# Diffusion Limits in the Online Subsequence Selection Problems

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## Abstract

In the stochastic sequential optimisation problems it is of interest to study features of strategies more delicate than just their performance measure. In this talk we focus on variations of the online monotone subsequence and bin packing problems, where it is possible to give a fairly explicit asymptotic description of the selection processes under strategies that are sufficiently close to optimality. We show that the transversal fluctuations of the shape and the length of selected subsequence approach Gaussian functional limits that are very different from their counterparts in the offline problem, where the full set of data can be used in selection algorithms.

**2012 ACM Subject Classification** Mathematics of computing → Markov processes

**Keywords and phrases** sequential optimisation, longest increasing subsequence, bin packing, fluctuations in the selection process, functional limit

**Digital Object Identifier** 10.4230/LIPIcs.AofA.2020.14

**Related Version** `https://arxiv.org/abs/2001.02249`

## 1 Introduction

In many stochastic optimisation problems units of the data become available in real time, whereas admissible decision strategies involve a series of irrevocable choices. The analysis of such problems is largely focused on finding optimal or near-optimal strategies to maximise a given performance criterion subject to constraints. Much less attention has been devoted to the structure of decison processes as a whole, in all intermediate states.

In this paper we mainly focus on the online monotone subsequence problem of Samuels and Steele [18]. Suppose i.i.d. marks drawn from the uniform distribution on $[0, 1]$ are observed, one by one, at times of an independent homogeneous Poisson process of intensity $\nu$ on $[0, 1]$. Each mark can be selected or rejected. The sequence of selected marks must increase. The task is to maximise the expected length of selected increasing subsequence using an online strategy. The online constraint requires that each decision becomes immediately irrevocable as the mark is observed, and must be based exclusively on the information accumulated previously without foresight of the future.

The optimal online strategy is defined recursively in terms of a variable acceptance window, which limits the difference between the next and previous selections. The strategy and its value can be found, in principle, by solving a dynamic programming equation, see [4, 6, 11] for properties of the solution and approximations. We are interested in the time evolution of increasing subsequences under online strategies that are within $O(1)$ gap from the optimum for large $\nu$.


31st International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms (AofA 2020).
Editors: Michael Drmota and Clemens Heuberger; Article No. 14; pp. 14:1–14:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Let $L_\nu(t)$ and $X_\nu(t)$, respectively, denote the length and the last element of the increasing subsequence selected by time $t \in [0,1]$ under the optimal strategy. The interest to date focused on the total length $L_\nu(1)$. Samuels and Steele derived the principal asymptotics $\mathbb{E}L_\nu(1) \sim \sqrt{2\nu}$, which was later found to be an upper bound with the optimality gap of order $\log \nu$ [6]. See [1, 2, 4, 6, 10] for refinements and generalisations. In a recent paper [11] we combined asymptotic analysis of the dynamic programming equation with a renewal approximation to the range of the process $Z_\nu(t) := \sqrt{\nu(1-t)(1-X_\nu(t))}$ to derive expansions for the mean

$$\mathbb{E}L_\nu(1) \sim \sqrt{2\nu} - \frac{1}{12}\log \nu + c_0^*, \quad \nu \to \infty, \tag{1}$$

and the variance

$$\mathrm{Var}\, L_\nu(1) \sim \frac{1}{3}\sqrt{2\nu} - \frac{1}{72}\log \nu + c_1^*, \quad \nu \to \infty, \tag{2}$$

where $c_0^*$ and $c_1^*$ are unknown constants. A central limit theorem for $L_\nu(1)$ was proved in [7] by analysis of a related martingale, and further extended in [11] to a larger class of asymptotically optimal strategies by the mentioned renewal theory approach.

The offline counterpart of the selection problem is the Ulam-Hammersley problem on the longest increasing subsequence of the Poisson scatter in the square $[0,1]^2$. Here, the well known principal asymptotics of the expected maximum length, $2\sqrt{\nu}$, is similar, but the second term of its asymptotic expansion and the principal term of the standard deviation are both of the order $\nu^{1/6}$. The limit law for the offline maximum length is the Tracy-Widom distribution from the random matrix theory. For survey and history see [17].

In the offline problem, some work has been done on the size of transversal fluctuations about the diagonal $x = t$ in $[0,1]^2$. Johansson [12] proved a measure concentration result asserting that, with probability approaching 1, every longest increasing subsequence (which is not unique) lies in a diagonal strip of width of the order $\nu^{-1/6+\epsilon}$. Duvergne, Nica and Virág [8] recently proved the existence and gave some description of the functional limit, which is not Gaussian. But for smaller exponent $-1/2 < \alpha < -1/6$, Joseph and Peled [15] showed that if the increasing sequence is restricted to lie within the strip of width $\nu^{-\alpha}$, the expected maximum length remains to be asymptotic to $2\sqrt{\nu}$, while the limit distribution of the length switches to normal.

