On the Approximability of Presidential Type Predicates

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Abstract

Given a predicate $P : \{-1, 1\}^k \to \{-1, 1\}$, let CSP($P$) be the set of constraint satisfaction problems whose constraints are of the form P. We say that P is approximable if given a nearly satisfiable instance of CSP(P), there exists a probabilistic polynomial time algorithm that does better than a random assignment. Otherwise, we say that P is approximation resistant.

In this paper, we analyze presidential type predicates, which are balanced linear threshold functions where all of the variables except the first variable (the president) have the same weight. We show that almost all presidential type predicates P are approximable. More precisely, we prove the following result: for any $\delta_0 > 0$, there exists a k_0 such that if $k \geq k_0$, $\delta \in (\delta_0, 1 - 2/k]$, and $\delta k + k - 1$ is an odd integer then the presidential type predicate $P(x) = \mathrm{sign}(\delta k x_1 + \sum_{i=2}^{k} x_i)$ is approximable. To prove this, we construct a rounding scheme that makes use of biases and pairwise biases. We also give evidence that using pairwise biases is necessary for such rounding schemes.

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1 Introduction

In constraint satisfaction problems (CSPs), we have a set of constraints and we want to satisfy as many of them as possible. Many fundamental problems in computer science are CSPs, including 3-SAT, MAX CUT, k-colorability, and unique games.

One fundamental question about CSPs is as follows. For a given type of CSP, is there a randomized polynomial time algorithm which is significantly better than randomly guessing an assignment? More precisely, letting r be the expected proportion of constraints satisfied by a random assignment, is there an $\epsilon > 0$ and a randomized polynomial time algorithm A such that given a CSP instance where at least $(1 - \epsilon)$ of the constraints can be satisfied, A returns an x which satisfies at least $(r + \epsilon)$ of the constraints in expectation? If so, we say that this type of CSP is approximable. If not, then we say that this type of CSP is approximation resistant.

For example, Håstad's 3-bit PCP theorem [10] proves that 3-XOR instances (where every constraint is a linear equation modulo 2 over 3 variables) are NP-hard to approximate. A direct corollary of Håstad's 3-bit PCP theorem is that 3-SAT is also NP-hard to approximate and this theorem has served as the basis for numerous other inapproximability results. On

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the other hand, Goemans and Williamson's [7] breakthrough algorithm for MAX CUT, which gives an approximation ratio of .878 for MAX CUT, shows that MAX CUT is approximable as a random cut would only cut half of the edges in expectation.

However, while the approximability or approximation resistance of CSPs has been extensively investigated, there is still much that is unknown. In this paper, we investigate CSPs where every constraint has the form of some fixed presidential type predicate P. We show that for almost all presidential type predicates P, this type of CSP is approximable.

1.1 Definitions

In order to better describe our results and their relationship to prior work, we need a few definitions.

▶ **Definition 1.** *A* Boolean predicate P *of arity* k *is a function* $P : \{-1, 1\}^k \to \{-1, 1\}$.

We remark that in general a predicate can be non-Boolean.

▶ **Definition 2.** *A* presidential type predicate *is a Boolean predicate of the form*

$$P(x_1, \ldots, x_k) = \mathrm{sign}(a \cdot x_1 + x_2 + \cdots + x_k),$$

where a *is an integer and* $a + k - 1$ *is odd.*

▶ Remark 3. In the definition above we require that a is an integer. This is not a serious restriction because if a is not an integer, then we can shift a up or down slightly to find another predicate with integer coefficient a' which is equivalent to the original predicate. We require $a + k - 1$ to be odd in order to prevent a tie.

We can think of the predicate as a vote where the vote of x_1, the "president", has weight a, while the votes of the remaining voters, the "citizens", have the same weight 1.

▶ Remark 4. Note that presidential type predicates are balanced linear threshold functions, i.e. functions of the form $sign(\sum_{i=1}^{k} c_i x_i)$ where $\forall i, c_i \in \mathbb{R}$ and $\forall x \in \{-1, 1\}^k, \sum_{i=1}^{k} c_i x_i \neq 0$ (so that the function is well-defined). Note that if a predicate P is a balanced linear threshold function, $P(-x) = -P(x)$ so exactly half of the assignments satisfy the predicate and thus a uniformly random assignment has expected value 0.

▶ **Definition 5.** *Given a Boolean predicate* $P : \{-1, 1\}^k \to \{-1, 1\}$, *an instance* Φ *of* $\mathrm{CSP}(P)$ *consists of a set of* n *variables* $x_1, \ldots, x_n$ *and* m *constraints* $C_1, \ldots, C_m$ *where each* C_i *has the form*

$$C_i(x_{i_1}, \ldots, x_{i_k}) = P(z_{i,1} x_{i_1}, \ldots, z_{i,k} x_{i_k})$$

for some distinct $i_1, \ldots, i_k \in [n]$ *and* $z_{i,1} \ldots z_{i,k} \in \{-1, 1\}$.

▶ **Definition 6.** *A Boolean predicate* P *is* approximable *if there exists a constant* $\epsilon > 0$ *and a polynomial time algorithm, possibly randomized, that on input* $\Phi \in \mathrm{CSP}(P)$ *such that* $\mathrm{OPT}(\Phi) \geq 1 - \epsilon$, *produces an assignment to* Φ*'s variables that in expectation satisfies* $r_P + \epsilon$ *fraction of the constraints in* Φ, *where* $r_P = \mathbb{E}_{x \in \{-1,1\}^k}[(1 + P(x))/2]$ *is the probability that a constraint in* Φ *is satisfied by a random assignment. Otherwise, we say* P *is* approximation resistant.

We say that a Boolean predicate P *is* weakly approximable *if there exists a constant* $\epsilon > 0$ *and a polynomial time algorithm, possibly randomized, that on input* $\Phi \in \mathrm{CSP}(P)$ *such that* $\mathrm{OPT}(\Phi) \geq 1 - \epsilon$, *produces an assignment to* Φ*'s variables that in expectation either satisfies at least* $r_P + \epsilon$ *fraction of the constraints in* Φ *or satisfies at most* $r_P - \epsilon$ *fraction of the constraints in* Φ. *Otherwise, we say that* P *is* strongly approximation resistant.

▶ Remark 7. For presidential type predicates, and in fact any odd predicate P (i.e. a predicate P where $P(-x) = -P(x)$), the notions of being approximable and being weakly approximable are equivalent.

1.2 Our Results

In this paper, we prove the following result.

▶ **Theorem 8.** *For any $\delta_0 > 0$, there exists a $k_0 \in \mathbb{N}$ such that if $k \geq k_0$, $\delta \in (\delta_0, 1 - 2/k]$, and $\delta k + k - 1$ is an odd integer then the presidential type predicate*

$$P(x) = \text{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$$

is approximable.

▶ Remark 9. Informally, this theorem says that if the weight of x_1 (the "president") is at least a constant times k, then the predicate is approximable for sufficiently large k. We have the condition $\delta \leq 1 - 2/k$ because if $\delta > 1 - 2/k$ and $\delta k + k - 1$ is an odd integer, then $\delta k \geq k$, which means the predicate is a dictator predicate which is trivially approximable.

We will prove this theorem by constructing a *rounding scheme* that makes use of *biases* and *pairwise biases*, which are given by a standard semi-definite program (see Section 2.2). Complementarily, we also give evidence that using pairwise biases is necessary for such rounding schemes. In particular, we show that for any fixed $\delta > 0$ and degree m, for sufficiently large k there is no rounding scheme for the predicate $P(x) = \text{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$ which has degree at most m and does not use pairwise biases (see Theorem 40).

1.3 Relationship to Prior Work

We now describe known criteria for determining whether a predicate P is approximable or approximation resistant and how our techniques compare to these criteria.

In 2008, Raghavendra [14] gave a characterization of which predicates are approximable and which predicates are approximation resistant. Raghavendra showed that either a standard semi-definite program (SDP) together with an appropriate rounding scheme gives a better approximation ratio than a random assignment or it is unique games hard to do so. However, this characterization leaves much to be desired because for a given predicate, it can be extremely hard to tell which case holds. In fact, it is not even known to be decidable!

Khot, Tulsiani, and Worah [12] gave a characterization of which predicates are weakly approximable which is based on whether there exist certain vanishing measures over a polytope which we call the KTW polytope (though similar polytopes were analyzed in some earlier papers, see e.g. [2, 3, 4, 11]). Unfortunately, it is also unknown whether this characterization is decidable.

Thus, if we want to determine if a given predicate P is approximable or approximation resistant, it is often better to use more direct criteria. For showing that predicates are hard to approximate, the following criterion, proved by Austrin and Mossel [5], is extremely useful.

