

Approximating k -Connected m -Dominating Sets

Zeev Nutov

The Open University of Israel, Raanana, Israel
nutov@openu.ac.il

Abstract

A subset S of nodes in a graph G is a **k -connected m -dominating set** ((k, m) -c ds) if the subgraph $G[S]$ induced by S is k -connected and every $v \in V \setminus S$ has at least m neighbors in S . In the k -CONNECTED m -DOMINATING SET ((k, m) -CDS) problem the goal is to find a minimum weight (k, m) -c ds in a node-weighted graph. For $m \geq k$ we obtain the following approximation ratios. For general graphs our ratio $O(k \ln n)$ improves the previous best ratio $O(k^2 \ln n)$ of [26] and matches the best known ratio for unit weights of [34]. For unit disk graphs we improve the ratio $O(k \ln k)$ of [26] to $\min\{\frac{m}{m-k}, k^{2/3}\} \cdot O(\ln^2 k)$ – this is the first sublinear ratio for the problem, and the first polylogarithmic ratio $O(\ln^2 k)/\epsilon$ when $m \geq (1 + \epsilon)k$; furthermore, we obtain ratio $\min\{\frac{m}{m-k}, \sqrt{k}\} \cdot O(\ln^2 k)$ for uniform weights. These results are obtained by showing the same ratios for the SUBSET k -CONNECTIVITY problem when the set of terminals is an m -dominating set.

2012 ACM Subject Classification Theory of computation \rightarrow Design and analysis of algorithms

Keywords and phrases k -connected graph, m -dominating set, approximation algorithm, rooted subset k -connectivity, subset k -connectivity

Digital Object Identifier 10.4230/LIPIcs.ESA.2020.73

Acknowledgements I thank an anonymous referee for many useful comments.

1 Introduction

All graphs in this paper are assumed to be simple, unless stated otherwise. A (simple) graph is **k -connected** if it has k pairwise internally node disjoint paths between every pair of its nodes; in this case the graph has at least $k + 1$ nodes. A subset S of nodes in a graph G is a **k -connected set** if the subgraph $G[S]$ induced by S is k -connected; S is an **m -dominating set** if every $v \in V \setminus S$ has at least m neighbors in S . If S is both k -connected and m -dominating set then S is a **k -connected m -dominating set**, or **(k, m) -c ds** for short. A graph is a **unit disk graph** if its nodes can be located in the Euclidean plane such that there is an edge between u and v iff the Euclidean distance between u and v is at most 1. We consider the following problem for $m \geq k$ both in general graphs and in unit disk graphs.

k -CONNECTED m -DOMINATING SET ((k, m) -CDS)

Input: A graph $G = (V, E)$ with node weights $\{w_v : v \in V\}$ and integers k, m .

Output: A minimum weight (k, m) -c ds $S \subseteq V$.

The problem generalizes several classic problems including SET-COVER ($k = 0, m = 1$), SET-MULTICOVER ($k = 0$), and CONNECTED DOMINATING SET ($k = m = 1$). The CONNECTED DOMINATING SET problem is closely related to the NODE WEIGHTED STEINER TREE problem, and both problems admit a tight ratio $O(\log n)$ [16, 12, 13]. In unit disk graphs, the problem is NP-hard [5], admits a PTAS for unit weights [3], and ratio $3 + 2.5\rho + \epsilon$ for arbitrary weights [33, 35], where ρ is the ratio for the edge-weighted STEINER TREE problem in general graphs. The (k, m) -CDS problem models (fault tolerant) virtual backbones in networks [7, 6], and it was studied extensively, c.f. [1, 3, 10, 12, 13, 21, 32, 26, 33, 31, 34, 35, 36] for the case $m \geq k$ and [2, 29] for the case $k = 2, m = 1$. For further motivation and history



© Zeev Nutov;

licensed under Creative Commons License CC-BY
28th Annual European Symposium on Algorithms (ESA 2020).

Editors: Fabrizio Grandoni, Grzegorz Herman, and Peter Sanders; Article No. 73; pp. 73:1–73:14

Leibniz International Proceedings in Informatics



LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

73:2 Approximating k -Connected m -Dominating Sets

survey we refer the reader to recent papers of Zhang, Zhou, Mo, and Du [31] and of Fukunaga [10], where they obtained in unit disk graphs ratios $O(k^3 \ln k)$ and $O(k^2 \ln k)$, respectively. This was improved to $O(k \ln k)$ in [26], where is also given ratio $O(k^2 \ln n)$ in general graphs.

Our main results is:

- **Theorem 1.** (k, m) -CDS with $m \geq k$ admits the following approximation ratios:
 - $O(k \ln n)$ in general graphs.
 - $\min \left\{ \frac{m}{m-k}, k^{2/3} \right\} \cdot O(\ln^2 k)$ in unit disk graphs.
 - $\min \left\{ \frac{m}{m-k}, \sqrt{k} \right\} \cdot O(\ln^2 k)$ in unit disk graphs with unit weights.

For general graphs our ratio $O(k \ln n)$ improves the previous ratio $O(k^2 \ln n)$ of [26] and matches (while using totally different techniques) the best known ratio for unit weights of Zhang et. al. [34]. For unit disk graphs our ratio $\min \left\{ \frac{k}{m-k}, k^{2/3} \right\} \cdot O(\ln^2 k)$ improves the previous best ratio $O(k \ln k)$ of [26]; this is the first sublinear ratio for the problem, and for any constant $\epsilon > 0$ and $m = k(1 + \epsilon)$ the first polylogarithmic ratio $O(\ln^2 k)/\epsilon$.

Let us say that a graph with a set T of terminals and a root $r \in T$ is **k - (T, r) -connected** if it has k internally node disjoint rt -paths for every $t \in T \setminus \{r\}$. Similarly, a graph is **k - T -connected** if it has k internally node disjoint st -paths for every $s, t \in T$. A reason why the case $m \geq k$ is easier than the case $m < k$ is given in the following statement (a proof can be found in many papers, c.f. [31, 10, 26]).

► **Lemma 2.** *Let T be a k -dominating set in a graph $H = (U, F)$. If H is k - (T, r) -connected then H is k - (U, r) -connected; if H is k - T -connected then H is k -connected.*

The above lemma implies that in the case $m \geq k$ (k, m) -CDS has the property that the union $T \cup S$ of a partial solution T that is just m -dominating and a node set S such that $G[T \cup S]$ is T - k -connected, is a feasible solution – this enables to construct the solution iteratively. Specifically, most algorithms for the case $m \geq k$ start by computing just an m -dominating set T ; the best ratios for m -DOMINATING SET are $\ln(\Delta + m)$ in general graphs [8] and $O(1)$ in unit disk graphs [10], where Δ is the maximum degree in G . By invoking just these ratios, Lemma 2 enables to reduce (k, m) -CDS with $m \geq k$ to following (node weighted) problem:

SUBSET k -CONNECTIVITY
Input: A graph $G = (V, E)$ with node-weights $\{w_v : v \in V\}$, a set $T \subseteq V$ of terminals, and an integer k .
Output: A minimum weight k - T -connected subgraph of G .

The ratios for this problem are usually expressed in terms of the best known ratio β for the following problem (in both problems we assume w.l.o.g. that $w_v = 0$ for all $v \in T$):

ROOTED SUBSET k -CONNECTIVITY
Input: A graph $G = (V, E)$ with node-weights $\{w_v : v \in V\}$, a set $T \subseteq V$ of terminals, a root node $r \in T$, and an integer k .
Output: A minimum weight k - (T, r) -connected subgraph of G .

