

# VC Density of Set Systems Definable in Tree-Like Graphs

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## Abstract

We study set systems definable in graphs using variants of logic with different expressive power. Our focus is on the notion of *Vapnik-Chervonenkis density*: the smallest possible degree of a polynomial bounding the cardinalities of restrictions of such set systems. On one hand, we prove that if  $\varphi(\bar{x}, \bar{y})$  is a fixed  $\text{CMSO}_1$  formula and  $\mathcal{C}$  is a class of graphs with uniformly bounded cliquewidth, then the set systems defined by  $\varphi$  in graphs from  $\mathcal{C}$  have VC density at most  $|\bar{y}|$ , which is the smallest bound that one could expect. We also show an analogous statement for the case when  $\varphi(\bar{x}, \bar{y})$  is a  $\text{CMSO}_2$  formula and  $\mathcal{C}$  is a class of graphs with uniformly bounded treewidth. We complement these results by showing that if  $\mathcal{C}$  has unbounded cliquewidth (respectively, treewidth), then, under some mild technical assumptions on  $\mathcal{C}$ , the set systems definable by  $\text{CMSO}_1$  (respectively,  $\text{CMSO}_2$ ) formulas in graphs from  $\mathcal{C}$  may have unbounded VC dimension, hence also unbounded VC density.

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## 1 Introduction

**VC dimension.** VC dimension is a widely used parameter measuring the complexity of set systems. Since its introduction in the 70s in the seminal work of Vapnik and Chervonenkis [17], it became a fundamental notion in statistical learning theory. VC dimension has also found multiple applications in combinatorics and in algorithm design, particularly in the area of approximation algorithms.

The original definition states that the VC dimension of a set system  $\mathcal{F} = (\mathcal{U}, \mathcal{S})$ , where  $\mathcal{U}$  is the universe and  $\mathcal{S}$  is the set family, is equal to the supremum of cardinalities of subsets of  $\mathcal{U}$  that are shattered by  $\mathcal{F}$ . Here, a subset  $X \subseteq \mathcal{U}$  is *shattered* by  $\mathcal{F}$  if the *restriction* of  $\mathcal{F}$  to  $X$  – defined as the set system  $\mathcal{F}[X] = (X, \{S \cap X : S \in \mathcal{S}\})$  – is the whole powerset of  $X$ .

In many applications, the boundedness of the VC dimension is exploited mainly through the *Sauer-Shelah Lemma* [14, 16], which states that a set system  $\mathcal{F}$  over a universe of size  $n$  and of VC dimension  $d$  contains only  $\mathcal{O}(n^d)$  different sets. As a bound on VC dimension is inherited under restrictions, this implies that for every subset  $A$  of the universe, the cardinality of the set system  $\mathcal{F}[A]$  is at most  $\mathcal{O}(|A|^d)$ .

However, for many set systems appearing in various settings, the bound provided by the Sauer-Shelah Lemma is far from optimum. This motivates the more refined notion of the *VC density* of a set system, which is informally defined as the lowest possible degree of a polynomial bounding the cardinalities of its restrictions; see Section 2 for a formal definition.



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This distinction is particularly important for applications in approximation algorithms, where having VC density equal to one (i.e. a linear bound in the Sauer-Shelah Lemma) implies the existence of  $\varepsilon$ -nets of size  $\mathcal{O}(\frac{1}{\varepsilon})$  [1], while a super-linear bound implied by the boundedness of the VC dimension gives only  $\varepsilon$ -nets of size  $\mathcal{O}(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon})$  (see e.g. [10]). This difference seems innocent at first glance, but shaving off the logarithmic factor actually corresponds to the possibility of designing constant-factor approximation algorithms [1].

**Defining set systems in logic.** In this work we concentrate on finding a precise understanding of the connection between the expressive power of different logical formalisms and structural properties of classes of graphs. In particular, our objective is to analyze which variants of logics only allow to define set systems of low VC-density in considered classes.

To make this idea concrete, we need a way to define a set system from a graph using a formula. Let  $\varphi(\bar{x}, \bar{y})$  be a formula of some logic  $\mathcal{L}$  (to be made precise later) in the vocabulary of graphs, where  $\bar{x}, \bar{y}$  are tuples of free vertex variables. Then  $\varphi$  defines in a graph  $G = (V, E)$  the set system of  $\varphi$ -definable sets:

$$S^\varphi(G) = ( V^{\bar{x}}, \{ \{ \bar{u} \in V^{\bar{x}} : G \models \varphi(\bar{u}, \bar{v}) \} : \bar{v} \in V^{\bar{y}} \} ),$$

where  $V^{\bar{x}}$  and  $V^{\bar{y}}$  denote the sets of valuations of variables of  $\bar{x}$  and  $\bar{y}$  in  $V$ , respectively.

For an example, if  $|\bar{x}| = |\bar{y}| = 1$  and  $\varphi(x, y)$  verifies whether the distance between  $x$  and  $y$  is at most  $d$ , for some  $d \in \mathbb{N}$ , then  $S^\varphi(G)$  is the set system whose universe is the vertex set of  $G$ , while the set family comprises all balls of radius  $d$  in  $G$ .

The situation when the considered logic  $\mathcal{L}$  is the First Order logic FO was recently studied by Pilipczuk, Siebertz, and Toruńczyk [11]. They showed that the simplicity of FO-definable set systems in graphs is tightly connected to their sparseness, as explained formally next. On one hand, if  $\mathcal{C}$  is a *nowhere dense* class of graphs, then every FO formula  $\varphi(\bar{x}, \bar{y})$  defines in graphs from  $\mathcal{C}$  set systems of VC density at most  $|\bar{y}|$ . On the other hand, if  $\mathcal{C}$  is not nowhere dense, but is closed under taking subgraphs, then there exists an FO formula that defines in graphs from  $\mathcal{C}$  set systems of arbitrarily high VC dimension.

In this work we are interested in similar dichotomy statements for more expressive variants of logic on graphs, namely  $\text{MSO}_1$  and  $\text{MSO}_2$ . The topic has been investigated by Grohe and Turán [9], who proved that if graphs from a graph class  $\mathcal{C}$  have uniformly bounded cliquewidth (i.e. there exists  $c \in \mathbb{N}$  such that each graph from  $\mathcal{C}$  has the cliquewidth at most  $c$ ), then every  $\text{MSO}_1$  formula defines in graphs from  $\mathcal{C}$  set systems with uniformly bounded VC dimension. They also gave a somewhat complementary lower bound showing that if  $\mathcal{C}$  contains graphs of arbitrarily high treewidth and is closed under taking subgraphs, then there exists a fixed  $\text{MSO}_1$  formula that defines in graphs from  $\mathcal{C}$  set systems with unbounded VC dimension.

**Our contribution.** We improve the results of Grohe and Turán [9] in two aspects. First, we prove tight upper bounds on the VC density of the considered set systems, and not only on the VC dimension. Second, we clarify the dichotomy statements by showing that the boundedness of the VC parameters for set systems definable in  $\text{MSO}_1$  is tightly connected to the boundedness of cliquewidth, and there is a similar connection between the complexity of set systems definable in  $\text{MSO}_2$  and the boundedness of treewidth. Formal statements follow.

For the upper bounds, our results are captured by the following theorem. Here,  $\text{CMSO}_1$  and  $\text{CMSO}_2$  are extensions of  $\text{MSO}_1$  and  $\text{MSO}_2$ , respectively, by *modular predicates* of the form  $|X| \equiv a \pmod{p}$ , where  $X$  is a monadic variable and  $a, p$  are integers. Also,  $\text{C}_2\text{MSO}_1$  is a restriction of  $\text{CMSO}_1$  where we allow only modular predicates with  $p = 2$ , that is, checking the parity of the cardinality of a set.

