $y^*$ in $X$. For each interval $Y = [y',\, y'')$ of $X$, let

$$\gamma(Y) = \mathit{len}(Y) - n_Y\,\rho_{\min}. \tag{3}$$

Then, for a separable interval $Y = [y',\, y'')$ of $X$ (thus, $Y$ is a minimal interval with respect to valuations in $X$ by Definition 8), we have

$$n_Y\,\rho_{\min} < \mathit{len}(Y) < (n_Y + 1)\,\rho_{\min} \quad \text{and} \quad 0 < \gamma(Y) < \rho_{\min}. \tag{4}$$

Actually, $\rho(Y) = \frac{\mathit{len}(Y)}{n_Y} > \rho_{\min}$ by the definition of a minimal interval $X$ of minimum density $\rho_{\min}$ and $\mathit{len}(Y) < (n_Y + 1)\,\rho_{\min}$ since $Y \subset X$ is a separable interval of $X$. Let

$$\gamma^* = \min_{Y \in \mathcal{Y}_{y^*}} \gamma(Y). \tag{5}$$

Clearly, $0 < \gamma^* < \rho_{\min}$ by Eqs.(2), (4). Let

$$\mathcal{Y}_{y^*}^{\gamma^*} = \{Y = [y^*,\, y'') \in \mathcal{Y}_{y^*} \mid \gamma(Y) = \gamma^*\} \text{ and } Z_{y^*}^{\gamma^*} = \{y'' \mid Y = [y^*,\, y'') \in \mathcal{Y}_{y^*}^{\gamma^*}\}. \tag{6}$$

That is, $\mathcal{Y}_{y^*}^{\gamma^*}$ is the set of separable intervals $Y = [y^*,\, y'')$ in $\mathcal{Y}_{y^*}$ with $\gamma(Y) = \gamma^*$ and $Z_{y^*}^{\gamma^*}$ is the set of right endpoints of the separable intervals in $\mathcal{Y}_{y^*}^{\gamma^*}$. Let $J = |Z_{y^*}^{\gamma^*}|$ and assume

$$Z_{y^*}^{\gamma^*} = \{z_1^*, z_2^*, \ldots, z_J^*\}, \quad z_1^* < z_2^* < \cdots < z_J^*, \quad Y_j = [y^*, z_j^*) \text{ for } j = 1, 2, \ldots, J. \tag{7}$$

For simplicity, we also consider $z_0^* = y^* + \gamma^*$ and $Y_0 = [y^*,\, z_0^*)$. Then we have the following.

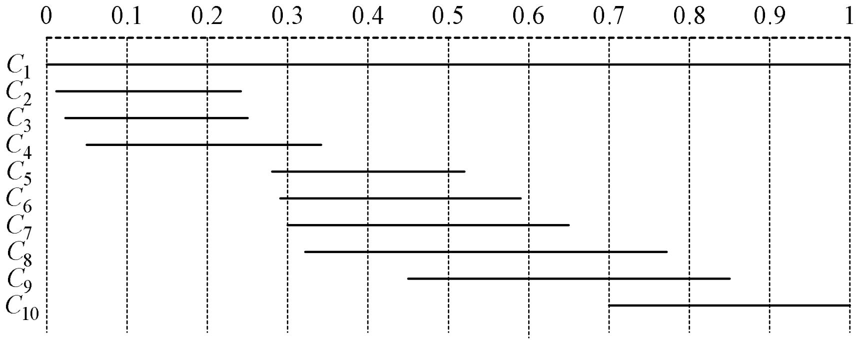


**Figure 3** Players $N = \{1, 2, \ldots, 10\}$ and their valuation intervals $C_1 = [0, 1)$, $C_2 = [0.01, 0.24)$, $C_3 = [0.02, 0.25)$, $C_4 = [0.05, 0.34)$, $C_5 = [0.28, 0.52)$, $C_6 = [0.29, 0.59)$, $C_7 = [0.3, 0.65)$, $C_8 = [0.32, 0.77)$, $C_9 = [0.45, 0.85)$, $C_{10} = [0.7, 1)$. $X = [0, 1)$ is a minimal interval of minimum density $\rho_{\min} = 0.1$, and there are several separable intervals of $X = [0, 1)$ such as $[0.01, 0.25)$, $[0.01, 1)$, $[0.28, 0.65)$, $[0.28, 0.77)$, $[0.28, 0.85)$. The largest left endpoint $y^*$ of the separable intervals in $X$ is 0.28 and the set of separable intervals with the largest left endpoint $y^* = 0.28$ is $\mathcal{Y}_{y^*} = \{[0.28, 0.65), [0.28, 0.77), [0.28, 0.85)\}$. Thus, $\mathcal{Y}^{\gamma^*}_{y^*} = \{[0.28, 0.65), [0.28, 0.85)\}$, $Z^{\gamma^*}_{y^*} = \{0.65, 0.85\}$, $J = 2$, $z^*_1 = 0.65 < z^*_2 = 0.85$ (and $z_0 = 0.35$, $Y_0 = [0.28, 0.35)$, $Y_1 = [0.35, 0.65)$ $Y_2 = [0.65, 0.85)$).