We refer the reader to recent surveys [28, 27] on approximation algorithm for node-connectivity problems and to [16, 22, 25, 9, 24] on approximation algorithm for various node-weighted connectivity problems and their generalizations. Currently, $\beta = O(k^2 \ln |T|)$ [22]. From previous work it can be deduced that SUBSET k -CONNECTIVITY with $|T| \geq k$

admits ratio $\beta + k^2$. Add a new root node r connected to a set $R \subseteq T$ of k nodes by edges of cost zero. Then compute a β -approximate solution to the obtained ROOTED SUBSET k -CONNECTIVITY instance. Finally, augment this solution by computing for every $u, v \in R$ a min-weight set of k internally disjoint uv -paths. For the (k, m) -CDS problem with $m \geq k$ this already gives ratio $\beta + k^2 = O(k^2 \ln |T|)$ in general graphs. For the special case when T is a k -dominating set the ratio $\beta + k^2$ was improved in [26] to $\beta + k - 1$, since then in the final step it is sufficient to compute a min-weight set of k internally disjoint uv -paths only for pairs that form a forest on R (by the Critical Cycle Theorem of Mader [20]).

We now consider unit disk graphs. Zhang et al. [31] showed that any k -connected unit disk graph has a k -connected spanning subgraph of maximum degree $\leq 5k$. This implies that the node weighted case is reduced with a loss of factor $O(k)$ to the case of node induced edge costs – when $c_{uv} = w_u + w_v$ for every edge $e = uv \in E$. The edge costs version of SUBSET k -CONNECTIVITY admits ratio $O(k^2 \ln k)$, which gives ratio $O(k^3 \ln k)$ for (k, m) -CDS with $m \geq k$ in unit disk graphs. Independently, Fukunaga [10] obtained ratio $O(k^2 \ln k)$ using a different approach – he considered the ROOTED SUBSET CONNECTIVITY AUGMENTATION problem, when $G[T]$ is ℓ - (T, r) -connected and we seek a minimum weight $S \subseteq V \setminus T$ such that $G[T \cup S]$ is $(\ell + 1)$ - (T, r) -connected. In [22] it is shown that the augmentation problem decomposes into $O(k)$ “uncrossable” subproblems (for precise definitions, see Definition 20 in Section 3), and Fukunaga [10] designed a primal-dual $O(1)$ -approximation algorithm for such an uncrossable subproblem in unit disk graphs. This gives ratio $O(\ell)$ for ROOTED SUBSET CONNECTIVITY AUGMENTATION in unit disk graphs. Furthermore, using the so called “backward augmentation analysis” [11] Fukunaga showed that since his approximation is w.r.t. an LP, then sequentially increasing the T -connectivity by 1 invokes only a factor of $O(\ln k)$, thus obtaining ratio $O(k \ln k)$ for ROOTED SUBSET CONNECTIVITY AUGMENTATION. He then combined this result with a decomposition of the SUBSET k -CONNECTIVITY problem into k ROOTED SUBSET k -CONNECTIVITY problems, and obtained ratio $O(k^2 \ln k)$. As was mentioned, in [26] it is proved that ratio β for ROOTED SUBSET k -CONNECTIVITY implies ratio $\beta + k - 1$ for (k, m) -CDS with $m \geq k$, which improves the ratio to $O(k \ln k)$.

However, it seems that previous reductions and methods alone do not enable to obtain ratio better than $O(k^2 \ln |T|)$ in general graphs, or a sublinear ratio in unit disk graphs. These algorithms rely on the ratios and decompositions for the ROOTED/SUBSET k -CONNECTIVITY problems from [22, 23, 19], but these do not consider the specific feature relevant to (k, m) -CDS with $m \geq k$ – that the set T of terminals is a k -dominating set; note that then SUBSET k -CONNECTIVITY is equivalent to the problem of finding the lightest k -connected subgraph containing T , by Lemma 2. Here we change this situation by asking the following question:

If the set T of terminals is an m -dominating set with $m \geq k$, what approximation ratio can be achieved for (node weighted) SUBSET k -CONNECTIVITY?

Our answer to this question is given in the following theorem, which is of independent interest, and note that it implies Theorem 1.

► **Theorem 3.** *The (node weighted) SUBSET k -CONNECTIVITY problem such that T is an m -dominating set with $m \geq k$ admits the following approximation ratios: $O(k \ln n)$ in general graphs, $O(\ln^2 k) \cdot \min \left\{ \frac{m}{m-k+1}, k^{2/3} \right\}$ in unit disk graphs, and $O(\ln^2 k) \cdot \min \left\{ \frac{m}{m-k+1}, \sqrt{k} \right\}$ in unit disk graphs with unit weights.*

In the proof of Theorem 3 we use several results and ideas from previous works [22, 23, 31, 10, 26]. As was mentioned, the best ratios for the SUBSET k -CONNECTIVITY are derived via reductions of [23, 26] from the ratios for the ROOTED SUBSET k -CONNECTIVITY problem,

so we will consider the latter problem; the currently best known ratio for this problem is $O(k^2 \ln |T|)$ [22]. The algorithm of [22] has k iterations, where at iteration $\ell = 0, \dots, k - 1$ it considers the augmentation problem of increasing the connectivity from ℓ to $\ell + 1$. This is equivalent to “covering” a certain family \mathcal{F} of “deficient sets” (see Section 2 for precise definitions), and the algorithm of [22] decomposes this problem into $O(\ell)$ uncrossable family covering problems; the ratio for covering each uncrossable family is $O(\ln n)$ in general graphs [22] and $O(1)$ in unit disk graphs [10].

However, a more careful analysis of the [22] algorithm reveals that in fact the number of uncrossable families is $O(\ell/q) + 1$, where q is the minimum number of terminals in a deficient set. Instances with $q \geq \ell + 1$ are often called “ T -independence-free” (see Lemmas 6 and 7). In T -independence-free instances the entire family of deficient sets is uncrossable, hence such instances admit ratio $O(\ln n)$ in general graphs and $O(1)$ in unit disk graphs. The algorithm of [22] has an “inflation phase” that works towards reaching $q \geq \ell + 1$ – to make the instance T -independence-free, by repeatedly covering $O(\ell/q)$ uncrossable families to double q . Hence if q_0 is the initial value of q , the total number of uncrossable families that the algorithm covers is 1 plus order of $\frac{\ell}{q_0} (1 + \frac{1}{2} + \frac{1}{4} + \dots) = O(\ell/q_0)$. Note that a large part of the uncrossable families are covered when q is small. One of our contributions is designing different “lighter” inflation algorithms for increasing the parameter q . These algorithms just aim to cover the inclusion minimal deficient sets (a.k.a. “cores” – for precise definition see Definition 14), by adding a light set S of nodes, and then add S to the set T of terminals; if T is a k -dominating set then adding any set S to T does not make the problem harder, by Lemma 2.

Our algorithms for covering inclusion minimal deficient sets reduce the problem to a set covering type problem. In the case of general graphs the reduction is to a special case considered in [18] of the SUBMODULAR COVERING problem; the ratio invoked by this procedure is only $O(\ln n)$ and if we apply it $p = \max\{2k - m - 1, 1\}$ times then we get $q \geq m - \ell + p(k - \ell) \geq k$ for all $\ell = 0, \dots, k - 1$. In fact, we apply this procedure before considering the augmentation problems, but it guarantees that $q \geq k$ through all augmentation iterations. The same procedure applies in the case of unit disk graphs, but to avoid the dependence on n in the ratio we use a different procedure. Specifically, we use the result of Zhang et. al. [31] that a minimally k -connected unit disk graph has maximum degree $\leq 5k$, to reduce the problem of covering the family of deficient sets to the SET COVER problem with soft capacities. This approach gives ratio $\min\left\{\frac{m}{m-k}, k^{2/3}\right\} \cdot O(\ln^2 k)$.

