Almost-Monochromatic Sets and the Chromatic Number of the Plane

Nóra Frankl
Department of Mathematics, London School of Economics and Political Science, UK
Laboratory of Combinatorial and Geometric Structures at MIPT, Moscow, Russia
n.frankl@lse.ac.uk

Tamás Hubai
MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics,
Eötvös Loránd University (ELTE), Budapest, Hungary
htamas@cs.elte.hu

Dömötör Pálvölgyi
MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics,
Eötvös Loránd University (ELTE), Budapest, Hungary
dom@cs.elte.hu

Abstract

In a colouring of $\mathbb{R}^d$ a pair (S, s_0) with $S \subseteq \mathbb{R}^d$ and with $s_0 \in S$ is *almost-monochromatic* if $S \setminus \{s_0\}$ is monochromatic but S is not. We consider questions about finding almost-monochromatic similar copies of pairs (S, s_0) in colourings of $\mathbb{R}^d$, $\mathbb{Z}^d$, and of $\mathbb{Q}$ under some restrictions on the colouring.

Among other results, we characterise those (S, s_0) with $S \subseteq \mathbb{Z}$ for which every finite colouring of $\mathbb{R}$ without an infinite monochromatic arithmetic progression contains an almost-monochromatic similar copy of (S, s_0). We also show that if $S \subseteq \mathbb{Z}^d$ and s_0 is outside of the convex hull of $S \setminus \{s_0\}$, then every finite colouring of $\mathbb{R}^d$ without a monochromatic similar copy of $\mathbb{Z}^d$ contains an almost-monochromatic similar copy of (S, s_0). Further, we propose an approach based on finding almost-monochromatic sets that might lead to a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$.

2012 ACM Subject Classification Mathematics of computing → Graph coloring; Theory of computation → Computational geometry

Keywords and phrases discrete geometry, Hadwiger-Nelson problem, Euclidean Ramsey theory

Digital Object Identifier 10.4230/LIPIcs.SoCG.2020.47

Related Version A full version of this paper is available at [6] `https://arxiv.org/abs/1912.02604`.

Funding *Nóra Frankl*: Research was part of project 2018-2.1.1-UK_GYAK-2018-00024 of the summer internship for Hungarian students studying in the UK, supported by the National Research, Development and Innovation Office. She also acknowledges the financial support from the Ministry of Education and Science of the Russian Federation in the framework of MegaGrant no 075-15-2019-1926. The research was also partially supported by the National Research, Development, and Innovation Office, NKFIH Grant K119670.
Tamás Hubai: Research supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under grant number LP2017-19/2017.
Dömötör Pálvölgyi: Research supported by the Lendület program of the Hungarian Academy of Sciences (MTA), under grant number LP2017-19/2017.

Acknowledgements We thank Konrad Swanepoel and the anonymous referees for helpful suggestions on improving the presentation of the paper. We also thank the participants of the Polymath16 project for discussions and consent to publish these related results that were obtained "offline" and separately from the main project.

36th International Symposium on Computational Geometry (SoCG 2020).
Editors: Sergio Cabello and Danny Z. Chen; Article No. 47; pp. 47:1–47:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

A colouring $\varphi : \mathbb{R}^2 \to [k]$ is a (unit-distance-avoiding) *proper k-colouring* of the plane, if $\|p - q\| = 1$ implies $\varphi(p) \neq \varphi(q)$. The *chromatic number* $\chi(\mathbb{R}^2)$ *of the plane* is the smallest k for which there exists a proper k-colouring of the plane. Determining the exact value of $\chi(\mathbb{R}^2)$, also known as the Hadwiger-Nelson problem, is a difficult problem. In 2018 Aubrey de Grey [1] showed that $\chi(\mathbb{R}^2) \geq 5$, improving the long standing previous lower bound $\chi(\mathbb{R}^2) \geq 4$ which was first noted by Nelson (see [13]). The best known upper bound $\chi(\mathbb{R}^2) \leq 7$ was first observed by Isbell (see [13]), and it is widely conjectured that $\chi(\mathbb{R}^2) = 7$. For history and related results we refer the reader to Soifer's book [13].

A graph $G = (V, E)$ is a *unit-distance graph* in the plane if $V \subseteq \mathbb{R}^2$ such that if $(v, w) \in E$ then $\|v - w\| = 1$. De Grey constructed a unit-distance graph G with 1581 vertices, and checked that $\chi(G) \geq 5$ by a computer program. Following his breakthrough, a polymath project, Polymath16 [2] was launched with the main goal of finding a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$. Following ideas proposed in Polymath16 by the third author [10], we present an approach that might lead to a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$.

We call a collection of unit circles $C = C_1 \cup \cdots \cup C_n$ having a common point O a *bouquet through O*. For a given colouring of $\mathbb{R}^2$, the bouquet C is *smiling* if there is a colour, say blue, such that every circle C_i has a blue point, but O is not blue.

▶ **Conjecture 1.** *For every bouquet C, every colouring of the plane with finitely many but at least two colours contains a smiling congruent copy of C.*

In Section 4 we show that the statement of Conjecture 1 would provide a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$. We prove the conjecture for a specific family of bouquets.

▶ **Theorem 2.** *Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O and for every i let O_i be the centre of C_i. If O and $O_1, \ldots, O_n$ are contained in $\mathbb{Q}^2$, further O is an extreme point of $\{O, O_1, \ldots, O_n\}$, then Conjecture 1 is true for C for every proper colouring.*

In Section 4.2 we prove a more general statement which implies Theorem 2. We also prove a statement similar to that of Conjecture 1 for concurrent lines. We call a collection of lines $L = L_1 \cup \cdots \cup L_n$ with a common point O a *pencil through O*. The pencil L is *smiling* if there is a colour, say blue, such that every line L_i has a blue point, but O is not blue.

▶ **Theorem 3.** *For every pencil L, every colouring of the plane with finitely many but at least two colours contains a smiling congruent copy of L.*

1.1 Almost-monochromatic sets

Let $S \subseteq \mathbb{R}^d$ be a finite set with $|S| \geq 3$, and let $s_0 \in S$. In a colouring of $\mathbb{R}^d$ we call S *monochromatic*, if every point of S has the same colour. A pair (S, s_0) is *almost-monochromatic* if $S \setminus \{s_0\}$ is monochromatic but S is not.

We call a colouring a *finite colouring*, if it uses finitely many colours. An *infinite arithmetic progression* in $\mathbb{R}^d$ is a similar copy of $\mathbb{N}$. A colouring is *arithmetic progression-free* if it does not contain a monochromatic infinite arithmetic progression. Motivated by its connections to the chromatic number of the plane,[1] we propose to study the following problem.

[1] The connection is described in details in the the proof of Theorem 24.

▶ **Problem 4.** *Characterise those pairs (S, s_0) with $S \subseteq \mathbb{R}^d$ and with $s_0 \in S$ for which it is true that every arithmetic progression-free finite colouring of $\mathbb{R}^d$ contains an almost-monochromatic similar copy of (S, s_0).*

Note that finding an almost-monochromatic *congruent* copy of a given pair (S, s_0) was studied by Erdős, Graham, Montgomery, Rothschild, Spencer, and Strauss [4]. We solve Problem 4 in the case when $S \subseteq \mathbb{Z}^d$. A point $s_0 \in S$ is called an *extreme point* of S if $s_0 \notin \text{conv}(S \setminus \{s_0\})$. From now on we will use the abbreviations AM for almost-monochromatic and AP for arithmetic progression.

▶ **Theorem 5.** *Let $S \subseteq \mathbb{Z}^d$ and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{R}^d$ without an AM similar copy of (S, s_0) if and only if $|S| > 3$ and s_0 is not an extreme point of S.*

We prove Theorem 5 in full generality in Section 3.1. The 'only if' direction follows from a stronger statement, Theorem 16. In Section 2 we consider the 1-dimensional case. We prove some statements similar to Theorem 5 for $d = 1$, and illustrate the ideas that are used to prove the theorem in general.