► **Theorem 1.** *Let  $\mathcal{C}$  be a class of graphs and  $\varphi(\bar{x}, \bar{y})$  be a partitioned formula. Additionally, assume that one of the following assertions holds:*

- (i)  $\mathcal{C}$  has uniformly bounded cliquewidth and  $\varphi(\bar{x}, \bar{y})$  is a  $\text{CMSO}_1$ -formula; or
- (ii)  $\mathcal{C}$  has uniformly bounded treewidth and  $\varphi(\bar{x}, \bar{y})$  is a  $\text{CMSO}_2$ -formula.

*Then there is a constant  $c \in \mathbb{N}$  such that for every graph  $G \in \mathcal{C}$  and non-empty  $A \subseteq V(G)$ ,*

$$|S^\varphi(G)[A]| \leq c \cdot |A|^{|\bar{y}|}.$$

Note that one cannot expect lower VC density than  $|\bar{y}|$  for any non-trivial logic  $\mathcal{L}$ , because the formula  $\alpha(x, \bar{y}) = \bigvee_{i=1}^{|\bar{y}|} (x = y_i)$  defines a set system of VC density  $|\bar{y}|$  in any structure.

Theorem 1 provides much better bounds on the cardinalities of restrictions of the considered set systems than bounding the VC dimension and using the Sauer-Shelah Lemma, as was done in [9]. In fact, as argued in [9, Theorem 12], even in the case of defining set systems over words, the VC dimension can be tower-exponential high with respect to the size of the formula. In contrast, Theorem 1 implies that the VC density will be actually much lower: at most  $|\bar{y}|$ . This improvement has an impact on some asymptotic bounds in learning-theoretical corollaries discussed by Grohe and Turán, see e.g. [9, Theorem 1].

For lower bounds, we work with labelled graphs. For a finite label set  $\Lambda$ , a  $\Lambda$ -*v-labelled graph* is a graph whose vertices are labelled using labels from  $\Lambda$ , while in a  $\Lambda$ -*ve-labelled graph* we label both the vertices and the edges using  $\Lambda$ . For a graph class  $\mathcal{C}$ , by  $\mathcal{C}^{\Lambda,1}$  we denote the class of all  $\Lambda$ -v-labelled graphs whose underlying unlabeled graphs belong to  $\mathcal{C}$ , while  $\mathcal{C}^{\Lambda,2}$  is defined analogously for  $\Lambda$ -ve-labelled graphs. The discussed variants of MSO work over labelled graphs in the obvious way.

► **Theorem 2.** *There exists a finite label set  $\Lambda$  such that the following holds. Let  $\mathcal{C}$  be a class of graphs and  $\mathcal{L}$  be a logic such that either*

- (i)  $\mathcal{C}$  contains graphs of arbitrarily large cliquewidth and  $\mathcal{L} = \text{C}_2\text{MSO}_1$ ; or
- (ii)  $\mathcal{C}$  contains graphs of arbitrarily large treewidth and  $\mathcal{L} = \text{MSO}_2$ .

*Then there exists a partitioned  $\mathcal{L}$ -formula  $\varphi(x, y)$  in the vocabulary of graphs from  $\mathcal{C}^{\Lambda,t}$ , where  $t = 1$  if (i) holds and  $t = 2$  if (ii) holds, such that the family*

$$\{ S^\varphi(G) : G \in \mathcal{C}^{\Lambda,t} \},$$

*contains set systems with arbitrarily high VC dimension.*

Thus, the combination of Theorem 1 and Theorem 2 provides a tight understanding of the usual connections between  $\text{MSO}_1$  and cliquewidth, and between  $\text{MSO}_2$  and treewidth, also in the setting of definable set systems. As for Theorem 2, part (ii) was essentially observed by Grohe and Turán in [9, Corollary 20], whereas part (i) seems new, but can be proved using a very similar argument. Thus, Theorem 2 follows from a right combination of tools available in the literature, and we provide it mostly for the sake of clarification.

As argued by Grohe and Turán in [9, Example 21], some mild technical conditions, like closedness under labelings with a finite label set, is necessary for a result like Theorem 2 to hold. Also, the fact that in the case of unbounded cliquewidth we need to rely on logic  $\text{C}_2\text{MSO}_1$  instead of plain  $\text{MSO}_1$  is connected to the longstanding conjecture of Seese [15] about decidability of  $\text{MSO}_1$  in classes of graphs.

## 2 Preliminaries

**Vapnik-Chervonenkis parameters.** We first recall the main definitions related to the Vapnik-Chervonenkis parameters. We only provide a terse summary of the relevant concepts and results, and refer to the work of Mustafa and Varadarajan [10] for a broader context.

## 78:4 VC Density of Set Systems Definable in Tree-Like Graphs

A *set system* is a pair  $\mathcal{F} = (\mathcal{U}, \mathcal{S})$ , where  $\mathcal{U}$  is the *universe* or *ground set*, while  $\mathcal{S}$  is a family of subsets of  $\mathcal{U}$ . While a set system is formally defined as the pair  $(\mathcal{U}, \mathcal{S})$ , we will often use that term with a family  $\mathcal{S}$  alone, and then  $\mathcal{U}$  is implicitly taken to be  $\bigcup_{S \in \mathcal{S}} S$ . The *size* of a set system is  $|\mathcal{F}| := |\mathcal{S}|$ .

For a set system  $\mathcal{F} = (\mathcal{U}, \mathcal{S})$  and  $X \subseteq \mathcal{U}$ , the *restriction* of  $\mathcal{S}$  to  $X$  is the set system  $\mathcal{F}[X] := (X, \mathcal{S} \cap X)$ , where  $\mathcal{S} \cap X := \{S \cap X : S \in \mathcal{S}\}$ . We say that  $X$  is *shattered* by  $\mathcal{F}$  if  $\mathcal{S} \cap X$  is the whole powerset of  $X$ . Then the *VC dimension* of  $\mathcal{F}$  is the supremum of cardinalities of sets shattered by  $\mathcal{F}$ . As we are mostly concerned with the asymptotic behavior of restrictions of set systems, the following notion will be useful.

► **Definition 3.** *The growth function of a set system  $\mathcal{F} = (\mathcal{U}, \mathcal{S})$  is the function*

$$\pi_{\mathcal{F}}(n) := \max \{ |\mathcal{S} \cap X| : X \subseteq \mathcal{U}, |X| = n \} \quad \text{for } n \in \mathbb{N}.$$

Clearly, for any set system  $\mathcal{F}$  we have that  $\pi_{\mathcal{F}}(n) \leq 2^n$ , but many interesting set systems admit asymptotically polynomial bounds. This is in particular implied by the boundedness of the VC dimension, via the Sauer-Shelah Lemma stated below.

► **Lemma 4** (Sauer–Shelah Lemma [14, 16]). *If  $\mathcal{F}$  is a set system of VC dimension  $d$ , then*

$$\pi_{\mathcal{F}}(n) \leq \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{d} \leq \mathcal{O}(n^d).$$

As noted before, the upper bound given by the Sauer–Shelah Lemma is often weak for many natural set systems. Therefore, we will study the following quantity instead.