▶ **Definition 10.** *We say that a Boolean predicate P has a balanced pairwise independent distribution of solutions if there exists a distribution D on $\{-1, 1\}^k$ such that*

1. *D is supported on $\{x \in \{-1, 1\}^k : P(x) = 1\}$ (D is a distribution of solutions to P)*
2. *For all $i \in [k]$, $\mathbb{E}_{x \in D}[x_i] = 0$ and for all $i < j \in [k]$, $\mathbb{E}_{x \in D}[x_i x_j] = 0$*

▶ **Theorem 11.** *If P has a balanced pairwise independent distribution of solutions then P is unique games hard to approximate.*
This criterion captures most but not all predicates which are known to be unique games hard to approximate. One example of a predicate which is not captured by this criterion is the predicate which was recently constructed by Potechin [13] which is unique games hard to approximate and is a balanced linear threshold function. [1]

For approximation resistance which does not rely on the hardness of unique games, Chan [6] gave the following stricter criterion which implies NP-hardness of approximation.

▶ **Theorem 12.** *If a predicate P has a balanced pairwise independent subgroup of solutions then P is NP-hard to approximate.*

For showing that predicates are approximable, the general technique is as follows:
1. Run Raghavendra's SDP to obtain biases $\{b_i : i \in [n]\}$ and pairwise biases $\{b_{ij} : i < j \in [n]\}$ for the variables.
2. Construct a rounding scheme which takes these biases and pairwise biases and gives us a solution x such that if the SDP "thinks" that almost all of the constraints are satisfiable then x satisfies significantly more constraints than a random assignment in expectation.

Based on rounding schemes which are essentially linear in the biases and pairwise biases, Hast [9] obtained the following criterion for when predicates are approximable:

▶ **Theorem 13** (Hast's criterion)**.** *Given a predicate $P : \{-1,+1\}^k \to \{-1,+1\}$,*
1. *Define $P_1 : \{-1,+1\}^k \to \mathbb{R}$ to be $P_1(x) = \sum_{i=1}^{k} \hat{P}_{\{i\}} x_i$*
2. *Define $P_2 : \{-1,+1\}^k \to \mathbb{R}$ to be $P_2(x) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \hat{P}_{\{i,j\}} x_i x_j$.*

If there are constants c_1, c_2 such that $c_2 \geq 0$ and $c_1 P_1(x) + c_2 P_2(x) > 0$ for all x such that $P(x) = 1$ then P is approximable.

Aside from Hast's criterion, most of the known approximability results are ad-hoc. Some such results are as follows.
1. Austrin, Benabbas, and Magen [1] showed that the monarchy predicate $P(x_1, \cdots, x_k) = \text{sign}((k-2)x_1 + \sum_{i=2}^{k} x_i)$ is approximable and that any predicate P which is a balanced symmetric quadratic threshold function is approximable.
2. Potechin [13] showed that the almost monarchy predicate $P(x_1, \cdots, x_k) = \text{sign}((k-4)x_1 + \sum_{i=2}^{k} x_i)$ is approximable for sufficiently large k.

In this paper, we prove that almost all presidential type predicates are approximable by generalizing the ideas Potechin [13] used to prove that the almost monarchy predicate is approximable for sufficiently large k and making these ideas more systematic. Our work compares to previous criteria as follows.

1. Raghavendra's criterion and the KTW criterion give a space of rounding schemes which should be considered but don't provide an efficient way to search for the best rounding scheme in this space. For our techniques, we take full advantage of this space of rounding schemes while also providing a way to systematically construct the rounding scheme which we need.

[1] There were previously known predicates, such as the GLST predicate [8] $P(x_1, x_2, x_3, x_4) = \frac{1+x_1}{2} x_2 x_3 + \frac{1-x_1}{2} x_2 x_4$, which are unique games hard (in fact NP-hard) to approximate yet do not have a balanced pairwise independent distribution of solutions. However, the hardness of these predicates can be reduced to the hardness of predicates which do have a balanced pairwise independent distribution of solutions, so Austrin and Mossel's criterion can still be used for these predicates.

2. Like Hast's criterion, we need to check that a certain expression is positive for all x such that $P(x) = 1$. However, there are two key differences between our techniques and Hast's criterion. First, as noted above, we use a larger space of rounding schemes. In particular, we use rounding schemes which are very much non-linear in the biases and pairwise biases. Second, because these rounding schemes are nonlinear in the biases and pairwise biases, it is not sufficient to check all x such that $P(x) = 1$. Instead, we need to check over the entire KTW polytope.

2 Techniques for Analyzing Boolean Predicates

In this section, we recall techniques for analyzing the approximability of boolean predicates.

2.1 Fourier Analysis

In this paper, we will make extensive use of the Fourier expansion of boolean predicates. The Fourier expansion of a k-ary boolean predicate P is of the form $P(x) = \sum_{I \subset [k]} \hat{P}_I x_I$, where $x_I = \prod_{i \in I} x_i$ and $\{\hat{P}_I : I \subseteq [k]\}$ are the Fourier coefficients $\hat{P}_I = \mathbb{E}_{x \in \{-1,1\}^k}[P(x) x_I]$ of P. We have the following lemma for the Fourier coefficients of presidential type predicates, the proof of which can be found in the full version.

▶ **Lemma 14** (Fourier coefficients of presidential type predicates)**.** *Let $P(x_1, \ldots, x_k) = \mathrm{sign}(a \cdot x_1 + x_2 + \cdots + x_k)$ be a presidential type predicate where $a \leq k-2$ and $a + k - 1$ is an odd integer. Let $\hat{P}_{tC}$ denote the Fourier coefficient of a set of t citizens (indices from 2 to k) and $\hat{P}_{P+tC}$ denote the Fourier coefficient of a set of t citizens together with the president (index 1). Let $\tau = \lfloor (k-a-1)/2 \rfloor$. We have*

(1) $\hat{P}_P = 1 - \frac{1}{2^{k-2}} \sum_{l=0}^{\tau} \binom{k-1}{l}$,

(2) $\hat{P}_{tC} = \frac{1}{2^{k-2}} \sum_{i=0}^{\tau} \sum_{j=0}^{\tau-i} (-1)^j \binom{k-t-1}{i} \binom{t}{j}$, $\quad \forall t (1 \leq t \leq k-1 \wedge t \text{ is odd})$,

(3) $\hat{P}_{P+tC} = -\frac{1}{2^{k-2}} \sum_{i=0}^{\tau} \sum_{j=0}^{\tau-i} (-1)^j \binom{k-t-1}{i} \binom{t}{j}$, $\quad \forall t (2 \leq t \leq k-1 \wedge t \text{ is even})$.

2.2 Choosing Rounding Schemes

Our approximation algorithms for presidential type predicates work as follows. We first run the standard SDP given by Raghavendra [14]. This standard SDP gives us biases $\{b_i : i \in [n]\}$ and pairwise biases $\{b_{ij} : i < j \in [n]\}$ such that for each constraint, these biases and pariwise biases give us a point in the KTW polytope for that constraint, which is defined below and which plays a crucial role in Khot, Tulsiani and Worah's [12] characterization of which predicates are weakly approximable. For a more detailed discussion of this standard SDP and the KTW polytope, see the full version of this paper.

▶ **Definition 15.** *Given $x \in \{-1,1\}^k$, let $p(x) \in \{-1,1\}^{k+\binom{k}{2}}$ be the vector obtained by concatenating x and $(x_1x_2, x_1x_3, \ldots, x_{k-1}x_k)$. Define*

$$KTW_P = \mathrm{Conv}(\{p(x) \mid x \in \{-1,1\}^k, P(x) = 1\}),$$

where $P : \{-1,1\}^k \to \{-1,1\}$ is a boolean predicate and $\mathrm{Conv}(S)$ is the convex hull of S.

Once we have these biases $\{b_i : i \in [n]\}$ and pairwise biases $\{b_{ij} : i < j \in [n]\}$, we use a probabilistic rounding scheme to obtain an actual $x \in \{-1,1\}^n$. To choose this rounding scheme, we choose $\mathbb{E}[x_I]$ for each monomial $x_I = \prod_{i \in I} x_i$. However, we do not have complete freedom for these choices. Intuitively, $\mathbb{E}[x_I]$ should obey the following constraints:

1. $\mathbb{E}[x_I]$ is a function of $\{b_i \mid i \in I\}$ and $\{b_{ij} \mid i,j \in I\}$.
2. $\mathbb{E}[x_I]$ is invariant under permutations of the indices in I.
3. If we flip the sign of any variable x_i where $i \in I$ (by flipping the signs of b_i and $\{b_{ij} : j \in I, j \neq i\}$), then the sign of $\mathbb{E}[x_I]$ should be flipped as well.