▶ **Lemma 10.** *Let $X = [x', x'')$ be a minimal interval of minimum density $\rho_{\min}$ in cake $C$. Let $Y = [y^*, z) \subset X$ be an interval such that there exists $C_i = [\alpha_i, \beta_i) \in \mathcal{C}_{N(X)}$ with $y^* \leq \alpha_i$ and $z = \beta_i$. Then $\gamma(Y) = \gamma^*$ for $z \in Z^{\gamma^*}_{y^*}$ and $\gamma(Y) > \gamma^*$ for $z \notin Z^{\gamma^*}_{y^*}$. Furthermore, if $z \notin Z^{\gamma^*}_{y^*}$ and $z > z^*_j$ for some $Y_j = [y^*, z^*_j)$ $(j = 0, 1, \ldots, J)$, then $z - z^*_j > \rho_{\min}(n_Y - n_{Y_j})$.*

**Proof.** It is clear that $\gamma(Y) = len(Y) - n_Y \rho_{\min} = z - y^* - n_Y \rho_{\min}$ by the definition of $\gamma(Y)$ of $Y = [y^*, z)$. Similarly, if $z \in Z^{\gamma^*}_{y^*}$ then $\gamma(Y) = \gamma^*$ by the definitions of $\mathcal{Y}^{\gamma^*}_{y^*}$ and $Z^{\gamma^*}_{y^*}$. Therefore, we can assume $z \notin Z^{\gamma^*}_{y^*}$ below.

(i) We first consider the case when $Y = [y^*, z)$ is not a separable interval. If $Y = [y^*, z)$ is a minimal interval with respect to valuations then $len(Y) \geq (n_Y + 1)\, \rho_{\min}$ and $\gamma(Y) = len(Y) - n_Y \rho_{\min} \geq \rho_{\min} > \gamma^*$. Otherwise (i.e., if $Y = [y^*, z)$ is not a minimal interval with respect to valuations), let $y' = \min_{C_j = [\alpha_j, \beta_j) \subset Y : C_j \in \mathcal{C}_{N(X)}} \alpha_j$. Then $y' > y^*$, since the valuation interval $C_i = [\alpha_i, \beta_i) \in \mathcal{C}_{N(X)}$ satisfies $\alpha_i \geq y^*$, $z = \beta_i$, and $C_i \subset Y = [y^*, z)$. Let $C_j = [\alpha_j, \beta_j) \in \mathcal{C}_{N(X)}$ satisfy $\alpha_j = y'$ and $C_j \subset Y$. Let $Y' = [y', z) \subset Y = [y^*, z)$. Then $Y'$ is a minimal interval with respect to valuations and $n_{Y'} = n_Y$. Note that, $Y'$ is not a separable interval since $y^*$ is the largest left endpoint of separable intervals. Thus, $len(Y') \geq (n_{Y'} + 1)\, \rho_{\min}$ and $len(Y) = z - y^* > z - y' = len(Y') \geq (n_{Y'} + 1)\, \rho_{\min} = (n_Y + 1)\, \rho_{\min}$ and we have $\gamma(Y) = len(Y) - n_Y \rho_{\min} > \rho_{\min} > \gamma^*$.

(ii) We next consider the case when $Y = [y^*, z)$ is a separable interval. Thus, $n_Y \rho_{\min} < len(Y) < (n_Y + 1)\rho_{\min}$. By the definition of $Z^{\gamma^*}_{y^*} = \{y'' \mid Y = [y^*, y'') \in \mathcal{Y}^{\gamma^*}_{y^*}\}$ and Eq.(5), we have $\gamma(Y) = len(Y) - n_Y \rho_{\min} > \gamma^*$ since $z \notin Z^{\gamma^*}_{y^*}$.

Furthermore, if $z \notin Z^{\gamma^*}_{y^*}$ and $z > z^*_j$ for some $Y_j = [y^*, z^*_j)$ $(j = 0, 1, \ldots, J)$, then $z - z^*_j = len(Y) - len(Y_j)$ and we have $z - z^*_j = len(Y) - len(Y_j) = \rho_{\min} n_Y + \gamma(Y) - (\rho_{\min} n_{Y_j} + \gamma(Y_j)) = (\gamma(Y) - \gamma(Y_j)) + \rho_{\min}(n_Y - n_{Y_j}) > \rho_{\min}(n_Y - n_{Y_j})$ by $\gamma(Y_j) = \gamma^* < \gamma(Y)$. ◀

The following lemma can be obtained by the same argument as in Proof of Lemma 1.

▶ **Lemma 11.** *Let $X = [x', x'')$ be a minimal interval of minimum density $\rho_{\min}$ in cake $C$. Then the valuation intervals $\mathcal{C}_{N(Y_j)}$ for each $Y_j = [y^*, z^*_j)$ $(j = 1, 2, \ldots, J)$ is solid.*

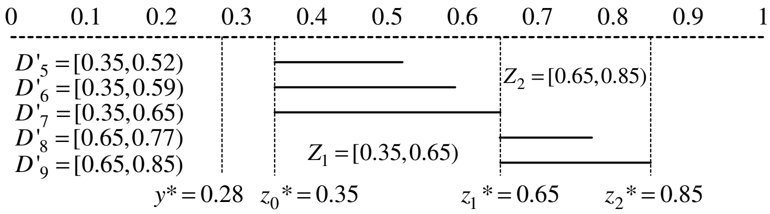


**Figure 4** In the example in Figure 3, $Z_1 = [0.35, 0.65)$, $S(Z_1) = \{5, 6, 7\}$, $\mathcal{D}'_{S(Z_1)} = (D'_5 = [0.35, 0.52), D'_6 = [0.35, 0.59), D'_7 = [0.35, 0.65])$, and $Z_2 = [0.65, 0.85)$, $S(Z_2) = \{8, 9\}$, $\mathcal{D}'_{S(Z_2)} = (D'_8 = [0.65, 0.77), D'_9 = [0.65, 0.85])$.

