Reaching a Consensus on Random Networks: The Power of Few

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Abstract

A community of n individuals splits into two camps, Red and Blue. The individuals are connected by a social network, which influences their colors. Everyday, each person changes his/her color according to the majority of his/her neighbors. Red (Blue) wins if everyone in the community becomes Red (Blue) at some point.

We study this process when the underlying network is the random Erdos-Renyi graph $G(n, p)$. With a balanced initial state ($n/2$ persons in each camp), it is clear that each color wins with the same probability.

Our study reveals that for any constants p and ε, there is a constant c such that if one camp has $\frac{n}{2} + c$ individuals at the initial state, then it wins with probability at least $1 - \varepsilon$. The surprising fact here is that c *does not* depend on n, the population of the community. When $p = 1/2$ and $\varepsilon = .1$, one can set $c = 6$, meaning one camp has $\frac{n}{2} + 6$ members initially. In other words, it takes only 6 extra people to win an election with overwhelming odds. We also generalize the result to $p = p_n = o(1)$ in a separate paper.

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Supplementary Material The simulation source code for the random process described in the paper is available at https://github.com/thbl2012/majority-dynamics-simulation

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1 Introduction

1.1 The opinion exchange dynamics

Building mathematical models to explain how collective opinions are formed is an important and interesting task (see [12] for a survey on the topic, with examples from various fields, economy, sociology, statistical physics, to mention a few).

Obviously, our opinions are influenced by people around us, and this motivates the study of the following natural and simple model. A community of n individuals splits into two camps, Red and Blue, representing two competing opinions, which can be on any topic such

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as brand competition, politics, ethical issues, etc. The individuals are connected by a social network, which influences their opinion on a daily basis (by some specific rule). We say that Red (respectively Blue) *wins* if everyone in the community becomes Red (respectively Blue) at some point.

We study this process when the underlying network is random. In this paper, we focus on the Erdos-Renyi random graph $G(n, p)$, which is the most popular model of random graphs [4, 10]. We use the majority rule, which is a natural choice. When a new day comes, a vertex scans its neighbors' colors in the previous day and adopts the dominant one. If there is a tie, it keeps its color.

▶ **Definition 1.** *The random graph $G(n, p)$ on $n \in \mathbb{N}$ vertices and density $p \in (0, 1)$ is obtained by putting an edge between any two vertices with probability p, independently.*

1.2 Results

With a balanced initial state ($n/2$ persons in each camp), by symmetry, each color wins with the same probability $q < 1/2$, regardless of p. (Notice that there are graphs, such as the empty and complete graphs, on which no one wins.)

Our study reveals that for any given p and ε, there is a constant c such that if one camp has $\frac{n}{2} + c$ individuals at the initial state, then it wins with probability at least $1 - \varepsilon$. The surprising fact here is that c does not depend on n, the population of the community. When $p = 1/2$ and $\varepsilon = .1$, one can set c as small as 6.

▶ **Theorem 2** (The power of few)**.** *Consider the (majority) process on $G(n, 1/2)$. Assume that the Red camp has at least $\frac{n}{2} + 6$ vertices at the initial state, where $n \geq 300$. Then Red wins after the fourth day with probability at least 90%.*

This result can be stated without the Erd́os-Renyi model; one can state an equivalent theorem by choosing the network uniformly, from the set of all graphs on n vertices.

This result reveals an interesting phenomenon, which we call "the power of few". The collective outcome can be extremely sensitive, as a modification of the smallest scale in the initial setting leads to the opposite outcome.

Our result applies in the following equivalent settings.

Model 1. We fix the two camps of size $n/2+6$ and $n/2-6$, respectively, and draw a random graph on their union.

Models 2. Draw a random graph first, let Red be a random subset of $n/2+6$ vertices (chosen uniformly from all subsets of that size), and Blue be the rest.

Model 3. Split the society into two camps of size $n/2$ each. Draw the random graph on their union, then recolor 6 random selected Blue vertices to Red.

Model 4. Split the society into two camps (Red and Blue) of size $n/2 - 6$ each and a "swinging" group (with no color yet) of 12 individuals. Draw the random graph on their union. Now let the swinging group join the Red camp.

With Model 3, we can imagine a balanced election process at the beginning. Then 6 persons change camp. This tiny group already guarantees the final win with an overwhelming odds. Similarly, Model 4 asserts that a swinging group of size 12 decides the outcome.

Our result can also be used to model the phenomenon that outcomes in seemingly identical situations become opposite. Consider two communities, each has exactly n individuals, sharing the same social network. In the first community, Red camp has size $n/2 + c$, and Blue camp has $n/2 - c$. In the second community, Blue camp has $n/2 + c$ and Red camp has $n/2 - c$. If

n is large, there is no way to tell the difference between the two communities. Even if we record everyone's initial opinion, clerical errors will surely swallow the tiny difference of $2c$. However, at the end, the collective opinion will be opposite, with high probability.

Now we state the general result for arbitrary constant density p.

▶ **Theorem 3** (Asymptotic bound)**.** *Let p be a constant in $(0, 1)$ and c_n be a positive integer which may depend on n . Assume that Red has $n/2 + c_n$ individuals in day zero and the random graph is $G(n, p)$. Then Red wins after the fourth day with probability at least $1 - K(p) \max\{n^{-1}, c_n^{-2}\}$, where $K(p)$ depends only on p.*

Both results follow from Theorem 6, which, in a slightly technical form, describes how the process evolves day by day. Our results can be extended to cover the case when there are more than 2 opinions; details will appear in a later paper [14].

1.3 Related results

Our problem is related to a well studied class of *opinion exchange dynamics* problems. In the field of Computer Science, loosely-related processes are studied in *population protocols* [2, 1], where individuals/agents/nodes choose their next state based on that of their neighbors. The most separating difference is the network, as connections in these models often randomly change with time, while our study concerns a fixed network (randomly generated before the process begins).

The survey by Mossel and Tamuz [12] discussed severals models for these problems, including the *DeGroot model* [6], where an individual's next state is a weighted average of its neighbors' current states, the *voter model* [5], where individuals change states by emulating a random neighbor each day. The *majority dynamics* model is in fact the same as ours, and is also more popular than the other two, having been studied in [11, 8, 3]. The key difference, as compared to our study, is in the set-ups. In these earlier papers, each individual chooses his/her initial color uniformly at random. The central limit theorem thus guarantees that with high probability, the initial difference between the two camps is of order $\Theta(\sqrt{n})$. Therefore, these papers did not touch upon the "power of few" phenomenon, which is our key message. On the other hand, they considered sparse random graphs where the density $p = p_n$ goes to zero as $n \to +\infty$.

In [3], Benjamini, Chan, O'Donnell, Tamuz, and Tan considered random graphs with $p \geq \lambda n^{-1/2}$, where λ is a sufficiently large constant, and showed that the dominating color wins with probability at least .4 [3, Theorem 1.2], while conjecturing that this probability in fact tends to 1 as $n \to \infty$. This conjecture was proved by Fountoulakis, Kang, and Makai [8, Theorem 1.1].

▶ **Theorem 4.** *For any $0 < \varepsilon \leq 1$ there is $\lambda = \lambda(\varepsilon)$ such that the following holds for $p \geq \lambda n^{-1/2}$. With probability at least $1 - \varepsilon$, over the choice of the random graph $G(n, p)$ and the choice of the initial state, the dominating color wins after four days.*

For related results on random regular graphs, see [11, 12].

1.4 Extension for sparse random graphs

Note that the results presented in this paper only applies for a constant p, which, in the context of $G(n, p)$, produces *dense graphs*. For *sparse graphs*, i.e. when $p = p_n$ depends on n and tends to 0 as $n \to +\infty$, the main ideas in this paper can be used, but with slightly different algebraic techniques, to obtain a similar result.