It turns out that for determining whether a predicate P is *weakly approximable* (which is the same as approximable for presidential type predicates), these are the only constraints on $\mathbb{E}[x_I]$. More precisely, we have the following theorem from [13], which is also implicit in [12]:

▶ **Theorem 16** (Theorem 5.1 in [13]). *Let $\{b_i \mid i \in [k]\}$ and $\{b_{ij} \mid i,j \in [k], i < j\}$ be biases and pairwise biases produced by the standard SDP. For every $a \in [k]$, let $f_a : [-1,1]^{a+\binom{a}{2}} \to [-1,1]$ be a continuous function satisfying the following symmetric requirements.*
1. *For all permutations $\sigma \in S_a$,*

$$f_a(b_{i_{\sigma(1)}}, \ldots, b_{i_{\sigma(a)}}, b_{i_{\sigma(1)}i_{\sigma(2)}}, \ldots, b_{i_{\sigma(a-1)}i_{\sigma(a)}}) = f_a(b_{i_1}, \ldots, b_{i_a}, b_{i_1 i_2}, \ldots, b_{i_{a-1}i_a})$$

2. *For all signs $s_{i_1}, \ldots, s_{i_a} \in \{-1,1\}^a$,*

$$f_a(s_{i_1}b_{i_1}, \ldots, s_{i_a}b_{i_a}, s_{i_1}s_{i_2}b_{i_1 i_2}, \ldots, s_{i_{a-1}}s_{i_a}b_{i_{a-1}i_a}) = f_a(b_{i_1}, \ldots, b_{i_a}, b_{i_1 i_2}, \ldots, b_{i_{a-1}i_a}) \cdot \prod_{j=1}^{a} s_{i_j}$$

Then there exists a sequence of rounding schemes $\{R_q\}$ and coefficients $\{c_q\}$ such that for all subsets $I = \{i_1, \ldots, i_a\}$ of size at most k,

$$\sum_q c_q \mathbb{E}_{R_q}[x_I] = f_a(b_{i_1}, \ldots, b_{i_a}, b_{i_1 i_2}, \ldots, b_{i_{a-1}i_a}),$$

where $\mathbb{E}_{R_q}[x_I]$ is the expected value of x_I given by rounding scheme R_q. Moreover, this sum can be taken to be globally convergent.

▶ Remark 17. This theorem gives us a linear combination of rounding schemes. The coefficients c_q can be thought of as a probability distribution of rounding schemes, but there are two problems:
1. $\sum_q |c_q|$ may not be 1. One fix to this issue is to scale f by an appropriate constant ϵ.
2. c_q may be negative. In general, this can be a real issue but here the predicates we consider are odd, which means if c_q is negative we can simply flip the rounding scheme R_q and take it with probability $-c_q$.

▶ **Example 18.** This theorem says the following about $\mathbb{E}[x_i]$ and $\mathbb{E}[x_i x_j]$.
1. We can take $\mathbb{E}[x_i] \sim f_1(b_i)$ for any continuous function f_1 such that $f_1(b_i) = -f_1(-b_i)$ (i.e. f_1 is odd).
2. We can take $\mathbb{E}[x_i x_j] \sim f_2(b_i, b_j, b_{ij})$ for any continuous function f_2 such that $f_2(b_i, b_j, b_{ij}) = f_2(b_j, b_i, b_{ij}) = -f_2(-b_i, b_j, -b_{ij})$. The first equality corresponds to exchanging i and j while the second equality corresponds to flipping x_i.

▶ **Example 19.** Some examples of possible functions f_3 are as follows:
1. We can take $\mathbb{E}[x_i x_j x_k] \sim x_i x_j x_k$
2. As discussed in the following subsections, we will take $\mathbb{E}[x_i x_j x_k] \sim (b_i b_{jk} + b_j b_{ik} + b_k b_{ij})$
3. Potechin [13] found a simpler rounding scheme for the monarchy predicate where $\mathbb{E}[x_i x_j x_k] \sim sign(x_i x_j x_k) \max\{|x_i|, |x_j|, |x_k|\}$

In choosing the rounding scheme, our goal is as follows. For each constraint, the standard SDP could give us any point in the KTW polytope. We need to show that no matter which point in the KTW polytope we are given, the probability that the rounding scheme satisfies the constraint is better than a random guess. Equivalently, we need to show that for all points in the KTW polytope, $\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I] > 0$.

▶ **Example 20.** Consider the majority predicate $P(x_1, \ldots, x_k) = \text{sign}(x_1 + \ldots + x_k)$. If we take $E[x_i] = f_1(b_i) = \epsilon b_i$ and take $f_a = 0$ whenever $a > 1$ then

$$\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I] = \epsilon \hat{P}_{\{1\}} \sum_{i=1}^{k} b_i$$

Since $\sum_{i=1}^{k} x_i \geq 1$ for every satisfying assignment, for any point in the KTW polytope, $\sum_{i=1}^{k} b_i \geq 1$ and thus $\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I] \geq \epsilon \hat{P}_{\{1\}} > 0$.

3 Techniques for Approximating Presidential Type Predicates

In this section, we describe our techniques for approximating presidential type predicates. These techniques are a generalization of the techniques used in [13] to show that the almost monarchy predicate is approximable for sufficiently large k.

3.1 High Level Overview

To approximate the presidential type predicate $P(x) = \text{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$, we use the following type of rounding scheme.

1. $f_1(b_i) = c_1 b_i$.
2. $f_{2l+1}(b_{i_1}, \ldots, b_{i_{2l+1}}, b_{i_1 i_2}, \ldots, b_{i_{2l} i_{2l+1}}) = c_{2l+1}\left(b_{i_1} b_{i_2 i_3} \cdots b_{i_{2l} i_{2l+1}} + \text{symmetric terms}\right)$

where we need to carefully choose the coefficients $c_1, c_3, \ldots$ so that for all points in the KTW polytope,

$$\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I] > 0.$$

Because of the symmetry of presidential type predicates P, we can analyze $\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I]$ in terms of a few key functions of the biases and pairwise biases.

▶ **Definition 21.** *Given biases $\{b_i : i \in [k]\}$ and pairwise biases $\{b_{ij} : i < j \in [k]\}$, we make the following definitions:*

1. *We define $\alpha = b_1$*
2. *We define $\beta = \sum_{i=2}^{k} b_i$*
3. *We define $S_{\{\{i_1,i_2\}\}} = \sum_{1 < i < j \in [k]} b_{ij}$. We then write $S_{\{\{i_1,i_2\}\}} = E(1 + \Delta)$ where $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$ is the value we expect for $S_{\{\{i_1,i_2\}\}}$ and Δ measures how far $S_{\{\{i_1,i_2\}\}}$ is from this expected value.*

With these definitions, we can approximate $\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I]$ in terms of α, β, and Δ. Our strategy is now as follows:

1. We choose a polynomial $h(x) = \sum_{l=1}^{m} a_l x^l$ so that $h(1 + \Delta) \approx 1$ except near $\Delta = -1$ as we must have that $h(0) = 0$. More precisely, we choose h to satisfy certain properties (see Lemma 36).

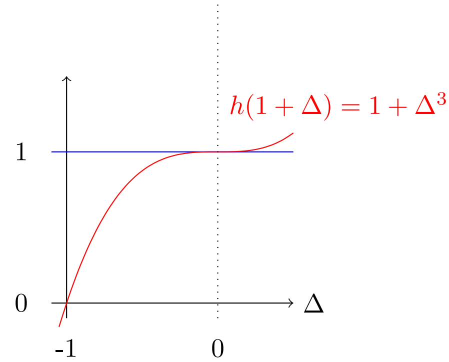

Figure 1 Plot of $h(1+\Delta)=1+\Delta^3$.

▶ Remark 22. A reasonably good choice for h is $h(1+\Delta)=1+\Delta^3$, which was used to give an approximation algorithm for the almost monarchy predicate for sufficiently large k [13]. In fact, while we don't prove it here, for quasi-monarchy predicates of the form $P(x_1,\dots,x_k)=sign\left((k-2c)x_1+\sum_{i=2}^{k}x_i\right)$ for a fixed constant c, $h(1+\Delta)=1+\Delta^3$ is sufficient to give an approximation algorithm for sufficiently large k. However, this h is not sufficient to give an approximation algorithm for more general presidential type predicates because $h(1+\Delta)=1+\Delta^3$ is far from 1 if Δ is much larger than 0.

2. We choose the coefficients $\{c_1\}\cup\{c_{2l+1}:l\in[m]\}$ so that

$$\begin{aligned}\sum_{I\subseteq[k]:I\neq\emptyset}\hat{P}_I\mathbb{E}[x_I]&=\left(\delta k^2+\frac{k}{\delta}\right)\alpha+k\left(\beta-\frac{\alpha}{\delta}\right)h(1+\Delta)+O(k)\cdot\Delta+O(1)\\&=k(\delta k\alpha+\beta)+k\left(\beta-\frac{\alpha}{\delta}\right)(h(1+\Delta)-1)+O(k)\cdot\Delta+O(1)\end{aligned}$$

3. Since for every satisfying assignment, $\delta kx_1+\sum_{i=2}^{k}x_i\geq 1$, for every point in the KTW polytope,

$$\delta kb_1+\sum_{i=2}^{k}b_i=\delta k\alpha+\beta\geq 1$$

and thus $k(\delta k\alpha+\beta)\geq k$. If we could show that the remaining terms $k\left(\beta-\frac{\alpha}{\delta}\right)(h(1+\Delta)-1)+O(k)\cdot\Delta+O(1)$ are $o(k)$, then we would be done. Unfortunately, this may not be true when $|\Delta|$ is large.