► **Definition 5.** *The VC density of a set system  $\mathcal{F}$  is the quantity*

$$\inf \{ \alpha \in \mathbb{R}^+ : \text{there exists } c \in \mathbb{R} \text{ such that } \pi_{\mathcal{F}}(n) \leq c \cdot n^\alpha \text{ for all } n \in \mathbb{N} \}.$$

Note that the definition of the VC density of  $\mathcal{F}$  makes little sense when the universe of  $\mathcal{F}$  is finite, as then the growth function ultimately becomes 0, allowing a polynomial bound of arbitrary small degree. Therefore, we extend the definition of VC density to *classes* of finite set systems (i.e., families of finite set systems) as follows: the VC density of a class  $\mathcal{C}$  is the infimum over all  $\alpha \in \mathbb{R}^+$  for which there is  $c \in \mathbb{R}$  such that  $\pi_{\mathcal{F}}(n) \leq c \cdot n^\alpha$  for all  $\mathcal{F} \in \mathcal{C}$  and  $n \in \mathbb{N}$ . This is equivalent to measuring the VC density of the set system obtained by taking the union of all set systems from  $\mathcal{C}$  on disjoint universes. Similarly, the VC dimension of a class of set systems  $\mathcal{C}$  is the supremum of the VC dimensions of the members of  $\mathcal{C}$ . Clearly, Sauer-Shelah Lemma implies that the VC density of a class of set systems is never larger than its VC dimension. However, it might be significantly smaller.

**Set systems definable in logic.** We assume basic familiarity with relational structures. The *domain* (or *universe*) of a relational structure  $\mathbb{A}$  will be denoted by  $\text{dom}(\mathbb{A})$ . For a tuple of variables  $\bar{x}$  and a subset  $S \subseteq \text{dom}(\mathbb{A})$ , by  $S^{\bar{x}}$  we denote the set of all *valuations* of  $\bar{x}$  in  $S$ , that is, functions mapping the variables of  $\bar{x}$  to elements of  $S$ . A *class* of structures is a set of relational structures over the same signature.

Consider a logic  $\mathcal{L}$  over some relational signature  $\Sigma$ . A *partitioned formula* is an  $\mathcal{L}$ -formula of the form  $\varphi(\bar{x}, \bar{y})$ , where the free variables are partitioned into *object variables*  $\bar{x}$  and *parameter variables*  $\bar{y}$ . Then for a  $\Sigma$ -structure  $\mathbb{A}$ , we can define the set system of  $\varphi$ -*definable sets* in  $\mathbb{A}$ :

$$S^\varphi(\mathbb{A}) := \left( \text{dom}(\mathbb{A})^{\bar{x}}, \{ \{ \bar{u} \in \text{dom}(\mathbb{A})^{\bar{x}} : \mathbb{A} \models \varphi(\bar{u}, \bar{v}) \} : \bar{v} \in \text{dom}(\mathbb{A})^{\bar{y}} \} \right).$$

If  $\mathcal{C}$  is a class of  $\Sigma$ -structures, then we define the class of set systems  $S^\varphi(\mathcal{C}) := \{ S^\varphi(\mathbb{A}) : \mathbb{A} \in \mathcal{C} \}$ .

Note that the universe of  $S^\varphi(\mathbb{A})$  is  $\text{dom}(\mathbb{A})^{\bar{x}}$ , so the elements of  $S^\varphi(\mathbb{A})$  can be interpreted as tuples of elements of  $\mathbb{A}$  of length  $|\bar{x}|$ . When measuring the VC parameters of set systems  $S^\varphi(\mathbb{A})$  it will be convenient to somehow still regard  $\text{dom}(\mathbb{A})$  as the universe. Hence, we introduce the following definition: a *k-tuple set system* is a pair  $(\mathcal{U}, \mathcal{S})$ , where  $\mathcal{U}$  is a universe and  $\mathcal{S}$  is a family of sets of  $k$ -tuples of elements of  $\mathcal{U}$ . Thus,  $S^\varphi(\mathbb{A})$  can be regarded as an  $|\bar{x}|$ -tuple set system with universe  $\text{dom}(\mathbb{A})$ .

When  $\mathcal{F} = (\mathcal{U}, \mathcal{S})$  is a  $k$ -tuple set system, for a subset of elements  $X \subseteq \mathcal{U}$  we define

$$\mathcal{S} \cap X := \{S \cap X^k : S \in \mathcal{S}\}.$$

This naturally gives us the definition of a restriction:  $\mathcal{F}[X] := (X, \mathcal{S} \cap X)$ . We may now lift all the relevant definitions – of shattering, of the VC dimension, of the growth function, and of the VC density – to  $k$ -tuple set systems using only such restrictions: to subsets  $X \subseteq \mathcal{U}$ .

**MSO and transductions.** Recall that Monadic Second Order logic (MSO) is an extension of the First Order logic (FO) that additionally allows quantification over subsets of the domain (i.e. unary predicates), represented as *monadic variables*. Sometimes we will also allow *modular predicates* of the form  $|X| \equiv a \pmod p$ , where  $X$  is a monadic variable and  $a, p$  are integers, in which case the corresponding logic shall be named CMSO. If only parity predicates may be used (i.e.  $p = 2$ ), we will speak about  $C_2$ MSO logic.

For a logic  $\mathcal{L}$  (usually a variant of MSO) and a signature  $\Sigma$ , by  $\mathcal{L}[\Sigma]$  we denote the logic comprising all  $\mathcal{L}$ -formulas over  $\Sigma$ . Then deterministic  $\mathcal{L}$ -transductions are defined as follows.

► **Definition 6.** Fix two relational signatures  $\Sigma$  and  $\Sigma' = (R_1, \dots, R_k)$ . A *deterministic  $\mathcal{L}$ -transduction  $l$  from  $\Sigma$ -structures to  $\Sigma'$ -structures* is a sequence of  $\mathcal{L}[\Sigma]$ -formulas:  $\gamma(x), \theta_{R_1}(\bar{x}_1), \dots, \theta_{R_k}(\bar{x}_k)$ , where the length of  $\bar{x}_i$  matches the arity of  $R_i$ .

The semantics we associate with this definition is as follows. Let  $\mathbb{A}$  be a  $\Sigma$  structure and  $D = \{u : u \in \text{dom}(\mathbb{A}), \mathbb{A} \models \gamma(u)\}$ . Then  $l(\mathbb{A})$  is a  $\Sigma'$  structure given by:

$$\langle D, \{\bar{u}_1 : \bar{u}_1 \in D^{\bar{x}_1}, \mathbb{A} \models \theta_{R_1}(\bar{x}_1)\}, \dots, \{\bar{u}_k : \bar{u}_k \in D^{\bar{x}_k}, \mathbb{A} \models \theta_{R_k}(\bar{x}_k)\} \rangle.$$

We will also use *non-deterministic transductions*, which are the following generalization.

► **Definition 7.** Fix two relational signatures  $\Sigma$  and  $\Sigma'$ . A *non-deterministic  $\mathcal{L}$ -transduction  $l$  from  $\Sigma$ -structures to  $\Sigma'$ -structures* is a pair consisting of: a finite signature  $\Gamma(l)$  consisting entirely of unary relation symbols, which is disjoint from  $\Sigma \cup \Sigma'$ ; and a deterministic  $\mathcal{L}$ -transduction  $l'$  from  $\Sigma \cup \Gamma(l)$ -structures to  $\Sigma'$ -structures. Transduction  $l'$  is called the deterministic part of  $l$ .