Let $X = [x', x'')$ be a minimal interval of minimum density $\rho_{\min}$ in cake $C$ and let $S = N(X)$ and $\mathcal{D}_S = (D_i = C_i : C_i \in \mathcal{C}_N, C_i \subseteq X) = \mathcal{C}_{N(X)}$. For each $j = 1, 2, \ldots, J$, let

$$Z_j = [z^*_{j-1}, z^*_j),\ S(Z_j) = \{i \in S \mid D_i \in \mathcal{D}_S, D_i \subseteq Y_j, D_i \not\subseteq Y_{j-1}\},\ n'_{Z_j} = |S(Z_j)|, \tag{8}$$

$$\mathcal{D}_{S(Z_j)} = (D_i \in \mathcal{D}_S : i \in S(Z_j)), \tag{9}$$

$$\mathcal{D}'_{S(Z_j)} = (D'_i = D_i \setminus Y_{j-1} : D_i \in \mathcal{D}_{S(Z_j)}) \tag{10}$$

(Figure 4). Note that $\mathcal{D}_{S(Z_j)} = (D_i \in \mathcal{D}_S : D_i \subseteq Y_j) \setminus (D_i \in \mathcal{D}_S : D_i \subseteq Y_{j-1})$. Note also that, $D'_i = D_i \setminus Y_{j-1} \in \mathcal{D}'_{S(Z_j)}$ is always contained in $Z_j = [z^*_{j-1}, z^*_j)$, although valuation interval $D_i = [d'_i, d''_i) \in \mathcal{D}_{S(Z_j)}$ may not be in $Z_j = [z^*_{j-1}, z^*_j)$ (i.e., $d'_i < z^*_{j-1}$ may happen). Of course, $y^* \le d'_i$ and $z^*_{j-1} < d''_i \le z^*_j$ hold. We consider the cake-cutting problem for cake $Z_j$, players $S(Z_j)$, solid valuation intervals $\mathcal{D}'_{S(Z_j)}$. Note that, there is no valuation interval of $\mathcal{D}_S = \mathcal{C}_{N(X)}$ contained in $Y_0 = [y^*, z^*_0) \subset X$ ($n_{Y_0} = 0$), since if there were a valuation interval $D_i \in \mathcal{D}_S$ contained in $Y_0$, then $D_i$ would be a minimal interval with respect to valuations and $n_{D_i} \ge 1$ and $\rho(D_i) \le len(Y_0) = z^*_0 - y^* = \gamma^* < \rho_{\min}$, a contradiction that $X$ is a minimal interval of minimum density $\rho_{\min}$. Thus, we have the following lemma.

▶ **Lemma 12.** *Each interval $Z_j = [z^*_{j-1}, z^*_j)$ ($j = 1, 2, \ldots, J$) is a minimal interval of minimum density $\rho'_{\min} = \rho_{\min}$ for the cake-cutting problem for cake $Z_j$, players $S(Z_j)$, valuation intervals $\mathcal{D}'_{S(Z_j)}$ with density $\rho'$. Furthermore, the valuation intervals $\mathcal{D}'_{S(Z_j)}$ is solid (thus, $\bigcup_{D'_i \in \mathcal{D}'_{S(Z_j)}} D'_i = Z_j$ holds).*

**Proof.** Since we set $S = N(X)$ and $\mathcal{D}_S = (D_i = C_i : C_i \in \mathcal{C}_N, C_i \subseteq X) = \mathcal{C}_{N(X)}$, we have $S(Y_j) = \{i \in S \mid D_i \in \mathcal{D}_S, D_i \subseteq Y_j\} = N(Y_j)$ and $\mathcal{D}_{S(Y_j)} = (D_i : D_i \in \mathcal{D}_S, D_i \subseteq Y_j) = \mathcal{C}_{N(Y_j)}$ for each $Y_j = [y^*, z^*_j)$ ($j = 1, 2, \ldots, J$). Thus, by Lemma 11, $\mathcal{D}_{S(Y_j)}$ is solid, and thus, for each point $z \in Z_j = [z^*_{j-1}, z^*_j) = Y_j \setminus Y_{j-1}$, there is a valuation interval $D_i \in \mathcal{D}_{S(Y_j)}$ containing $z$. The interval $D_i$ is not in $\mathcal{D}_{S(Y_{j-1})}$, since $z \notin Y_{j-1}$. Thus, $z$ is in $D'_i = D_i \setminus Y_{j-1} \in \mathcal{D}'_{S(Z_j)}$. This implies that the valuation intervals $\mathcal{D}'_{S(Z_j)}$ is solid.

Thus, we will show below that each $Z_j = [z^*_{j-1}, z^*_j)$ ($j = 1, 2, \ldots, J$) is a minimal interval of minimum density $\rho'_{\min} = \rho_{\min}$. It is clear that $\rho'(Z_j) = \frac{len(Z_j)}{n'_{Z_j}} = \rho_{\min}$, since $Y_j = [y^*, z^*_j)$, $Y_{j-1} = [y^*, z^*_{j-1})$, $Z_j = Y_j \setminus Y_{j-1}$, $S(Z_j) = S(Y_j) \setminus S(Y_{j-1})$, $len(Y_j) = \rho_{\min} n_{Y_j} + \gamma^*$, $len(Y_{j-1}) = \rho_{\min} n_{Y_{j-1}} + \gamma^*$, $n'_{Z_j} = |S(Z_j)| = |S(Y_j)| - |S(Y_{j-1})| = n_{Y_j} - n_{Y_{j-1}}$ and $len(Z_j) = len(Y_j) - len(Y_{j-1}) = \rho_{\min}(n_{Y_j} - n_{Y_{j-1}}) = \rho_{\min} n'_{Z_j}$.

Let $Z = [z', z'')$ be a proper subinterval of $Z_j$ (i.e., $Z \subset Z_j$) such that $z''$ is a right endpoint of some valuation interval in $\mathcal{D}'_{S(Z_j)}$ and that $z' = z^*_{j-1}$ or $z' \ne z^*_{j-1}$ and $z'$ is a left endpoint of some valuation interval in $\mathcal{D}'_{S(Z_j)}$. Thus, $Z = [z', z'') \subset Z_j$ is a minimal interval with respect to valuations in cake $Z_j$.