▶ **Theorem 5.** *For any $0 < \varepsilon \leq 1$ there is $c = c(\varepsilon)$ such that the following holds for $p \geq (2 + o(1))(\log n)/n$. Assume that Red camp has size at least $n/2 + c/p$ initially, then it wins with probability at least $1 - \varepsilon$.*

The technical changes needed to prove this theorem require rewriting entire proofs with new computations, so we leave the proof to our future paper [14]. Additional information such as the length of the process and the explicit relation between the bound with p and c will also be discussed there. Notice that when p is a constant, this result covers the "Power of Few" phenomenon as a special case, albeit with c much larger than 6. Therefore, the techniques and results in this paper still have merit since they achieve a specific, surprisingly small constant. Theorem 5 no longer holds for $p < (\log n)/n$ as in this case there are, with high probability, isolated vertices. Any of these vertices keeps it original color forever. In this case, the number of Blue vertices converges with time, and we obtain a bound on the limit in [14].

One can use Theorem 5 to derive a "delayed" version (in which Red may need more than 4 days) of Theorem 4, by first proving that with high probability, one side gains an advantage of size at least $C\sqrt{n}$ after the first day, for some constant C. This "majority side" then wins with high probability given $p \geq \lambda n^{-1/2}$ (which satisfies the requirement $p \geq (2 + \mathrm{o}(1))(\log n)/n)$ with λ sufficiently large so that $\lambda C = pC\sqrt{n}$ is large. The detailed argument is in Appendix A.3.

1.5 Notation

- R_t, B_t: Respectively the sets of Red and Blue vertices after day t. (At this point each person has updated their color t times.)
- $\mathrm{I}_t(u) \stackrel{\text{def}}{=} \mathbf{1}_{\{u \in R_t\}}$: $\{0,1\}$-indicator of the event that u is Red after day t.
- $\mathrm{J}_t(u) \stackrel{\text{def}}{=} 2\mathrm{I}_t(u) - 1$: $\{-1,1\}$-indicator of the same event.
- $u \sim v \equiv (u,v) \in E$: Event that u and v are adjacent.
- $\Gamma(v) \stackrel{\text{def}}{=} \{u : u \sim v\}$: The neighborhood of v.
- $W_{uv} \stackrel{\text{def}}{=} \mathbf{1}_{\{u \sim v\}}$ - Indicator of the adjacency between u and v.
- $\mathcal{N}(\mu, \sigma^2)$: The Normal Distribution with mean μ and variance σ^2.
- $\Phi(a,b) \stackrel{\text{def}}{=} (2\pi)^{-1/2} \int_a^b e^{-\frac{x^2}{2}} dx$ and $\Phi(a) \stackrel{\text{def}}{=} \Phi(-\infty, a)$, $\Phi_0(a) \stackrel{\text{def}}{=} \Phi(0, a)$.

1.6 Main Theorem

The main theorem concerns dense graphs, where p is at least a constant. When given appropriate values for the parameters, it implies the "Power of Few" phenomenon in Theorem 2. Before stating the theorem, we define some expressions.

$$
\begin{aligned}
& C_0 \stackrel{\text{def}}{=} 0.56, \quad C_1 \stackrel{\text{def}}{=} \sqrt{3\log 2}, \quad \sigma = \sigma(p) \stackrel{\text{def}}{=} \sqrt{p(1-p)}, \\
P_1 = P_1(n,p,c,\varepsilon_2) &\stackrel{\text{def}}{=} \frac{1/4 + 4C_0^2\left(1 - 2\sigma^2\right)^2 \cdot \frac{n-1}{n-2}}{\left(\sqrt{n-1}\Phi_0\left(\frac{2pc + \min\{p, 1-p\}}{\sigma\sqrt{n-1}}\right) - C_0\frac{1-2\sigma^2}{\sigma} - \frac{C_1 + \varepsilon_2}{2p} - \frac{1}{2\sqrt{n}}\right)^2}, \\
P_2 = P_2(n, \varepsilon_2, \varepsilon_1) &\stackrel{\text{def}}{=} \frac{1}{n}\exp\left(-n\left[(1 - 2\varepsilon_1)\varepsilon_2 - \varepsilon_1\right]\right), \\
P_3 = P_3(n, p, \varepsilon_1) &\stackrel{\text{def}}{=} \frac{1}{n}\exp\left(-\tfrac{1}{2}p^3(2\varepsilon_1 n - 1)^2 + 2n\log 2\right), \\
P_4 = P_4(n,p) &\stackrel{\text{def}}{=} n\exp\left(-\tfrac{2}{9}p^2(n-1)\right).
\end{aligned}
$$

▶ **Theorem 6.** *Let $p \in (0,1)$, $c \in \mathbb{N}$, $n \in \mathbb{N}$, and $\varepsilon_1, \varepsilon_2 > 0$. Define C_0, C_1, σ, P_1, P_2, P_3, P_4 as above. Assume that*

$$2\sqrt{n-1}\Phi_0\left(\frac{2pc+\min\{p,1-p\}}{\sigma\sqrt{n-1}}\right) > \frac{C_1+\varepsilon_2}{p} + \frac{1}{\sqrt{n}} \quad \text{and} \quad 2\varepsilon_1 n > 1. \tag{1}$$

With n^R, n^B being integers such that $n^R + n^B = n$ and $1 \le n^B \le \frac{n}{2} - c$, the election process on $G \sim G(n,p)$ with $|B_0| = n^B$ satisfies the following

1. *With $n_0^B = \frac{n}{2} - c$, $n_1^B = \frac{n-1}{2} - \left(\frac{C_1+\varepsilon_2}{2p}\right)\sqrt{n}$, $n_2^B = \left(\frac{1}{2} - \varepsilon_1\right)n$, $n_3^B = \frac{1}{3}p(n-1)$, $n_4^B = 0$, we have $\mathbf{P}\left(|B_t| \le n_t^B \;\middle|\; |B_{t-1}| \le n_{t-1}^B\right) \ge 1 - P_t$ for each $t = 1,2,3,4$.*
2. *$\mathbf{P}\left(R_4 = V(G) \;\middle|\; |B_0| = n^B\right) \ge 1 - (P_1 + P_2 + P_3 + P_4)$.*

Intuitively, P_i is a upper bound on the probability of some abnormal event happening on Day i. If none of these "catastrophes" occur, the whole population becomes Red after Day Four. Note that if one let $n \to +\infty$ while fixing all other parameters, P_2, P_3 and P_4 all tend to 0, leaving P_1 as the main asymptotic component of the probability bound.

The proof for this theorem has two main parts corresponding to the next two sections. In Section 2, we apply a concentration bound to the number of Red vertices after Day 1 to show that with probability at least $1 - P_1$, this number is at least $n - n_1^B$. In Section 3, we show that with high probability, this $\Omega(n^{1/2})$ advantage leads Red to win, using a *shrinking argument* that bypasses the dependency of the coloring on the current day.

From Theorem 6, one can deduce Theorems 2 and 3 in a few steps. Detailed proofs appear in Appendix A.1.

1.7 Open questions

Let $\rho(k,n)$ be the probability that Red win if its camp has size $n/2 + k$ in the beginning, when $p = .5$. Theorem 2 shows that $\rho(6,n) \ge .9$ (given that n is sufficiently large). In other words, six defectors guarantee Red's victory with an overwhelming odd. In fact, we have $\rho(4,n) \ge .7$ by plugging in the same values for ε_1 and ε_2 with $c = 4$ in Theorem 2's proof. We conjecture that one defector already brings a non-trivial advantage.