4. To handle this, we show that if $|\Delta|$ is large then we can obtain a considerably better bound on $\delta k\alpha+\beta$ More precisely, we proceed as follows:
   a. When $\Delta\geq-0.55$, we show that $\delta k\alpha+\beta\geq\frac{(\delta^2k-1)|\Delta|}{4}+\frac{1}{2}$ (see Lemma 37). As long as $h(1+\Delta)$ is sufficiently close to 1, this allows us to show that $k(\delta k\alpha+\beta)+k\left(\beta-\frac{\alpha}{\delta}\right)(h(1+\Delta)-1)+O(k)\cdot\Delta+O(1)$ is positive.
   b. When $\Delta<-0.55$, we show that we must have $\alpha>0$. In this case, we rewrite $\sum_{I\subseteq[k]:I\neq\emptyset}\hat{P}_I\mathbb{E}[x_I]=\left(\delta k^2+\frac{k}{\delta}\right)\alpha+k\left(\beta-\frac{\alpha}{\delta}\right)h(1+\Delta)+O(k)\cdot\Delta+O(1)$ as

$$k(\delta k\alpha+\beta)h(1+\Delta)+\left(\delta k^2+\frac{k}{\delta}\right)\alpha(1-h(1+\Delta))+O(k)\cdot\Delta+O(1).$$

   and show that the sum of the first two terms is positive and $\Omega(k^2)$.

3.2 Sums of Products of Biases and Pairwise Biases

In order to implement this strategy, we need some notations related to biases and pairwise biases. Note that similar definitions were also used in [13].

▶ **Definition 23.** *For $E_1 \subseteq [k]$ and $E_2 \subseteq \binom{[k]}{2}$, define*

$$B_{E_1,E_2} = \prod_{i \in E_1} b_i \prod_{\substack{\{i,j\} \in E_2 \\ i<j}} b_{ij}.$$

▶ **Definition 24.** *Let $V = \{\alpha, i_1, i_2, \ldots, i_{k-1}\}$. Let $H = H_1 \cup H_2$ where $H_1 \subseteq V$ and $H_2 \subseteq \binom{V}{2}$. Define*

$$S_H = \sum_{\substack{E_1,E_2:\exists \sigma:V \to [k] \text{ bijective} \\ \sigma(\alpha)=1, \sigma(H_1)=E_1, \sigma(H_2)=E_2}} B_{E_1,E_2},$$

where $\sigma(H_1) = \{\sigma(i) \mid i \in H_1\}$, $\sigma(H_2) = \{\{\sigma(i), \sigma(j)\} \mid \{i,j\} \in H_2\}$.

Intuitively, S_H is the sum of products B_{E_1,E_2} where $E_1 \cup E_2$ has the form H. One particularly important such sum in our algorithms is $S_{\{\{i_1,i_2\}\}}$, which is the sum of pairwise biases with indices in $[2, k]$.

▶ **Definition 25.** *We define the following shorthand notations for some important sums.*

$$\begin{aligned}
S_{1,l} &= S_{\{i_1,\{i_2,i_3\},\{i_4,i_5\},\ldots,\{i_{2l},i_{2l+1}\}\}}, \\
S_{2,l} &= S_{\{\alpha,\{i_1,i_2\},\{i_3,i_4\},\ldots,\{i_{2l-1},i_{2l}\}\}}, \\
S_{3,l} &= S_{\{i_1,\{\alpha,i_2\},\{i_3,i_4\},\ldots,\{i_{2l-1},i_{2l}\}\}}.
\end{aligned}$$

▶ **Example 26.** In the case where $k = 4, l = 1$, we have

$$\begin{aligned}
S_{1,l} &= b_2 b_{34} + b_3 b_{24} + b_4 b_{23}, \\
S_{2,l} &= b_1 b_{23} + b_1 b_{24} + b_1 b_{34}, \\
S_{3,l} &= b_2 b_{13} + b_2 b_{14} + b_3 b_{12} + b_3 b_{14} + b_4 b_{12} + b_4 b_{13}.
\end{aligned}$$

The reason that these sums are important is because they are the main terms which appear when we evaluate $\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I]$.

▶ **Proposition 27.** *If we take*

$$f_{2l+1}(b_{i_1}, \ldots, b_{i_{2l+1}}, b_{i_1 i_2}, \ldots, b_{i_{2l} i_{2l+1}}) = c_{2l+1} \left(b_{i_1} b_{i_2 i_3} \cdots b_{i_{2l} i_{2l+1}} + \textit{symmetric terms}\right)$$

then

$$\sum_{|I|=2l+1} \hat{P}_I \mathbb{E}[X_I] = c_{2l+1} \left(\hat{P}_{(2l+1)C} S_{1,l} + \hat{P}_{P+(2l)C}(S_{2,l} + S_{3,l})\right).$$

To approximate these sums, we use the following proposition (recall that we set $S_{\{\{i_1,i_2\}\}} = E(1 + \Delta)$). The proof of this proposition can be found in the appendix.

▶ **Proposition 28.** *For every* $l \geq 1$,

$$\frac{l!}{E^l}S_{1,l} = \beta(1+\Delta)^l - \frac{S_{\{i_1,\{i_1,i_2\}\}}}{E}l(1+\Delta)^{l-1} - \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2}l(l-1)(1+\Delta)^{l-2} + O\left(\frac{1}{k}\right),$$

$$\frac{l!}{E^l}S_{2,l} = \alpha(1+\Delta)^l + O\left(\frac{1}{k}\right),$$

$$\frac{l!}{E^l}S_{3,l} = \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E}l(1+\Delta)^{l-1} + O\left(\frac{1}{k}\right),$$

where the hidden constants in big-O may depend on l.

4 Proof of Theorem 8

In this section, we prove Theorem 8.

▶ **Theorem 8.** *For any* $\delta_0 > 0$*, there exists a* $k_0 \in \mathbb{N}$ *such that if* $k \geq k_0$*,* $\delta \in (\delta_0, 1-2/k]$*, and* $\delta k + k - 1$ *is an odd integer then the presidential type predicate*

$$P(x) = \operatorname{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$$

is approximable.

In particular, we prove that for sufficiently large k and a carefully chosen polynomial h, the following rounding scheme approximates the presidential type predicate $P(x) = \operatorname{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$.

▶ **Definition 29.** *Given a polynomial* $h(x) = \sum_{l=1}^{m} a_l x^l$*, we define* $R_{k,\delta,h}$ *to be the rounding scheme such that setting* $u = \frac{1+\delta}{2}k$*,* $v = \frac{1-\delta}{2}k$*, and* $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$*,*

1. $f_1(b_i) = \left(\delta k^2 + \frac{k}{\delta}\right) b_i$
2. *For all* $l \in [m]$*,*

$$f_{2l+1}(b_{i_1}, \ldots, b_{i_{2l+1}}, b_{i_1 i_2}, \ldots, b_{i_{2l} i_{2l+1}}) = c_{2l+1}\left(b_{i_1} b_{i_2 i_3} \cdots b_{i_{2l} i_{2l+1}} + \textit{symmetric terms}\right)$$

where $c_{2l+1} = a_l \cdot \frac{2^{k-2}(u-1)!(v-1)!}{(k-2l-2)!\delta^{2l}k^{2l-1}E^l}$, .

▶ **Theorem 30.** *For all* $\delta_0 > 0$*, if* $h = \sum_{l=1}^{m} a_l x^l$ *is a polynomial such that*

1. $h'(1) = h''(1) = 0$*,*
2. *For all* $\Delta \in [-0.55, \frac{1}{\delta_0^2}]$*,* $|h(1+\Delta) - 1| \leq \frac{\delta_0^2|\Delta|}{5}$*,*
3. *For all* $\Delta \in [-1, -0.55]$*,* $0 \leq h(1+\Delta) \leq 1$*,*

then there exists a $k_0 \in \mathbb{N}$ *such that for all* $\delta \geq \delta_0$ *and* $k \geq k_0$ *where* $\delta k + k - 1$ *is an odd integer,* $R_{k,\delta,h}$ *approximates the presidential type predicate* $P(x) = \operatorname{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$.

▶ Remark 31. As described in Section 3.1, our proof contains a case analysis of Δ. The value -0.55 is chosen because when $\Delta < -0.55$, the bias α of the president is always positive.

This section is organized as follows. We first compute the expected value of the rounding scheme in terms of h. Then, we show that if h has the required properties, then the expected value is positive over the entire polytope, which implies that our predicate is approximable. Finally, we find such a polynomial with the desired properties.

4.1 Evaluating the Rounding Scheme

In this subsection, we analyze $\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I]$ in terms of h. We have the following lemma for the Fourier coefficients, the proof of which can be found in the full version.