We associate the following semantics with this definition. If  $\mathbb{A}$  is a  $\Sigma$ -structure, then  $\mathbb{A}^{\Gamma(l)}$  denotes the set of all possible  $\Sigma \cup \Gamma(l)$ -structures obtained by adding valuations of the unary predicates from  $\Gamma(l)$  to  $\mathbb{A}$ . Then we define  $l(\mathbb{A}) := l'(\mathbb{A}^{\Gamma(l)})$ , which is again a set of structures.

If  $\mathcal{C}$  is a class of  $\Sigma$ -structures and  $l$  is a transduction (deterministic or not), then by  $l(\mathcal{C})$  we denote the union of images of  $l$  over elements of  $\mathcal{C}$ . Also, if  $\Gamma$  is a signature consisting of unary relation names that is disjoint from  $\Sigma$ , then we write  $\mathcal{C}^\Gamma := \{\mathbb{A}^\Gamma : \mathbb{A} \in \mathcal{C}\}$  for the class of all possible  $\Sigma \cup \Gamma$ -structures that can be obtained from the structures from  $\mathcal{C}$  by adding valuations of the unary predicates from  $\Gamma$ .

An important property of deterministic transductions is that  $\mathcal{L}$  formulas working over the output structure can be “pulled back” to  $\mathcal{L}$  formulas working over the input structure that select exactly the same tuples. This translation is formally encapsulated in the following result, and we will denote the formula  $\psi$  provided by the Lemma by  $l^{-1}(\varphi)$ .

► **Lemma 8** (Backwards Translation Lemma, [2]). *Let  $I$  be a deterministic transduction from  $\Sigma$ -structures to  $\Sigma'$ -structures and  $\mathcal{L} \in \{\text{MSO}, \text{CMSO}, \text{C}_2\text{MSO}\}$ . Then for every  $\mathcal{L}[\Sigma']$ -formula  $\varphi(\bar{x})$  there is an  $\mathcal{L}[\Sigma]$ -formula  $\psi(\bar{x})$  such that for every  $\Sigma$ -structure  $\mathbb{A}$  and  $\bar{u} \in \text{dom}(\mathbb{A})^{\bar{x}}$ ,*

$$\mathbb{A} \models \psi(\bar{u}) \quad \text{if and only if} \quad \bar{u} \in \text{dom}(I(\mathbb{A}))^{\bar{x}} \quad \text{and} \quad I(\mathbb{A}) \models \varphi(\bar{u}).$$

Finally, we remark that in the literature there is a wide variety of different notions of logical transductions and interpretations; we chose one of the simplest, as it will be sufficient for our needs. We refer a curious reader to a survey of Courcelle [2].

**MSO on graphs.** We will work with two variants of MSO on graphs:  $\text{MSO}_1$  and  $\text{MSO}_2$ . Both these variants are defined as the standard notion of MSO logic, but applied to two different encodings of graphs as relational structures. When we talk about  $\text{MSO}_1$ -formulas, we mean MSO-formulas over graphs represented as structures with the domain consisting of vertices and a single binary relation representing adjacency. The second variant,  $\text{MSO}_2$ , encompasses MSO-formulas over structures representing graphs using both vertices and edges as members of the domain, and a binary incidence relation that selects all pairs  $(e, v)$  such that  $e$  is an edge and  $v$  is one of its endpoints. These two encodings of graphs will be called the *adjacency encoding* and the *incidence encoding*, respectively. Practically speaking, in  $\text{MSO}_1$  we may only quantify over subsets of vertices, while in  $\text{MSO}_2$  we allow quantification both over subsets of vertices and over subsets of edges. We may extend  $\text{MSO}_1$  and  $\text{MSO}_2$  with modular predicates in the natural way, thus obtaining logic  $\text{CMSO}_1, \text{C}_2\text{MSO}_1$ , etc.

If  $G$  is a graph and  $\varphi(\bar{x}, \bar{y})$  is an  $\mathcal{L}$ -formula over graphs, where  $\mathcal{L}$  is any of the variants of MSO discussed above, then we may define the  $|\bar{x}|$ -tuple set system  $S^\varphi(G)$  as before, where the universe of  $S^\varphi(G)$  is the vertex set of  $G$ . We remark that in case of  $\text{MSO}_2$ , despite the fact that formally an  $\text{MSO}_2$ -formula works over a universe consisting of both vertices and edges, in the definition of  $S^\varphi(G)$  we consider only the vertex set  $V$  as the universe. That is, the parameter variables  $\bar{y}$  range over  $V$  and each evaluation  $\bar{v} \in V^{\bar{y}}$  defines the set of valuations  $\bar{u} \in V^{\bar{x}}$  satisfying  $G \models \varphi(\bar{u}, \bar{v})$  which is included in  $S^\varphi(G)$ .

**MSO and tree automata.** When proving upper bounds we will use the classical connection between MSO and tree automata. Throughout this paper, all trees will be finite, rooted, and binary: every node may have a left child and a right child, though one or both of them may be missing. Trees are represented as relational structures where the domain consists of the nodes and there are two binary relations: the left child and the right child. In case of labeled trees, the signature is extended with a unary predicate for each label.

► **Definition 9.** *Let  $\Sigma$  be a finite alphabet. A (deterministic) tree automaton is a tuple  $(Q, F, \delta)$  where  $Q$  is a finite set of states,  $F$  is a subset of  $Q$  denoting the accepting states, while  $\delta: (Q \cup \{\perp\})^2 \times \Sigma \rightarrow Q$  is the transition function.*

A run of a tree automaton  $\mathcal{A} = (Q, F, \delta)$  over a  $\Sigma$ -labeled tree  $T$  is the labeling of its nodes  $\rho: V(T) \rightarrow Q$  which is computed in a bottom-up manner using the transition function. That is, if a node  $v$  bears symbol  $a \in \Sigma$  and the states assigned by the run to the children of  $v$  are  $q_1$  and  $q_2$ , respectively, then the state assigned to  $v$  is  $\delta(q_1, q_2, a)$ . In case  $x$  has no left or right child, the corresponding state  $q_t$  is replaced with the special symbol  $\perp$ . In particular, the state in every leaf is determined as  $\delta(\perp, \perp, a)$ , where  $a \in \Sigma$  is the label of the leaf. We say that a tree automaton  $\mathcal{A}$  *accepts* a finite tree  $T$  if  $\rho(\text{root}(T)) \in F$ .

The following claim expresses the equivalence of CMSO and finite automata over trees.

► **Lemma 10** ([12]). *For every CMSO sentence  $\varphi$  over the signature of  $\Sigma$ -labeled trees there exists a tree automaton  $\mathcal{A}_\varphi$  which is equivalent to  $\varphi$  in the following sense: for every  $\Sigma$ -labeled tree  $T$ ,  $T \models \varphi$  if and only if  $\mathcal{A}_\varphi$  accepts  $T$ .*

Since we are actually interested in formulas with free variables and not only sentences, we will need to change this definition slightly. Let  $T$  be a  $\Sigma$ -labelled tree and consider a tuple of variables  $\bar{x}$  along with its valuation  $\bar{u} \in V(T)^{\bar{x}}$ . We can encode  $\bar{u}$  in  $T$  by defining an *augmented tree*  $T_{\bar{u}}$  as a  $\Sigma \times \{0, 1\}^{\bar{x}}$ -labelled tree that is obtained from  $T$  by, for each node  $v$ , assigning a label corresponding to a product of its label in  $T$  and the function  $f_v \in \{0, 1\}^{\bar{x}}$  such that we have  $f_v(x) = 1$  if and only if  $v = \bar{u}(x)$ . As observed by Grohe and Turán [9], CMSO formulas can be translated to equivalent tree automata working over augmented trees.