Problem 4 is related to and motivated by Euclidean Ramsey theory, a topic introduced by Erdős, Graham, Montgomery, Rothschild, Spencer, and Strauss [3]. Its central question asks for finding those finite sets $S \subseteq \mathbb{R}^d$ for which the following is true. *For every k if d is sufficiently large, then every colouring of $\mathbb{R}^d$ using at most k colours contains a monochromatic congruent copy of S.* Characterising sets having the property described above is a well-studied difficult question, and is in general wide open. For a comprehensive overview see Graham's survey [7].

The nature of the problem significantly changes if instead of a monochromatic congruent copy we ask for a monochromatic similar copy, or a monochromatic homothetic copy. A *(positive) homothetic copy* (or *(positive) homothet*) of a set $H \subseteq \mathbb{R}^d$ is a set $c + \lambda H = \{c + \lambda h : h \in H\}$ for some $c \in \mathbb{R}^d$ and some (positive) $\lambda \in \mathbb{R} \setminus \{0\}$. Gallai proved that if $S \subseteq \mathbb{R}^d$ is a finite set, then every colouring of $\mathbb{R}^d$ using finitely many colours contains a monochromatic positive homothetic copy of S. This statement first appeared in the mentioned form in the book of Graham, Rothschild, and Spencer [8].

A direct analogue of Gallai's theorem for AM sets is not true: there is no AM similar copy of any (S, s_0) if the whole space is coloured with one colour only. However, there are pairs (S, s_0) for which a direct analogue of Gallai's theorem is true for colourings of $\mathbb{Q}$ with more than one colour. In particular, we prove the following result in the full version of the paper [6]. (This result is not used elsewhere in the paper.)

▶ **Theorem 6.** *Let $S = \{0, 1, 2\}$ and $s_0 = 0$. Then every finite colouring of $\mathbb{Q}$ with more than one colour contains an AM positive homothet of (S, s_0).*

In general, we could ask whether every non-monochromatic colouring of $\mathbb{R}^d$ with finitely many colours contains an AM similar copy of every (S, s_0). This, however, is false, as shown by the following example from [4]. Let $S = \{1, 2, 3\}$ and $s_0 = 2$. If $\mathbb{R}_{>0}$ is coloured red and $\mathbb{R}_{\leq 0}$ is coloured blue, we obtain a colouring of $\mathbb{R}$ without an AM similar copy of (S, s_0). Restricting the colouring to $\mathbb{N}$, using the set of colours $\{0, 1, 2\}$ and colouring every $n \in \mathbb{N}$ with n modulo 3, we obtain a colouring without an AM similar copy of (S, s_0). However, notice that in both examples each colour class contains an infinite monochromatic AP.

Therefore, our reason, apart from its connections to the Hadwiger-Nelson problem, for finding AM similar copies of (S, s_0) in AP-free colourings was to impose a meaningful condition to exclude "trivial" colourings.

2 The line

In this section we prove a statement slightly weaker than Theorem 5 for $d = 1$. The main goal of this section to illustrate some of the ideas that we use to prove Theorem 5, but in a simpler case. Note that in $\mathbb{R}$ the notion of similar copy and homothetic copy is the same.

▶ **Theorem 7.** *Let $S \subseteq \mathbb{Z}$ and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{N}$ and of $\mathbb{R}$ without an AM positive homothetic copy of (S, s_0) if and only if $|S| > 3$ and s_0 is not an extreme point of S.*

To prove Theorem 7 it is sufficient to prove the "if" direction only for $\mathbb{R}$ and the "only if" direction only for $\mathbb{N}$. Thus it follows from the three lemmas below, that consider cases of Theorem 7 depending on the cardinality of S and on the position of s_0.

▶ **Lemma 8.** *If s_0 is an extreme point of S, then every finite AP-free colouring of $\mathbb{N}$ contains an AM positive homothetic copy of (S, s_0).*

▶ **Lemma 9.** *If $|S| = 3$, then every AP-free finite colouring of $\mathbb{N}$ contains an AM positive homothetic copy of (S, s_0).*

▶ **Lemma 10.** *If $S \subseteq \mathbb{R}$, $|S| > 3$ and s_0 is not an extreme point of S, then there is an AP-free finite colouring of $\mathbb{R}$ without an AM positive homothetic copy of (S, s_0).*

Before turning to the proofs, recall Van der Waerden's theorem [14] and a corollary of it. A colouring is a *k-colouring* if it uses at most k colours.

▶ **Theorem 11** (Van der Waerden [14]). *For every $k, l \in \mathbb{N}$ there is an $N(k, l) \in \mathbb{N}$ such that every k-colouring of $\{1, \dots, N(k, l)\}$ contains an l-term monochromatic AP.*

▶ **Corollary 12** (Van der Waerden [14]). *For every $k, l \in \mathbb{N}$ and for every k-colouring of $\mathbb{N}$ there is a $t \leq N(k, l)$ such that there are infinitely many monochromatic l-term AP of the same colour with difference t.*

Proof of Lemma 8. Let $S = \{p_1, \dots, p_n\}$ with $1 < p_1 < \dots < p_n$ and φ be an AP-free colouring of $\mathbb{N}$. If s_0 is an extreme point of S, then either $s_0 = p_1$ or $s_0 = p_n$.

Case 1: $s_0 = p_n$. By Theorem 11 φ contains a monochromatic positive homothet $M + \lambda([1, p_n) \cap \mathbb{N})$ of $[1, p_n) \cap \mathbb{N}$ of colour, say, blue. Observe that since φ is AP-free there is a $q \in M + \lambda([p_n, \infty) \cap \mathbb{N})$ which is not blue. Let $M + q\lambda$ be the smallest non-blue element in $M + \lambda([p_n, \infty) \cap \mathbb{N})$. Then $(\lambda(q - p_n) + M + \lambda S, M + \lambda q)$ is an AM homothet of (S, s_0).

Case 2: $s_0 = p_1$. By Corollary 12 there is a $\lambda \in \mathbb{N}$ such that φ contains infinitely many monochromatic congruent copies of $\lambda((1, p_n] \cap \mathbb{N})$, say of colour blue. Without loss of generality, we may assume that infinitely many of these monochromatic copies are contained in $\lambda\mathbb{N}$. Since φ is AP-free, $\lambda\mathbb{N}$ is not monochromatic, and thus there is an i such that $i\lambda$ and $(i + 1)\lambda$ are of different colours. Consider a blue interval $M + \lambda((1, p_n] \cap \mathbb{N})$ such that $M + \lambda > i\lambda$, and let q be the largest non-blue element of $[1, M + \lambda) \cap \lambda\mathbb{N}$. This largest element exists since λi and $\lambda(i + 1)$ are of different colour. Then $(q - \lambda p_1 + \lambda S, q)$ is an AM homothet of (S, s_0). ◀

Proof of Lemma 9. Let $S = \{p_1, p_2, p_3\}$ with $1 < p_1 < p_2 < p_3$ and φ be an AP-free colouring of $\mathbb{N}$. We may assume that $s_0 = p_2$, otherwise we are done by Lemma 8.

There is an $r \in \mathbb{Q}_{>0}$ such that $\{q_1, q_2, q_3\}$ is a positive homothet of S if and only if $q_2 = rq_1 + (1 - r)q_3$. Fix an $M \in \mathbb{N}$ for which $Mr \in \mathbb{N}$. We say that I is an interval of $c + \lambda\mathbb{N}$ of length ℓ if there is an interval $J \subseteq \mathbb{R}$ such that $I = J \cap (c + \lambda\mathbb{N})$ and $|I| = \ell$.