▶ **Lemma 32.** *Let $\delta_0 > 0$ be a constant. Let $P(x_1, \ldots, x_k) = \mathrm{sign}(\delta \cdot kx_1 + x_2 + \cdots + x_k)$ where $\delta \in [\delta_0, 1)$ such that $\delta k + k - 1$ is an odd integer. Let $u = \frac{1+\delta}{2}k$ and $v = \frac{1-\delta}{2}k$. Let $\hat{P}_{tC}$ denote the Fourier coefficient of a set of t citizens and $\hat{P}_{P+tC}$ denote the Fourier coefficient of a set of t citizens together with the president. We have the following:*

1. $\hat{P}_P = 1 - \frac{1}{2^{k-2}} \sum_{l=0}^{v-1} \binom{k-1}{l}$.
2. *If t is an odd integer, then*
$$\hat{P}_{tC} = \frac{1}{2^{k-2}} \cdot \frac{(k-t-1)!}{(u-1)!(v-1)!} \left(\delta^{t-1} k^{t-1} - \frac{(t-1)(t-2)}{2} \delta^{t-3} k^{t-2} + O(k^{t-3}) \right).$$
3. *If t is an even integer, then*
$$\hat{P}_{P+tC} = -\frac{1}{2^{k-2}} \cdot \frac{(k-t-1)!}{(u-1)!(v-1)!} \left(\delta^{t-1} k^{t-1} - \frac{(t-1)(t-2)}{2} \delta^{t-3} k^{t-2} + O(k^{t-3}) \right).$$

Here, the constants inside the big Os grows with t but not with δ.

▶ Remark 33. The lemma allows δ to depend on k as long as $\delta = \Omega(1)$. In particular, we can take $\delta = 1 - \frac{2c}{k}$ for any constant $c \geq 1$. Also, when δ is at least a constant we have that $\hat{P}_P$ is exponentially larger than $\hat{P}_C$.

Recall that we set $S_{\{\{i_1,i_2\}\}} = E(1+\Delta)$ where $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$ (see Definition 21). The reason for this choice for E is as follows. We expect that the cases which are most difficult to round are the two cases where $\delta k \alpha + b = 1$:

1. The president and $\frac{1-\delta}{2}k$ citizens vote 1, others vote -1. In this case,
$$\sum_{i<j \in [2,k]} x_i x_j = \binom{k-1}{2} - 2\frac{(1-\delta)k}{2}\left(\frac{(1+\delta)k}{2} - 1\right)$$
2. The president and $\frac{1+\delta}{2}k$ citizens vote -1, others vote 1. In this case,
$$\sum_{i<j \in [2,k]} x_i x_j = \binom{k-1}{2} - 2\frac{(1+\delta)k}{2}\left(\frac{(1-\delta)k}{2} - 1\right)$$

For both of these cases, $\sum_{i<j \in [2,k]} x_i x_j$ is approximately $\frac{\delta^2 k^2}{2}$. Taking the average of these two cases we have $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$. Note that since $\delta > \delta_0$ is at least a constant, we have $E = \Omega(k^2)$.

The following lemma analyzes $\sum_{I \subseteq [k]: |I| \geq 3} \hat{P}_I \mathbb{E}[x_I]$ in terms of h. Its proof can be found in the appendix.

▶ **Lemma 34.** *Assume that we have $h(x) = \sum_{l=1}^{m} a_l x^l$ and coefficients*
$$c_{2l+1} = a_l \cdot \frac{2^{k-2}(u-1)!(v-1)!}{(k-2l-2)!\delta^{2l} k^{2l-1} E^l}$$
where $u = \frac{1+\delta}{2}k$, $v = \frac{1-\delta}{2}k$, and $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$. The contribution of degree ≥ 3 terms is
$$k\left(\beta - \frac{\alpha}{\delta}\right) h(1+\Delta) - \frac{2(1+\Delta)^2 \beta h''(1+\Delta)}{\delta^2} + \frac{(1+\Delta)\beta h'(1+\Delta)}{\delta^2}$$
$$- k\left(\frac{S_{\{i_1,\{i_1,i_2\}\}}}{E} h'(1+\Delta) + \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2} h''(1+\Delta) + \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E\delta} h'(1+\Delta)\right) + O(1)$$

▶ **Corollary 35.** *If we have $h(x) = \sum_{l=1}^{m} a_l x^l$ such that $h'(1) = h''(1) = 0$ and choose coefficients*

$$c_{2l+1} = a_l \cdot \frac{2^{k-2}(u-1)!(v-1)!}{(k-2l-2)!\delta^{2l}k^{2l-1}E^l}$$

then the contribution of degree ≥ 3 terms is

$$k\left(\beta - \frac{\alpha}{\delta}\right) h(1+\Delta) + O(k) \cdot \Delta + O(1).$$

Proof. This follows from the fact that $E = \Omega(k^2), \beta = O(k), S_{\{i_1,\{i_1,i_2\}\}} = O(k^2)$, $S_{\{\{i_1,i_2\},\{i_1,i_3\}\}} = O(k^3)$ and $S_{\{\{\alpha,i_1\}\}} = O(k)$. ◀

4.2 Conditions on the Rounding Polynomial

▶ **Lemma 36.** *For all $\delta_0 > 0$, if $h = \sum_{l=1}^{m} a_l x^l$ is a polynomial such that*

1. $h'(1) = h''(1) = 0$
2. *For all $\Delta \in [-0.55, \frac{1}{\delta_0^2}]$, $|h(1+\Delta) - 1| \leq \frac{\delta_0^2|\Delta|}{5}$*
3. *For all $\Delta \in [-1, -0.55]$, $0 \leq h(1+\Delta) \leq 1$*

then there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and all $\delta \geq \delta_0$, the rounding scheme $R_{k,\delta,h}$ has positive expected value over the entire KTW polytope.

To prove this, we need the following lemma about points in the KTW polytope for P:

▶ **Lemma 37.** *For sufficiently large k we have*

$$\delta k\alpha + \beta \geq \frac{(\delta^2 k - 1)|\Delta|}{4} + \frac{1}{2}.$$

The proof of Lemma 37 can be found in the appendix.

Proof of Lemma 36. Since $h'(1) = h''(1) = 0$, by Corollary 35 the contribution of degree ≥ 3 terms becomes

$$k\left(\beta - \frac{\alpha}{\delta}\right) h(1+\Delta) + O(k) \cdot \Delta + O(1).$$

Now we add in the contribution of degree 1 terms. Since $\hat{P}_P$ is extremely close to 1 and $\hat{P}_C$ is exponentially small, the contribution of degree 1 terms is extremely close to $c_1\alpha = (\delta k^2 + k/\delta)\alpha$. Adding this to the contribution from the higher degree terms, we get that

$$\sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I] = k\left(\beta - \frac{\alpha}{\delta}\right) h(1+\Delta) + \left(\delta k^2 + \frac{k}{\delta}\right)\alpha + O(k) \cdot \Delta + O(1). \qquad (*)$$

To establish the theorem, we need to show that $(*)$ is positive over the entire KTW polytope. We proceed with a case analysis on Δ. Note that the range of Δ is approximately $(-1 - O(1/k), 1/\delta^2)$. We have the following cases.

1. $\Delta \geq -0.55$. In this case we have

$$\begin{aligned}(*) &= k\left(\beta - \frac{\alpha}{\delta}\right) h(1+\Delta) + \left(\delta k^2 + \frac{k}{\delta}\right)\alpha + O(k) \cdot \Delta + O(1)\\ &= k(\delta k\alpha + \beta) + k\left(\beta - \frac{\alpha}{\delta}\right)(h(1+\Delta) - 1) + O(k) \cdot \Delta + O(1)\\ &\geq k\left(\frac{(\delta^2 k - 1)|\Delta|}{4} + \frac{1}{2}\right) + k\left(\beta - \frac{\alpha}{\delta}\right)(h(1+\Delta) - 1) + O(k) \cdot \Delta + O(1)\end{aligned}$$

The last inequality is due to Lemma 37. Here the two terms which are quadratic in k are $\delta^2 k^2|\Delta|/4$ (note that $\delta > \delta_0$ is at least a constant) and $k\beta(h(1+\Delta)-1)$. Since $|\beta| < k$ and $|h(1+\Delta)-1| \le \frac{\delta_0^2|\Delta|}{5}$, the above quantity is positive when k is sufficiently large.

2. $\Delta < -0.55$. Note that if $x_1 = -1$ then the minimum value of Δ is about 0. If $x_1 = 1$, then the minimum value of Δ is about -1. This means that when $\Delta < -0.55$, with probability > 0.5 we have $x_1 = 1$, which implies $\alpha > 0$ and is $\Omega(1)$. We can write $(*)$ as

$$(*) = k(\delta k\alpha + \beta)h(1+\Delta) + \left(\delta k^2 + \frac{k}{\delta}\right)\alpha(1 - h(1+\Delta)) + O(k)\cdot\Delta + O(1).$$

If $\Delta \ge -1$, then $h(1+\Delta) \in [0,1]$ and both the first two terms are positive and at least one of the two terms is $\Omega(k^2)$. If $\Delta \le -1$ then since $\Delta \ge -1 - O(1/k)$ we know that the first term is $O(k)$ and the second term is $\Omega(k^2)$ and positive. Either way, we get a positive value when k is sufficiently large. ◀

4.3 Choosing the Rounding Polynomial

To finish the proof of our main theorem, we need to construct a polynomial that satisfies the conditions in Lemma 36. We claim that $h(x) = 1 - (1-x)^3 \exp(-Bx)$ works for some constant B except that it's not a polynomial. However, by truncating the Taylor expansion of this function, we can get a polynomial which also works. See full version for the proof of these claims.