▶ **Conjecture 7** (The power of one)**.** *There is a constant $\delta > 0$ such that $\rho(1,n) \ge 1/2 + \delta$ for all sufficiently large n.*

In the following numerical experiment, we run $T = 10000$ independent trials. In each trial, we fix a set of $N = 10000$ nodes with 5001 Red and 4999 Blue (meaning $c = 1$), generate a graph from $G(N, 1/2)$, and simulate the process on the resulting graph. We record the number of wins and the number of days to achieve the win in percentage in Table 1. Among others, we see that Red wins within 3 days with frequency more than .9. The source code for the simulation along with execution instructions can be found online at `https://github.com/thbl2012/majority-dynamics-simulation`.

Imagine that people defect from Blue camp to Red camp one by one. The *value* of the ith defector is defined as $v(i,n) = \rho(i,n) - \rho(i-1,n)$ (where we take $\rho(n,0) = 1/2$). It is intuitive to think that the values of the defectors decrease. (Clearly defector number $n/2$ adds no value.)

▶ **Conjecture 8** (Values of defectors)**.** *For any fixed i and sufficiently large n, we have $v(i,n) \ge v(i+1,n)$.*

It is clear that the Conjecture 8 implies Conjecture 7, with $\delta = \frac{4}{5} = .08$, although the simulation results above suggests that δ can be at least .43.

Table 1 Winners and winning days with their frequencies.

T	p	Red	Blue	Winner	Last day	Count	Frequency
10^4	1/2	5001	4999	Blue	3	496	4.96 %
10^4	1/2	5001	4999	Blue	4	77	0.77 %
10^4	1/2	5001	4999	Blue	5	3	0.03 %
10^4	1/2	5001	4999	Blue	7	1	0.01 %
10^4	1/2	5001	4999	Red	2	25	0.25 %
10^4	1/2	5001	4999	Red	3	9313	93.13 %
10^4	1/2	5001	4999	Red	4	85	0.85 %

2 Day One

At day one, the number of Red and Blue neighbors of each node v are both binomial random variables, with means roughly $n/2+c$ and $n/2-c$ respectively. The central limit theorem then implies that most of their masses are concentrated within an interval of length $\Theta(\sqrt{n})$ around their respective expectations. A subinterval of constant length in this interval will have $\Theta(n^{-1/2})$ mass. Therefore, one expects that the probability that the number of Red exceeds the number of Blues (in that particular neighborhood) is $1/2+\Omega(n^{-1/2})$. Thus, the expectation of Red nodes after the first day is $n/2+\Omega(n^{1/2})$. We consolidate this intuition in the main result of this section, Theorem 9.

Firstly, let us recall a few terms defined in Section 1.6.

$$\sigma \stackrel{\text{def}}{=} \sqrt{p(1-p)}, \quad \text{and } P_1 \stackrel{\text{def}}{=} \frac{\frac{1}{4}+4C_0^2\left(1-2\sigma^2\right)^2\cdot\frac{n-1}{n-2}}{\left(\sqrt{n-1}\Phi_0\left(\frac{2pc+\min\{p,1-p\}}{\sigma\sqrt{n-1}}\right)-\frac{C_0(1-2\sigma^2)}{\sigma}-\frac{C_1+\varepsilon_2}{2p}-\frac{1}{2\sqrt{n}}\right)^2}.$$

Define a new term Q by

$$Q=Q(n,p,c,d) \stackrel{\text{def}}{=} \frac{\frac{1}{4}+4C_0^2\left(1-2\sigma^2\right)^2\cdot\frac{n-1}{n-2}}{\left(\sqrt{n-1}\Phi_0\left(\frac{2pc+\min\{p,1-p\}}{\sigma\sqrt{n-1}}\right)-\frac{C_0(1-2\sigma^2)}{\sigma}-d-\frac{1}{2\sqrt{n}}\right)^2}.$$

Observe that $Q\left(n,p,c,\frac{C_1+\varepsilon_2}{2p}\right)=P_1(n,p,c,\varepsilon_2)$. The following result thus covers the first day in Theorem 6 by just plugging in $d=(C_1+\varepsilon_2)/(2p)$.

▶ **Theorem 9.** *Let $p\in(0,1)$ and c be constants and σ and Q be defined above. Then if $n,n^R,n^B\in\mathbb{N}$ such that $n^R+n^B=n$, $1\le n^B\le\frac{n}{2}-c$. Then for all $d\in\mathbb{R}_{>0}$ and $n\in\mathbb{N}$ such that*

$$\sqrt{n-1}\Phi_0\left(\frac{2pc+\min\{p,1-p\}}{\sigma\sqrt{n-1}}\right)-\frac{C_0(1-2\sigma^2)}{\sigma}>d+\frac{1}{2\sqrt{n}}, \tag{2}$$

we have

$$\mathbf{P}\left(|B_1|>\frac{n-1}{2}-d\sqrt{n}\;\middle|\;|B_0|=n^B\right)\le Q\left(n,p,c,d\right).$$

The crux of the proof relies on some preliminary results regarding the difference of two binomial random variables, which we discuss next.

2.1 Background on difference of Binomial Random Variables

The difference of two binomial random variables with the same probability p be written as a sum of independent random variables, each of which is either a $\text{Bin}(1, p)$ variable or minus of one. A natural way to bound this sum is done via a Berry-Esseen normal approximation.

▶ **Theorem 10** (Berry-Esseen). *Let n be any positive integer. If $X_1, X_2, X_3, \ldots, X_n$ are random variables with zero means, variances $\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2 > 0$, and absolute third moments $\mathbf{E}\left[|X_i|^3\right] = \rho_i < \infty$, we have:*

$$\sup_{x\in\mathbb{R}} \left| \mathbf{P}\left(\sum_{i=1}^{n} X_i \leq x\right) - \Phi\left(\frac{x}{\sigma_X}\right) \right| \leq C_0 \cdot \frac{\sum_{i=1}^{n} \rho_i}{\sigma_X^3},$$

where $\sigma_X = \left(\sum_{i=1}^{n} \sigma_i^2\right)^{1/2}$ and $C_0 = .56$ is a constant.

The original proof by Esseen [7] yielded $C_0 = 7.59$, and this constant has been improved a number of times. The latest work by Shevtsova [13] achieved $C_0 = .56$, which will be used for the rest of the paper. A direct application of this theorem gives the following lemma.

▶ **Lemma 11.** *For $p \in (0, 1)$, $\sigma = \sqrt{p(1-p)}$ and $n_1, n_2 \in \mathbb{N}$ such that $n_1 > n_2$. let $Y_1 \sim Bin(n_1, p)$, $Y_2 \sim Bin(n_2, p)$ be independent random variables. Then for any $d \in \mathbb{R}$,*

$$\mathbf{P}\left(Y_1 > Y_2 + d\right) \geq \frac{1}{2} + \Phi_0\left(\frac{p(n_1 - n_2) - d}{\sigma\sqrt{n_1 + n_2}}\right) - \frac{C_0\left(1 - 2\sigma^2\right)}{\sigma\sqrt{n_1 + n_2}}.$$