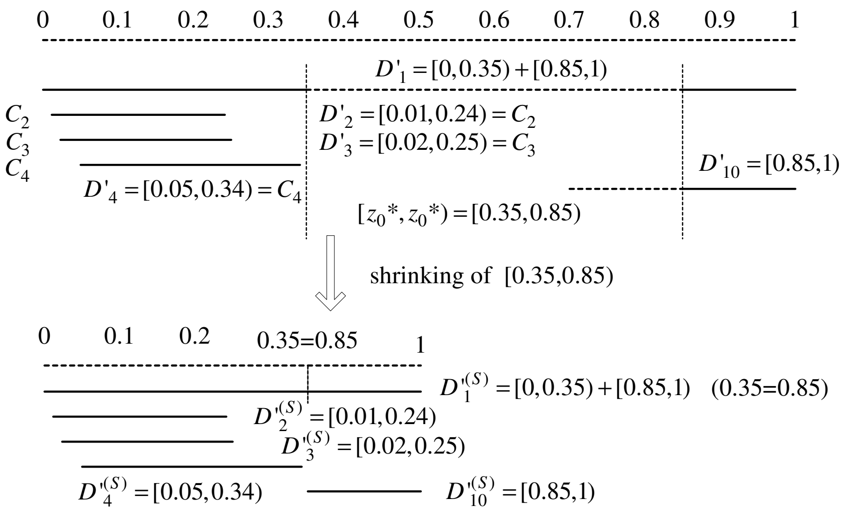


**Figure 5** In the example in Figure 3, $S' = \{1, 2, 3, 4, 10\}$, $X' = [0, 0.35) + [0.85, 1)$, $\mathcal{D}'_{S'} = (D'_1, D'_2, D'_3, D'_4, D'_{10})$ and the cake-cutting problem obtained by shrinking of $[0.35, 0.85)$.

If $z' \neq z^*_{j-1}$ then $\rho'(Z) = \rho(Z) > \rho_{\min}$, since $Z \subset X$ ($Z \neq X$) and $X$ is a minimal interval of minimum density $\rho_{\min}$. Thus, we assume $z' = z^*_{j-1} < z'' < z^*_j$ since $Z \subset Z_j$. Now consider the intervals $Y'_j = [\,y^*, z'')$ and $Y_{j-1} = [\,y^*, z^*_{j-1}) \subset Y'_j = [\,y^*, z'')$. Thus, $Z = Y'_j \setminus Y_{j-1}$. Let $n'_Z = |\mathcal{D}_{S(Y'_j)} \setminus \mathcal{D}_{S(Y_{j}-1)}|$. Then $n'_Z = n_{Y'_j} - n_{Y_{j-1}}$. By Lemma 10, we have $len(Z) = z'' - z^*_{j-1} > \rho_{\min}(n_{Y'_j} - n_{Y_{j-1}}) = \rho_{\min}\, n'_Z$ and $\rho'(Z) = \frac{len(Z)}{n'_Z} > \rho_{\min}$. Thus, we have $Z_j = [z^*_{j-1}, z^*_j)$ is a minimal interval of minimum density $\rho'_{\min} = \rho_{\min}$. ◀

Next, we consider the remaining cake-cutting problem after deletion of the interval $[z^*_0, z^*_J) = Z_1 + Z_2 + \cdots + Z_J$. Let $S([z^*_0, z^*_J)) = S(Z_1) + S(Z_2) + \cdots + S(Z_J)$. Thus, $S([z^*_0, z^*_J))$ is the set of players whose valuation intervals are in $Y^*_J = [\,y^*, z^*_J)$. Let

$$S' = S \setminus S([z^*_0, z^*_J)),\ X' = X \setminus [z^*_0, z^*_J),\ \mathcal{D}'_{S'} = (D'_i = D_i \setminus [z^*_0, z^*_J) : D_i \in \mathcal{D}_S, D_i \not\subseteq Y_J)$$

(Figure 5). Then, we reduce the cake-cutting problem for cake $X'$, players $S'$ and valuations $\mathcal{D}'_{S'}$ with density $\rho'$ by shrinking of $[z^*_0, z^*_J)$ to the cake-cutting problem for cake $X'^{(S)}$, players $S'$ and solid valuation intervals $\mathcal{D}'^{(S)}_{S'}$ with density $\rho'$, where $X'^{(S)}$, $D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}$ and $\mathcal{D}'^{(S)}_{S'}$ are obtained from $X'$, $D'_i \in \mathcal{D}'_{S'}$ and $\mathcal{D}'_{S'}$ by shrinking of $[z^*_0, z^*_J)$, respectively.

▶ **Lemma 13.** *$X'^{(S)}$ is a minimal interval of minimum density $\rho'_{\min} = \rho_{\min}$ in the cake-cutting problem for cake $X'^{(S)}$, players $S'$, and solid valuation intervals $\mathcal{D}'^{(S)}_{S'}$ with density $\rho'$. (Thus, this can be solved by calling CutMinInterval($S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'}$) recursively.)*

**Proof.** We will show that $X'^{(S)}$ is a minimal interval of minimum density $\rho_{\min}$. We can obtain $\rho'(X'^{(S)}) = \rho_{\min}$ by almost the same argument as in Lemma 12.

Let $Z = [z', z'') \not\subseteq [\,y^*, z^*_J)$ be an interval in $X$ such that $Z'^{(S)}$, obtained from $Z' = Z \setminus [z^*_0, z^*_J)$ by shrinking of $[z^*_0, z^*_J)$, is a proper subinterval in $X'^{(S)}$ (i.e., $Z'^{(S)} \subset X'^{(S)}$). Thus, $z' < y^*$ or $z'' > z^*_J$. We will show that $\rho'(Z'^{(S)}) > \rho_{\min}$ by dividing into two subcases: (i) the case of $z' < y^*$ and (ii) the case of $y^* \leq z'$ and $z'' > z^*_J$. As mentioned before, there is no valuation interval of $\mathcal{D}_S = (D_i = C_i : C_i \in \mathcal{C}_N, C_i \subseteq X) = \mathcal{C}_{N(X)}$ which is contained in $[\,y^*, z^*_0)$. Thus, there is no valuation interval of $\mathcal{D}'_{S'}$ (and of $\mathcal{D}'^{(S)}_{S'}$) contained in $[\,y^*, z^*_0)$.