Our algorithms are simple and combinatorial. We omit the running time analysis, but it is polynomial and dominated by that of finding k times an approximate solution to ROOTED SUBSET k -CONNECTIVITY in T -independence-free instances [22, 10]. Apart from significantly improving approximation ratios for (k, m) -CDS and special instances of SUBSET/ROOTED k -CONNECTIVITY, which are extensively studied important fundamental problems, we also have the following contribution. The framework of approximating k -connectivity problems via independence-free graphs was very successful in [22] and [4, 30] (see also [14] for the first paper that used this framework, for an exact algorithm), but these are the only papers that succeeded to apply it. In general, it is not clear how to find a cheap partial solution to make the residual instance independence-free. In [22] this was achieved by “merging” deficient sets, so that at each iteration the minimum number of terminals in a deficient set is doubled. The method used in [4, 30] constructs an independence free instance in just two iterations, but it is tailor made for the k -CONNECTED SUBGRAPH problem considered there. We use a different method that reaches an almost independence-free instance by just repeatedly solving

a SET COVER (or a SUBMODULAR COVER) problem. We believe that the method used here can be also applied for other problems, maybe for activation network design problems [24, 9], that generalize node weighted network design problems.

In the rest of the paper we prove Theorem 3; Section 2 considers general graphs and Section 3 considers unit disk graphs.

2 General graphs

In order to prove our results we need to characterize k -connectivity in terms of “cuts” rather than in terms of paths. While edge-cuts of a graph correspond to node subsets, a natural way to represent a node-cut of a graph is by a pair of sets called a “biset”.

► **Definition 4.** An ordered pair $\mathbb{A} = (A, A^+)$ of subsets of V with $A \subseteq A^+$ is called a **biset**; the set $\partial\mathbb{A} = A^+ \setminus A$ is called the **cut** of \mathbb{A} . We say that \mathbb{A} is a **(T, r) -biset** if $A \cap T \neq \emptyset$ and $r \in V \setminus A^+$. For an edge set/graph J let $d_J(\mathbb{A})$ denote the number of edges in J that have one end in A and the other in $V \setminus A^+$.

Let $\kappa_G(t, r)$ denote the maximum number of pairwise internally disjoint tr -paths in G . In biset terms, the node connectivity version of Menger’s Theorem (that applies also for non-simple graphs) says that $\kappa_G(t, r)$ equals $\min\{|\partial\mathbb{A}| : t \in A, r \in V \setminus A^+\}$ plus the number of tr -edges. Here $\partial\mathbb{A}$ is a node cut that separates t from r in the graph obtained from G by removing the tr -edges. It is not hard to verify that if J is an edge set that contains the tr -edges then $\kappa_G(t, r) = \min\{|\partial\mathbb{A}| : t \in A, r \in V \setminus A^+\} + d_J(\mathbb{A})$. In particular, we have:

► **Lemma 5.** Let $G = (V, E)$ be a graph and let $\{t, r\} \subseteq T \subseteq V$. Then

$$\kappa_G(t, r) = \min_{\mathbb{A}} \{|\partial\mathbb{A}| + d_{G[T]}(\mathbb{A}) : t \in A, r \in V \setminus A^+\}.$$

Here the nodes in $\partial\mathbb{A}$ and the edges in $G[T]$ that go from A to $V \setminus A^+$ form a “mixed” st -cut of G that contains both nodes and edges. The original Menger’s Theorem is the case $T = \{r, t\}$, while the case $T = V$ is also widely used in the literature, c.f. [28].

From Lemma 5 we get that G is k - (T, r) -connected iff $|\partial\mathbb{A}| + d_{G[T]}(\mathbb{A}) \geq k$ holds for every (T, r) -biset \mathbb{A} . Given a ROOTED SUBSET k -CONNECTIVITY instance, we say that a (T, r) -biset \mathbb{A} is a **deficient biset** if $|\partial\mathbb{A}| + d_{G[T]}(\mathbb{A}) \leq k - 1$. We use the algorithm from [22] for ROOTED SUBSET k -CONNECTIVITY. Two deficient bisets \mathbb{A}, \mathbb{B} are **T -independent** if $A \cap T \subseteq \partial\mathbb{B}$ or $B \cap T \subseteq \partial\mathbb{A}$. A ROOTED SUBSET k -CONNECTIVITY instance is **T -independence-free** if no pair of deficient bisets are T -independent. We have the following from previous work [22].

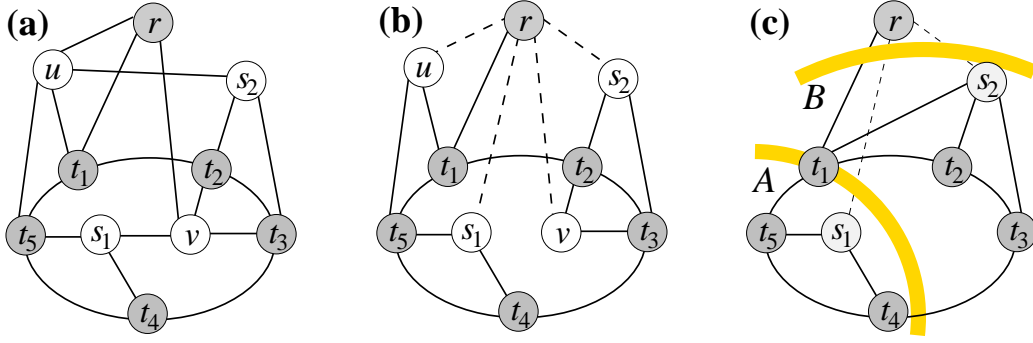
► **Lemma 6** ([22]). T -independence-free ROOTED SUBSET k -CONNECTIVITY instances admit ratio $O(k \ln |T|)$.

Clearly, a sufficient condition for an instance to be T -independence-free is:

► **Lemma 7.** If for a ROOTED SUBSET k -CONNECTIVITY instance $|A \cap T| \geq k$ holds for every deficient biset \mathbb{A} , then the instance is T -independence-free.

In the next two lemmas we show how to find an $O(k \ln n)$ -approximate set $S \subseteq V \setminus T$ such that adding S to T result in a T -independence-free instance.

► **Lemma 8** (Inflation Lemma for general graphs). There exists a polynomial time algorithm that given an instance of ROOTED SUBSET k -CONNECTIVITY finds $S \subseteq V \setminus T$ such that $|A \cap S| \geq k - (|\partial\mathbb{A}| + d_{G[T \cup S]}(\mathbb{A}))$ holds for any (T, r) -biset \mathbb{A} , and $w(S) = O(\ln \Delta) \cdot \text{opt}$.



■ **Figure 1** Illustration to Lemma 8 Algorithm 1 for $k = 3$. (a) ROOTED SUBSET k -CONNECTIVITY instance; nodes in T are shown by gray circles. (b) CENTERED ROOTED SUBSET k -CONNECTIVITY instance constructed in Algorithm 1; added edges are shown by dashed lines. (c) The set $S = \{s_1, s_2\}$ returned by the algorithm and two bisets \mathbb{A}, \mathbb{B} in $G[T \cup S]$ (the dashed edges do not belong to $G[T \cup S]$), where $A = \{t_5, t_4, s_1\}$, $\partial\mathbb{A} = \{t_1\}$ and $B = (T \cup S) \setminus \{r\}$, $\partial\mathbb{B} = \emptyset$.

Proof. The CENTERED ROOTED SUBSET k -CONNECTIVITY problem is a particular case of the ROOTED SUBSET k -CONNECTIVITY problem when all nodes of positive weight are neighbors of the root. This problem admits ratio $O(\ln \Delta)$ [18], where here Δ is the maximum degree of a neighbor of the root. We use this in our algorithm as follows (see Fig. 1):

■ **Algorithm 1** ($G = (V, E), w, r, T, k$).