To extend the parallels and gain further insight into the optimal selection it is of considerable interest to examine fluctuations of the processes $L_\nu$ and $X_\nu$ as a whole. On this path, one is lead to study the following scaled and centred versions of the running maximum and length processes:

$$\widetilde{X}_\nu(t) := \nu^{1/4}(X_\nu(t) - t), \qquad \widetilde{L}_\nu(t) = \nu^{1/4}\left(\frac{L_\nu(t)}{\sqrt{2\nu}} - t\right), \quad t \in [0,1]. \tag{3}$$

To compare, in the offline problem by similar centring the critical transversal and longitudinal scaling factors appear to be $\nu^{1/6}$ and $\nu^{1/3}$, respectively. Our central result (Theorem 4) is a functional limit theorem which entails that the process $(\widetilde{X}_\nu, \widetilde{L}_\nu)$ converges weakly to a simple two-dimensional Gaussian diffusion. In particular, $\widetilde{X}_\nu$ approaches a Brownian bridge. The limit of $\widetilde{L}_\nu$ is a non-Markovian process with the covariance function

$$(s,t) \mapsto \frac{2s(2-t) - (2-s-t)\log(1-s)}{6\sqrt{2}}, \quad 0 \le s \le t \le 1,$$

which corresponds to a correlated sum of a Brownian motion and a Brownian bridge.

The question about functional limits for $L_\nu$ and $X_\nu$ has been initiated by Bruss and Delbaen [7]. They employed the Doob-Meyer decomposition to compensate the processes, and in an analytic tour de force showed that the scaled martingales jointly converge to a correlated Brownian motion in two dimensions. However, the compensation keeps out of sight a drift component absorbing much of the fluctuations immanent to the selection process, let alone that the compensators themselves are nonlinear integral transforms of $X_\nu$. Looking at the generator of (3) we shall recognise the limit process without difficulty, showing that this is a Gaussian diffusion driven by the same Brownian motion as in [7]. But in order to justify the weak convergence in the Skorokhod space on the closed interval $[0, 1]$ we will need to circumvent a difficulty caused by pole singularities of the control function and the drift coefficient at the right endpoint.

A mathematically equivalent problem appears if the monotonicity condition is replaced by the constraint that the sum of of selected marks cannot exceed 1. A more general form of the latter "online bin-packing" problem has also been studied in the literature [9, 16], under the assumption that (positive) marks are sampled from some distribution with regular behaviour near 0. In the last section we sketch functional limit results for this bin-packing problem.

## 2 Selection strategies

It will be convenient to extend slightly the underlying framework by considering a homogeneous Poisson random measure $\Pi$ with intensity $\nu$ in the halfplane $\mathbb{R}_+ \times \mathbb{R}$, along with the filtration induced by restricting $\Pi$ to $[0, t] \times \mathbb{R}$ for $t \geq 0$. We interpret the generic atom $(t, x)$ of $\Pi$ as random mark $x$ observed at time $t$. A sequence $(t_1, x_1), \dots, (t_\ell, x_\ell)$ of atoms is said to be increasing if it is a chain in two dimensions, i.e. $t_1 < \cdots < t_\ell,\ x_1 < \cdots < x_\ell$.

For a given bounded measurable *control* function $\psi : [0, 1] \times \mathbb{R} \to \mathbb{R}_+$, an online strategy selecting such increasing sequence is defined by the following intuitive rule. Let $x$ be the last mark selected before time $t$, or some given $x_0$ if no selection has been made. Given the next mark $x'$ is observed at time $t$, this mark is selected if and only if $x < x' \leq x + \psi(t, x)$. One can think of more general online strategies, with the acceptance window shaped differently from an interval or possibly depending on the history in a more complex way. Yet the considered class is sufficient for the sake of optimisation and can be further reduced to controls of a special type.

For a given control $\psi$, define $X(t)$ and $L(t)$ to be, respectively, the last mark selected and the number of marks selected within the time interval $[0, t]$. The process $X = (X(t),\ t \in [0, 1])$, which we call the *running maximum*, is a time-inhomogeneous Markov process, jumping from the generic state $x$ at rate $\psi(t, x)$ to another state uniformly distributed on $[x, x+\psi(t, x)]$. The *length process* $L = (L(t),\ t \in [0, 1])$ just counts the jumps of $X$, hence the bivariate process $(X, L)$ is also Markovian. Moreover, the conditional distribution of $((X(t), L(t)),\ t \geq s)$ depends on the pre-$s$ history only through $X(s)$.