(i): We only discuss the case of $z' < y^* < z_0^* < z'' \leq z_J^*$. Since shrinking of $[z_0^*, z_J^*)$ is performed, we can consider $z'' = z_J^*$ and $Y_J = [y^*, z_J^*) \subset Z = [z', z_J^*)$, $len(Z) = z_J^* - z'$, $\rho_{\min} n_{Y_J} = z_J^* - z_0^*$. Thus, after shrinking of $[z_0^*, z_J^*)$, $Z' = Z \setminus [z_0^*, z_J^*)$ becomes $Z'^{(S)} = [z', z_0^*)$ and we have $n_{Z'^{(S)}} = n_Z - n_{Y_J}$, $len(Z'^{(S)}) = z_0^* - z'$, and $\rho'(Z'^{(S)}) = \frac{len(Z'^{(S)})}{n_{Z'^{(S)}}} = \frac{z_0^* - z'}{n_{Z'^{(S)}}} > \rho_{\min}$ by $\rho(Z) = \frac{len(Z)}{n_Z} = \frac{z_J^* - z_0^* + z_0^* - z'}{n_{Z'^{(S)}} + n_{Y_J}} = \frac{\rho_{\min} n_{Y_J} + z_0^* - z'}{n_{Z'^{(S)}} + n_{Y_J}} > \rho_{\min}$ since $Z \subset X$ and $X$ is a minimal interval of minimum density $\rho_{\min}$.

(ii): We only discuss the case of $z_0^* \leq z' < z_J^* < z''$. By Lemma 10 for $j = J$, $Y_J = [y^*, z_J^*)$ and $Y = [y^*, z'')$, we have $Z'^{(S)} = Z' = Z \setminus [z_0^*, z_J^*) = [z_J^*, z'')$, $len(Z'^{(S)}) = z'' - z_J^* = len(Y) - len(Y_J) > \rho_{\min}(n_Y - n_{Y_J})$ and $n_{Z'^{(S)}} \leq n_Y - n_{Y_J}$ (note that $D_i = [\alpha_i, \beta_i)$ with $y^* \leq \alpha_i < z_0^*$ and $z_J^* < \beta_i \leq z''$ is in $Y = [y^*, z'')$, but not in $Y_J = [y^*, z_J^*)$, and thus, $D_i' = D_i \setminus [z_0^*, z_J^*) = [\alpha_i, z_0^*) + [z_J^*, \beta_i)$ is not contained in $Z'^{(S)} = Z' = Z \setminus [z_0^*, z_J^*) = [z_J^*, z'')$). Thus, $\rho'(Z'^{(S)}) = \frac{len(Z'^{(S)})}{n_{Z'^{(S)}}} \geq \frac{len(Z'^{(S)})}{n_Y - n_{Y_J}} > \rho_{\min}$.

Thus, we have shown that $X'^{(S)}$ is a minimal interval of minimum density $\rho'_{\min} = \rho_{\min}$.

Finally, we can show that the valuation intervals $\mathcal{D}'^{(S)}_{S'}$ is solid, by almost the same argument (and we omit its proof). ◀

Based on Lemmas 9, 12 and 13, we can write CutMinInterval$(S, X, \mathcal{D}_S)$ as follows.

**Procedure** CutMinInterval$(S, X, \mathcal{D}_S)$.

---

**if** $X = [x', x'')$ is nonseparable **then** AllocateInterval$(S, X, \mathcal{D}_S)$;
  // this finds an allocation of $X$ to players $S$ by Lemma 9
**else** // there is a separable interval in $X$
  Find $y^*$, $\gamma^*$, $\mathcal{Y}_{y^*}^{\gamma^*}$, and $Z_{y^*}^{\gamma^*}$ defined by Eqs. (2), (5), (6), and (7), respectively;
  Let $Z_{y^*}^{\gamma^*} = \{z_1^*, z_2^*, \ldots, z_J^*\}$ and assume $z_0^* = y^* + \gamma^* < z_1^* < z_2^* < \cdots < z_J^*$;
  **for** $j = 1$ **to** $J$ **do**
    $Z_j = [z_{j-1}^*, z_j^*)$; cut cake $X$ at both endpoints $z_{j-1}^*, z_j^*$ of $Z_j = [z_{j-1}^*, z_j^*)$;
    let $S(Z_j)$ and $\mathcal{D}'_{S(Z_j)}$ be defined in Eqs. (8) and (10);
    CutMinInterval$(S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)})$;
  $S' = S \setminus S([z_0^*, z_J^*))$; $X' = X \setminus [z_0^*, z_J^*)$;
  **if** $S' \neq \emptyset$ **then**
    $\mathcal{D}'_{S'} = \emptyset$; **for** each $D_i \in \mathcal{D}_S$ with $i \in S'$ **do** $D_i' = D_i \setminus [z_0^*, z_J^*)$; $\mathcal{D}'_{S'} = \mathcal{D}'_{S'} + \{D_i'\}$;
    Perform shrinking of $[z_0^*, z_J^*)$;
    Let $X'^{(S)}$, $D_i'^{(S)} \in \mathcal{D}'^{(S)}_{S'}$ and $\mathcal{D}'^{(S)}_{S'}$ be obtained from $X'$, $D_i' \in \mathcal{D}'_{S'}$ and $\mathcal{D}'_{S'}$
      by shrinking of $[z_0^*, z_J^*)$, respectively;
    CutMinInterval$(S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'})$; Perform inverse shrinking of $[z_0^*, z_J^*)$;

---

We have the following lemma for CutMinInterval$(S, X, \mathcal{D}_S)$.