Proof. By definition, the difference $Y = Y_1 - Y_2$ can be expressed as

$$X = X_1 + X_2 + X_3 + \cdots + X_{n_1+n_2},$$

where all X_i's are independent and either $X_i \sim \text{Bin}(1, p)$ or $-X_i \sim \text{Bin}(1, p)$. Then $\mathbf{E}[X] = \sum_i \mathbf{E}[X_i] = p(n_1 - n_2)$. For all i, $\mathbf{Var}[X_i] = \sigma^2$ and $\mathbf{E}\left[|X_i - \mathbf{E}[X_i]|^3\right] = p(1-p)^3 + (1-p)p^3 = \sigma^2(1 - 2\sigma^2)$ Applying Theorem 10, we have

$$\begin{aligned}
\mathbf{P}\left(Y_1 \leq Y_2 + d\right) &= \mathbf{P}\left(X - \mathbf{E}[X] \leq d - p(n_1 - n_2)\right) \\
&\leq \Phi\left(\frac{d - p(n_1 - n_2)}{\sigma_X}\right) + C_0 \frac{\sum_i \mathbf{E}\left[|X_i - \mathbf{E}[X_i]|^3\right]}{\sigma_X^3} \\
&= \Phi\left(\frac{d - p(n_1 - n_2)}{\sigma\sqrt{n_1 + n_2}}\right) + C_0 \frac{\sigma^2(1 - 2\sigma^2)(n_1 + n_2)}{\sigma^3(n_1 + n_2)^{3/2}} \\
&= \frac{1}{2} - \Phi_0\left(\frac{p(n_1 - n_2) - d}{\sigma\sqrt{n_1 + n_2}}\right) + \frac{C_0(1 - 2\sigma^2)}{\sigma\sqrt{n_1 + n_2}},
\end{aligned}$$

and the claim follows by taking the complement event. ◀

▶ **Lemma 12.** *Let $p \in (0, 1)$ be a constant and $\sigma = \sqrt{p(1-p)}$, $X_1 \sim Bin(n_1, p)$ and $X_2 \sim Bin(n_2, p)$ be independent r.v.s. Then for any integer d,*

$$\mathbf{P}\left(X_1 = X_2 + d\right) \leq \frac{2C_0\left(1 - 2\sigma^2\right)}{\sigma\sqrt{n_1 + n_2}}.$$

Proof. Let $n = n_1 + n_2$ and $\mu = \mathbf{E}[X_1] - \mathbf{E}[X_2] = p(n_1 - n_2)$. Fix $\varepsilon \in (0, 1)$, by the same computations in Lemma 11, we have

$$\mathbf{P}\left(X_1 - X_2 \leq d - \varepsilon\right) \geq \Phi\left(\frac{d - \mu - \varepsilon}{\sigma\sqrt{n}}\right) - \frac{C_0\left(1 - 2\sigma^2\right)}{\sigma\sqrt{n}},$$

$$\mathbf{P}\left(X_1 - X_2 < d + \varepsilon\right) \leq \Phi\left(\frac{d - \mu + \varepsilon}{\sigma\sqrt{n}}\right) + \frac{C_0\left(1 - 2\sigma^2\right)}{\sigma\sqrt{n}}.$$

It follows that

$$\mathbf{P}(X_1 = X_2 + d) \;\le\; \mathbf{P}(d - \varepsilon < X_1 - X_2 < d + \varepsilon)$$
$$\le\; \Phi\left(\frac{d-\mu+\varepsilon}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{d-\mu-\varepsilon}{\sigma\sqrt{n}}\right) + \frac{2C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n}}.$$

Letting $\varepsilon \to 0$, we obtain the desired claim. ◀

Proof of Theorem 9

Recall that $\left|R_1\right| = n - \left|B_1\right|$. Our goal is to lower-bound the probability that $\left|R_1\right| < \frac{n+1}{2} + d\sqrt{n}$ for any given constant d. Recall the indicator $\mathrm{I}_1(v)$ which is 1 if v is Red after Day One and 0 otherwise. We have:

$$\left|R_1\right| = \sum_{v\in V} \mathrm{I}_1(v).$$

Since the indicators are not independent, a natural choice for bounding their sum is to use Chebysev's inequality. We proceed in two steps:

1. Lower-bound $\mathbf{E}\left[\left|R_1\right|\right]$ by lower-bounding each term $\mathbf{E}\left[\mathrm{I}_1(v)\right]$.
2. Upper-bound $\mathbf{Var}\left[\left|R_1\right|\right]$ by upper-bounding each $\mathbf{Var}\left[\mathrm{I}_1(v)\right]$ and $\mathbf{Cov}\left[\mathrm{I}_1(v), \mathrm{I}_1(v')\right]$.

▷ Claim 13. $\mathbf{E}\left[\left|R_1\right|\right] \;\ge\; \dfrac{n+1}{2} + \left(T(n,p,c,d) + d\right)\sqrt{n},$

where $T(n,p,c) \;\stackrel{\text{def}}{=}\; \sqrt{n-1}\,\Phi_0\left(\dfrac{2pc + \min\{p, 1-p\}}{\sigma\sqrt{n-1}}\right) - \dfrac{C_0\left(1-2\sigma^2\right)}{\sigma}.$

Proof. For each vertex v, let $d_0^R(v)$ and $d_0^B(v)$ respectively be the numbers of its Red and Blue neighbors before the first day. By our rule, the event $\{v \in R_1\}$ is equivalent to $d_0^R(v) > d_0^B(v)$ if $v \in B_0$, i.e. $\mathrm{I}_0(v) = 0$, and to $d_0^R(v) \ge d_0^B(v)$ if $v \in R_0$, i.e. $\mathrm{I}_0(v) = 1$. This implies

$$\forall v \in V.\left[v \in R_1 \;\Leftrightarrow\; d_0^R(v) + \mathrm{I}_0(v) > d_0^B(v)\right]. \tag{3}$$

Note that $d_0^R(v) \sim \mathrm{Bin}\left(n^R - \mathrm{I}_0(v), p\right)$ and $d_0^B(v) \sim \mathrm{Bin}\left(n^B + \mathrm{I}_0(v) - 1, p\right)$. By Lemma 11, we have:

$$\mathbf{E}\left[\mathrm{I}_1(v)\right] = \mathbf{P}\left(v \in R_1\right) = \mathbf{P}\left(d_0^R(v_i) + \mathrm{I}_0(v_1) > d_0^B(v_i)\right)$$
$$\ge \frac{1}{2} + \Phi_0\left(\frac{p\left(n^R - n^B + 1 - 2\mathrm{I}_0(v)\right) + \mathrm{I}_0(v)}{\sigma\sqrt{n^R + n^B - 1}}\right) - \frac{C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n^R+n^B-1}}$$
$$= \frac{1}{2} + \Phi_0\left(\frac{2pc + p_v}{\sigma\sqrt{n-1}}\right) - \frac{C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n-1}},$$

where $p_v \stackrel{\text{def}}{=} p\left(1 - \mathrm{I}_0(v)\right) + (1-p)\mathrm{I}_0(v) \ge \min\{p, 1-p\}$. Now

$$\mathbf{E}\left[\left|R_1\right|\right] = \sum_{v\in V} \mathbf{E}\left[\mathrm{I}_1(v)\right] \ge \sum_{v\in V} \Phi_0\left(\frac{2pc + \min\{p,1-p\}}{\sigma\sqrt{n-1}}\right) + n\left(\frac{1}{2} - \frac{C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n-1}}\right)$$
$$\ge \frac{n}{2} + \left[\sqrt{n-1}\,\Phi_0\left(\frac{2pc + \min\{p,1-p\}}{\sigma\sqrt{n-1}}\right) - \frac{C_0\left(1-2\sigma^2\right)}{\sigma}\right]\sqrt{n}$$
$$= \frac{n}{2} + \left(T(n,p,c,d) + d + \frac{1}{2\sqrt{n}}\right)\sqrt{n} = \frac{n+1}{2} + \left(T(n,p,c,d) + d\right)\sqrt{n},$$

The proof is complete. ◁

▷ Claim 14. $\mathbf{Var}\left[\left|R_1\right|\right] \;\leq\; \dfrac{n}{4} + 4C_0^2\left(1-2\sigma^2\right)^2 \cdot \dfrac{n(n-1)}{n-2}$.