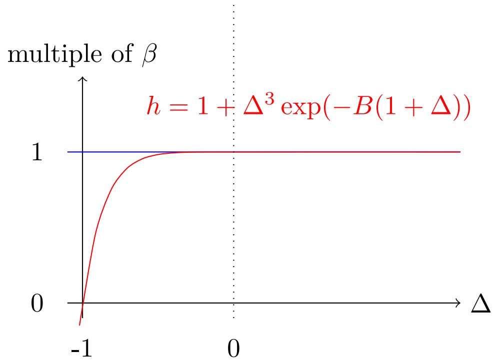

Figure 2 Plot of $h = 1 + \Delta^3 \exp(-B(1+\Delta))$.

5 Evidence for the Necessity of Pairwise Biases

We have now given rounding schemes for almost all presidential predicates. These rounding schemes crucially use the pairwise biases $\{b_{ij} : i < j \in [k]\}$. A natural question is whether this is necessary or it is possible to only use the biases $\{b_i : i \in [k]\}$. If there is a rounding scheme which only uses the biases, then instead of using a semidefinite program, it is sufficient to use a linear program, which is much faster. Indeed, such rounding schemes exist for predicates which are close to the majority function [9] and for the monarchy predicate [5, 13].

In this section, we give evidence that this is not possible for more general presidential type predicates and it is necessary to use the pairwise biases. In particular, we prove the following theorem.

Recall that we choose a rounding scheme by specifying $f_a(b_{i_1}, \ldots, b_{i_a}, b_{i_1 i_2}, \ldots, b_{i_{a-1} i_a})$ for each $a \in [k]$.

▶ **Definition 38.** *We say that a rounding scheme has degree m if $f_m \neq 0$ and $f_a = 0$ for all $a > m$.*

▶ **Definition 39.** *We say that a rounding scheme does not use pairwise biases if for all $a \in [k]$, $f_a(b_{i_1}, \ldots, b_{i_a}, b_{i_1 i_2}, \ldots, b_{i_{a-1} i_a})$ only depends on $\{b_{i_1}, \ldots, b_{i_a}\}$.*

▶ **Theorem 40.** *For all $\delta_0 > 0$ and all $m \in \mathbb{N}$, there exists a k_0 such that for all $k \geq k_0$ and $\delta \in (\delta_0, 1 - 4/k]$ where $\delta k + k - 1$ is an odd integer, the presidential type predicate $P(x) = \text{sign}\left(\delta k x_1 + \sum_{i=2}^{k} x_i\right)$ cannot be approximated by any rounding scheme of degree at most m which does not use pairwise biases.*

Proof. Let us consider a two-player zero-sum game where Alice chooses a point $b \in [-1, 1]^k$ in the KTW polytope[2] of P and Bob chooses a rounding scheme R of degree at most m. The objective of Alice is to minimize $R(b) = \sum_{I \subseteq [k]: I \neq \emptyset} \hat{P}_I \mathbb{E}[x_I]$, the expected value of $P(x)$ if we are given the point b in the KTW polytope and apply the rounding scheme R.

The lemma will follow if we can show a mixed strategy for Alice, which is a distribution μ over points in KTW polytope, such that for any rounding scheme R, $\mathbb{E}_{b \sim \mu}[R(b)] = 0$. Recalling that for each $a \in [m]$ and monomial $x_{i_1} x_{i_2} \ldots x_{i_a}$ of degree a, $\mathbb{E}[x_{i_1} x_{i_2} \ldots x_{i_a}] = f_a(b_{i_1}, \ldots, b_{i_a})$, it suffices to have the sum of degree a terms be zero for every $a \in [m]$, i.e.,

$$\mathbb{E}_{b \sim \mu}\left[\sum_{\substack{I \subset [k], |I| = a \\ I = \{i_1, \ldots, i_a\}}} \hat{P}_I f_a(b_{i_1}, \ldots, b_{i_a})\right] = 0, \quad \forall a \in [m].$$

Now let us construct such a distribution μ. By Lemma 14, $\hat{P}_P$, the Fourier coefficient of the president x_1, is exponentially larger than $\hat{P}_C$, and $\lim_{k \to \infty} \hat{P}_{P+(t-1)C} / \hat{P}_{tC} = -1$ for every odd integer $t \leq m$. For concreteness, let us assume that $m = 5$. Then we will have the following distribution for μ:

Probability	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\cdots$	x_k
p_1	0	1	0	0	0	0	0	$\cdots$	0
p_2	0	1	1	-1	0	0	0	$\cdots$	0
p_3	0	1	1	1	0	0	0	$\cdots$	0
p_4	0	1	1	1	1	-1	0	$\cdots$	0
p_5	-1	1	1	1	1	1	1	$\cdots$	1

First of all, it is easy to check that all these points are inside the KTW polytope for $P(x)$. The following is a table of contribution of each degree from each of these points.

Degrees \ Points	1st type	2nd type	3rd type	4th type	5th type
$f_1(1)$	$p_1 \hat{P}_C$	$p_2 \hat{P}_C$	$3 p_3 \hat{P}_C$	$3 p_4 \hat{P}_C$	$p_5(-\hat{P}_P + (k-1)\hat{P}_C)$
$f_3(1,1,1)$	0	$-p_2 \hat{P}_{3C}$	$p_3 \hat{P}_{3C}$	$-2 p_4 \hat{P}_{3C}$	$p_5(\binom{k-1}{2}\hat{P}_{P+2C} + \binom{k-1}{3}\hat{P}_{3C})$
$f_5(1,1,1,1,1)$	0	0	0	$-p_4 \hat{P}_{5C}$	$p_5(\binom{k-1}{4}\hat{P}_{P+4C} + \binom{k-1}{5}\hat{P}_{5C})$

To balance degree 1 terms, we need

$$p_1 \cdot \hat{P}_C + p_2 \cdot \hat{P}_C + p_3 \cdot 3\hat{P}_C + p_4 \cdot 3\hat{P}_C + p_5 \cdot (-\hat{P}_P + (k-1)\hat{P}_C) = 0.$$

[2] The KTW polytope of P actually has dimension $k + \binom{k}{2}$, but since pairwise biases play no role here, we omit those coordinates for simplicity.

Notice that every point in this distribution has a positive contribution from citizens (i.e., variables $x_2, \dots, x_k$), so we need a negative contribution from x_1. Since $\hat{P}_P$ is exponentially larger than $\hat{P}_C$, we can achieve the balance by having p_5 be exponentially small in k. Then we balance degree 5 terms, for which we need

$$-p_4 \cdot \hat{P}_{5C} + p_5 \cdot \left(\binom{k-1}{4}\hat{P}_{P+4C} + \binom{k-1}{5}\hat{P}_{5C}\right) = 0.$$

Recall that $\lim_{k\to\infty} \hat{P}_{P+4C}/\hat{P}_{5C} = -1$, so we can achieve the balance by having $p_4 = poly(k) \cdot p_5$, where $poly(k)$ is a polynomial in k. For degree 3 terms, we need

$$-p_2 \cdot \hat{P}_{3C} + p_3 \cdot \hat{P}_{3C} - 2p_4 \cdot \hat{P}_{3C} + p_5 \cdot \left(\binom{k-1}{2}\hat{P}_{P+2C} + \binom{k-1}{3}\hat{P}_{3C}\right) = 0.$$

We can then use either the second type or the third type to balance degree 3 terms. Again we will only use $poly(k) \cdot p_5$ amount of probability. When k is sufficiently large, $p_2+p_3+p_4+p_5 \le 1$ and we let the first type of points take up the remaining probability. This method can be easily extended to handle the case where m is any fixed positive integer. ◀

▶ Remark 41. For the monarchy predicate $P(x) = sign\left((k-2)x_1 + \sum_{i=2}^{k} x_i\right)$, this argument fails for the following reason. The only satisfying assignment to the monarchy predicate where $x_1 = -1$ is when all of the other x_i are 1. This implies that for all $i \in [2, k]$, $b_i \ge -b_1$, which means that the point $b = (0, 1, 1, -1, 0, \dots, 0)$ and similar points are not in the KTW polytope.

▶ Remark 42. This theorem rules out any fixed degree rounding schemes that use only biases, but it does not rule out the possibility that a rounding scheme might be able to succeed with just biases if its degree grows with k.