▶ **Lemma 14.** *CutMinInterval$(S, X, \mathcal{D}_S)$ returns an envy-free allocation $(A_i : i \in S)$ of $X$ to players $S$ such that $A_i \subseteq D_i \in \mathcal{D}_S$, $len(A_i) = \rho_{\min}$ for each $i \in S$, $\sum_{i \in S} A_i = X$ and runs in $O(s^3)$ time where $s = |S|$, and the number of cuts made over $X$ is at most $2s - 2$.*

**Proof.** We first show that CutMinInterval$(S, X, \mathcal{D}_S)$ satisfies the following (a) – (c).

**(a)** CutMinInterval$(S, X, \mathcal{D}_S)$ returns an envy-free allocation $(A_i : i \in S)$ of $X$ to players $S$ such that $A_i \subseteq D_i \in \mathcal{D}_S$, $len(A_i) = \rho_{\min}$ for each $i \in S$ and $\sum_{i \in S} A_i = X$.

**(b)** CutMinInterval$(S, X, \mathcal{D}_S)$ runs in $O(s^3)$ time where $s = |S|$.

**(c)** The number of cuts made over $X$ by CutMinInterval$(S, X, \mathcal{D}_S)$ is at most $2s - 2$.

If $X$ is nonseparable, then, by Lemma 9, AllocateInterval$(S, X, \mathcal{D}_S)$ finds an allocation $(A_i : i \in S)$ of $X$ to players $S$ such that $A_i \subseteq D_i \in \mathcal{D}_S$, $len(A_i) = \rho_{\min}$ for each $i \in S$ and $\sum_{i\in S} A_i = X$. Thus, $V_i(A_i) = len(A_i \cap D_i) = len(A_i) = \rho_{\min} = len(A_j) \geq len(A_j \cap D_i) = V_i(A_j)$ for each $i, j \in S$. The number of cuts made by AllocateInterval$(S, X, \mathcal{D}_S)$ is $s - 1$. A perfect matching of a bipartite graph in Lemma 9 with $2s$ vertices can be obtained in $O(s^3)$ time (a much faster algorithm can be obtained based on the structures of this bipartite graph and a greedy plane sweep method). Thus, (a) – (c) hold in this case.

If $X$ is separable, then (a) – (c) can be shown by induction on the number of recursive calls of CutMinInterval$(\cdot,\cdot,\cdot)$ in CutMinInterval$(S, X, \mathcal{D}_S)$ in total.

Assume that (a) – (c) hold when CutMinInterval$(S, X, \mathcal{D}_S)$ contains at most $k \geq 0$ recursive calls in total. Now we consider when CutMinInterval$(S, X, \mathcal{D}_S)$ contains $k + 1$ recursive calls in total. Thus, $X$ has a separable interval and CutMinInterval$(S, X, \mathcal{D}_S)$ contains $J$ recursive calls CutMinInterval$(S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)})$ $(j = 1, 2, \ldots, J)$ each for cake $Z_j$ which is a minimal interval of minimum density $\rho_{\min}$ by Lemma 12 and a recursive call CutMinInterval$(S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'})$ for cake $X'^{(S)}$ which is a minimal interval of minimum density $\rho_{\min}$ by Lemma 13. Note that, each of CutMinInterval$(S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)})$ $(j = 1, 2, \ldots, J)$ and CutMinInterval$(S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'})$ has at most $k$ recursive calls in total. By the induction hypothesis, CutMinInterval$(S(Z_j), Z_j, \mathcal{D}'_{S(Z_j)})$ finds an allocation $(A_i : i \in S(Z_j))$ of $Z_j$ to players $S(Z_j)$ such that $A_i \subseteq D'_i \in \mathcal{D}'_{S(Z_j)}$ (thus, $A_i \subseteq D_i \in \mathcal{D}_S$), $len(A_i) = \rho_{\min}$ for each $i \in S(Z_j)$ and $\sum_{i\in S(Z_j)} A_i = Z_j$ for each $j = 1, 2, \ldots, J$. Furthermore, CutMinInterval$(S', X'^{(S)}, \mathcal{D}'^{(S)}_{S'})$ finds an allocation $(A'^{(S)}_i : i \in S')$ of $X'^{(S)}$ to players $S'$ such that $A'^{(S)}_i \subseteq D'^{(S)}_i \in \mathcal{D}'^{(S)}_{S'}$, $len(A'^{(S)}_i) = \rho_{\min}$ for each $i \in S'$ and $\sum_{i\in S'} A'^{(S)}_i = X'^{(S)}$. By inverse shrinking of $[z^*_0, z^*_J)$, we have the allocation $(A_i : i \in S')$ of $X' = X \setminus [z^*_0, z^*_J)$ to players $S'$ such that $A_i \subseteq D'_i \in \mathcal{D}'_{S'}$ (thus, $A_i \subseteq D_i \in \mathcal{D}_S$), $len(A_i) = \rho_{\min}$ for each $i \in S'$ and $\sum_{i\in S'} A_i = X'$.

Thus, we can obtain that CutMinInterval$(S, X, \mathcal{D}_S)$ returns an allocation $(A_i : i \in S)$ of $X$ to players $S$ such that $A_i \subseteq D_i \in \mathcal{D}_S$, $len(A_i) = \rho_{\min}$ for each $i \in S$ and $\sum_{i\in S} A_i = X$. Since $A_i \subseteq D_i$, $len(A_i) = \rho_{\min}$ and $V_i(A_i) = len(A_i \cap D_i) = len(A_i) = \rho_{\min} = len(A_j) \geq len(A_j \cap D_i) = V_i(A_j)$ for each $i, j \in S$, the allocation $(A_i : i \in S)$ is envy-free. Thus, (a) is obtained.