Proof. We first have: $\mathbf{Var}\left[\left|R_1\right|\right] = \sum_{i=1}^{k} \mathbf{Var}\left[\mathrm{I}_1(v_1)\right] + 2\sum_{v_1 \neq v_2} \mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right]$.
The variances $\mathbf{Var}\left[\mathrm{I}_1(v)\right]$ are easy due to $\mathrm{I}_1(v)$ being a Bernoulli r.v.:

$$\mathbf{Var}\left[\mathrm{I}_1(v)\right] = \mathbf{E}\left[\mathrm{I}_1(v)\right]\left(1 - \mathbf{E}\left[\mathrm{I}_1(v)\right]\right) = \frac{1}{4} - \left(\mathbf{E}\left[\mathrm{I}_1(v)\right] - \frac{1}{2}\right)^2 \leq \frac{1}{4}. \tag{4}$$

Bounding the covariance $\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right]$ for two distinct vertices v_1, v_2 requires a bit more care, as the indicators are not independent. By definition

$$\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] \;=\; \mathbf{P}\left(v_1, v_2 \in R_1\right) - \mathbf{P}\left(v_1 \in R_1\right)\mathbf{P}\left(v_2 \in R_1\right).$$

Consider the event $\{v_1, v_2 \in R_1\}$; $\mathbf{P}\left(v_1, v_2 \in R_1\right)$ can be written as

$$\mathbf{P}\left(v_1, v_2 \in R_1 | v_1 \sim v_2\right)\mathbf{P}(v_1 \sim v_2) + \;\mathbf{P}\left(v_1, v_2 \in R_1 | v_1 \not\sim v_2\right)\mathbf{P}(v_1 \not\sim v_2).$$

Notice that after we specify the adjacency between v_1 and v_2, the remaining vertices in the neighborhoods of v_1 and v_2 are independent. Letting $a_i = \mathbf{P}(v_i \in R_1 \mid v_1 \sim v_2)$, $b_i = \mathbf{P}(v_i \in R_1 \mid v_1 \not\sim v_2)$ and using shorthand $q \overset{\text{def}}{=} 1-p$, we have

$$\mathbf{P}\left(v_1, v_2 \in R_1\right) = pa_1a_2 + (1-p)b_1b_2.$$

Now consider $\mathbf{P}\left(v_1 \in R_1\right)\mathbf{P}\left(v_2 \in R_1\right)$. Splitting up the two events by $\{v_1 \sim v_2\}$ gives $\mathbf{P}\left(v_1 \in R_1\right) = pa_1 + qb_1$ and $\mathbf{P}\left(v_2 \in R_1\right) = pa_2 + qb_2$. Putting everything together, we have

$$\begin{aligned} \mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] &= pa_1a_2 + qb_2b_2 - \left(pa_1 + qb_1\right)\left(pa_2 + qb_2\right) \\ &= pq(a_1 - b_1)(a_2 - b_2) = \sigma^2(a_1 - b_1)(a_2 - b_2). \end{aligned} \tag{5}$$

We next analyze the relationship between a_1 and b_1. (The analysis for a_2 and b_2 is similar). Define

$$\begin{aligned} m^R &\overset{\text{def}}{=} \left|R_0 \setminus \{v_1, v_2\}\right| &&= n^R - \left(\mathrm{I}_0(v_1) + \mathrm{I}_0(v_2)\right), \\ m^B &\overset{\text{def}}{=} \left|B_0 \setminus \{v_1, v_2\}\right| &&= n^B + \left(\mathrm{I}_0(v_1) + \mathrm{I}_0(v_2)\right) - 2, \\ d^R &\overset{\text{def}}{=} \left|\left(\Gamma(v_1) \cap R_0\right) \setminus \{v_2\}\right| &&= d_0^R(v_1) - \mathrm{I}_0(v_2)W_{v_1v_2} \\ d^B &\overset{\text{def}}{=} \left|\left(\Gamma(v_1) \cap B_0\right) \setminus \{v_2\}\right| &&= d_0^B(v_1) + \left(\mathrm{I}_0(v_2) - 1\right)W_{v_1v_2} \end{aligned}$$

We have $m^R + m^B = n-2$, $d^R \sim \mathrm{Bin}(m^R, p)$, $d^B \sim \mathrm{Bin}(m^B, p)$ and $d_0^R(v_1) - d_0^B(v_1) = d^R - d^B + \mathrm{J}_0(v_2)W_{v_1v_2}$. Now we can rewrite a_1 and b_1 using (3) in terms of the above:

$$\begin{aligned} a_1 &= \mathbf{P}\left(d_0^R(v_1) + \mathrm{I}_0(v_1) > d_0^B(v_1) \;\middle|\; v_1 \sim v_2\right) = \mathbf{P}\left(d^R - d^B > -\mathrm{J}_0(v_2) - \mathrm{I}_0(v_1)\right), \\ b_1 &= \mathbf{P}\left(d_0^R(v_1) + \mathrm{I}_0(v_1) > d_0^B(v_1) \;\middle|\; v_1 \not\sim v_2\right) = \mathbf{P}\left(d^R - d^B > -\mathrm{I}_0(v_1)\right). \end{aligned}$$

Case analysis on $\mathrm{J}_0(v_2)$: $\begin{cases} \mathrm{J}_0(v_2) = -1 & \Longrightarrow \quad a_1 - b_1 = \mathbf{P}\left(d^R - d^B = -\mathrm{I}_0(v_1)\right) \\ \mathrm{J}_0(v_2) = 1 & \Longrightarrow \quad b_1 - a_1 = \mathbf{P}\left(d^R - d^B = -1 - \mathrm{I}_0(v_1)\right). \end{cases}$

In any case, by Lemma 12 we have $\left|a_1 - b_1\right| \leq \dfrac{2C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{m^R + m^B}} = \dfrac{2C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n-2}}$.
The same analysis for a_2 and b_2, and Equation (5) then imply

$$\mathbf{Cov}\left[\mathrm{I}_1(v_1), \mathrm{I}_1(v_2)\right] \leq \sigma^2 \cdot \frac{2C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n-2}} \cdot \frac{2C_0\left(1-2\sigma^2\right)}{\sigma\sqrt{n-2}} = \frac{4C_0^2\left(1-2\sigma^2\right)^2}{n-2}. \tag{6}$$

Equations (4) and (6) together yield

$$\mathbf{Var}\left[|R_1|\right] \le \frac{1}{4}\cdot n + 2\cdot \frac{4C_0^2\left(1-2\sigma^2\right)^2}{n-2}\cdot\binom{n}{2} = \frac{n}{4} + 4C_0^2\left(1-2\sigma^2\right)^2\cdot\frac{n(n-1)}{n-2}. \tag{7}$$

The proof is complete. ◁

From Claims 13 and 14, a standard Chebyshev's inequality gives

$$\begin{aligned}\mathbf{P}\Big(|B_1| > \tfrac{n-1}{2} - d\sqrt{n}\Big) &= \mathbf{P}\Big(|R_1| < \tfrac{n+1}{2} + d\sqrt{n}\Big) \le \frac{\mathbf{Var}\left[|R_1|\right]}{\left(\mathbf{E}\left[|R_1|\right] - \tfrac{n+1}{2} - d\sqrt{n}\right)^2} \\ &\le \frac{\tfrac{n}{4} + 4C_0^2\left(1-2\sigma^2\right)^2\cdot\tfrac{n(n-1)}{n-2}}{T(n,p,c,d)^2 n} = Q(n,p,c,d).\end{aligned}$$

The proof of Theorem 9 is complete. This theorem forms the first part of Theorem 6, which shrinks the Blue camp from size $\frac{n}{2}-c$ to $\frac{n}{2}-\Omega(\sqrt{n})$. We state explicitly the relevant result to wrap up this section.