► **Lemma 11** ([9]). *For every CMSO formula  $\varphi(\bar{x})$  over the signature of  $\Sigma$ -labeled trees there is a tree automaton  $\mathcal{A}_\varphi$  over  $\Sigma \times \{0, 1\}^{\bar{x}}$ -labelled trees that is equivalent to  $\varphi(\bar{x})$  as follows: for every  $\Sigma$ -labelled tree  $T$  and  $\bar{u} \in V(T)^{\bar{x}}$ ,  $T \models \varphi(\bar{u})$  if and only if  $\mathcal{A}_\varphi$  accepts  $T_{\bar{u}}$ .*

### 3 Upper bounds

In this section we prove Theorem 1. We start with investigating the case of CMSO-definable set systems in trees. This case will be later translated to the case of classes with bounded treewidth or cliquewidth by means of CMSO-transductions.

**Trees.** Recall that labelled binary trees are represented as structures with domains containing their nodes, two successor relations – one for the left child, and one for the right – and unary predicates for labels. It turns out that CMSO-definable set systems over labelled trees actually admit optimal upper bounds for VC density. This improves the result of Grohe and Turán [9] showing that such set systems have bounded VC dimension.

► **Theorem 12.** *Let  $\mathcal{C}$  be a class of finite binary trees with labels from a finite alphabet  $\Sigma$ , and  $\varphi(\bar{x}, \bar{y})$  be a partitioned CMSO-formula over the signature of  $\Sigma$ -labeled binary trees. Then there is a constant  $c \in \mathbb{N}$  such that for every tree  $T \in \mathcal{C}$  and a non-empty subset of nodes  $A$ ,*

$$|S^\varphi(T)[A]| \leq c \cdot |A|^{|\bar{y}|}.$$

**Proof.** By Lemma 11,  $\varphi(\bar{x}, \bar{y})$  is equivalent to a tree automaton  $\mathcal{A} = (Q, F, \delta)$  over an alphabet of  $\Sigma \times \{0, 1\}^{\bar{x}} \times \{0, 1\}^{\bar{y}}$ . We will now investigate how the choice of parameters  $\bar{y}$  can affect the runs of  $\mathcal{A}$  over  $T$ .

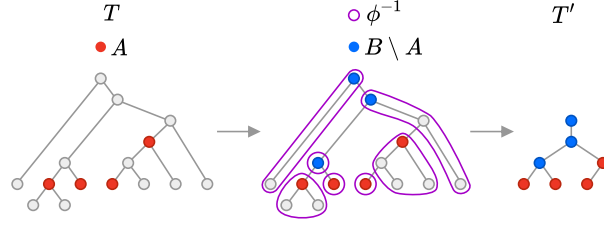
Since we are really considering  $T$  over the alphabet extended with binary markers for  $\bar{x}$  and  $\bar{y}$ , we will use  $T_{\diamond\circ}$  to denote the extension of the labeling of  $T$  where all binary markers are set to 0. That is,  $T_{\diamond\circ}$  is the tree labeled with alphabet  $\Sigma \times \{0, 1\}^{\bar{x}} \times \{0, 1\}^{\bar{y}}$  obtained from  $T$  by extending each symbol appearing in  $T$  with functions that map all variables of  $\bar{x}$  and  $\bar{y}$  to 0. Tree  $T_{\diamond\bar{q}}$  is defined analogously, where the markers for  $\bar{y}$  are set according to the valuation  $\bar{q}$ , while the markers for  $\bar{x}$  are all set to 0.

In  $T$  we have natural ancestor and descendant relations; we consider every node its own ancestor and descendant as well. Let  $B$  be the subset of nodes of  $T$  that consists of:

- the root of  $T$  and all the nodes of  $A$ ; and
- all nodes  $u \notin A$  such that both the left child and right child of  $u$  have a descendant that belongs to  $A$ .

Note that  $|B| \leq 1 + |A| + (|A| - 1) = 2|A|$ . For convenience, let  $\phi: V(T) \rightarrow B$  be a function that maps every node  $u$  of  $T$  to the least ancestor of  $u$  that belongs to  $B$ .





■ **Figure 1** Definitions of  $B$ ,  $\phi$ , and  $T'$ .

We define a tree  $T'$  with  $B$  as the set of nodes as follows. A node  $v \in B$  is the left child of a node  $u \in B$  in  $T'$  if the following holds in  $T$ :  $v$  is a descendant of the left child of  $u$  and no internal vertex on the unique path from  $u$  to  $v$  belongs to  $B$ . Note that every node  $u \in B$  has at most one left child in  $T'$ , for if it had two left children  $v, v'$ , then the least common ancestor of  $v$  and  $v'$  would belong to  $B$  and would be an internal vertex on both the  $u$ -to- $v$  path and the  $u$ -to- $v'$  path. The right child relation in  $T'$  is defined analogously. The reader may think of  $T'$  as of  $T$  with  $\phi^{-1}(u)$  contracted to  $u$ , for every  $u \in B$ ; see Figure 1.

Note that we did not define any labeling on the tree  $T'$ . Indeed, we treat  $T'$  as an unlabeled tree, but will consider different labelings of  $T'$  induced by various augmentations of  $T$ . For this, we define alphabet

$$\Delta = \{0, 1\}^{\bar{x}} \rightarrow ((Q^2 \rightarrow Q) \cup (Q \rightarrow Q) \cup Q),$$

where  $X \rightarrow Y$  denotes the set of functions from  $X$  to  $Y$ . Now, for a fixed valuation of parameter variables  $\bar{q} \in V(T)^{\bar{y}}$  and object variables  $\bar{p} \in V(T)^{\bar{x}}$ , we define the  $\Delta$ -labeled tree  $T'_{\bar{p}\bar{q}}$  as follows. Consider any node  $u \in B$  and let  $T'_{\bar{p}\bar{q}}[u]$  be the *context* of  $u$ : a tree obtained from  $T'_{\bar{p}\bar{q}}$  by restricting it to the descendants of  $u$ , and, for every child  $v$  of  $u$  in  $T'$ , replacing the subtree rooted at  $v$  by a single special node called a *hole*. The automaton  $\mathcal{A}$  can be now run on the context  $T'_{\bar{p}\bar{q}}[u]$  provided that for every hole of  $T'_{\bar{p}\bar{q}}[u]$  we prescribe a state to which this hole should evaluate. Thus, running  $\mathcal{A}$  on  $T'_{\bar{p}\bar{q}}[u]$  defines a state transformation  $\delta'_{\bar{p}\bar{q}}[u]$ , which maps tuples of states assigned to the holes of  $T'_{\bar{p}\bar{q}}[u]$  to the state assigned to  $u$ . Intuitively,  $\delta'_{\bar{p}\bar{q}}[u]$  encodes the compressed transition function of  $\mathcal{A}$  when run over the subtree of  $T'_{\bar{p}\bar{q}}$  induced by  $\phi^{-1}(u)$ , where it is assumed that on the input we are given the states to which the children of  $u$  in  $T'$  are evaluated. Note that the domain of  $\delta'_{\bar{p}\bar{q}}[u]$  consists of pairs of states if  $u$  has two children in  $T'$ , of one state if  $u$  has one child in  $T'$ , and of zero states if  $u$  has no children in  $T'$ . Thus

$$\delta'_{\bar{p}\bar{q}}[u] \in ((Q^2 \rightarrow Q) \cup (Q \rightarrow Q) \cup Q).$$