Similalry, (b) and (c) can be obtained. For example, for (c), by induction hypothesis, we have the number of cuts on each $Z_j$ is at most $2|S(Z_j)| - 2$, the number of cuts on $X'^{(S)}$ (on $X'$) is at most $2|S'| - 2$ and the number of cuts on $X$ to obtain all $Z_j$ $(j = 1, 2, \ldots, J)$ is exactly $J + 1$. Thus, in total, the number of cuts is at most $\sum_{j=1}^{J}(2|S(Z_j)| - 2) + (2|S'| - 2) + (J + 1) = 2s - J - 1 \leq 2s - 2$ since $J \geq 1$. For (b), note that, all the separable intervals can be found in $O(s^2)$ time, since a separable interval is a minimal interval with respect to valuations and there are at most $s^2$ minimal intervals $[y', y'')$ in $X$ with respect to valuations (since $y'$ is the left endpoint of a valuation interval in $\mathcal{D}_S$ and $y''$ is the right endpoint of a valuation interval in $\mathcal{D}_S$ and there are exactly $s$ valuation intervals in $\mathcal{D}_S$). Thus, $O(s^3)$ time can be obtained by a naive analysis. ◀

We have the following lemma for CutMaxInterval$(R, H, \mathcal{D}_R)$ and CutCake$(P, D, \mathcal{D}_P)$.

▶ **Lemma 15.** *CutMaxInterval$(R, H, \mathcal{D}_R)$ returns an envy-free allocation $(A_i : i \in R)$ of $H$ to players $R$ with $A_i \subseteq D_i \in \mathcal{D}_R$, $len(A_i) = \rho_{\min}$ for each $i \in R$ and $\sum_{i\in R} A_i = H$, and runs in $O(r^3)$ time where $r = |R|$, and the number of cuts made over $H$ is at most $2r - 2$.*

*CutCake$(P, D, \mathcal{D}_P)$ returns an envy-free and truthful allocation $(A_i : i \in P)$ of $D$ to players $P$ such that $A_i \subseteq D_i \in \mathcal{D}_P$, $len(A_i) \geq \rho_{\min}$ for each $i \in P$ and $\sum_{i\in P} A_i = D$, and runs in $O(p^3)$ time where $p = |P|$, and the number of cuts made over $D$ is at most $2p - 2$.*

This Lemma can be proved by almost the same argument as in Proof of Lemma 14. Actually, for cake $H$ which is a maximal interval of minimum density $\rho_{\min}$, players $R$, and solid valuation intervals $\mathcal{D}_R = (D_i : i \in R)$ with $\bigcup_{i \in R} D_i = H$, we can show that CutMaxInterval$(R, H, \mathcal{D}_R)$ satisfies the following (a) – (c).

**(a)** CutMaxInterval$(R, H, \mathcal{D}_R)$ returns an envy-free allocation $(A_i : i \in R)$ of $H$ to players $R$ with $A_i \subseteq D_i \in \mathcal{D}_R$, $len(A_i) = \rho_{\min}$ for each $i \in R$ and $\sum_{i \in R} A_i = H$.

**(b)** CutMaxInterval$(R, H, \mathcal{D}_R)$ runs in $O(r^3)$ time where $r = |R|$.

**(c)** The number of cuts made over $H$ by CutMaxInterval$(R, H, \mathcal{D}_R)$ is at most $2r - 2$.

Similarly, for cake $D$, players $P$, and solid valuation intervals $\mathcal{D}_P = (D_i : i \in P)$ with $\bigcup_{i \in P} D_i = D$, CutCake$(P, D, \mathcal{D}_P)$ satisfies the following (a) – (d).

**(a)** CutCake$(P, D, \mathcal{D}_P)$ returns an envy-free allocation $(A_i : i \in P)$ of $D$ to players $P$ such that $A_i \subseteq D_i \in \mathcal{D}_P$, $len(A_i) \geq \rho_{\min}$ for each $i \in P$ and $\sum_{i \in P} A_i = D$.

**(b)** CutCake$(P, D, \mathcal{D}_P)$ runs in $O(p^3)$ time where $p = |P|$.

**(c)** The number of cuts made over $D$ by CutCake$(P, D, \mathcal{D}_P)$ is at most $2p - 2$.

**(d)** The allocation $(A_i : i \in R)$ returned by CutCake$(P, D, \mathcal{D}_P)$ is truthful.

Note that, (d) can be obtained in a similar way as in papers [9] and [18], since (a) holds. By Lemmas 6, 7, and 15, we have the following theorem.

▶ **Theorem 16.** *Mechanism 1 correctly finds, in $O(n^3)$ time, an envy-free and truthful allocation $A_N = (A_i : i \in N)$ of cake $C$ to $n$ players $N$ with $A_i \subseteq C_i$ for each player $i \in N$ and $\sum_{i \in N} A_i = C$, and the number of cuts made over $C$ by Mechanism 1 is at most $2n - 2$.*

## 5 Concluding Remarks

We gave a new envy-free and truthful mechanism with a small number of cuts (i.e, using at most $2(n-1)$ cuts) based on the ideas of the structural properties of intervals of minimum density and of separable intervals in a minimal interval of minimum density. Our mechanism can be extended to the related pie-cutting problem [7]. We believe that, if we replace