-
- 1 construct a CENTERED ROOTED SUBSET k -CONNECTIVITY instance ($G' = (V, E'), w, T, r, k$), where G' is obtained from G by removing edges in $G[(V \setminus T) \cup \{r\}]$ and adding an rv -edge for each $v \in V \setminus T$ (see Fig. 1(a,b))
 - 2 compute an $O(\ln \Delta)$ -approximate solution $S \subseteq V \setminus T$ for the obtained CENTERED ROOTED SUBSET k -CONNECTIVITY instance (see Fig. 1(c))
 - 3 **return** S
-

Let S^* and S_c^* be optimal solutions to ROOTED SUBSET k -CONNECTIVITY and the constructed CENTERED ROOTED SUBSET k -CONNECTIVITY instances, respectively. For every $t \in T$ fix some set of k internally disjoint rt -paths in the graph $G[T \cup S^*]$, and obtain a set P_t by picking for each path the node in S^* that is closest to t on this path, if such a node exists. Let $P = \cup_{t \in T} P_t$. Then P is a feasible solution to the constructed CENTERED ROOTED SUBSET k -CONNECTIVITY instance, since for each $t \in T$, G' has $|P_t|$ internally disjoint rt -paths of length 2 each that go through P_t , and $k - |P_t|$ paths that have all nodes in T . Furthermore, since $P \subseteq S^*$, $w(P) \leq w(S^*)$. Thus $w(S_c^*) \leq w(P) \leq w(S^*)$, implying that $w(S) = O(\ln \Delta) \cdot w(S^*)$.

Now let \mathbb{A} be a (T, r) -biset on $T \cup S$. Then:

- $d_{G'[T \cup S]}(\mathbb{A}) = |A \cap S| + d_{G[T \cup S]}(\mathbb{A})$ by the construction.
 - $|\partial\mathbb{A}| + d_{G'[T \cup S]}(\mathbb{A}) \geq k$ since $G'[T \cup S]$ is k - (T, r) -connected.
- Combining we get that $|\partial\mathbb{A}| + d_{G[T \cup S]}(\mathbb{A}) + |A \cap S| \geq k$, as claimed. ◀

Note that Lemma 8 does *not* assume that T has any domination properties, and it does *not* imply that $G[T \cup S]$ has higher (T, r) -connectivity than $G[T]$ – see the example in Fig. 1. The lemma just states that for every biset \mathbb{A} in $G[T \cup S]$, $|A \cap S|$ is at least the “deficiency” $k - (|\partial\mathbb{A}| + d_{G[T \cup S]}(\mathbb{A}))$ of \mathbb{A} . E.g., in the example in Fig. 1(c) we have:

- $A \cap S = \{s_1\}$, $k - (|\partial\mathbb{A}| + d_{G[T \cup S]}(\mathbb{A})) = 3 - (1 + 1) = 1$.
- $B \cap S = \{s_1, s_2\}$, $k - (|\partial\mathbb{B}| + d_{G[T \cup S]}(\mathbb{B})) = 3 - (0 + 1) = 2$.

Hence if we add S to T and $T \leftarrow T \cup S$ will become the new set of terminals, then the new ROOTED SUBSET k -CONNECTIVITY instance will be “closer” to being T -independence-free than the original instance. And if also T is a k -dominating set (this is not the case in Fig. 1), then adding S to T does not increase the optimal solution value, by Lemma 2.

Our algorithms use the following simple procedure – Algorithm 2, that sequentially adds p sets S_1, \dots, S_p to an m -dominating set $T = T_0$ with $m \geq k$; in the case of general graphs considered in this section, each S_i is as in Lemma 8.

■ **Algorithm 2** ($G = (V, E), c, r, T = T_0, k, 1 \leq p \leq k - 1$).

```

1 for  $i = 1$  to  $p$  do
2    $T \leftarrow T \cup S_i$ 
3 return  $T$ 

```

► **Lemma 9.** *Suppose that we are given a ROOTED SUBSET k -CONNECTIVITY instance such that T is an m -dominating set in G with $m \geq k$. If at each iteration i at step 2 of Algorithm 2 we add to T a set $S = S_i$ as in Lemma 8, then at the end of the algorithm $w(T \setminus T_0) = O(p \ln \Delta) \cdot \text{opt}$, and $|A \cap T| \geq m - \ell + p(k - \ell)$ holds for any biset \mathbb{A} on T with $|\partial \mathbb{A}| + d_{G[T]}(\mathbb{A}) = \ell \leq k - 1$. In particular, if $p \geq \max\{2k - m - 1, 1\}$ then the resulting instance is T -independence-free.*

Proof. The bound $w(T \setminus T_0) = O(p \ln \Delta) \cdot \text{opt}$ follows from Lemma 2 and the bound $w(S) = O(\ln \Delta) \cdot \text{opt}$ in Lemma 8.

Let \mathbb{A} be a biset as in the lemma. Let $T_i = T_0 \cup S_1 \cup \dots \cup S_i$ be the set stored in T at the end of iteration i , where T_0 is the initial set. Applying Lemma 8 on T_{i-1} and S_i we get

$$|A \cap S_i| \geq k - (|\partial \mathbb{A} \cap T_{i-1}| + d_{G[T_i]}(\mathbb{A})) \geq k - (|\partial \mathbb{A}| + d_{G[T]}(\mathbb{A})) = k - \ell.$$

In particular $A \cap S_1 \neq \emptyset$. Any $v \in A \cap S_1$ has in $G[T]$ at least m neighbors in T_0 , and at most ℓ of them are not in A ; thus v has at least $m - \ell$ neighbors in $A \cap T_0$, so $|A \cap T_0| \geq m - \ell$. Since T_0, S_1, \dots, S_p are pairwise disjoint we get $|A \cap T| \geq |A \cap T_0| + \sum_{i=1}^p |A \cap S_i| \geq m - \ell + p(k - \ell)$. If $p \geq \max\{2k - m - 1, 1\}$ then $m - \ell + p(k - \ell) \geq k$; thus, by Lemma 7, the resulting instance is T -independence-free. ◀

The proof of the following known statement can be found in [17], and the second part follows from Mader’s Undirected Critical Cycle Theorem [20].

► **Lemma 10.** *Let $H_r = (U, F)$ be a k - (U, r) -connected graph and R the set of neighbors of r in H_r . The graph $H = H_r \setminus \{r\}$ can be made k -connected by adding a set J of new edges on R , and if J is inclusion minimal then J is a forest.*

Note that an inclusion minimal edge set J as in Lemma 10 can be computed in polynomial time, by starting with J being a clique on R and repeatedly removing from J an edge e if $H \cup (J \setminus e)$ remains k -connected.

Our algorithm for general graphs is as follows.

Algorithm 3 ($G = (V, E), w, T$) general graphs.

- 1 construct a graph G_r by adding to G and to T a new node r connected to a set $R \subseteq T$ of k nodes by a set $F_r = \{rv : v \in R\}$ of new edges
 - 2 apply the Lemma 9 algorithm with $p = \max\{2k - m - 1, 0\}$
 - 3 use the algorithm from Lemma 6 to compute an $O(k \ln n)$ -approximate set $S \subseteq V \setminus T$ such that $H_r = G_r[T \cup S]$ is k -(T, r)-connected
 - 4 let $H = H_r \setminus \{r\} = G[T \cup S]$ and let J be a forest of new edges on R as in Lemma 10 such that the graph $H \cup J$ is k -connected
 - 5 for every $uv \in J$ find a minimum weight node set P_{uv} such that $G[T \cup S \cup P_{uv}]$ has k internally disjoint uv -paths; let $P = \bigcup_{uv \in J} P_{uv}$
 - 6 return $T \cup S \cup P$
-

Except step 2, the algorithm is identical to the algorithm of [26] – the only difference is that step 2 improves the factor invoked by step 3. In [26] it is also proved that at the end of the algorithm $T \cup S \cup P$ is a k -connected set. The dominating terms in the ratio are invoked by steps 2 and 3, and they are both $O(k \ln n)$, while step 5 invokes just ratio $k - 1$; thus the overall ratio is $O(k \ln n)$.