▶ **Proposition 13.** *Let I_1 and I_3 be intervals of $\lambda\mathbb{N}$ of length $2M$ and M respectively such that $\max I_1 < \min I_3$. Then there is an interval $I_2 \subseteq \lambda\mathbb{N}$ of length M such that $\max I_1 < \max I_2 < \max I_3$, and for every $q_2 \in I_2$ there are $q_1 \in I_1$ and $q_3 \in I_3$ such that $\{q_1, q_2, q_3\}$ is a positive homothetic copy of S.*

Proof. Without loss of generality we may assume that $\lambda = 1$. Let I_1^L be the set of the M smallest elements of I_1. By the choice of M for any $q_3 \in \mathbb{N}$ the interval $rI_1^L + (1-r)q_3$ contains at least one natural number. Let q_3 be the smallest element of I_3 and $q_1 \in I_1^L$ such that $rq_1 + (1-r)q_3 \in \mathbb{N}$. Then $I_2 = \{r(q_1+i) + (1-r)(q_3+i) : 0 \le i < M\}$ is an interval of $\mathbb{N}$ of length M satisfying the requirements, since $q_1 + i \in I_1$ and $q_3 + i \in I_3$. ◀

We now return to the proof of Lemma 9. Let I be an interval of $\mathbb{N}$ of length $2M$. By Theorem 12 there is a $\lambda \in \mathbb{N}$ such that φ contains infinitely many monochromatic copies of λI of the same colour, say of blue. Moreover, by the pigeonhole principle there is a $c \in \mathbb{N}$ such that infinitely many of these blue copies are contained in $c + \lambda\mathbb{N}$, and without loss of generality we may assume that $c = 0$.

Consider a blue interval $[a\lambda, a\lambda + 2M\lambda - \lambda]$ of $\lambda\mathbb{N}$ of length $2M$. Since φ is AP-free, $[a\lambda + 2M\lambda, \infty) \cap \lambda\mathbb{N}$ is not completely blue. Let $q\lambda$ be its smallest element which is not blue and let $I_1 = [q\lambda - 2M\lambda, (q-1)\lambda] \cap \lambda\mathbb{N}$. Let I_3 be the blue interval of length M in $\lambda\mathbb{N}$ with the smallest possible $\min I_3$ for which $\max I_1 < \min I_3$. Then Proposition 13 provides an AM positive homothet of (S, s_0).

Indeed, consider the interval I_2 given by the proposition. There exists a $q_2 \in I_2$ which is not blue, otherwise every point of I_2 is blue, contradicting the minimality of $\min I_3$. But then there are $q_1 \in I_1$, $q_3 \in I_3$ such that $(\{q_1, q_2, q_3\}, q_2)$ is an AM homothet copy of (S, s_0). ◀

Proof of Lemma 10. S contains a set S' of 4 points with $s_0 \in S'$ such that s_0 is not an extreme point of S'. Thus we may assume that $S = \{p_1, p_2, p_3, p_4\}$ with $p_1 < p_2 < p_3 < p_4$ and that $s_0 = p_2$ or $s_0 = p_3$. We construct the colouring for these two cases separately. First we construct a colouring φ_1 of $\mathbb{R}_{>0}$ for the case of $s_0 = p_3$, and a colouring φ_2 of $\mathbb{R}_{\ge 0}$ for the case of $s_0 = p_2$. Then we extend the colouring in both cases to $\mathbb{R}$.

Construction of φ_1 ($s_0 = p_3$): Fix K such that $K > \frac{p_4 - p_2}{p_2 - p_1} + 1$ and let $\{0, 1, 2\}$ be the set of colours. We define φ_1 as follows. Colour $(0, 1)$ with colour 2, and for every $i \in \mathbb{N} \cup \{0\}$ colour $[K^i, K^{i+1})$ with i modulo 2. It is not hard to check that φ_1 defined this way is AP-free. Thus we only have to show that it does not contain an AM positive homothet of (S, s_0).

Consider a positive homothet $c + \lambda S = \{r_1, r_2, r_3, r_4\}$ of S with $r_1 < r_2 < r_3 < r_3$. If $\{r_1, r_2, r_3, r_4\} \cap [0, 1) \neq \emptyset$, then $\{r_1, r_2, r_3, r_4\}$ cannot be AM. Thus we may assume that $\{r_1, r_2, r_3, r_4\} \cap [0, 1) = \emptyset$.

Note that by the choice of K we have

$$Kr_2 > r_2 + \frac{p_4 - p_2}{p_2 - p_1} r_2 = r_2 + \frac{p_4 - p_2}{p_2 - p_1} \left(\lambda(p_2 - p_1) + r_1\right) \ge r_2 + \lambda(p_4 - p_2) = r_4.$$

Hence $\{r_2, r_3, r_4\}$ is contained in the union of two consecutive intervals of the form $[K^i, K^{i+1})$. This means that $(\{r_1, \ldots, r_4\}, r_3)$ cannot be AM since either $\{r_2, r_3, r_4\}$ is monochromatic, or r_2 and r_4 have different colours.

Construction of φ_2 ($s_0 = p_2$): Fix K such that $K > \frac{p_4 - p_2}{p_2 - p_1} + 1$, let $L = K \cdot \left\lceil \frac{p_3 - p_1}{p_4 - p_3} \right\rceil$ and let $\{0, \ldots, 2L\}$ be the set of colours. We define φ_2 as follows. For each odd $i \in \mathbb{N} \cup \{0\}$, divide the interval $[L \cdot K^i, L \cdot K^{i+1})$ into L equal half-closed intervals, and colour the j-th of them with colour j. For even $i \in \mathbb{N} \cup \{0\}$ divide the interval $[L \cdot K^i, L \cdot K^{i+1})$ into L equal half-closed intervals, and colour the j-th of them with colour $L+j$. That is, for $j=1,\dots,L$ we colour $[L\cdot K^i+(j-1)(K^{i+1}-K^i), L\cdot K^i+j(K^{i+1}-K^i))$ with colour j if i is odd, and with colour $j+L$ if i is even. Finally, colour the points in $[0,L)$ with colour 0.

It is not hard to check that φ_2 defined this way is AP-free, thus we only have to show it does not contain an AM positive homothetic copy of (S,s_0).

Consider a positive homothet $c+\lambda S=\{r_1,r_2,r_3,r_4\}$ of S with $r_1<r_2<r_3<r_4$. If $\{r_1,r_2,r_3,r_4\}\cap[0,L)\neq\emptyset$, then $(\{r_1,r_2,r_3,r_4\},r_2)$ cannot be AM, thus we may assume that $\{r_1,r_2,r_3,r_4\}\cap[0,L)=\emptyset$. Note that by the choice of K we again have

$$Kr_2>r_2+\frac{p_4-p_2}{p_2-p_1}r_2=r_2+\frac{p_4-p_2}{p_2-p_1}(\lambda(p_2-p_1)+r_1)\geq r_2+\lambda(p_4-p_2)=r_4.$$

This means that $\{r_2,r_3,r_4\}$ is contained in the union of two consecutive intervals of the form $[L\cdot K^i, L\cdot K^{i+1})$, which implies that if $(\{r_1,r_2,r_3,r_4\},r_2)$ is AM, then $\{r_3,r_4\}$ is contained in an interval $[L\cdot K^i+(j-1)(K^{i+1}-K^i), L\cdot K^i+j(K^{i+1}-K^i))$ for some $1\leq j\leq L$. However, then by the choice of L we have that r_1 is either contained in the interval $[L\cdot K^i, L\cdot K^{i+1})$ or in the interval $[L\cdot K^{i-1}, L\cdot K^i)$. Indeed,

$$\begin{aligned} r_3-r_1&\leq\left\lceil\frac{r_3-r_1}{r_4-r_3}\right\rceil(r_4-r_3)\leq\left\lceil\frac{r_3-r_1}{r_4-r_3}\right\rceil(K^{i+1}-K^i)\\ &=\left\lceil\frac{p_3-p_1}{p_4-p_3}\right\rceil(K^{i+1}-K^i)=L(K^i-K^{i-1}).\end{aligned}$$

Thus, if r_1 has the same colour as r_3 and r_4, then r_1 is also contained in the interval $[L\cdot K^i+(j-1)(K^{i+1}-K^i), L\cdot K^i+j(K^{i+1}-K^i))$, implying that $(\{r_1,r_2,r_3,r_4\},r_2)$ is monochromatic.