**if** $X = [x', x'')$ is nonseparable **then** AllocateInterval$(S, X, \mathcal{D}_S)$;
**else** // there is a separable interval in $X$

in Procedure CutMinInterval$(S, X, \mathcal{D}_S)$ (where $S = N(X)$ and $\mathcal{D}_S = \mathcal{C}_{N(X)}$) with

construct the bipartite graph $G_{N(X)}$ defined in Lemma 9;
**if** $G_{N(X)}$ has a perfect matching **then** AllocateInterval$(S, X, \mathcal{D}_S)$;
**else** // there is a separable interval in $X$

then we can decrease the number of cuts in many cases without increasing time complexity since we can decide whether $G_{N(X)}$ has a perfect matching quite efficiently based on the structures of $G_{N(X)}$ and a greedy plane sweep method. Note that, if $G_{N(X)}$ has no perfect matching then $X = [x', x'')$ is separable by Lemma 9. By this modification, for the example in Figure 3, we have allocation $(A_1 = [0, 0.04) + [0.34, 0.35) + [0.85, 0.9)$, $A_2 = [0.04, 0.14)$, $A_3 = [0.14, 0.24)$, $A_4 = [0.24, 0.34)$, $A_5 = [0.35, 0.45)$, $A_6 = [0.45, 0.55)$, $A_7 = [0.55, 0.65)$, $A_8 = [0.65, 0.75)$, $A_9 = [0.75, 0.85)$, $A_{10} = [0.9, 1))$ with 11 cuts, while without modification, we have allocation $(A_1 = [0, 0.04) + [0.34, 0.35) + [0.85, 0.9)$, $A_2 = [0.05, 0.15)$, $A_3 = [0.15, 0.25)$, $A_4 = [0.04, 0.05) + [0.25, 0.34)$, $A_5 = [0.39, 0.49)$, $A_6 = [0.49, 0.59)$, $A_7 = [0.35, 0.39) + [0.59, 0.65)$, $A_8 = [0.67, 0.77)$, $A_9 = [0.65, 0.67) + [0.77, 0.85)$, $A_{10} = [0.9, 1))$ with 14 cuts.

We also believe that our mechanism could be extended for solving the more general cake-cutting problem where valuation functions of players are all piecewise uniform.

## References


1. Reza Alijani, Majid Farhadi, Mohammad Ghodsi, Masoud Seddighin, and Ahmad S. Tajik. Envy-free mechanisms with minimum number of cuts. In *31st AAAI Conference on Artificial Intelligence, 2017*, pages 312–318, 2017.
2. Takao Asano and Hiroyuki Umeda. An envy-free and truthful mechanism for the cake-cutting problem. In *RIMS Kôkyûroku 2154, Kyoto University, April, 2020*, pages 54–91, 2020.
3. Haris Aziz and Simon Mackenzie. A discrete and bounded envy-free cake cutting protocol for any number of agents. In *57th Annual Symposium on Foundations of Computer Science, 2016*, pages 416–427, 2016.
4. Haris Aziz and Chun Ye. Cake cutting algorithms for piecewise constant and piecewise uniform valuations. In *10th International Conference on Web and Internet Economics, 2014*, pages 1–14, 2014.
5. Xiaohui Bei, Ning Chen, Xia Hua, Biaoshuai Tao, and Endong Yang. Optimal proportional cake cutting with connected pieces. In *26th AAAI Conference on Artificial Intelligence, 2012*, pages 1263–1269, 2012.
6. Steven J. Brams, Michal Feldman, John K. Lai, Jamie Morgenstern, and Ariel D. Procaccia. On maxsum fair cake divisions. In *26th AAAI Conference on Artificial Intelligence, 2012*, pages 1285–1291, 2012.
7. Steven J. Brams, Michael Jones, and Christian Klamler. Proportional pie-cutting. *International Journal of Game Theory*, 36:353–367, 2008.
8. Ioannis Caragiannis, John K. Lai, and Ariel D. Procaccia. Towards more expressive cake cutting. In *22nd International Joint Conference on Artificial Intelligence, 2011*, pages 127–132, 2011.
9. Yiling Chen, John K. Lai, David C. Parkes, and Ariel D. Procaccia. Truth, justice, and cake cutting. *Games and Economic Behavior*, 77:284–297, 2013.
10. Xiaotie Deng, Qi Qi, and Amin Saberi. Algorithmic solutions for envy-free cake cutting. *Operations Research*, 60:1461–1476, 2012.
11. Jeff Edmonds and Kirk Pruhs. Balanced allocations of cake. In *47th Annual Symposium on Foundations of Computer Science, 2006*, pages 623–634, 2006.
12. Paul W. Goldberg, Alexandros Hollender, and Warut Suksompong. Contiguous cake cutting: hardness results and approximation algorithms. In *34th AAAI Conference on Artificial Intelligence, 2020*, pages 1990–1997, 2020.
13. P. Hall. On representations of subsets. *Journal of London Mathematical Society*, 10:26–30, 1934.
14. David Kurokawa, John K. Lai, and Ariel D. Procaccia. How to cut a cake before the party ends. In *27th AAAI Conference on Artificial Intelligence, 2013*, pages 555–561, 2013.
15. Avishay Maya and Noam Nisan. Incentive compatible two player cake cutting. In *8th International Conference on Web and Internet Economics, 2012*, pages 170–183, 2012.
16. Ariel D. Procaccia. Cake cutting: not just child's play. *Commun. ACM*, 56:78–87, 2013.
17. Jack M. Robertson and William A. Webb. *Cake Cutting Algorithms: Be Fair If you Can.* A.K. Peters, 1998.
18. Masoud Seddighin, Majid Farhadi, Mohammad Ghodsi, Reza Alijani, and Ahmad S. Tajik. Expand the shares together: envy-free mechanisms with a small number of cuts. *Algorithmica*, 81:1728–1755, 2019.
19. Hugo Steinhaus. The problem of fair division. *Econometrica*, 16:101–104, 1948.
20. Walter Stromquist. How to cut a cake fairly. *American Mathematical Monthly*, 87:640–644, 1980.
21. Walter Stromquist. Envy-free cake divisions cannot be found by finite protocols. *The Electronic Journal of Combinatorics*, 15:#R11 (pp.1–10), 2008.
22. Francis E. Su. Rental harmony: Sperner's lemma in fair division. *American Mathematical Monthly*, 106:930–942, 1999.