This concludes the proof of Theorem 3 for general graphs.

3 Unit disk graphs

Our goal in this section is to prove the following:

► **Lemma 11.** *Consider a SUBSET k -CONNECTIVITY instance on a unit disk graph $G = (V, E)$ where T is an (ℓ, m) -cds in G (so $G[T]$ is ℓ -connected and every $v \in V \setminus T$ has at least m neighbors in T), $m \geq k \geq \ell + 1$. Then for any $1 \leq p \leq \ell + 1$ there exists a polynomial time algorithm that computes $S \subseteq V \setminus T$ such that $G[T \cup S]$ is $(\ell + 1)$ -connected and*

$$\frac{w(S)}{\text{opt}} = \frac{O(\ln k)}{k - \ell} \left(p + \frac{(m + p)^2}{(m + p - \ell)^2} \right).$$

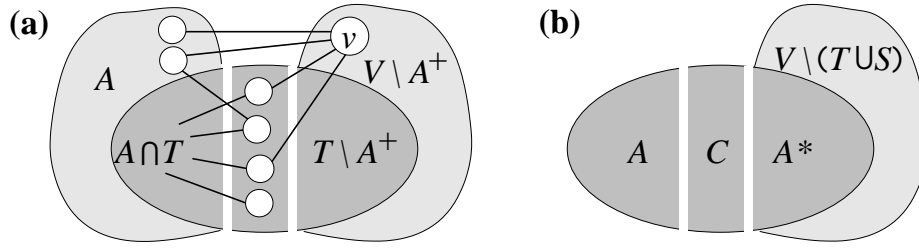
Furthermore, in the case of unit weights $\frac{w(S)}{\text{opt}} = \frac{O(\ln k)}{k - \ell} \left(p + \frac{m + p}{m + p - \ell} \right)$.

Let us show that Lemma 11 implies the unit disk part of Theorem 3. We can apply the Lemma 11 algorithm sequentially, starting with an $O(1)$ -approximate m -dominating set $T = T_0$, and at iteration $\ell = 0, \dots, k - 1$ add to T a set $S = S_\ell$ as in the lemma. In the case of arbitrary weights choosing $p = 1$ if $m - \ell \geq \ell^{2/3}$ and $p = \ell^{2/3}$ otherwise gives $\frac{w(S_\ell)}{\text{opt}} = \frac{O(\ln \ell)}{k - \ell} \min \left\{ \frac{m}{m - \ell}, \ell^{2/3} \right\}$. Then denoting $S = S_0 \cup \dots \cup S_{k-1}$ we get:

$$\frac{w(S)}{\text{opt}} = \sum_{\ell=0}^{k-1} \frac{O(\ln \ell)}{k - \ell} \min \left\{ \frac{m}{m - \ell}, \ell^{2/3} \right\} = O(\ln^2 k) \cdot \min \left\{ \frac{m}{m - k + 1}, k^{2/3} \right\}$$

In the case of unit weights, choosing $p = 1$ if $m - \ell \geq \sqrt{\ell}$ and $p = \sqrt{\ell}$ otherwise gives $\frac{w(S_\ell)}{\text{opt}} = \frac{O(\ln \ell)}{k - \ell} \min \left\{ \frac{m}{m - \ell}, \sqrt{\ell} \right\}$, and then by a similar analysis we get that

$$\frac{w(S)}{\text{opt}} = O(\ln^2 k) \cdot \min \left\{ \frac{m}{m - k + 1}, \sqrt{k} \right\}.$$



■ **Figure 2** (a) Illustration to the definition of a biset $\mathbb{A} \in \mathcal{D}_T$ and a node v that covers \mathbb{A} . (b) Illustration to the proof that if $G[T \cup S]$ is not $(\ell + 1)$ -connected then S does not cover \mathcal{D}_T .

In the rest of this section we prove Lemma 11, so let G , T , and ℓ be as in the lemma; in particular, $G[T]$ is ℓ -connected. Define the following family of biset on V (see Fig. 2)

$$\mathcal{D}_T = \{ \mathbb{A} : A \cap T \neq \emptyset \neq T \setminus A^+, d_{G[T]}(\mathbb{A}) = 0, |\partial \mathbb{A}| = \ell \} .$$

Let $\mathbb{A} \in \mathcal{D}_T$. Then $G[T] \setminus \partial \mathbb{A}$ is disconnected (since $d_{G[T]}(\mathbb{A}) = 0$), hence $\partial \mathbb{A} \cap T$ is a node cut of $G[T]$ that separates between the non-empty sets $A \cap T$ and $T \setminus A^+$. Since $|\partial \mathbb{A}| = \ell$ and since G is ℓ -connected, $\partial \mathbb{A}$ must be a minimum node cut of $G[T]$. Thus we have:

- **Corollary 12.** *A biset \mathbb{A} on V belongs to \mathcal{D}_T if and only if the following holds:*
- $\partial \mathbb{A} \subseteq T$ and $\partial \mathbb{A}$ is a minimum node cut (of size ℓ) of the graph $G[T]$.
 - $A \cap T$ is a union of some, but not all, connected components of $G[T] \setminus \partial \mathbb{A}$.

We say that a **node v covers $\mathbb{A} \in \mathcal{D}_T$** (see Fig. 2(a)) if $v \in \Gamma(A) \setminus T$, where $\Gamma(A) \subseteq V \setminus A$ is the set of neighbors of A in G ; $S \subseteq V \setminus T$ covers $\mathcal{F} \subseteq \mathcal{D}_T$ if every $\mathbb{A} \in \mathcal{F}$ is covered by some $v \in S$. Using Menger's Theorem and Lemma 2, one can see the following (see also [10, Section 5.2]).

► **Lemma 13.** *Let T be an $(\ell, \ell + 1)$ -cds in a graph $G = (V, E)$. Let $S \subseteq V \setminus T$. Then $G[T \cup S]$ is $(\ell + 1)$ -connected if and only if S covers \mathcal{D}_T .*

Proof. Let $\mathbb{A} \in \mathcal{D}_T$. Then $\partial \mathbb{A} \subseteq T$ is a node cut of size ℓ of the graph $G[T]$ that separates between the nonempty sets $A \cap T$ and $T \setminus A^+$. If S does not cover \mathbb{A} , then $S \cap \Gamma(A) = \emptyset$, and then $\partial \mathbb{A}$ is also a node cut of size ℓ of the graph $G[T \cup S]$ that separates between $A \cap T$ and $(T \setminus A^+) \cup S$; thus $G[T \cup S]$ is not ℓ -connected. Consequently, if $G[T \cup S]$ is $(\ell + 1)$ -connected then S must cover \mathcal{D}_T .