We now extend the colouring to $\mathbb{R}$ in the case of $s_0=p_3$. Let φ_2' be a colouring of $\mathbb{R}_{\geq 0}$ isometric to the reflection of φ_2 over 0. Then φ_2' contains no AM positive homothet of (S,s_0). If further we assume that φ_1 and φ_2' use disjoint sets of colours, it is not hard to check that the union of φ_1 and φ_2' is an AP-free colouring of $\mathbb{R}$ containing no AM positive homothet of (S,s_0). We can extend the colouring similarly in the case of $s_0=p_2$. ◀

3 Higher dimensions

In this section we prove Theorem 5.

3.1 Proof of "if" direction of Theorem 5

Let $S\subseteq\mathbb{R}^d$ such that $|S|>3$ and s_0 is not an extreme point of S. To prove the 'if' direction of Theorem 5, we prove that there is an AP-free colouring of $\mathbb{R}^d$ without an AM similar copy of (S,s_0). (Note that for the proof of Theorem 5, it would be sufficient to prove this for $S\subseteq\mathbb{Z}^d$.)

Recall that $C\subseteq\mathbb{R}^d$ is a *convex cone* if for every $x,y\in C$ and $\alpha,\beta\geq 0$, the vector $\alpha x+\beta y$ is also in C. The *angle* of C is $\sup_{x,y\in C\setminus\{o\}}\angle(x,y)$.

We partition $\mathbb{R}^d$ into finitely many convex cones $C_1\cup\dots\cup C_m$, each of angle at most $\alpha=\alpha(d,S)$, where $\alpha(d,S)$ will be set later. We colour the cones with pairwise disjoint sets of colours as follows. First, we describe a colouring φ of the closed circular cone $C=C(\alpha)$ of angle α around the line $x_1=\dots=x_d$. Then for each i we define a colouring φ_i of C_i using pairwise disjoint sets of colours in a similar way. More precisely, let f_i be an isometry with $f_i(C_i)\subseteq C$, and define φ_i such that it is isometric to φ on $f_i(C_i)$.

It is not hard to see that it is sufficient to find an AP-free colouring φ of C without an AM similar copy of (S, s_0). Indeed, since the cones C_i are coloured with pairwise disjoint sets of colours, any AP or AM similar copy of (S, s_0) is contained in one single C_i.

We now turn to describing the colouring φ of C. Note that by choosing α sufficiently small we may assume that $C \subseteq \mathbb{R}^d_{\geq 0}$. For $x \in \mathbb{R}^d$ let $\|x\|_1 = |x_1| + \cdots + |x_d|$. Then for any $x \in \mathbb{R}^d$ we have

$$\|x\| \leq \|x\|_1 \leq \sqrt{d}\|x\|. \tag{1}$$

Let $S = \{p_1, \ldots, p_n\}$ and fix K such that

$$K > 1 + 2\sqrt{d} \max_{p_i, p_j, p_l, p_\ell \in S, p_i \neq p_j} \frac{\|p_k - p_\ell\|}{\|p_i - p_j\|}.$$

For a sufficiently large L, to be specified later, we define $\varphi : C \to \{0, 1, \ldots, 2L\}$ as

$$\varphi(x) = \begin{cases} 0 & \text{if } \|x\|_1 < L \\ j & \text{if for some even } i \in \mathbb{N} \text{ and } j \in [L] \text{ we have} \\ & \qquad \|x\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)) \\ L + j & \text{if for some odd } i \in \mathbb{N} \text{ and } j \in [L] \text{ we have} \\ & \qquad \|x\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)). \end{cases}$$

It is not hard to check that φ is AP-free. Thus we only have to show that it does not contain an AM similar copy of (S, s_0). Let $(\{r_1, \ldots, r_n\}, q_0)$ be a similar copy of (S, s_0), with

$$\|r_1\|_1 \leq \|r_2\|_1 \leq \cdots \leq \|r_n\|_1. \tag{2}$$

▷ Claim 14. $\{\|r_2\|_1, \ldots, \|r_n\|_1\}$ is contained in the union of two consecutive intervals of the form $[L \cdot K^j, L \cdot K^{j+1})$.

Proof. For any r_i with $i \geq 2$ we have

$$\begin{aligned} \|r_i\|_1 &\leq \|r_2\|_1 + \|r_i - r_2\|_1 \\ &\leq \|r_2\|_1 + \sqrt{d}\|r_i - r_2\| && \text{by (1)} \\ &= \|r_2\|_1 + \sqrt{d}\frac{\|r_i - r_2\|}{\|r_2 - r_1\|}\|r_2 - r_1\| \\ &< \|r_2\|_1 + \frac{K-1}{2} \cdot \|r_2 - r_1\| && \text{by the definition of } K \\ &\leq \|r_2\|_1 + \frac{K-1}{2}(\|r_2\| + \|r_1\|) && \text{by the triangle inequality} \\ &\leq \|r_2\|_1 + \frac{K-1}{2}(\|r_2\|_1 + \|r_1\|_1) && \text{by (1)} \\ &\leq K\|r_2\|_1 && \text{by (2).} \end{aligned}$$

◁

Assume now that $(\{r_1, \ldots, r_n\}, q_0)$ is AM. Note that $\|x\|_1$ is $\sqrt{d}$ times the length of the projection of x on the $x_1 = \cdots = x_d$ line for $x \in \mathbb{R}^d_{\geq 0}$. Thus for any similar copy $\psi(S)$ of S we have $\|\psi(s_0)\|_1 \in \operatorname{conv}\{\|p\|_1 : p \in \psi(S \setminus \{s_0\})\}$, and we may assume that $q_0 \neq r_1, r_n$. This means that $\varphi(r_1) = \varphi(r_n)$, and there is exactly one $i \in \{2, \ldots, n-1\}$ with $\varphi(r_i) \neq \varphi(r_1)$. This, by Claim 14 and by the definition of φ, is only possible if $i = 2$ and there are $i \in \mathbb{N}$ and $j \in [L]$ such that

$$\|r_3\|_1, \ldots, \|r_{n-1}\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)).$$

The following claim finishes the proof.

▷ Claim 15. If L is sufficiently large and α is sufficiently small, then $\|r_1\|_1$ is contained in $[L \cdot K^{i-1}, L \cdot K^i) \cup [L \cdot K^i, L \cdot K^{i+1})$.

The claim indeed finishes the proof. By the definition of φ then $\varphi(r_1) = \varphi(r_n)$ implies

$$\|r_1\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i)).$$

But then we have

$$\|r_2\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i))$$

as well, contradicting $\varphi(r_2) \neq \varphi(r_1)$.