6 Conclusions

In this paper, we showed that almost all presidential type predicates are approximable. To do this, we carefully constructed rounding schemes which have positive expected value over the entire KTW polytope. These rounding schemes use both the biases $\{b_i : i \in [k]\}$ and the pairwise biases $\{b_{ij} : i < j \in [k]\}$ and have relatively high (but still constant) degree.

This work raises a number of open questions, including the following:

1. Which other types of predicates can this technique be applied to? For example, can we show that almost all oligarchy-type predicates are approximable, where oligarchy-type predicates are balanced LTFs where all but a few of the inputs have the same weight? As another example, can we extend the result of Austrin, Bennabas, and Magen that all symmetric quadratic threshold functions with no constant term are approximable to show that almost all quadratic threshold functions with no constant term which are symmetric with respect to all but one variable are approximable or at least weakly approximable?
2. Can we show that for almost all presidential type predicates, there is no rounding scheme which only uses the biases $\{b_i : i \in [k]\}$? Note that by Theorem 40, such rounding schemes would have to have degree which increases with k.
3. Our results only hold if k is sufficiently large. Is it true that all presidential type predicates are approximable? Less ambitiously, can we either extend our techniques or develop new techniques to handle presidential type predicates where k is relatively small?

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A Proof of Proposition 28

▶ **Proposition 28.** *For every $l \geq 1$,*

$$\frac{l!}{E^l} S_{1,l} = \beta(1+\Delta)^l - \frac{S_{\{i_1,\{i_1,i_2\}\}}}{E} l(1+\Delta)^{l-1} - \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2} l(l-1)(1+\Delta)^{l-2} + O\left(\frac{1}{k}\right),$$

$$\frac{l!}{E^l} S_{2,l} = \alpha(1+\Delta)^l + O\left(\frac{1}{k}\right),$$

$$\frac{l!}{E^l} S_{3,l} = \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E} l(1+\Delta)^{l-1} + O\left(\frac{1}{k}\right),$$

where the hidden constants in big-O may depend on l.

Proof. Here we only prove the first equality since the other two can be proved similarly. Recall that $S_{\{\{i_1,i_2\}\}} = E(1+\Delta)$ and $E = \Theta(k^2)$. The first equality is equivalent to

$$\begin{aligned}l!S_{1,l} =&\beta(S_{\{\{i_1,i_2\}\}})^l - S_{\{i_1,\{i_1,i_2\}\}}l(S_{\{\{i_1,i_2\}\}})^{l-1} - \beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}l(l-1)(S_{\{\{i_1,i_2\}\}})^{l-2}\\ &+ O\left(k^{2l-1}\right).\end{aligned}$$

Let's analyze the term $\beta(S_{\{\{i_1,i_2\}\}})^l$, by definition, it's equal to

$$\left(\sum_{i\geq 2} b_i\right)\left(\sum_{2\leq i<j} b_{ij}\right)^l = \sum_{\substack{j_1,j_2,\ldots,j_{2l+1}\in\{2,3,\ldots,k\}\\ j_2<j_3,j_4<j_5,\cdots,j_{2l}<j_{2l+1}}} b_{j_1}b_{j_2j_3}b_{j_4j_5}\cdots b_{j_{2l}j_{2l+1}}.$$

Let's call the sum on the right hand side T. We classify the terms in T according to number of repetitions in indices. If there is no repetition, then the term $b_{j_1}b_{j_2j_3}b_{j_4j_5}\cdots b_{j_{2l}j_{2l+1}}$ is also in $S_{1,l}$. Note that in $S_{1,l}$ the order of the l pairwise biases can be arbitrary, so the sum of terms with no repeated indices is equal to $l!S_{1,l}$. If there are two or more repetitions, then the number of distinct indices is at most $2l-1$, and the contribution of such terms is $O(k^{2l-1})$. If there is exact one repetition, then there are two cases.

1. j_1 is equal to some j_t for $t \geq 2$. Without loss of generality consider the terms where the only repetition is $j_1 = j_2$ or $j_1 = j_3$ (note that $j_2 < j_3$). The contribution of these terms are

$$\begin{aligned}&\sum_{\substack{j_1,j_2,\ldots,j_{2l+1}\in\{2,3,\ldots,k\}\\ j_2<j_3,j_4<j_5,\cdots,j_{2l}<j_{2l+1}\\ j_1=j_2 \text{ or } j_1=j_3\\ j_2,j_3,\ldots,j_{2l+1} \text{ distinct}}} b_{j_1}b_{j_2j_3}b_{j_4j_5}\cdots b_{j_{2l}j_{2l+1}}\\ =&\sum_{\substack{j_1,j_2,\ldots,j_{2l+1}\in\{2,3,\ldots,k\}\\ j_2<j_3,j_4<j_5,\cdots,j_{2l}<j_{2l+1}\\ j_1=j_2 \text{ or } j_1=j_3}} b_{j_1}b_{j_2j_3}b_{j_4j_5}\cdots b_{j_{2l}j_{2l+1}} + O(k^{2l-1})\\ =&S_{\{i_1,\{i_1,i_2\}\}}(S_{\{\{i_1,i_2\}\}})^{l-1} + O(k^{2l-1}).\end{aligned}$$

This is because the terms where $j_2, j_3, \ldots, j_{2l+1}$ are not distinct have at most $2l-1$ distinct indices and contribute $O(k^{2l-1})$. So the contribution of this case is

$$lS_{\{i_1,\{i_1,i_2\}\}}(S_{\{\{i_1,i_2\}\}})^{l-1} + O(k^{2l-1}).$$

2. $j_s = j_t$ for some $s, t \geq 2$. Note that in this case s and t cannot appear in the same pairwise bias. Without loss of generality assume $s \in \{2,3\}$ and $t \in \{4,5\}$. We have

$$\begin{aligned}&\sum_{\substack{j_1,j_2,\ldots,j_{2l+1}\in\{2,3,\ldots,k\}\\ j_2<j_3,j_4<j_5,\cdots,j_{2l}<j_{2l+1}\\ \text{one repetition in } j_2,j_3,j_4,j_5\\ \text{other indices distinct}}} b_{j_1}b_{j_2j_3}b_{j_4j_5}\cdots b_{j_{2l}j_{2l+1}}\\ =&\sum_{\substack{j_1,j_2,\ldots,j_{2l+1}\in\{2,3,\ldots,k\}\\ j_2<j_3,j_4<j_5,\cdots,j_{2l}<j_{2l+1}\\ \text{one repetition in } j_2,j_3,j_4,j_5}} b_{j_1}b_{j_2j_3}b_{j_4j_5}\cdots b_{j_{2l}j_{2l+1}} + O(k^{2l-1})\\ =&\quad 2\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}(S_{\{\{i_1,i_2\}\}})^{l-2} + O(k^{2l-1}).\end{aligned}$$

So the contribution of this case is

$$\begin{aligned}&\binom{l}{2}\cdot\left(2\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}(S_{\{\{i_1,i_2\}\}})^{l-2} + O(k^{2l-1})\right)\\ =&\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}l(l-1)(S_{\{\{i_1,i_2\}\}})^{l-2} + O\left(k^{2l-1}\right).\end{aligned}$$

We conclude that

$$\beta(S_{\{\{i_1,i_2\}\}})^l = l!S_{1,l} + lS_{\{i_1,\{i_1,i_2\}\}}(S_{\{\{i_1,i_2\}\}})^{l-1} + \beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}l(l-1)(S_{\{\{i_1,i_2\}\}})^{l-2} + O\left(k^{2l-1}\right).$$

We get the desired equality by shifting the terms. ◀

B Proof of Lemma 34

▶ **Lemma 34.** *Assume that we have $h(x) = \sum_{l=1}^{m} a_l x^l$ and coefficients*

$$c_{2l+1} = a_l \cdot \frac{2^{k-2}(u-1)!(v-1)!}{(k-2l-2)!\delta^{2l}k^{2l-1}E^l}$$

where $u = \frac{1+\delta}{2}k$, $v = \frac{1-\delta}{2}k$, and $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$. The contribution of degree ≥ 3 terms is

$$k\left(\beta - \frac{\alpha}{\delta}\right)h(1+\Delta) - \frac{2(1+\Delta)^2\beta h''(1+\Delta)}{\delta^2} + \frac{(1+\Delta)\beta h'(1+\Delta)}{\delta^2}$$
$$- k\left(\frac{S_{\{i_1,\{i_1,i_2\}\}}}{E}h'(1+\Delta) + \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2}h''(1+\Delta) + \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E\delta}h'(1+\Delta)\right) + O(1)$$