Suppose that $G[T \cup S]$ is not $(\ell + 1)$ -connected (see Fig. 2(b)). Then there is a node cut C of size ℓ of $G[T \cup S]$ that separates some $A \subseteq T \cup S$ from $A^* = (T \cup S) \setminus (A \cup C)$. Since T is an $(\ell + 1)$ -dominating set in $G[T \cup S]$ and since $|C| = \ell$, we must have $A \cap T \neq \emptyset$; otherwise, if $A \cap T = \emptyset$, then for any $u \in A$, there must exist $u' \in A^*$ with $uu' \in E$ since T is an $(\ell + 1)$ -dominating set, which contradicts that C is a node cut separating A from A^* . By a similar argument, $A^* \cap T \neq \emptyset$. Let $\mathbb{A} = (A, A \cup C)$. Then $A \cap T \neq \emptyset \neq T \setminus A^+$, $d_{G[T]}(\mathbb{A}) = 0$ (since $d_{G[T \cup S]}(\mathbb{A}) = 0$), and $|\partial \mathbb{A}| = |C| = \ell$. Thus $\mathbb{A} \in \mathcal{D}_T$. Furthermore, in $G[T]$, $C \cap T$ separates between the nonempty node sets $A \cap T$ and $T \setminus A^+$, hence $C = \partial \mathbb{A} = \Gamma(A) \subseteq T$, since $G[T]$ is ℓ -connected. Consequently, S does not cover \mathbb{A} , and the proof is complete. ◀

Thus we have the following LP-relaxation for the problem of finding a min-weight cover of \mathcal{D}_T (a similar LP was used by Fukunaga in [10, Section 5.2]):

$$\begin{aligned} \tau(\mathcal{D}_T) = \min \quad & \sum_{v \in V \setminus T} w_v x_v \\ \text{s.t.} \quad & \sum_{v \in \Gamma(A) \setminus T} x_v \geq 1 \quad \forall A \in \mathcal{D}_T \\ & x_v \geq 0 \quad \forall v \in V \setminus T \end{aligned}$$

73:10 Approximating k -Connected m -Dominating Sets

Note that if $\mathbb{A} \in \mathcal{D}_T$ then $\Gamma(A) \setminus T = \Gamma(A) \setminus \partial \mathbb{A}$, and thus the constraint $\sum_{v \in \Gamma(A) \setminus T} x_v \geq 1$ is equivalent to $\sum_{v \in \Gamma(A) \setminus \partial \mathbb{A}} x_v \geq 1$.

► **Definition 14.** We say that \mathbb{A} *contains* \mathbb{B} and write $\mathbb{A} \subseteq \mathbb{B}$ if $A \subseteq B$ and $A^+ \subseteq B^+$. Inclusion minimal members of a biset family \mathcal{F} are called **\mathcal{F} -cores**. The **intersection** and the **union** of two bisets \mathbb{A}, \mathbb{B} are defined by

$$\mathbb{A} \cap \mathbb{B} = (A \cap B, A^+ \cap B^+) \quad \mathbb{A} \cup \mathbb{B} = (A \cup B, A^+ \cup B^+).$$

► **Lemma 15.** Let \mathcal{C} be the family of \mathcal{D}_T -cores. Then $C \cap C' = \emptyset$ for any distinct $C, C' \in \mathcal{C}$ or $|\mathcal{C}| \leq \ell(\ell + 1)$.

Proof. let \mathcal{F} be the family of “deficient bisets” of the ℓ -connected graph $G[T]$; namely, \mathcal{F} is the family of those bisets \mathbb{A} on T such that $\partial \mathbb{A}$ is a minimum node cut (of size ℓ) of the graph $G[T]$, and A is a union of some, but not all, connected components of $G[T] \setminus \partial \mathbb{A}$. Note that from Corollary 12 it follows that:

- $\mathcal{F} = \{\mathbb{A} \in \mathcal{D}_T : A \subseteq T\}$.
- Every \mathcal{D}_T -core belongs to \mathcal{F} , hence \mathcal{C} coincides with the family of \mathcal{F} -cores.

Thus it is sufficient to prove the lemma with \mathcal{D}_T replaced by \mathcal{F} . The family \mathcal{F} has the following well known “symmetry” and “crossing” properties, c.f. [30].

- (i) If $\mathbb{A} \in \mathcal{F}$ then $(V \setminus A^+, V \setminus A) \in \mathcal{F}$.
- (ii) If $\mathbb{A}, \mathbb{B} \in \mathcal{F}$ and $A \cap B, T \setminus (A^+ \cup B^+)$ are non-empty then $\mathbb{A} \cap \mathbb{B}, \mathbb{A} \cup \mathbb{B} \in \mathcal{F}$.

In [[15], Lemma 3.5, Case II] it is proved that if $C \cap C' \neq \emptyset$ for some distinct $C, C' \in \mathcal{C}$ then there is $P \subseteq V$ with $|P| \leq \ell + 1$ such that $P \cap C \neq \emptyset$ for every $C \in \mathcal{C}$. In this case \mathcal{F} has at most $\ell(\ell + 1)$ distinct cores, since:

- For every $C \in \mathcal{C}$, there is $s \in P \cap C$, and there is $t \in P \cap (V \setminus C^+)$, by (i).
- For each $(s, t) \in P \times P$ there is at most one such C , by (ii).

Hence if $C \cap C' \neq \emptyset$ for some distinct $C, C' \in \mathcal{C}$ then $|\mathcal{C}| \leq |P|(|P| - 1) \leq \ell(\ell + 1)$. ◀

To obtain an approximation ratio that depends on k rather than on n , we will need the following result.

► **Theorem 16** (Zhang, Zhou, Mo, & Du [31]). *Any k -connected unit disk graph has a k -connected spanning subgraph of maximum degree $\leq 5k$.*

► **Lemma 17** (Inflation Lemma for unit disk graphs). *There exists a polynomial time algorithm that computes $S \subseteq V \setminus T$ that covers the family \mathcal{C} of cores of \mathcal{D}_T and $w(S) = O(\ln k) \cdot \frac{\text{opt}}{k - \ell}$.*

Proof. The problem of covering \mathcal{C} is essentially a (weighted) SET COVER problem where for each $v \in V \setminus T$ the corresponding set has weight w_v and consists of the cores covered by v . Then the greedy algorithm for SET COVER computes a solution of weight $O(\ln |\mathcal{C}|)$ times the value of the standard SET COVER LP

$$\begin{aligned} \tau(\mathcal{C}) = \min & \sum_{v \in V \setminus T} w_v x_v \\ \text{s.t.} & \sum_{v \in \Gamma(A) \setminus T} x_v \geq 1 \quad \forall A \in \mathcal{C} \\ & x_v \geq 0 \quad \forall v \in V \setminus T \end{aligned}$$

For any $S' \subseteq V \setminus T$ such that $G[T \cup S']$ is k -connected, any set A has at least $k - \ell$ neighbors in $G[T \cup S']$, hence if x' is a characteristic vector of S' then $\frac{x'}{k - \ell}$ is a feasible solution to the LP. Consequently, $\tau(\mathcal{C}) \leq \frac{\text{opt}}{k - \ell}$.

In the case $|\mathcal{C}| \leq \ell(\ell + 1)$ we get a solution of weight $O(\ln \ell) \cdot \tau(\mathcal{C}) = O(\ln \ell) \cdot \frac{\text{opt}}{k - \ell}$.