Proof of Claim 15. It is sufficient to show that $\|r_{n-1}\|_1 - \|r_1\|_1 < LK^i - LK^{i-1}$. We have

$$\|r_{n-1}\|_1 - \|r_1\|_1 \leq \sqrt{d}\|r_{n-1} - r_1\| = \sqrt{d}\|r_n - r_{n-1}\| \frac{\|r_{n-1} - r_n\|}{\|r_n - r_{n-1}\|} < \|r_{n-1} - r_n\| \frac{K-1}{2},$$

by (1) and by the definition of K. Let H_1 and H_2 be the hyperplanes orthogonal to the line $x_1 = \cdots = x_d$ at distance $\frac{1}{\sqrt{d}}(L \cdot K^i + (j-1)(K^{i+1} - K^i))$ and $\frac{1}{\sqrt{d}}(L \cdot K^i + j(K^{i+1} - K^i))$ from the origin respectively. Since $\|r_n\|_1, \|r_{n-1}\|_1 \in [L \cdot K^i + (j-1)(K^{i+1} - K^i), L \cdot K^i + j(K^{i+1} - K^i))$ we have that r_n and r_{n-1} are contained in the intersection T of $C(\alpha)$ and the slab bounded by the hyperplanes H_1 and H_2.

Thus $\|r_{n-1} - r_n\|$ is bounded by the length of the diagonal of the trapezoid which is obtained as the intersection of T and the 2-plane through r_n, r_{n-1} and the origin. Scaled by $\sqrt{d}$, this is shown in Figure 1.

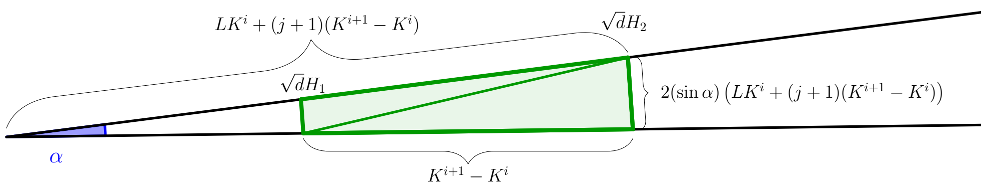

Figure 1 $T \cap C(\alpha)$.

From this, by the triangle inequality we obtain

$$\begin{aligned}\|r_{n-1} - r_n\| &\leq \frac{1}{\sqrt{d}}\left(K^{i+1} - K^i + 2(\sin\alpha)\left(LK^i + (j+1)(K^{i+1} - K^i)\right)\right) \\ &\leq \frac{1}{\sqrt{d}}\left(K^{i+1} - K^i + 2\sin\alpha \cdot LK^{i+1}\right) \leq \frac{2}{\sqrt{d}}\left(K^{i+1} - K^i\right),\end{aligned}$$

where the last inequality holds if α is sufficiently small. Combining these inequalities and choosing $L = \frac{K^2}{\sqrt{d}}$ we obtain the desired bound $\|r_{n-1}\|_1 - \|r_n\|_1 < LK^i - LK^{i-1}$, finishing the proof of the claim. ◁

3.2 Proof of "only if" direction of Theorem 5

The "only if" direction follows from Theorem 7 in the case of $d = 1$, and from the following stronger statement for $d \geq 2$ (since in this case s_0 is an extreme point of S).

▶ **Theorem 16.** *Let $S \subseteq \mathbb{Z}^d$ and $s_0 \in S$ be an extreme point of S. Then for every k there is a constant $\Lambda = \Lambda(d, S, k)$ such that the following is true. Every k-colouring of $\mathbb{Z}^d$ contains either an AM similar copy of (S, s_0) or a monochromatic similar copy of $\mathbb{Z}^d$ with an integer scaling ratio $1 \leq \lambda \leq \Lambda$.*

Before the proof we need some preparation.

▶ **Lemma 17.** *There is an $R > 0$ such that for any ball D of radius at least R the following is true. For every $p \in \mathbb{Z}^d$ at distance at most 1 from D there is a similar copy (S', s'_0) of (S, s_0) in $\mathbb{Z}^d$ such that $s'_0 = p$ and $S' \setminus \{s'_0\} \subset D$.*

Let $\mathbb{Q}_N = \left\{\frac{a}{b} : a, b \in \mathbb{Z}, b \leq N\right\} \subseteq \mathbb{Q}$. Lemma 17 follows from the next lemma.

▶ **Lemma 18.** *There is an $\varepsilon > 0$ and an $N \in \mathbb{N}$ such that for any ball D of radius 1 the following is true. For all $p \in \mathbb{Q}_N^d$ at distance at most ε from D there is a similar copy (S', s'_0) of (S, s_0) in $\mathbb{Q}_N^d$ such that $s'_0 = p$ and $S' \setminus \{s'_0\} \subset D$.*

Proof. Let D be a ball of radius 1. Since s_0 is an extreme point of S, there is a hyperplane that separates s_0 from $S \setminus \{s_0\}$. With this it is not hard to see that there are $\delta, \varepsilon > 0$ with the following property. If p is ε close to D, then there is a congruent copy (S'', s''_0) of $\delta(S, s_0)$ with $s''_0 = p$ and such that every point of $S'' \setminus \{s''_0\}$ is contained in D at distance at least ε from the boundary of D. Now we use the fact that $O(\mathbb{R}^n) \cap \mathbb{Q}^{n \times n}$, the set of rational rotations, is dense in $O(\mathbb{R}^n)$ (see for example [12]). By this and the compactness of $O(\mathbb{R})^n$, we can find an $N = N(\delta, \varepsilon) \in \mathbb{N}$ and (S', s'_0) in $\mathbb{Q}_N^d$ which is a rotation of (S'', s''_0) around p, ε-close to (S'', s''_0). With this $S' \setminus \{s'_0\}$ is contained in D. Therefore N and ε satisfy the requirements. ◀

The proof of the following variant of Gallai's theorem can be found in the Appendix of the full version of the paper [6].

▶ **Theorem 19** (Gallai). *Let $S \subseteq \mathbb{Z}^d$ be finite. Then there is a $\lambda(d, S, k) \in \mathbb{Z}$ such that every k-colouring of $\mathbb{Z}^d$ contains a monochromatic positive homothet of S with an integer scaling ratio bounded by $\lambda(d, S, k)$.*

Proof of Theorem 16. Let R be as in Lemma 17 and let H be the set of points of $\mathbb{Z}^d$ contained in a ball of radius R. By Theorem 19 there is a monochromatic, say blue, homothetic copy $H_0 = c + \lambda H$ of H for some integer $\lambda \leq \lambda(d, H, k)$. Without loss of generality we may assume that $H_0 = B(O, \lambda R) \cap \lambda\mathbb{Z}^d$ for some $O \in \mathbb{Z}^d$, where $B(O, \lambda R)$ is the ball of radius λR centred at O.

Consider a point $p \in \lambda\mathbb{Z}^d \setminus H$ being at distance at most λ from H_0. If p is not blue then using Lemma 17 we can find an AM similar copy of (S, s_0). Thus we may assume that any point $p \in \lambda\mathbb{Z}^d \setminus H_0$ which is λ close to H_0 is blue as well.

By repeating a similar procedure, we obtain that there is either an AM similar copy of (S, s_0), or every point of $H_i = B(O, \lambda R + i\lambda) \cap \lambda\mathbb{Z}^d$ is blue for every $i \in \mathbb{N}$. But the latter means $\lambda\mathbb{Z}^d$ is monochromatic, which finishes the proof. ◀

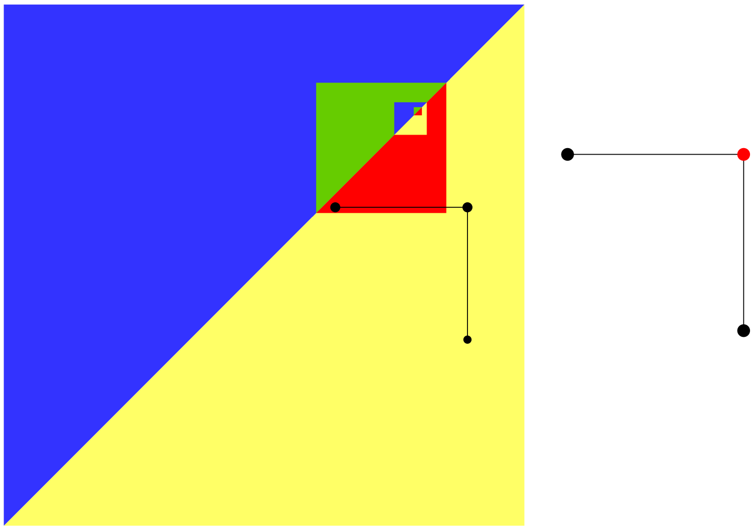

Figure 2 A 4-colouring avoiding AM homothets of (S, s_0).