Note that for fixed  $\bar{q}$  and  $u$ ,  $\delta'_{\bar{p}\bar{q}}[u]$  is uniquely determined by the subset of variables of  $\bar{x}$  that  $\bar{p}$  maps to  $u$ . This is because  $\bar{p} \in A^{\bar{x}}$ , while  $u$  is the only node of  $\phi^{-1}(u)$  that may belong to  $A$ . Hence, with  $u$  we can associate a function  $f_u \in \Delta$  that given  $\bar{t} \in \{0, 1\}^{\bar{x}}$ , outputs the transformation  $\delta'_{\bar{p}\bar{q}}[u]$  for any (equivalently, every)  $\bar{p} \in A^{\bar{x}}$  satisfying  $\bar{t}(x) = 1$  iff  $\bar{p}(x) = u$ , for all  $x \in \bar{x}$ . Then we define the  $\Delta$ -labeled tree  $T'_q$  as  $T'$  with labeling  $u \mapsto f_u$ . Note that the above construction can be applied to  $\bar{q} = \diamond$  in the same way.

Now, for  $\bar{p} \in A^{\bar{x}} \cup \{\diamond\}$  we define the  $\Delta \times \{0, 1\}^{\bar{x}}$ -labeled tree  $(T'_q)_{\bar{p}}$  by augmenting  $T'_q$  with markers for the valuation  $\bar{p}$ ; note that this is possible because  $A$  is contained in the node set of  $T'$ . We also define an automaton  $\mathcal{A}'$  working on  $\Delta \times \{0, 1\}^{\bar{x}}$ -labeled trees as follows.  $\mathcal{A}'$  uses the same state set as  $\mathcal{A}$ , while its transition function is defined by taking the binary valuation for  $\bar{x}$  in a given node  $u$ , applying it to the  $\Delta$ -label of  $u$  to obtain a



state transformation, verifying that the arity of this transformation matches the number of children of  $u$ , and finally applying that transformation to the input states. Then the following claim follows immediately from the construction.

▷ **Claim 13.** For all  $\bar{p} \in A^{\bar{x}} \cup \{\diamond\}$  and  $\bar{q} \in B^{\bar{y}} \cup \{\diamond\}$ , the run of  $\mathcal{A}'$  on  $(T'_q)_{\bar{p}}$  is equal to the restriction of the run of  $\mathcal{A}$  on  $T_{\bar{p}\bar{q}}$  to the nodes of  $B$ .

From Claim 13 it follows that if for two tuples  $\bar{q}, \bar{q}'$  we have  $T'_{\bar{q}} = T'_{\bar{q}'}$ , then for every  $\bar{p} \in A^{\bar{x}}$ ,  $\mathcal{A}$  accepts  $T_{\bar{p}\bar{q}}$  if and only if  $\mathcal{A}$  accepts  $T_{\bar{p}\bar{q}'}$ . As  $\mathcal{A}$  is equivalent to the formula  $\varphi(\bar{x}, \bar{y})$  in the sense of Lemma 11, this implies that

$$\{\bar{p} \in A^{\bar{x}} : T \models \varphi(\bar{p}, \bar{q})\} = \{\bar{p} \in A^{\bar{x}} : T \models \varphi(\bar{p}, \bar{q}')\}.$$

In other words,  $\bar{q}$  and  $\bar{q}'$  define the same element of  $S^\varphi(T)[A]$ . We conclude that the cardinality of  $S^\varphi(T)[A]$  is bounded by the number of different trees  $T'_q$  that one can obtain by choosing different  $\bar{q} \in V(T)^{\bar{y}}$ .

Observe that for each  $\bar{q} \in V(T)^{\bar{y}}$ , tree  $T'_q$  differs from  $T'_\diamond$  by changing the labels of at most  $|\bar{y}|$  nodes. Indeed, from the construction of  $T'_q$  it follows that for each  $u \in B$ , the labels of  $u$  in  $T'_q$  and in  $T'_\diamond$  may differ only if  $\bar{q}$  maps some variable of  $\bar{y}$  to a node belonging to  $\phi^{-1}(u)$ ; this can happen for at most  $|\bar{y}|$  nodes of  $B$ . Recalling that  $|B| \leq 2|A|$  and  $|\Delta| \leq |Q|^{2^{|\bar{x}|} \cdot (|Q|^2 + |Q| + 1)}$ , the number of different trees  $T'_q$  is bounded by

$$\sum_{i=0}^{|\bar{y}|} \binom{|B|}{i} \cdot \left(|Q|^{2^{|\bar{x}|} \cdot (|Q|^2 + |Q| + 1)}\right)^{|\bar{y}|} \leq c \cdot |A|^{|\bar{y}|},$$

where  $c := 2^{|\bar{y}|} \cdot (|\bar{y}| + 1) \cdot \left(|Q|^{2^{|\bar{x}|} \cdot (|Q|^2 + |Q| + 1)}\right)^{|\bar{y}|}$ . As argued, this number is also an upper bound on the cardinality of  $S^\varphi(T)[A]$ , which concludes the proof. ◀

**Classes with bounded treewidth or cliquewidth.** We now exploit the known connections between trees and graphs of bounded treewidth or cliquewidth, expressed in terms of the existence of suitable MSO-transductions, to lift Theorem 12 to more general classes of graphs, thereby proving Theorem 1. In fact, we will not rely on the original combinatorial definitions of these parameters, but on their logical characterizations proved in subsequent works.

The first parameter of interest is the *cliquewidth* of a graph, introduced by Courcelle and Olariu [6]. We will use the following well-known logical characterization of cliquewidth.

▶ **Theorem 14** ([5, 8]). *For every  $k \in \mathbb{N}$  there is a finite alphabet  $\Sigma_k$  and a deterministic MSO-transduction  $\mathfrak{l}_k$  such that for every graph  $G$  of cliquewidth at most  $k$  there exists a  $\Sigma_k$ -labeled binary tree  $T$  satisfying the following:  $\mathfrak{l}_k(T)$  is the adjacency encoding of  $G$ .*

Thus, one may think of graphs of bounded cliquewidth as of graphs that are MSO-interpretable in labeled trees. By combining Theorem 14 with Theorem 12 we can prove part (i) of Theorem 1 as follows.

Fix a class  $\mathcal{C}$  with uniformly bounded cliquewidth and a partitioned CMSO-formula  $\varphi(\bar{x}, \bar{y})$  over the signature of  $\mathcal{C}$ . Let  $k$  be an upper bound on the cliquewidth of graphs from  $\mathcal{C}$ , and let  $\Sigma_k$  and  $\mathfrak{l}_k$  be the alphabet and the deterministic MSO-transduction provided by Theorem 14 for  $k$ . Then for every  $G \in \mathcal{C}$ , we can find a  $\Sigma_k$ -labeled tree  $T$  such that  $\mathfrak{l}_k(T)$  is the adjacency encoding of  $G$ . Note that  $V(G) \subseteq V(T)$ . Observe that for every  $A \subseteq V(G)$ , we have

$$S^\varphi(G)[A] \subseteq S^{\mathfrak{l}_k^{-1}(\varphi)}(T)[A],$$

where  $\mathsf{I}_k^{-1}(\varphi)$  is the formula  $\varphi$  pulled back through the transduction  $\mathsf{I}_k$ , as given by Lemma 8. As by Theorem 12 we have  $|S^{\mathsf{I}_k^{-1}(\varphi)}(T)[A]| \leq c \cdot |A|^{|y|}$  for some constant  $c$ , the same upper bound can be also concluded for the cardinality of  $S^\varphi(G)[A]$ . This proves Theorem 1, part (i).