Proof. We have the following computation:

$$\sum_{l=1}^{m}\sum_{|I|=2l+1}\hat{P}_I\mathbb{E}[X_I]$$
$$= \sum_{l=1}^{m} c_{2l+1}\left(\hat{P}_{(2l+1)C}S_{1,l} + \hat{P}_{P+(2l)C}(S_{2,l}+S_{3,l})\right)$$
$$= \sum_{l=1}^{m} a_l\left(\left(k - l(2l-1)\delta^{-2} + O\left(\frac{1}{k}\right)\right)\cdot\right.$$
$$\left(\beta(1+\Delta)^l - \frac{S_{\{i_1,\{i_1,i_2\}\}}}{E}l(1+\Delta)^{l-1} - \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2}l(l-1)(1+\Delta)^{l-2} + O\left(\frac{1}{k}\right)\right) -$$
$$\left(1 - \frac{2l+1}{k}\right)\cdot\left(\frac{k}{\delta} - \frac{(l-1)(2l-1)}{\delta^3} + O\left(\frac{1}{k}\right)\right)\cdot$$
$$\left.\left(\alpha(1+\Delta)^l + \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E}l(1+\Delta)^{l-1} + O\left(\frac{1}{k}\right)\right)\right)$$
$$= \sum_{l=1}^{m} a_l\left(k\left(\beta - \frac{\alpha}{\delta}\right)(1+\Delta)^l - \frac{l(2l-1)\beta(1+\Delta)^l}{\delta^2}\right.$$
$$\left. - kl(1+\Delta)^{l-1}\left(\frac{S_{\{i_1,\{i_1,i_2\}\}}}{E} + \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2(1+\Delta)}(l-1) + \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E\delta}\right) + O(1)\right)$$
$$= k\left(\beta - \frac{\alpha}{\delta}\right)h(1+\Delta) - \frac{2(1+\Delta)^2\beta h''(1+\Delta)}{\delta^2} + \frac{(1+\Delta)\beta h'(1+\Delta)}{\delta^2}$$
$$- k\left(\frac{S_{\{i_1,\{i_1,i_2\}\}}}{E}h'(1+\Delta) + \frac{\beta S_{\{\{i_1,i_2\},\{i_1,i_3\}\}}}{E^2}h''(1+\Delta) + \frac{\beta S_{\{\{\alpha,i_1\}\}}}{E\delta}h'(1+\Delta)\right) + O(1).$$ ◀

C Proof of Lemma 37

▶ **Lemma 37.** *For sufficiently large k we have*

$$\delta k\alpha + \beta \geq \frac{(\delta^2 k - 1)|\Delta|}{4} + \frac{1}{2}.$$

Proof. Since $|\Delta|$ is a convex function on the KTW polytope, it suffices to check that for each satisfying assignment, $\delta k\alpha + \beta \geq \frac{\delta k|\Delta|}{4} + \frac{1}{2}$. Letting t be the number of ones in $x_2, \ldots, x_k$, we have that $\beta = t - (k-1-t) = 2t - k + 1$ and

$$\sum_{2\leq i<j} x_i x_j = \binom{t}{2} + \binom{k-1-t}{2} - t(k-1-t) = 2t^2 - 2(k-1)t + \binom{k-1}{2},$$

Recalling that $E = \frac{\delta^2 k^2}{2} - \frac{k}{2} + 1$, this implies that

$$\Delta = \frac{\sum_{2\leq i<j} x_i x_j - E}{E} = \frac{1}{E}\left(2t^2 - 2(k-1)t + \binom{k-1}{2} - \frac{\delta^2 k^2}{2} + \frac{k}{2} - 1\right)$$
$$= \frac{1}{E}\left(2t^2 - 2(k-1)t + \frac{(1-\delta^2)k^2}{2} - k\right)$$

Since $E > \frac{\delta^2 k^2}{2} - \frac{k}{2}$, we have

$$\frac{\delta^2 k - 1}{4}|\Delta| = \frac{\delta^2 k - 1}{4E}\left|2t^2 - 2(k-1)t + \frac{(1-\delta^2)k^2}{2} - k\right|$$
$$< \frac{1}{2k}\left|2t^2 - 2(k-1)t + \frac{(1-\delta^2)k^2}{2} - k\right|$$
$$= \left|\frac{t^2}{k} - \frac{(k-1)t}{k} + \frac{(1-\delta^2)k}{4} - \frac{1}{2}\right|$$

We will show that $\delta k\alpha + \beta \geq \left|\frac{t^2}{k} - \frac{(k-1)t}{k} + \frac{(1-\delta^2)k}{4} - \frac{1}{2}\right| + \frac{1}{2}$, from which our lemma will follow. To this end, we show that $\delta k\alpha + \beta \geq \left(\frac{t^2}{k} - \frac{(k-1)t}{k} + \frac{(1-\delta^2)k}{4} - \frac{1}{2}\right) + \frac{1}{2}$ and $\delta k\alpha + \beta \geq -\left(\frac{t^2}{k} - \frac{(k-1)t}{k} + \frac{(1-\delta^2)k}{4} - \frac{1}{2}\right) + \frac{1}{2}$.

1. $\delta k\alpha + \beta \geq \left(\frac{t^2}{k} - \frac{(k-1)t}{k} + \frac{(1-\delta^2)k}{4} - \frac{1}{2}\right) + \frac{1}{2}$.
   We have two cases, $\alpha = 1$ or $\alpha = -1$. If $\alpha = 1$, then $\delta k\alpha + \beta = \delta k + 2t - k + 1$ and since it's a satisfying assignment we have $t \geq \frac{1-\delta}{2}k$. The inequality becomes

$$\frac{t^2}{k} - \frac{(k-1)t}{k} - 2t + \frac{(1-\delta^2)k}{4} - \delta k + k - 1 \leq 0.$$

   The left hand side is a quadratic function on t with positive leading coefficient, and to check it's non-positive we simply need to check its values on $t = \frac{1-\delta}{2}k$ and $t = k-1$, the boundary points of t's domain. When $t = \frac{1-\delta}{2}k$,

$$\frac{t^2}{k} - \frac{(k-1)t}{k} - 2t + \frac{(1-\delta^2)k}{4} - \delta k + k - 1 =$$
$$\left(\frac{t^2}{k} - t + \frac{(1-\delta^2)k}{4}\right) + (-2t - \delta k + k) + \left(\frac{t}{k} - 1\right) =$$
$$\left(\frac{(1-\delta)^2 k}{4} - \frac{(1-\delta)k}{2} + \frac{(1-\delta^2)k}{4}\right) + 0 + \left(\frac{1-\delta}{2} - 1\right) =$$
$$\frac{1-\delta}{2} - 1 < 0$$

When $t = k - 1$,

$$\frac{t^2}{k} - \frac{(k-1)t}{k} - 2t + \frac{(1-\delta^2)k}{4} - \delta k + k - 1 =$$
$$\left(\frac{t^2}{k} - \frac{(k-1)t}{k}\right) + (-t + k - 1) + \left(-t - \delta k + \frac{(1-\delta^2)k}{4}\right) =$$
$$-\frac{3 - 4\delta + \delta^2}{4}k + 1$$

which is negative when k is sufficiently large (note that $\delta \leq 1 - \frac{3}{k}$).
If $\alpha = -1$, then $\delta k\alpha + \beta = -\delta k + 2t - k + 1$ and we have that $t \geq \frac{1+\delta}{2}k$. The inequality becomes

$$\frac{t^2}{k} - \frac{(k-1)t}{k} - 2t + \frac{(1-\delta^2)k}{4} + \delta k + k - 1 \leq 0.$$

We check the value of LHS on $t = \frac{1+\delta}{k}$ and $t = k-1$. Following exactly the same argument we used for $\alpha = 1$ except that δ is replaced by $-\delta$, when $t = \frac{1+\delta}{2}k$,

$$\frac{t^2}{k} - \frac{(k-1)t}{k} - 2t + \frac{(1-\delta^2)k}{4} + \delta k + k - 1 = \frac{1+\delta}{2} - 1 < 0$$

and when $t = k - 1$,

$$\frac{t^2}{k} - \frac{(k-1)t}{k} - 2t + \frac{(1-\delta^2)k}{4} + \delta k + k - 1 = -\frac{3 - 4\delta + \delta^2}{4}k + 1 < 0$$

2. $\delta k\alpha + \beta \geq -\left(\frac{t^2}{k} - \frac{(k-1)t}{k} + \frac{(1-\delta^2)k}{4} - \frac{1}{2}\right) + \frac{1}{2}$.
We again have two cases, $\alpha = 1$ or $\alpha = -1$. If $\alpha = 1$, we have $\delta k\alpha + \beta = \delta k + 2t - k + 1$ and the inequality becomes

$$\frac{t^2}{k} + \left(2 - \frac{k-1}{k}\right)t + \frac{(1-\delta^2)k}{4} + \delta k - k \geq 0.$$

The left hand side is a quadratic function that achieves minimum when t is negative, so we simply need the inequality to hold when $t = \frac{1-\delta}{2}k$, at which point the value of LHS is $\frac{1-\delta}{2} \geq 0$.
If $\alpha = -1$, the inequality becomes

$$\frac{t^2}{k} + \left(2 - \frac{k-1}{k}\right)t + \frac{(1-\delta^2)k}{4} - \delta k - k \geq 0.$$

Again, we simply need it to hold when $t = \frac{1+\delta}{2}k$, at which point the value of LHS is $\frac{1+\delta}{2} \geq 0$.

This completes our proof. ◀