3.3 Finding an AM positive homothet

The following statement shows that it is not possible to replace an AM similar copy of (S, s_0) with a positive homothet of (S, s_0) in the "only if" direction of Theorem 5.

▶ **Proposition 20.** *Let $S \subseteq \mathbb{Z}^d$ such that S is not contained in a line and $s_0 \in S$. Then there is an AP-free colouring of $\mathbb{R}^d$ without an AM positive homothet of (S, s_0).*

Proof. We may assume that $|S| = 3$ and thus $S \subseteq \mathbb{R}^2$. Since the problem is affine invariant, we may further assume that $S = \{(0,1),(1,1),(1,0)\}$ with $s_0 = (1,1)$, $s_1 = (0,1)$ and $s_2 = (1,0)$. First we describe a colouring of $\mathbb{R}^2$ and then we extend it to $\mathbb{R}^d$.

For every $i \in \mathbb{N}$ let Q_i be the square $[-4^i, 4^{i-1}] \times [-4^i, 4^{i-1}]$, and $Q_0 = \emptyset$. Further let H_+ be the open half plane $x < y$ and H_- be the closed half plane $x \geq y$. We colour $\mathbb{R}^2$ using four colours, green, blue, red and yellow as follows (see also Figure 2).

- **Green:** For every odd $i \in \mathbb{N}$ colour $(Q_i \setminus Q_{i-1}) \cap H_+$ with red.
- **Blue:** For every even $i \in N$ colour $(Q_i \setminus Q_{i-1}) \cap H_+$ with yellow.
- **Red:** For every odd $i \in \mathbb{N}$ colour $(Q_i \setminus Q_{i-1}) \cap H_-$ with green.
- **Yellow:** For every even $i \in \mathbb{N}$ colour $(Q_i \setminus Q_{i-1}) \cap H_-$ with blue.

It is not hard to see that this colouring φ_1 is AP-free. Thus we only have to check that it contains no AM positive homothet of (S, s_0). Let S' be a positive homothet of S. First note that we may assume that S' is contained in one of the half planes bounded by the $x = y$ line, otherwise it is easy to see that it cannot be AM. Thus by symmetry we may assume that $s'_0 \in Q_i \setminus Q_{i-1} \cap H_+$ for some $i \in \mathbb{N}$.

If the y-coordinate of s'_0 is smaller than -4^{i-1}, then $s'_1 \in Q_i \setminus Q_{i-1}$, and hence S' cannot be AM. On the other hand, if the y-coordinate of s'_0 is at least -4^{i-1}, then $\|s'_0 - s'_1\| \leq 2 \cdot 4^{i-1}$. This means that the y-coordinate of s'_2 is at least $-(4^{i-1} + 2 \cdot 4^{i-1}) > -4^i$. Thus, in this case s'_2 is contained in $s'_1 \in Q_i \setminus Q_{i-1}$, and hence S' cannot be monochromatic.

To finish the proof, we extend the colouring to $\mathbb{R}^d$. Let $T \cong \mathbb{R}^{d-2}$ be the orthogonal complement of $\mathbb{R}^2$. Fix an AP-free colouring φ of T using the colour set $\{1, 2\}$. Further let φ_2 be a colouring of $\mathbb{R}^2$ isometric to φ_1, but using a disjoint set of colours. For every $t \in T$ colour $\mathbb{R}^2 + t$ by translating φ_i if $\varphi(t) = i$. It is not hard to check that this colouring is AP-free and does not contain any AM positive homothet of (S, s_0). ◀

4 Smiling bouquets and the chromatic number of the plane

For a graph $G = (V, E)$ with a given origin (distinguished vertex) $v_0 \in V$ a colouring φ with $\varphi : V \setminus \{v_0\} \to \binom{[k]}{1}$ and $\varphi(v_0) \in \binom{[k]}{2}$ is a proper k-colouring with *bichromatic origin* v_0, if $(v, w) \in E$ implies $\varphi(v) \cap \varphi(w) = \emptyset$. There are unit-distance graphs with not too many vertices that do not have a 4-colouring with a certain bichromatic origin. Figure 3 shows such an example, the 34-vertex graph G_{34}, posted by the second author [9] in Polymath16. It is the first example found whose chromatic number can be verified quickly without relying on a computer. Finding such graphs has been motivated by an approach to find a human-verifiable proof of $\chi(\mathbb{R}^5) \geq 5$, proposed by the third author [10] in Polymath16.

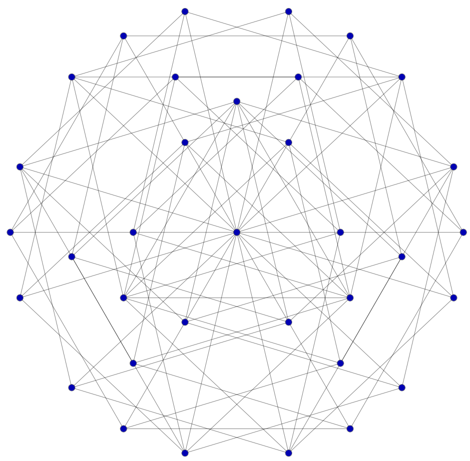

Figure 3 A 34 vertex graph without a 4-colouring if the origin is bichromatic.

Theorem 21 with $G = G_{34}$ shows that a human-verifiable proof of Conjecture 1 for $k = 4$ would provide a human-verifiable proof of $\chi(\mathbb{R}^2) \geq 5$. Note that G_{34} was found by a computer search, and for finding other similar graphs one might rely on a computer program. Thus, the approach we propose, is human-verifiable, however it might be computer-assisted.

For a graph G with origin v_0 let $\{C_1, \ldots, C_n\}$ be the set of unit circles whose centres are the neighbours of v_0, and let $C(G, v_0) = C_1 \cup \cdots \cup C_n$ be the bouquet through v_0.

▶ **Theorem 21.** *If there is a unit-distance graph $G = (V, E)$ with $v_0 \in V$ which does not have a proper k-colouring with bichromatic origin v_0, and Conjecture 1 is true for $C(G, v_0)$, then $\chi(\mathbb{R}^2) \geq k + 1$.*

Proof of Theorem 21. Assume for a contradiction that there is a proper k-colouring φ of the plane. Using φ we construct a proper k-colouring of G with bichromatic origin $v_0 \in V$.

Let $v_1, \ldots, v_n$ be the neighbours of the origin v_0, and C_j be the unit circle centred at v_j. Then $C = C_1 \cup \cdots \cup C_n$ is a bouquet through v_0. If Conjecture 1 is true for φ, then there is a smiling congruent copy $C' = C'_1 \cup \ldots C'_n$ of C through v'_0. That is, there are points $p_1 \in C'_1, \ldots, p_n \in C'_n$ with $\ell = \varphi(p_1) = \cdots = \varphi(p_n) \neq \varphi(v'_0)$.