To transfer these result to the case of  $\text{CMSO}_2$  over graphs of bounded treewidth, we use the following concept. The *incidence graph* of a graph  $G$  is the bipartite graph with vertex set  $V(G) \cup E(G)$ , where a vertex  $u$  is adjacent to an edge  $e$  if and only if  $u$  is an endpoint of  $e$ . The following result links  $\text{CMSO}_2$  on a graph with  $\text{CMSO}_1$  on its incidence graph.

► **Lemma 15** ([3, 4]). *Let  $G$  be a graph of treewidth  $k$ . Then the cliquewidth of the incidence graph of  $G$  is at most  $k + 3$ . Moreover, with any  $\text{CMSO}_2$ -formula  $\varphi(\bar{x})$  one can associate a  $\text{CMSO}_1$ -formula  $\psi(\bar{x})$  such that for any graph  $H$  and  $\bar{a} \in V(H)^{\bar{x}}$  we have  $H \models \varphi(\bar{a})$  if and only if  $H' \models \psi(\bar{a})$ , where  $H'$  is the incidence graph of  $H$ .*

Now Lemma 15 immediately reduces part (ii) of Theorem 1 to part (i). Indeed, for every partitioned  $\text{CMSO}_2$ -formula  $\varphi(\bar{x}, \bar{y})$ , the corresponding  $\text{CMSO}_1$ -formula  $\psi(\bar{x}, \bar{y})$  provided by Lemma 15 satisfies the following: for every graph  $H$  and its incidence graph  $H'$ , we have  $S^\varphi(H) \subseteq S^\psi(H')$ . Observe that by Lemma 15, if a graph class  $\mathcal{C}$  has uniformly bounded treewidth, then the class  $\mathcal{C}'$  comprising the incidence graphs of graphs from  $\mathcal{C}$  has uniformly bounded cliquewidth. Hence we can apply part (i) of Theorem 1 to the class  $\mathcal{C}'$  and obtain an upper bound of the form  $|S^\psi(H')[A]| \leq c \cdot |A|^{|y|}$  for any  $A \subseteq V(H')$ , where  $c$  is a constant. By the above containment of set systems, this upper bound carries over to restrictions of  $S^\varphi(H)$ . This concludes the proof of part (ii) of Theorem 1.

## 4 Lower bounds

We now turn to proving Theorem 2. As in the work of Grohe and Turán [9], the idea is that structures responsible for unbounded VC dimension of  $\text{MSO}$ -definable set systems are *grids*. The first step is to prove a suitable unboundedness result for the class of grids, which was done explicitly by Grohe and Turán in [9, Example 19]. Second, if the considered graph class  $\mathcal{C}$  has unbounded treewidth (resp., cliquewidth), then we give a deterministic  $\text{MSO}_2$ -transduction (resp.  $\text{C}_2\text{MSO}_1$ -transduction) from  $\mathcal{C}$  to the class of grids. Such transductions are present in the literature and follow from known forbidden-structures theorems for treewidth and cliquewidth. Then we can combine these two steps into the proof of Theorem 2 using the following statement. In the following, we say that logic  $\mathcal{L}$  has *unbounded VC dimension* on a class of structures  $\mathcal{C}$  if there exists a partitioned  $\mathcal{L}$ -formula  $\varphi(\bar{x}, \bar{y})$  over the signature of  $\mathcal{C}$  such that the class of set systems  $S^\varphi(\mathcal{C})$  has infinite VC dimension.

► **Lemma 16.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be two classes of structures and  $\mathcal{L} \in \{\text{MSO}, \text{CMSO}, \text{C}_2\text{MSO}\}$ . Suppose that there exists a deterministic  $\mathcal{L}$ -transduction  $\mathsf{I}$  with input signature being the signature of  $\mathcal{C}$  and the output signature being the signature of  $\mathcal{D}$  such that  $\mathsf{I}(\mathcal{C}) \supseteq \mathcal{D}$ . Then if  $\mathcal{L}$  has unbounded VC dimension on  $\mathcal{D}$ , then  $\mathcal{L}$  also has unbounded VC dimension on  $\mathcal{C}$ .*

**Proof.** Let formula  $\psi(\bar{x}, \bar{y})$  witness that  $\mathcal{L}$  has unbounded VC dimension on  $\mathcal{D}$ . Then it is easy to see that the formula  $\varphi := \mathsf{I}^{-1}(\psi)$ , provided by Lemma 8, witnesses that  $\mathcal{L}$  has unbounded VC dimension on  $\mathcal{C}$ . ◀

**Grids.** For  $n \in \mathbb{N}$ , we denote  $[n] := \{1, \dots, n\}$ . An  $n \times n$  *grid* is a relational structure over the universe  $[n] \times [n]$  with two successor relations. The horizontal successor relation  $H(\cdot, \cdot)$  selects all pairs of elements of the form  $(i, j), (i+1, j)$ , where  $i \in [n-1]$  and  $j \in [n]$ . Similarly, the vertical successor relation  $V(\cdot, \cdot)$  selects all pairs of elements the form  $(i, j), (i, j+1)$ , where  $i \in [n]$  and  $j \in [n-1]$ . Note that these relations are not symmetric: the second element in the pair must be the successor of the first in the given direction.

Grohe and Turán proved the following.

► **Theorem 17** (Example 19 in [9]). *MSO has unbounded VC dimension on the class of grids.*

The proof of Theorem 17 roughly goes as follows. The key idea is that for a given set of elements  $X$  it is easy to verify in MSO the following property:  $(i, j) \in X$  is true if and only if the  $i$ th bit of the binary encoding of  $j$  is 1. This can be done on the row-by-row basis, by expressing that elements of  $X$  in every row encode, in binary, a number that is one larger than what the elements of  $X$  encoded in the previous row. Using this observation, one can easily write a formula  $\varphi(x, y)$  that selects exactly pairs of the form  $((i, 0), (0, j))$  such that  $(i, j) \in X$ . Then  $\varphi(x, y)$  shatters the set  $\{(i, 0) : 1 \leq i \leq \lfloor \log n \rfloor\}$ , as the binary encodings of numbers from 1 to  $n$  give all possible bit vectors of length  $\lfloor \log n \rfloor$  when restricted to the first  $\lfloor \log n \rfloor$  bits. Consequently,  $\varphi(x, y)$  shatters a set of size  $\lfloor \log n \rfloor$  in an  $n \times n$  grid, which enables us to deduce the following slight strengthening of Theorem 17: MSO has unbounded VC dimension on any class of structures that contains infinitely many different grids.

For the purpose of using existing results from the literature, it will be convenient to work with *grid graphs* instead of grids. An  $n \times n$  *grid graph* is a graph on vertex set  $[n] \times [n]$  where two vertices  $(i, j)$  and  $(i', j')$  are adjacent if and only if  $|i - i'| + |j - j'| = 1$ . When speaking about grid graphs, we assume the adjacency encoding as relational structures. Thus, the difference between grid graphs and grids is that the former are only equipped with a symmetric adjacency relation without distinguishment of directions, while in the latter we may use (oriented) successor relations, different for both directions. Fortunately, grid graphs can be reduced to grids using a well-known construction, as explained next.