In the case $|\mathcal{C}| > \ell(\ell + 1)$, $C \cap C' = \emptyset$ for any $C, C' \in \mathcal{C}$, by Lemma 15. Then relying on Theorem 16 we modify this reduction such that every $v \in V \setminus T$ can cover at most $5k$ cores; this is essentially the SET COVER with (soft) capacities problem. Specifically, for each pair (v, J) where $v \in V \setminus S$ and J is a set of at most $5k$ edges incident to v , we add a new node v_J of weight w_v with corresponding copies of the edges in J . In the obtained SET COVER instance the maximum size of a set is at most $5k$, since the sets in $\{C : C \in \mathcal{C}\}$ are pairwise disjoint. Note that we do not need to construct this SET COVER instance explicitly to run the greedy algorithm – we just need to determine for each $v \in V$ the maximum number of at most $5k$ not yet covered cores that can be covered by v . Since the sets in $\{C : C \in \mathcal{C}\}$ are pairwise disjoint, this can be done in polynomial time. Note that during the greedy algorithm we may pick pairs (v, J) and (v, J') with distinct J, J' but with the same node v , but this only makes the solution lighter. Since in the SET COVER instance the maximum set size is $5k$, the computed solution has weight $O(\ln k) \cdot \tau$, where here τ is an optimal LP-value of the modified instance. Now we argue in the same way as before that $\tau \leq \frac{\text{opt}}{k-\ell}$. Consider a feasible solution $S' \subseteq V \setminus T$ and an edge J' such that $G[T] \cup S' \cup J'$ is a spanning k -connected subgraph of $G[T \cup S']$ and $\deg_{J'}(v) \leq 5k$ for all $v \in S'$; such J' exists by Theorem 16. Let x' be the characteristic vector of the pairs (v, J'_v) where $v \in S'$ and J'_v is the set of edges in J' incident to v . Any set A has at least $k - \ell$ neighbors in $G[T] \cup S' \cup J'$, hence $\frac{x'}{k-\ell}$ is a feasible solution to the LP. Consequently, $\tau \leq \frac{\text{opt}}{k-\ell}$. ◀

► **Corollary 18.** *If at step 2 of Algorithm 2 we add $S = S_i$ as in Lemma 17, then at the end of the algorithm $w(T \setminus T_0) = O(p \ln k) \cdot \tau^*$ and $|A \cap T| \geq m - \ell + p$ holds for any $A \in \mathcal{D}_T$.*

Proof. We have $|A \cap S_i| \geq 1$ for all i . In particular $A \cap S_1 \neq \emptyset$. Any $v \in A \cap S_1$ has in $G[T]$ at least m neighbors in T_0 , and at most ℓ of them are not in A ; thus v has at least $m - \ell$ neighbors in $A \cap T_0$, so $|A \cap T_0| \geq m - \ell$. Since T_0, S_1, \dots, S_p are pairwise disjoint we get $|A \cap T| \geq |A \cap T_0| + \sum_{i=1}^p |A \cap S_i| \geq m - \ell + p$. ◀

Now we decompose the problem of covering \mathcal{D}_T into several subproblems. For $r \in T$ let $\mathcal{D}_{(T,r)} = \{A \in \mathcal{D}_T : r \in T \setminus A^+\}$; we note the sets in $\{A : A \in \mathcal{D}_{(T,r)}\}$ are called “demand cuts” by Fukunaga [10, Section 5.2].

► **Theorem 19** ([23]). *Given an ℓ - T -connected graph with $|T| \geq \ell + 1$, one can find in polynomial time $R \subseteq T$ of size $|R| = O\left(\frac{|T|}{|T|-\ell} \ln \ell\right)$ such that $\mathcal{D}_T = \cup_{r \in R} \mathcal{D}_{(T,r)}$.*

We now describe how to cover the family $\mathcal{D}_{(T,r)}$ for given $r \in T$.

► **Definition 20.** *The biset $A \setminus B$ is defined by $A \setminus B = (A \setminus B^+, A^+ \setminus B)$. We say that a biset family \mathcal{F} is:*

- **uncrossable** if $A \cap B, A \cup B \in \mathcal{F}$ or if $A \setminus B, B \setminus A \in \mathcal{F}$ for all $A, B \in \mathcal{F}$.
- **T -intersecting** if $A \cap B, A \cup B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$ with $A \cap B \cap T \neq \emptyset$.
- **T -co-crossing** if $A \setminus B, B \setminus A \in \mathcal{F}$ for any $A, B \in \mathcal{F}$ with $A \cap B^* \cap T \neq \emptyset$ and $B \cap A^* \cap T \neq \emptyset$.

► **Lemma 21** ([22, 10]). *$\mathcal{D}_{(T,r)}$ is T -intersecting and T -co-crossing for any $r \in T$.*

► **Theorem 22** ([22]). *There exists a polynomial time algorithm that given a T -intersecting T -co-crossing biset family \mathcal{F} sequentially finds $O\left(\frac{q+\ell}{q}\right)$ T -intersecting uncrossable subfamilies of \mathcal{F} , such that the union of covers of these subfamilies covers \mathcal{F} , where $q = \min\{|A \cap T| : A \in \mathcal{F}\}$ and $\ell = \max_{A \in \mathcal{F}} |\partial A \cap T|$.*

73:12 Approximating k -Connected m -Dominating Sets

In [10, Theorem 3 and Proof of Corollary 3] Fukunaga proved the following.

► **Theorem 23** (Fukunaga [10]). *If \mathcal{F} is a T -intersecting uncrossable subfamily of $\mathcal{D}_{(T,r)}$ then there exists a polynomial time algorithm that computes a cover S of \mathcal{F} of weight $w(S) \leq \frac{15 \text{opt}}{k-\ell}$.*

Combining Lemma 21 with Theorems 22 and 23 we get:

► **Corollary 24.** *For any $r \in T$, there exists a polynomial time algorithm that computes a cover S_r of $\mathcal{D}_{(T,r)}$ such that if $q = \min\{|A \cap T| : \mathbb{A} \in \mathcal{D}_{(T,r)}\}$ then*

$$\frac{w(S_r)}{\text{opt}} = O\left(\frac{q + \ell}{q(k - \ell)}\right).$$

The algorithm for unit disk graphs is as follows.

■ **Algorithm 4** ($G = (V, E), w, T, p$) **unit disk graphs.**

-
- 1 apply Algorithm 2 where at step 2 each S_i is as in Lemma 17
 - 2 **if** $G[T]$ is $(\ell + 1)$ connected then $S \leftarrow \emptyset$
 - 3 **else** {comment: now $|T| \geq m + p$ and $|A \cap T| \geq m + p - \ell$ for all $\mathbb{A} \in \mathcal{D}_T$ }
 - 4 $\left[\begin{array}{l} \text{find a set } R \text{ of } O\left(\frac{|T|}{|T|-\ell} \ln \ell\right) \text{ roots as in Theorem 19} \\ \text{for each } r \in R \text{ compute a cover } S_r \text{ of } \mathcal{D}_{(T,r)} \text{ as in Corollary 24} \\ S \leftarrow \cup_{r \in R} S_r \end{array} \right.$
 - 5
 - 6
 - 7 **return** $T \cup S$
-

We bound the weight of each of the sets computed. Let T_0 denote the initial set stored in T . By Lemma 17, at the end of step 1 we have

$$\frac{w(T \setminus T_0)}{\text{opt}} = \frac{O(\ln k)}{k - \ell} \cdot p$$

Now we bound the weight of the set S computed in steps 3 to 6:

$$\frac{w(S)}{\text{opt}} = |R| \cdot O\left(\frac{q + \ell}{q(k - \ell)}\right) = \frac{O(\ln \ell)}{k - \ell} \frac{|T|}{|T| - \ell} \frac{q + \ell}{q} = \frac{O(\ln k)}{k - \ell} \frac{(m + p)^2}{(m + p - \ell)^2}$$

The last equation is since $q = \min\{|A \cap T| : \mathbb{A} \in \mathcal{F}\} \geq m - \ell + p$ by Corollary 18 and since $|T| \geq m + p$. The overall weight of the augmenting set computed is as claimed in Lemma 11.