For $i \in [n]$ let v_i' be the centre of C_i'. We define a colouring φ' of G as $\varphi'(v_0) = \{\varphi(v_0'), \ell\}$ and $\varphi'(v_i) = \varphi(v_i')$ for $v \in V \setminus \{v_0\}$. We claim that φ' is a proper k-colouring of G with a bichromatic origin v_0, contradicting our assumption.

Indeed, if $v_i \neq v_0 \neq v_j$ then for $(v_i, v_j) \in E$ we have $\varphi'(v_i) \neq \varphi'(v_j)$ because $\varphi(v_i') \neq \varphi(v_j')$. For $(v_0, v_i) \in E$, we have $\varphi'(v_i) \neq \varphi(v_0)$ because $\varphi(v_i') \neq \varphi(v_0')$, and $\varphi'(v_i) \neq \ell$ because $\varphi(v_i') \neq \ell$ since $\|v_i' - p_i\| = 1$. This finishes the proof of Theorem 21. ◀

4.1 Smiling pencils

In this section we prove Theorem 3. We start with the following simple claim.

▷ Claim 22. For every pencil L through O there is an $\varepsilon > 0$ for which the following is true. For any circle C or radius R if a point p is at distance at most εR from C, then there is a congruent copy L' of L through p such that every line of L' intersects C.

Proof. It is sufficient to prove the following. If C is a unit circle and p is sufficiently close to C, then there is a congruent copy L' of L through p such that every line of L' intersects C.

Note that if p is contained in the disc bounded by C, clearly every line of every congruent copy L' of L through p intersects C. Thus we may assume that p is outside the disc.

Let $0 < \alpha < \pi$ be the largest angle spanned by lines in L. If p is sufficiently close to C, then the angle spanned by the tangent lines of C through p is larger than α. Thus, any congruent copy L' of L through p can be rotated around p so that every line of the pencil intersects C. ◁

Proof of Theorem 3. Assume for contradiction that φ is a colouring using at least two colours, but there is a pencil L such that there is no congruent smiling copy of L.

First we obtain a contradiction assuming that there is a monochromatic, say red, circle C of radius r. We claim that then every point p inside the disc bounded by C is red. Indeed, translating L to a copy L' through p, each line L_i' will intersect C, and so have a red point. Thus p must be red.

A similar argument together with Claim 22 shows that if there is a non-red point at distance at most εr from C, we would find a congruent smiling copy of L through p. Thus there is a circle C' of radius $(1+\varepsilon)r$ concentric with C, such that every point of the disc bounded by C' is red. Repeating this argument, we obtain that every point of $\mathbb{R}^2$ is red contradicting the assumption that φ uses at least 2 colours.

To obtain a contradiction, we prove that there exists a monochromatic circle. For $1 \leq i \leq n$ let α_i be the angle of L_i and L_{i+1}. Fix a circle C, and let $a_1, \dots, a_n \in C$ be points such that if $c \in C \setminus \{a_1, \dots, a_n\}$, then the angle of the lines connecting c with a_i and c with a_{i+1} is α_i. By Gallai's theorem there is a monochromatic (say red) set $\{a_1', \dots, a_n'\}$ similar to $\{a_1, \dots, a_n\}$. Let C' be the circle that contains $\{a_1', \dots, a_n'\}$. Then C' is monochromatic. Indeed, if there is a point p on C' for which $\varphi(p)$ is not red, then by choosing L_j' to be the line connecting p with a_j' we obtain $L' = L_1' \cup \dots \cup L_n'$, a smiling congruent copy of L. ◀

4.2 Conjecture 1 for lattice-like bouquets

Using the ideas from the proof of Theorem 16, we prove Conjecture 1 for a broader family of bouquets.

4.2.1 Lattices

A *lattice* $\mathcal{L}$ generated by linearly independent vectors v_1 and v_2 is the set $\mathcal{L} = \mathcal{L}(v_1, v_2) = \{n_1v_1 + n_2v_2 : n_1, n_2 \in \mathbb{Z}\}$. We call a lattice $\mathcal{L}$ *rotatable* if for every $0 \leq \alpha_1 < \alpha_2 \leq \pi$ there is an angle $\alpha_1 < \alpha < \alpha_2$ and scaling factor $\lambda = \lambda(\alpha_2, \alpha_1)$ such that $\lambda\alpha(\mathcal{L}) \subset \mathcal{L}$, where $\alpha(L)$ is the rotated image of $\mathcal{L}$ by angle α around the origin. For example, $\mathbb{Z}^2$, the triangular grid, and $\left\{n_1(1,0) + n_2(0, \sqrt{2}) : n_1, n_2 \in \mathbb{Z}\right\}$ are rotatable, but $\mathcal{L} = \{n_1(1,0) + n_2(0,\pi) : n_1, n_2 \in \mathbb{Z}\}$ is not.[2]

The rotatability of $\mathcal{L}$ allows us to extend Lemma 17 from $\mathbb{Z}^2$ to $\mathcal{L}$. This leads to an extension of Theorem 16 to rotatable lattices.

▶ **Theorem 23.** *Let $\mathcal{L}$ be a rotatable lattice, $S \subseteq \mathcal{L}$ be finite and s_0 be an extreme point of S. Then for every $k \in N$ there exists a constant $\Lambda = \Lambda(\mathcal{L}, S, k)$ such that the following is true. In every k-colouring of $\mathcal{L}$ there is either an AM similar copy of (S, s_0) with a positive scaling factor bounded by Λ, or a monochromatic positive homothetic copy of $\mathcal{L}$ with an integer scaling factor $1 \leq \lambda \leq \Lambda$.*

The proof of extending Lemma 17 to rotatable lattices is analogous to the original one, so is the proof of Theorem 23 to the proof of Theorem 16. Therefore, we omit the details.

4.2.2 Lattice-like bouquets

Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O, and for $i \in [n]$ let O_i be the centre of C_i. We call C *lattice-like* if O is an extreme point of $\{O, O_1, \ldots, O_n\}$ and there is a rotatable lattice $\mathcal{L}$ such that $\{O, O_1, \ldots, O_n\} \subseteq \mathcal{L}$. Similarly, we call a unit-distance graph $G = (V, E)$ with an origin $v_0 \in V$ *lattice-like* if there is a rotatable lattice $\mathcal{L}$ such that v_0 and its neighbours are contained in $\mathcal{L}$, and v_0 is not in the convex hull of its neighbours.

Since $\mathbb{Z}^2$ is a rotatable lattice, Theorem 2 is a direct corollary of the result below.

▶ **Theorem 24.** *If C is a lattice-like bouquet, then every proper k-colouring of $\mathbb{R}^2$ contains a smiling congruent copy of C.*

This implies the following, similarly as Conjecture 1 implied Theorem 21.

▶ **Theorem 25.** *If there exists a lattice-like unit-distance graph $G = (V, E)$ with an origin v_0 that does not admit a proper k-colouring with bichromatic origin v_0, then $\chi(\mathbb{R}^2) \geq k + 1$.*

In the proof of Theorem 24, we need a simple geometric statement.

▶ **Proposition 26.** *Let $C = C_1 \cup \cdots \cup C_n$ be a bouquet through O, and let $\mathcal{O} = \{O_1, \ldots, O_n\}$, where O_j is the center of C_j. Then for every $0 < \lambda \leq 2$ there are n points $P_1, \ldots, P_n$ such that $P_j \in C_j$ and $\{P_1, \ldots, P_n\}$ is congruent to $\lambda\mathcal{O}$.*

Proof. For $\lambda = 2$ let P_j be the image of O reflected in O_j. Then $P_j \in C_j$, and $\{P_1, \ldots, P_n\}$ can be obtained by enlarging $\mathcal{O}$ from O with a factor of 2. For $\lambda < 2$, scale $\{P_1, \ldots, P_n\}$ by $\frac{\lambda}{2}$ from O obtaining $\{P'_1, \ldots, P'_n\}$. Then there is an angle α such that rotating $\{P'_1, \ldots, P'_n\}$ around O by α, the rotated image of each P'_j is on C_j. ◀

Proof of Theorem 24. Let $C = C_1 \cup \cdots \cup C_n$ be the lattice-like bouquet through O, O_i be the centre of C_i for $i \in [k]$, and $\mathcal{L}$ be the rotatable lattice containing $S = \{O, O_1, \ldots, O_n\}$. Consider a proper k-colouring φ of $\mathbb{R}^2$ and let $\delta \in \mathbb{Q}$ to be chosen later.