▶ **Corollary 15.** *For any $p\in(0,1)$, $c>0$, $\varepsilon_2>0$ and $n\in\mathbb{N}$, if the Blue side starts with at most $\frac{n}{2}-c=n_0^B$ members, it shrinks to size at most $\frac{n-1}{2}-\left(\frac{C_1+\varepsilon_2}{2p}\right)\sqrt{n}=n_1^B$ with probability at least $1-P_1(n,p,c,\varepsilon_2)$.*

3 Day Two and after

Next, we analyze the situation after the first day. Clearly, if one fixes the coloring after Day 1 and examine the graph, its distribution is no longer $G(n,p)$. Therefore, we cannot apply the same method in proving Theorem 9 for later days. Instead, we use "shrinking arguments" to argue that it is likely for the Blue camp to monotonously shrink to empty, regardless of the choice of its members, due to G's structure.

The core of our shrinking argument is *Hoeffding's inequality*, a classical result that gives exponentially small probability tails for sums of independent random variables.

▶ **Theorem 16** (Hoeffding's inequality)**.** *Let $\{X_i\}_{i=1}^n$ be independent random variables and $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$, such that for all $i=1,2,\cdots,n$, $a_i\le X_i\le b_i$ almost surely. Then for $X=X_1+X_2+\cdots+X_n$, we have*

$$\max\Big\{\mathbf{P}\left(X-\mathbf{E}\left[X\right]\ge t\right),\ \mathbf{P}\left(X-\mathbf{E}\left[X\right]\le -t\right)\Big\} \le \exp\left(-\frac{2t^2}{\sum_{i=1}^n(b_i-a_i)^2}\right).$$

The proof of Hoeffding's inequality is available in most graduate level probability textbooks, e.g. [15]. The original proof given by Hoeffding appeared in [9].

A simple yet useful shrinking argument is that, in the $G(n,p)$ model, it is with high probability that all vertices in G have many neighbors, so a small enough Blue camp will not be able to influence anyone by a majority, thus inevitably vanishes the next day.

▶ **Lemma 17.** *For $p\in(0,1)$ and $n\in\mathbb{N}_{>1}$, with probability at least*

$$1-n\exp\left(-\tfrac{2}{9}p^2(n-1)\right) = 1-P_4(n,p),$$

G is such that all vertices have more than $\frac{2}{3}p(n-1)$ neighbors, thus any choice of the Blue camp of at most $\frac{1}{3}p(n-1)=n_3^B$ members shrinks to $0=n_4^B$ the next day.

The proof is standard using the Hoeffding bound, so we refer to Appendix A.2 for details.

This simple lemma forms the first block of our shrinking argument. The overall aim is to argue that with high probability, G is such that a Blue camp of size $\frac{n}{2} - \mathrm{O}(\sqrt{n})$ in Day 1 will inevitably be reduced to size $\mathrm{O}(n)$ in Day 2, and then to size less than $\frac{1}{2}p(n-1)$ in Day 3, then vanishes in the fourth day by Lemma 17. The remaining blocks, which correspond to days before the fourth, will require a more complicated argument.

▶ **Lemma 18.** *Let $p \in (0,1)$, $n, n_0 \in \mathbb{N}$, $n_0 < \frac{n}{2}$. Then for all $m \in \mathbb{N}, m \leq n$, with probability at least*

$$1 - \frac{4^n}{n} \exp\left(-\frac{2p^2(n-2n_0-1)^2 m}{n+m-2}\right),$$

G is such that any choice of the Blue camp of at most n_0 members shrinks to below m in the next day.

Proof. Consider a subset S of V with m elements. We will first bound the probability that S entirely turn Blue the next day. Let (R, B) be the initial coloring with $|B| = n_0 < n - n_0 = |R|$. For each $v \in V$, let $\mathrm{dif}(v) \overset{\text{def}}{=} |\Gamma(v) \cap R| - |\Gamma(v) \cap B|$, and let $\mathrm{dif}(S) \overset{\text{def}}{=} \sum_{v \in S} \mathrm{dif}(v)$. We break down each $\mathrm{dif}(v)$ and $\mathrm{dif}(S)$ as follows:

$$\begin{aligned}
\mathrm{dif}(v) &= \sum_{u \in R \cap S} W_{vu} + \sum_{u \in R \setminus S} W_{vu} - \sum_{u \in B \cap S} W_{vu} - \sum_{u \in B \setminus S} W_{vu} \\
\mathrm{dif}(S) &= \sum_{v \in S} \sum_{u \in R \cap S} W_{vu} + \sum_{v \in S} \sum_{u \in R \setminus S} W_{vu} - \sum_{v \in S} \sum_{u \in B \cap S} W_{vu} - \sum_{v \in S} \sum_{u \in B \setminus S} W_{vu}.
\end{aligned} \tag{8}$$

We have

$$\sum_{v \in S} \sum_{u \in R \cap S} W_{vu} = \sum_{v \in R \cap S} \sum_{u \in R \cap S} W_{vu} + \sum_{v \in B \cap S} \sum_{u \in R \cap S} W_{vu} = \sum_{\{u,v\} \subset S \cap R} 2W_{uv} + \sum_{v \in B \cap S} \sum_{u \in R \cap S} W_{vu}.$$

Similarly, $\displaystyle\sum_{v \in S} \sum_{u \in B \cap S} W_{vu} = \sum_{\{u,v\} \subset S \cap B} 2W_{uv} + \sum_{v \in B \cap S} \sum_{u \in R \cap S} W_{vu}$.
Substituting back into Equation 8, we get

$$\mathrm{dif}(S) = \sum_{\{u,v\} \subset S \cap R} (2W_{uv}) - \sum_{\{u,v\} \subset S \cap B} (2W_{uv}) + \sum_{u \in S} \sum_{v \in R \setminus S} W_{uv} - \sum_{u \in S} \sum_{v \in R \setminus B} W_{uv}.$$

This is now a sum of independent variables, so we can apply Theorem 16. Firstly,

$$\mathbf{E}[\mathrm{dif}(S)] = p|S|(|R| - |B|) - p(|S \cap R| - |S \cap B|) \geq pm(n - 2n_0 - 1). \tag{9}$$

Moreover, each W_{uv} takes values in $[0,1]$ (a range of length 1) and $2W_{uv}$ takes values in $[0,2]$ (a range of length 2), so the sum of squares of these lengths are

$$\begin{aligned}
F &= 4\binom{|S \cap R|}{2} + 4\binom{|S \cap B|}{2} + |S||R \setminus S| + |S||B \setminus S| \\
&= |S|(n - 2 + |S|) - 4\,|S \cap R|\,|S \cap B| \leq m(n-2+m).
\end{aligned}$$

By Hoeffding's inequality:

$$\begin{aligned}
\mathbf{P}(S \subseteq B_1 \mid R_0, B_0) \;&\leq\; \mathbf{P}(\mathrm{dif}(S) \leq 0) \;=\; \mathbf{P}(\mathrm{dif}(S) - \mathbf{E}[\mathrm{dif}(S)] \leq -\mathbf{E}[\mathrm{dif}(S)]) \\
&\leq\; \exp\left(-\frac{\mathbf{E}[\mathrm{dif}(S)]^2}{F}\right) \;\leq\; \exp\left(-\frac{2p^2(n-2n_0-1)^2 m}{n-2+m}\right).
\end{aligned}$$

Applying a double union bound over choices of S and (R, B), noting that there are $\binom{n}{n_0}\binom{n}{m} \leq 4^n/n$ choices, we have

$$\mathbf{P}\left(\exists(R,B).\exists S.\ |B| = n_0, |S| = m, S \subset B_1\right) \leq \frac{4^n}{n}\exp\left(-\frac{2p^2(n-2n_0-1)^2 m}{n+m-2}\right).$$

Taking the complement event, we get the desired result. ◀

Lemma 18 turns out to be sufficient for the remaining blocks of our shrinking argument. The following lemmas are direct corollaries of Lemma 18.