In the case of unit weights, we add arbitrary ℓ nodes to T ; this step invokes an additive term of $O(1)$ to the ratio, and $|T| \geq 2\ell + 1$ holds after this step. Hence by Theorem 19 we will have $|R| = O\left(\frac{|T|}{|T|-\ell} \ln \ell\right) = O\left(\frac{2\ell+1}{\ell+1} \ln \ell\right) = O(\ln \ell)$ and thus

$$\frac{w(S)}{\text{opt}} = |R| \cdot O\left(\frac{q + \ell}{q(k - \ell)}\right) = \frac{O(\ln \ell)}{k - \ell} \frac{q + \ell}{q} = \frac{O(\ln k)}{k - \ell} \frac{m + p}{m + p - \ell}.$$

The overall weight of the augmenting set computed is as claimed in Lemma 11.

This concludes the proof of Lemma 11, and thus also the proof of Theorem 3.

References

- 1 C. Ambühl, T. Erlebach, M. Mihalák, and M. Nunkesser. Constant-factor approximation for minimum weight (connected) dominating sets in unit disk graphs. In *APPROX-RANDOM*, pages 3–14, 2006.
- 2 A. Belgi and Z. Nutov. An $\tilde{O}(\log^2 n)$ -approximation algorithm for 2-edge-connected dominating set. CoRR abs/1912.09662, 2019. [arXiv:1912.09662](https://arxiv.org/abs/1912.09662).
- 3 X. Cheng, X. Huang, D. Li, W. Weili, and D.-Z. Du. A polynomial-time approximation scheme for the minimum-connected dominating set in ad hoc wireless networks. *Networks*, 42(4):202–208, 2003.
- 4 J. Cheriyan and L Végé. Approximating minimum-cost k -node connected subgraphs via independence-free graphs. *SIAM J. Comput.*, 43(4):1342–1362, 2014. Preliminary version in FOCS 2013.
- 5 B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Mathematics*, 86(1-3):165–177, 1990.
- 6 B. Das and V. Bharghavan. Routing in ad-hoc networks using minimum connected dominating sets. In *IEEE International Conference on Communications, Montreal*, page 376–380, 1997.
- 7 A. Ephremides, J. E. Wieselthier, and D. J. Baker. A design concept for reliable mobile radio networks with frequency hopping signaling. *Proceedings of the IEEE*, 75(1):56–73, 1987.
- 8 K.-T. Förster. Approximating fault-tolerant domination in general graphs. In *ANALCO*, pages 25–32, 2013.
- 9 T. Fukunaga. Spider covers for prize-collecting network activation problem. *ACM Trans. Algorithms*, 13(4):49:1–49:31, 2017. preliminary version in SODA 2015.
- 10 T. Fukunaga. Approximation algorithms for highly connected multi-dominating sets in unit disk graphs. *Algorithmica*, 80(11):3270–3292, 2018.
- 11 M. Goemans, A. Goldberg, S. Plotkin, D. Shmoys, E. Tardos, and D. Williamson. Improved approximation algorithms for network design problems. In *SODA*, pages 223–232, 1994.
- 12 S. Guha and S. Khuller. Approximation algorithms for connected dominating sets. *Algorithmica*, 20(4):374–387, 1998. preliminary version in ESA 1996.
- 13 S. Guha and S. Khuller. Improved methods for approximating node weighted steiner trees and connected dominating sets. *Inf. Comput.*, 150(1):57–74, 1999.
- 14 B. Jackson and T. Jordán. Independence free graphs and vertex connectivity augmentation. *J. Comb. Theory, Ser. B*, 94(1):31–77, 2005. Preliminary version in IPCO 2001.
- 15 T. Jordán. On the optimal vertex-connectivity augmentation. *J. Combin. Theory Ser. B*, 63:8–20, 1995.
- 16 P. Klein and R. Ravi. A nearly best-possible approximation algorithm for node-weighted steiner trees. *J. Algorithms*, 19(1):104–115, 1995. Preliminary version in IPCO 1993.
- 17 G. Kortsarz and Z. Nutov. Approximating node connectivity problems via set covers. *Algorithmica*, 37:75–92, 2003. Preliminary version in APPROX 2000.
- 18 G. Kortsarz and Z. Nutov. Approximating source location and star survivable network problems. *Theoretical Computer Science*, 674:32–42, 2017.
- 19 B. Laekhanukit. An improved approximation algorithm for the minimum cost subset k -connected subgraph problem. *Algorithmica*, 72(3):714–733, 2015. Preliminary version in ICALP 2011.
- 20 W. Mader. Ecken vom grad n in minimalen n -fach zusammenhängenden graphen. *Archive der Mathematik*, 23:219–224, 1972.
- 21 M. V. Marathe, H. Breu, H. B. Hunt III, S. S. Ravi, and D. J. Rosenkrantz. Simple heuristics for unit disk graphs. *Networks*, 25(2):59–68, 1995.
- 22 Z. Nutov. Approximating minimum cost connectivity problems via uncrossable bifamilies. *ACM Transactions on Algorithms*, 9(1):1:1–1:16, 2012. Preliminary version in FOCS 2009.
- 23 Z. Nutov. Approximating subset k -connectivity problems. *J. Discrete Algorithms*, 17:51–59, 2012.

- 24 Z. Nutov. Activation network design problems. In T. F. Gonzalez, editor, *Handbook on Approximation Algorithms and Metaheuristics, Second Edition*, volume 2, chapter 12. Chapman & Hall/CRC, 2018.
- 25 Z. Nutov. Erratum: Approximating minimum cost connectivity problems via uncrossable bifamilies. *ACM Transactions on Algorithms*, 14(3):37:1–37:8, 2018.
- 26 Z. Nutov. Improved approximation algorithms for k -connected m -dominating set problems. *Information Processing Letters*, 140:30–33, 2018.
- 27 Z. Nutov. The k -connected subgraph problem. In T. F. Gonzalez, editor, *Handbook on Approximation Algorithms and Metaheuristics, Second Edition*, volume 2, chapter 15. Chapman & Hall/CRC, 2018.
- 28 Z. Nutov. Node-connectivity survivable network problems. In T. F. Gonzalez, editor, *Handbook on Approximation Algorithms and Metaheuristics, Second Edition*, volume 2, chapter 13. Chapman & Hall/CRC, 2018.
- 29 Z. Nutov. 2-node-connectivity network design. CoRR abs/2002.04048, 2020. [arXiv:2002.04048](https://arxiv.org/abs/2002.04048).
- 30 Z. Nutov. A $4 + \epsilon$ approximation for k -connected subgraphs. In *SODA*, pages 1000–1009, 2020.
- 31 Y. Shi, Z. Zhang, Y. Mo, and D.-Z. Du. Approximation algorithm for minimum weight fault-tolerant virtual backbone in unit disk graphs. *IEEE/ACM Transactions on networking*, 25(2):925–933, 2017.
- 32 M. Thai, N. Zhang, R. Tiwari, and X. Xu. On approximation algorithms of k -connected m -dominating sets in disk graphs. *Theoretical Computer Science*, 385:49–59, 2007.
- 33 J. Willson, Z. Zhang, W. Wu, and D.-Z. Du. Fault-tolerant coverage with maximum lifetime in wireless sensor networks. In *INFOCOM*, pages 1364–1372, 2015.
- 34 Z. Zhang, J. Zhou, S. Tang, X. Huang, and D.-Z. Du. Computing minimum k -connected m -fold dominating set in general graphs. *INFORMS Journal on Computing*, 30(2):217–224, 2018.
- 35 F. Zou, X. Li, S. Gao, and W. Weili. Node-weighted steiner tree approximation in unit disk graphs. *J. Combinatorial Optimization*, 18(4):342–349, 2009.
- 36 F. Zou, Y. Wang, X. XiaoHua, X. Li, D. Hongwei, P.-J. Wan, and W. Weili. New approximations for minimum-weighted dominating sets and minimum-weighted connected dominating sets on unit disk graphs. *Theoretical Computer Science*, 412(3):198–208, 2011.