[2] For another characterization of rotatable lattices, see `https://mathoverflow.net/a/319030/955`.

By Theorem 23, the colouring φ either contains an AM similar copy of $\delta(S, s_0)$ with a positive scaling factor bounded by $\lambda(\mathcal{L}, S, k)$, or a monochromatic similar copy of $\delta\mathcal{L}$ with an integer scaling factor bounded by $\lambda(\mathcal{L}, S, k)$.

If the first case holds and δ is chosen so that $\delta\lambda(\mathcal{L}, S, k) \leq 2$, Proposition 26 provides a smiling congruent copy of C. Now assume for contradiction that the first case does not hold. Then there is a monochromatic similar copy $\mathcal{L}'$ of $\delta\mathcal{L}$ with an integer scaling factor λ bounded by $\lambda(\mathcal{L}, S, k)$. However, if we choose $\delta = \frac{1}{\lambda(\mathcal{L},S,k)!}$, then for any $1 \leq \lambda \leq \lambda(\mathcal{L}, S, k)$ we have $\delta\lambda = \frac{1}{N_\lambda}$ for some $N_\lambda \in \mathbb{N}$. But this would imply that there are two points in the infinite lattice $\lambda\delta\mathcal{L}$ at distance 1, contradicting that φ is a proper colouring $\mathbb{R}^2$. ◀

5 Further problems and concluding remarks

Problems in the main focus of this paper are about finding AM sets *similar* to a given one. However, it is also interesting to find AM sets *congruent* to a given one. In this direction, Erdős, Graham, Montgomery, Rothschild, Spencer and Straus made the following conjecture.

▶ **Conjecture 27** (Erdős et al. [4]). *Let $s_0 \in S \subset \mathbb{R}^2$, $|S| = 3$. There is a non-monochromatic colouring of $\mathbb{R}^2$ that contains no AM congruent copy of (S, s_0) if and only if S is collinear and s_0 is not an extreme point of S.*

As noted in [4], the "if" part is easy; colour $(x, y) \in \mathbb{R}^2$ red if $y > 0$ and blue if $y \leq 0$. In fact, this colouring also avoids AM similar copies of such S.

Conjecture 27 was proved in [4] for the vertex set S of a triangle with angles $120°, 30°$, and $30°$ with any $s_0 \in S$. It was also proved for any isosceles triangle in the case when s_0 is one of the vertices on the base, and for an infinite family of right-angled triangles.

Much later, the same question was asked independently in a more general form by the third author [11]. In a comment to this question on the MathOverflow site, a counterexample (to both the MathOverflow question and Conjecture 27) was pointed out by user "fedja" [5], which we sketch in the full version of the paper [6].

Straightforward generalisations of our arguments from Section 4 would also imply lower bounds for the chromatic number of other spaces. For example, if C is a lattice-like bouquet of spheres, then every proper k-colouring of $\mathbb{R}^d$ contains a smiling congruent copy of C. This implies that if one can find a lattice-like unit-distance graph with an origin v_0 that does not admit a proper k-colouring with bichromatic origin v_0, then $\chi(\mathbb{R}^d) \geq k + 1$. Possibly one can even strengthen this further; in $\mathbb{R}^d$ it could be even true that there is a *d-smiling* congruent copy of any bouquet C, meaning that there are d colours that appear on each sphere of C. This would imply $\chi(\mathbb{R}^d) \geq k + d - 1$ if we could find a lattice-like unit-distance graph with an origin v_0 that does not admit a proper k-colouring with d-chromatic origin v_0.

On of our main questions is about characterising those pairs (S, s_0) for which in every colouring of $\mathbb{R}^d$ we either find an AM similar copy of (S, s_0) or an *infinite* monochromatic AP. However, regarding applications to the Hadwiger-Nelson problem the following, weaker version would also be interesting to consider: Determine those (S, s_0) with $S \subseteq \mathbb{R}^d$ and $s_0 \in S$ for which there is a $D = D(k, S)$ such that the following is true. For every n in every k-colouring of $\mathbb{R}^D$ there is an AM similar copy of (S, s_0) or an n-term monochromatic AP with difference $t \in \mathbb{N}$ bounded by D. Note that there are pairs for which the property above does not hold when colouring $\mathbb{Z}$. For example let $S = \{-2, -1, 0, 1, 2\}$, $s_0 = 0$, and colour $i \in \mathbb{Z}$ red if $\lfloor i/D \rfloor \equiv 0 \bmod 2$ and blue if $\lfloor i/D \rfloor \equiv 1 \bmod 2$.

References

1 A. D.N.J. de Grey. The chromatic number of the plane is at least 5. *Geombinatorics*, 28:18–31, 2018.

2 A. D.N.J. de Grey. Polymath proposal: finding simpler unit distance graphs of chromatic number 5. https://polymathprojects.org/2018/04/10/polymath-proposal-finding-simpler-unit-distance-graphs-of-chromatic-number-5/, 2018.

3 P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus. Euclidean Ramsey theorems. I. *Journal of Combinatorial Theory, Series A*, 14(3):341–363, 1973. doi:10.1016/0097-3165(73)90011-3.

4 P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, and E. G. Straus. Euclidean Ramsey theorems. III. In *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. I*, pages 559–583. North-Holland, Amsterdam, 1975.

5 [fedja]. Comment on to the problem “Almost monochromatic point set” on MathOverflow. https://mathoverflow.net/q/300604, 2018.

6 N. Frankl, T Hubai, and D. Pálvölgyi. Almost-monochromatic sets and the chromatic number of the plane, 2019. arXiv:1912.02604.

7 R. L. Graham. Euclidean Ramsey theory. In Joseph O’Rourke Jacob E. Goodman and Csaba D. Tóth, editors, *Handbook of Discrete and Computational Geometry, Third Edition*, pages 281–297. Chapman and Hall/CRC, 2017. doi:10.1201/9781420035315.ch11.

8 R. L. Graham, B. L. Rothschild, and J.H. Spencer. *Ramsey theory*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., New York, 1990.

9 T. Hubai. Comment in “polymath16, fourth thread: Applying the probabilistic method”. https://dustingmixon.wordpress.com/2018/05/05/polymath16-fourth-thread-applying-the-probabilistic-method/#comment-4391, 2018.

10 D. Pálvölgyi [domotorp]. Comment in “polymath16, fourth thread: Applying the probabilistic method”. https://dustingmixon.wordpress.com/2018/05/05/polymath16-fourth-thread-applying-the-probabilistic-method/#comment-4306, 2018.

11 D. Pálvölgyi [domotorp]. Proposing the problem “Almost monochromatic point sets” on MathOverflow. https://mathoverflow.net/q/300604, 2018.

12 I. Rivin. Answer to the question “Existence of rational orthogonal matrices” on MathOverflow. https://mathoverflow.net/q/90070, 2012.

13 A. Soifer. *The Mathematical Coloring Book*. Springer-Verlag New York, 2009. doi:10.1007/978-0-387-74642-5.

14 B. L. van der Waerden. Beweis einer baudetschen Vermutung. *Nieuw Arch. Wiskunde*, 15:212–216, 1927. URL: https://ci.nii.ac.jp/naid/10004588776/en/.