▶ **Lemma 19.** *For $p \in (0, 1)$, $\varepsilon_1 \in \left(0, \frac{1}{2}\right)$ and $n \in \mathbb{N}_{>1}$, with probability at least*

$$1 - \frac{1}{n}\exp\left(-\tfrac{1}{2}p^3(2\varepsilon_1 n - 1)^2 + 2n\log 2\right) = 1 - P_3(n, p, \varepsilon_1),$$

G is such that any choice of the Blue camp of at most $\left(\frac{1}{2} - \varepsilon_1\right)n = n_2^B$ members shrinks to size at most $\frac{1}{3}p(n-1) = n_3^B$ the next day.

▶ **Lemma 20.** *With $C_1 = (3\log 2)^{1/2}$ defined in Section 1.6, let $p \in (0, 1)$, $n \in \mathbb{N}$ and $\varepsilon_1, \varepsilon_2 > 0$, then with probability at least*

$$1 - \exp\left(-n\left[(1-2\varepsilon_1)\varepsilon_2 - \varepsilon_1\right] - \log n\right) = 1 - P_2(n, \varepsilon_2, \varepsilon_1), \tag{10}$$

G is such that any choice of the Blue camp of at most $\frac{n-1}{2} - \left(\frac{C_1+\varepsilon_2}{2p}\right)\sqrt{n} = n_1^B$ members shrinks to size at most $\left(\frac{1}{2} - \varepsilon_1\right)n = n_2^B$ the next day.

A routine calculation shows that for appropriate choices of values for ε_1 and ε_2 such that $\varepsilon_1 < \varepsilon_2/(1 + 2\varepsilon_2)$, the bound in Equation (10) tends to 1 as $n \to +\infty$. Detailed proofs of Lemmas 19 and 20 are in Appendix A.2. These lemmas and Lemma 17, together with Corollary 15 form the complete "chain of shrinking" for the number of Blue vertices to reach 0 in four days, hence wrapping up the proof of Theorem 6.

4 Conclusion

The majority dynamics scheme on a network of n individuals is a process where each person is assigned an initial color, which changes daily to match the majority among their neighbors. The main results in this paper reveal a surprising facts. When the underlying network is a random $G(n, p)$ graph, for any given constants p and ε, there is a constant $c = c(p, \varepsilon)$ such that if one color has $\frac{n}{2} + c$ members in the initial state, then with probability at least $1 - \varepsilon$, it covers the whole network in just four days, regardless of n.

Our main result, Theorem 6, yields an explicit lower-bound based on n, p and c for the probability that the side with the initial majority wins. It has two important implications. The first is the *Power of Few* phenomenon (Theorem 2), which shows that when $p = 1/2$ and $\varepsilon = .1$, c can be set to just 6, meaning six extra people is all it takes to win a large election with overwhelming odds. The second is an asymptotic dependency between the ε, n and c (Theorem 3), which shows that for any fixed p, there is a constant $K(p)$ such that choosing n and c both large enough so that $K(p)\max\{n^{-1}, c^{-2}\} < \varepsilon$ will ensure that the winning probability is at least $1 - \varepsilon$.

The main idea behind Theorem 6 involves shrinking the set of Blue, the side with the initial minority, from $(n/2 - c)$ to 0 members in the course of four days. This chain of shrinking goes from $(n/2 - c)$ through $(n/2 - \Omega(\sqrt{n}))$, $(1/2 - \Omega(1))n$ and $(1/2 - \Omega(1))pn$, eventually reaching 0 after Day 4. There is a small probability that the shrinking fails to occur on each day, and their sum is the bound we obtained in the theorem's statement.

Although the results in this paper only applies for dense $G(n,p)$ graphs, we do cover sparse graphs in a separate in-progress paper [14], where we obtain the Power of Few phenomenon for $p = \Omega((\log n)/n)$, and discuss the end result (other than a win) for lower values of p. We nevertheless included one of the main proven results of the upcoming paper (Theorem 5), and used it to prove the main theorem in the paper [8] by Fountoulakis, Kang and Makai in Appendix A.3.

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A Appendix

In this appendix we provided detailed proofs for claims and lemmas not proven in the main text. We begin with some useful general probabilistic lemmas that are used throughout these proofs.

A.1 Proofs of Theorems 2 and 3

Proof of Theorem 2. Assume Theorem 6. Observe that if the conditions in (1) hold for some value of n, then they hold for all larger values of n. Let $n = 300$, $\varepsilon_1 = .15$ and $\varepsilon_2 = .3$ (along with $p = 1/2$ and $c = 6$), we have the condition in Equation (1) satisfied. Furthermore, a routine calculation shows that

$$P_1 \le .08454, \quad P_2 < .00001, \quad P_3 < .00001, \quad P_4 \le .00001,$$

which implies that $\mathbf{P}(B_4 \neq \varnothing) < 0.1$ or equivalently that Red wins in the fourth day with probability at least .9 (conditioned on the event $|B_0| = n^B \le \frac{n}{2} - c$). ◀

Proof of Theorem 3. In this proof, only n and $c = c_n$ can vary. We can assume, without loss of generality, that $c_n \le n/2$. Assuming Theorem 6, we choose (constants) $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_2(1 - 2\varepsilon_1) - \varepsilon_1 > 0$, then a routine calculation shows that $P_2, P_3, P_4 = \mathrm{o}(n^{-2})$ and $P_1 = \Omega(n^{-1})$, so $P_1 + P_2 + P_3 + P_4 = P_1 + \mathrm{o}(1)$. We have, for sufficiently large c_n and sufficiently large n,

$$\sqrt{n-1}\ \Phi_0\left(\tfrac{2pc_n + \min\{p, 1-p\}}{\sigma\sqrt{n-1}}\right) - \tfrac{C_0(1-2\sigma^2)}{\sigma} - \tfrac{C_1+\varepsilon_2}{2p} - \tfrac{1}{2\sqrt{n}} \ \ge\ \tfrac{\sqrt{n}}{2}\ \Phi_0\left(\tfrac{2c_n\sqrt{p}}{\sqrt{n}}\right) \ =\ \tfrac{T(n)}{2},$$
$$1/4 + 4C_0^2(1-2\sigma^2)^2\tfrac{n-1}{n-2} \ \le\ 1/4 + 4\cdot 0.6^2 \cdot 1.5 \ <\ 3.$$

Thus, $P_1 \le 12T(n)^{-2}$. It then suffices to show $T(n) \ge H(p)\min\{c_n^{-2}, n^{-1}\}$ for some term $H(p)$ depending solely on p. Consider 2 cases:

If $c_n \ge \sqrt{n}$, then: $T(n) \ \ge\ \sqrt{n}\cdot\Phi_0(2\sqrt{p}) \ \ge\ \sqrt{n}\cdot 2\sqrt{p}\cdot\dfrac{\Phi_0(2)}{2} \ =\ \sqrt{p}\ \Phi_0(2)\sqrt{n}.$

If $c_n < \sqrt{n}$, then: $T(n) \ \ge\ \sqrt{n}\cdot\dfrac{2c_n\sqrt{p}}{\sqrt{n}}\cdot\dfrac{\Phi_0(2)}{2} \ =\ \sqrt{p}\ \Phi_0(2)\ c_n.$

In any case, $T(n) \ge H(p)\min\{c_n^{-2}, n^{-1}\}$, for $H(p) = \Phi_0(2)\sqrt{p}$, as desired. ◀

A.2 Proofs for lemmas in Day Two

We provide proofs for Lemmas 19 and 20 in Section 3.