► **Lemma 18.** *There exists a non-deterministic MSO transduction  $J$  from the adjacency encodings of graphs to grids such that for every class of graphs  $\mathcal{C}$  that contains arbitrarily large grid graphs, the class  $J(\mathcal{C})$  contains arbitrarily large grids.*

**Proof.** The transduction uses six additional unary predicates:  $\Gamma(J) = \{A_0, A_1, A_2, B_0, B_1, B_2\}$ . We explain how the transduction works on grid graphs, which gives rise to a formal definition of the transduction in a straightforward way.

Given an  $n \times n$  grid graph  $G$ , the transduction non-deterministically chooses the valuation of the predicates of  $\Gamma(J)$  as follows: for  $t \in \{0, 1, 2\}$ ,  $A_t$  selects all vertices  $(i, j)$  such that  $i \equiv t \pmod 3$  and  $B_t$  selects all vertices  $(i, j)$  such that  $j \equiv t \pmod 3$ . Then the horizontal successor relation  $H(\cdot, \cdot)$  can be interpreted as follows:  $H(u, v)$  holds if and only if  $u$  and  $v$  are adjacent in  $G$ ,  $u$  and  $v$  are both selected by  $B_s$  for some  $s \in \{0, 1, 2\}$ , and there is  $t \in \{0, 1, 2\}$  such that  $u$  is selected by  $A_t$  while  $v$  is selected by  $A_{t+1 \pmod 3}$ . The vertical successor relation is interpreted analogously.

It is easy to see that if  $G$  is an  $n \times n$  grid graph and the valuation of the predicates of  $\Gamma(J)$  is selected as above, then  $J$  indeed outputs an  $n \times n$  grid. This implies that if  $\mathcal{C}$  contains infinitely many different grid graphs, then  $J(\mathcal{C})$  contains infinitely many different grids. ◀

We may now combine Lemma 18 with Theorem 17 to show the following.

► **Lemma 19.** *Suppose  $\mathcal{L} \in \{\text{MSO}, \text{C}_2\text{MSO}, \text{CMSO}\}$  and  $\mathcal{C}$  is a class of structures such that there exists a non-deterministic  $\mathcal{L}$ -transduction  $\mathsf{l}$  from  $\mathcal{C}$  to adjacency encodings of graphs such that  $\mathsf{l}(\mathcal{C})$  contains infinitely many different grid graphs. Then there exists a finite signature  $\Gamma$  consisting only of unary relation names such that  $\mathcal{L}$  has unbounded VC dimension on  $\mathcal{C}^\Gamma$ .*

**Proof.** As non-deterministic transductions are closed under composition for all the three considered variants of logic (cf. [2]), from Lemma 18 we infer that there is a non-deterministic  $\mathcal{L}$ -transduction  $\mathsf{K}$  such that  $\mathsf{K}(\mathcal{C})$  contains infinitely many different grids. By definition, transduction  $\mathsf{K}$  has its deterministic part  $\mathsf{K}'$  such that  $\mathsf{K}(\mathcal{C}) = \mathsf{K}'(\mathcal{C}^{\Gamma(\mathsf{K})})$ . It now remains to take  $\Gamma := \Gamma(\mathsf{K})$  and use Lemma 16 together with Theorem 17 (and the remark after it). ◀

**Classes with unbounded treewidth and cliquewidth.** For part (ii) of Theorem 2 we will use the following standard proposition, which essentially dates back to the work of Seese [15].

► **Lemma 20.** *There exists a non-deterministic MSO-transduction  $\mathsf{l}$  from incidence encodings of graphs to adjacency encodings of graphs such that for every graph class  $\mathcal{C}$  whose treewidth is not uniformly bounded, the class  $\mathsf{l}(\mathcal{C})$  contains all grid graphs.*

**Proof.** Recall that a *minor model* of a graph  $H$  in a graph  $G$  is a mapping  $\phi$  from  $V(H)$  to connected subgraphs of  $G$  such that subgraphs  $\{\phi(u) : u \in V(H)\}$  are pairwise disjoint, and for every edge  $uv \in E(H)$  there is an edge in  $G$  with one endpoint in  $\phi(u)$  and the other in  $\phi(v)$ . Then  $G$  contains  $H$  as a *minor* if there is a minor model of  $H$  in  $G$ . By the Excluded Grid Minor Theorem [13], if a class of graphs  $\mathcal{C}$  has unbounded treewidth, then every grid graph is a minor of some graph from  $\mathcal{C}$ . Therefore, it suffices to give a non-deterministic MSO-transduction  $\mathsf{l}$  from incidence encodings of graphs to adjacency encodings of graphs such that for every graph  $G$ ,  $\mathsf{l}(G)$  contains all minors of  $G$ .

The transduction  $\mathsf{l}$  works as follows. Suppose  $G$  is a given graph and  $\phi$  is a minor model of some graph  $H$  in  $G$ . First, in  $G$  we non-deterministically guess three subsets:

- a vertex subset  $D$ , containing one arbitrary vertex from each subgraph  $\{\phi(u) : u \in V(H)\}$ ;
- an edge subset  $F$ , consisting of the union of spanning trees of subgraphs  $\{\phi(u) : u \in V(H)\}$ , where each spanning tree is chosen arbitrarily;
- an edge subset  $L$ , consisting of one edge connecting a vertex of  $\phi(u)$  and a vertex of  $\phi(v)$  for each edge  $uv \in E(H)$ , chosen arbitrarily.

Recall that graph  $G$  is given by its incidence encoding, hence these subsets can be guessed using three unary predicates in  $\Gamma(\mathsf{l})$ . With sets  $D, F, L$  in place, the adjacency encoding of the minor  $H$  can be interpreted as follows: the vertex set of  $H$  is  $D$ , while two vertices  $u, u' \in D$  are adjacent in  $H$  if and only if in  $G$  they can be connected by a path that traverses only edges of  $F$  and one edge of  $L$ . It is straightforward to express this condition in  $\text{MSO}_2$ . ◀

Observe that part (ii) of Theorem 2 follows immediately by combining Lemma 20 with Lemma 19. Indeed, from this combination we obtain a partitioned MSO-formula  $\varphi(\bar{x}, \bar{y})$  and a finite signature  $\Gamma$  consisting of unary relation names such that the class of set systems  $S^\varphi(\mathcal{C}^\Gamma)$  has infinite VC dimension. Here, we treat  $\mathcal{C}$  as the class of incidence encodings of graphs from  $\mathcal{C}$ . Now if we take the label set  $\Lambda$  to be the powerset of  $\Gamma$ , we can naturally modify  $\varphi(\bar{x}, \bar{y})$  to an equivalent formula  $\varphi'(\bar{x}, \bar{y})$  working over  $\Lambda$ -ve-labelled graphs, where the  $\Lambda$ -label of every vertex  $u$  encodes the subset of predicates of  $\Gamma$  that select  $u$ . Thus  $S^{\varphi'}(\mathcal{C}^{\Lambda, 2})$  has infinite VC dimension, which concludes the proof of part (ii) of Theorem 2.

To prove part (i) of Theorem 2 we apply exactly the same reasoning, but with Lemma 20 replaced with the following result of Courcelle and Oum [7].

► **Lemma 21** (Corollary 7.5 of [7]). *There exists a  $C_2$ MSO-transduction  $\mathsf{l}$  from adjacency encodings of graphs to adjacency encodings of graphs such that if  $\mathcal{C}$  is a class of graphs of unbounded cliquewidth, then  $\mathsf{l}(\mathcal{C})$  contains arbitrarily large grid graphs.*

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