Proof. In a $G(n,p)$ graph, $d(v)$ is a sum of $(n-1)$ $\mathrm{Bin}(1,p)$ random variables, so Theorem 16 implies that for any $u \in V$,

$$\mathbf{P}\big(d(u) \le \tfrac{2}{3}p(n-1)\big) \le \mathbf{P}\big(d(u) - \mathbf{E}[d(u)] \le -\tfrac{1}{3}p(n-1)\big) \le \exp\left(-\tfrac{2}{9}p^2(n-1)\right).$$

By a union bound, the probability that all vertices have more than $\frac{2}{3}p(n-1)$ neighbors is at least $1 - n\exp\left(-\frac{2}{9}p^2(n-1)\right) = 1 - P_4(n,p)$. Given this, a Blue camp of size $\frac{1}{3}p(n-1)$ surely vanishes the next day since it cannot form a majority in any vertex's neighborhood. The result then follows. ◀

Proof of Lemma 19. Let $n_2 \stackrel{\text{def}}{=} \left\lfloor\left(\frac{1}{2} - \varepsilon_1\right)n\right\rfloor$ and $m \stackrel{\text{def}}{=} \left\lceil\frac{1}{3}p(n-1)\right\rceil$. Lemma 18 implies that the G satisfies that every Blue set of at most n_2 vertices shrinks to size $m-1$ with probability at least

$$1 - \frac{4^n}{n}\exp\left(-\frac{2p^2(n-2n_2-1)^2m}{n+m-2}\right) = 1 - \frac{1}{n}\exp\left(-\frac{2p^2(n-2n_2-1)^2m}{n+m-2} + 2n\log 2\right).$$

Since $n_2 \leq \left(\frac{1}{2} - \varepsilon_1\right) n$, $(n - 2n_2 - 1)^2 \geq (2\varepsilon_1 n - 1)^2$. Furthermore, $m \geq \frac{1}{3}p(n-1)$ so $\frac{m}{n+m-2} \geq \frac{p(n-1)/3}{n+p(n-1)/3-1} = \frac{p}{3+p} \geq \frac{p}{4}$. Therefore

$$\frac{1}{n}\exp\left(-\frac{2p^2(n-2n_2-1)^2 m}{n+m-2} + 2n\log 2\right) \leq \frac{1}{n}\exp\left(-\frac{1}{2}p^3(2\varepsilon_1 n - 1)^2 + 2n\log 2\right).$$

The result then follows. ◀

Proof of Lemma 20. Let $n_2 \stackrel{\text{def}}{=} \left\lfloor \frac{n-1}{2} - \left(\frac{C_1+\varepsilon_2}{2p}\right)\sqrt{n} \right\rfloor$ and $m \stackrel{\text{def}}{=} \left\lceil \left(\frac{1}{2} - \varepsilon_1\right) n \right\rceil$. Lemma 18 implies that the G satisfies that every Blue set of at most n_2 vertices shrinks to size $m-1$ with probability at least

$$1 - \frac{4^n}{n}\exp\left(-\frac{2p^2(n-2n_2-1)^2 m}{n+m-2}\right) = 1 - \frac{1}{n}\exp\left[-\left(\frac{2p^2(n-2n_2-1)^2 m}{n+m-2} - 2n\log 2\right)\right].$$

Since $\frac{m}{n+m-2} \geq \frac{m}{n+m} \geq \frac{1-2\varepsilon_1}{3-2\varepsilon_1}$ and $n - 2n_2 - 1 \geq \left(\frac{C_1+\varepsilon_2}{p}\right)\sqrt{n}$, we can bound the exponent of the RHS of the above as follows

$$\begin{aligned}
\frac{2p^2(n-2n_2-1)^2 m}{n+m-2} - 2n\log 2 \ &\geq\ 2p^2\left(\frac{C_1+\varepsilon_2}{p}\right)^2 \frac{1-2\varepsilon_1}{3-2\varepsilon_1} n - \frac{2C_1^2}{3} n \\
=\ \frac{2n}{3-2\varepsilon_1}\left[(C_1+\varepsilon_2)^2(1-2\varepsilon_1) - \frac{C_1^2}{3}(3-2\varepsilon_1)\right] \ &\geq\ \frac{2n}{3}\left[\varepsilon_2(\varepsilon_2+2C_1)(1-2\varepsilon_1) - \frac{4}{3}\varepsilon_1 C_1^2\right] \\
\geq\ \frac{2n}{3}\left[2.8\varepsilon_2(1-2\varepsilon_1) - 2.8\varepsilon_1\right] \ =\ \frac{5.6n}{3}\left[\varepsilon_2(1-2\varepsilon_1) - \varepsilon_1\right] \ &\geq\ n\left[\varepsilon_2(1-2\varepsilon_1) - \varepsilon_1\right].
\end{aligned}$$

Note that we have used the facts that $C_1 > 1.4$ and $\log_2 < .7$. The proof is complete. ◀

A.3 Proof of Fountoulakis et al's Theorem from Theorem 5

We provide the proof of the main theorem in [8] (Theorem 4) with our Theorem 5.

Proof. Assume Theorem 5. Let R_0 and B_0 respectively be the initial Red and Blue camps. Fix a constant $0 < c' \leq \varepsilon/6$. $|R_0| \sim \mathrm{Bin}(n, 1/2)$ since it is a sum of $\mathrm{Bin}(1, 1/2)$ variables. An application of the Berry-Esseen theorem (Theorem 10; with $C_0 = .56$) implies that

$$\mathbf{P}\left(|R_0| - \frac{n}{2} \leq c'\sqrt{n}\right) \leq \Phi(2c') + \frac{C_0}{\sqrt{n}} \quad \text{and} \quad \mathbf{P}\left(|R_0| - \frac{n}{2} \leq -c'\sqrt{n}\right) \geq \Phi(-2c') - \frac{C_0}{\sqrt{n}},$$

Thus

$$\begin{aligned}
\mathbf{P}\left(\left||R_0| - \frac{n}{2}\right| \leq c'\sqrt{n}\right) \ &\leq\ \left(\Phi(2c') + \frac{C_0}{\sqrt{n}}\right) - \left(\Phi(-2c') - \frac{C_0}{\sqrt{n}}\right) \\
&\leq\ \Phi(-2c', 2c') + \frac{2C_0}{\sqrt{n}} \ \leq\ \frac{4c'}{\sqrt{2\pi}} + \frac{2C_0}{\sqrt{n}} \ \leq\ \frac{\varepsilon}{3} + \frac{2C_0}{\sqrt{n}} \leq \varepsilon/2,
\end{aligned}$$

for sufficiently large n.

On the other hand, if $\left||R_0| - n/2\right| > c'\sqrt{n}$, then one of the sides has more than $n/2 + c'\sqrt{n}$ initial members, which we call the *majority side*. Now we apply Theorem 5 with ε replaced by $\varepsilon/2$. Notice that in the setting of Theorem 4 if we have $p = \lambda n^{-1/2}$ for λ sufficiently large, then $c'\sqrt{n} \geq c/p$, where c is the constant in Theorem 5. Thus, by this theorem, the probability for the majority side to win is at least $1 - \varepsilon/2$, and we are done by the union bound. ◀