Intuitively, the bigger $\psi$ the faster $X$ and $L$ increase. To enable comparisons of selection processes with different controls it is very convenient to couple them by means of an additive representation through another Poisson random measure $\Pi^*$, thought of as a reserve of positive increments. The underlying properties of the planar Poisson process are translation invariance and spatial independence: $\Pi$ restricted to the shifted quadrant $(t, x) + \mathbb{R}_+^2$ is independent of $\Pi_{|[0,t]\times\mathbb{R}}$ and has the same distribution as the translation of $\Pi_{|\mathbb{R}_+^2}$ by vector $(t, x)$. So, letting $\Pi^*$ to be a distributional copy of $\Pi$, a solution to the system of stochastic differential equations

$$\mathrm{d}X(t) = \int_0^{\psi(t,X(t))} x\,\Pi^*(\mathrm{d}t\mathrm{d}x), \quad \mathrm{d}L(t) = \int_0^{\psi(t,X(t))} \Pi^*(\mathrm{d}t\mathrm{d}x) \tag{4}$$

with initial values $X(0) = x_0$ and $L(0) = 0$ will have the same distribution as $(X, L)$.

▶ **Lemma 1.** *For $i = 1, 2$ let $X_i$ be selection processes driven by controls $\psi_i$. By coupling via* (4)*, each time a process with smaller acceptance window jumps, the other process also has a jump of the same size.*

Conditionally on $(X(s), L(s)) = (x, \ell)$, the process $(X(s+\cdot) - x, L(s+\cdot) - \ell))$ has the same distribution as $(X^{(s,x)}, L^{(s,x)})$, which similarly to (4) is given by

$$dX^{(s,x)}(u) = \int_0^{\psi(s+u,x+X^{(s,x)}(u))} y\Pi^*(\mathrm{d}u\mathrm{d}y), \quad \mathrm{d}L^{(s,x)}(u) = \int_0^{\psi(s+u,x+X^{(s,x)}(u))} \Pi^*(\mathrm{d}u\mathrm{d}y).$$

Averaging, we obtain formulas for the predictable *compensators* of $X$ and $L$

$$C_X(t) := \frac{\nu}{2}\int_0^t \psi^2(s, X(s))\mathrm{d}s, \quad C_L(t) := \nu\int_0^t \psi(s, X(s))\mathrm{d}s, \tag{5}$$

so $X - C_X, L - C_L$ are zero-mean martingales.

With every control we may further relate a zero-mean martingale

$$M(t) := L(t) + \mathbb{E}\{L(1) - L(t)|X(t)\} - \mathbb{E}L(1) \tag{6}$$

with terminal value $L(1) - \mathbb{E}L(1)$. If $\psi$ does not depend on $x$, $L$ has independent increments and $M(t) = L(t) - \mathbb{E}L(t)$.

The selected increasing chain fits in the unit square if $X(1) \le 1$, which translates in terms of the control function as the condition of *feasibility*:

$$0 < \psi(t, x) \le 1 - x \quad \text{for} \quad (t, x) \in [0, 1]^2.$$

In the sequel, if not stated otherwise we set $x_0 = 0$ and only consider feasible controls.

## 2.1 Principal convergence of the moments

Let

$$p(t) := \mathbb{E}X(t) = \mathbb{E}C_X(t), \quad q(t) := \frac{\mathbb{E}L(t)}{\sqrt{2\nu}} = \frac{\mathbb{E}C_L(t)}{\sqrt{2\nu}}.$$

Some general relations between the moments follow straight from formulas for the compensators (5). For shorthand, write $\psi = \psi(X(s), s)$. We have

$$0 \le \mathbb{E}\int_0^t \left(1 \pm \sqrt{\nu/2}\,\psi\right)^2 \mathrm{d}s = t \pm 2q(t) + p(t),$$

where the right-hand side in increasing in $t$. It follows,

$$p(t) - t \ge 2\,(q(t) - t). \tag{7}$$

Using the Cauchy-Schwarz inequality

$$(p(t) - t)^2 = \left(\mathbb{E}\int_0^t \left(1 - \frac{\nu}{2}\psi^2\right)\mathrm{d}s\right)^2 \le (t + 2q(t) + p(t))(t - 2q(t) + p(t)). \tag{8}$$

Similarly

$$(q(t)-t)^2 = \left(\mathbb{E}\int_0^t 1\cdot\left(1-\sqrt{\nu/2}\,\psi\right)\mathrm{d}s\right)^2 \le t\,(t-2q(t)+p(t)) \tag{9}$$

The above relations did not use the feasibility constraint. For feasible control we have $p(1)<1$, hence from (7) also $q(1)<1$. Since all factors in the right-hand sides of (8), (9) are increasing, replacing them by their maximal values at $t=1$ we obtain

$$(p(t)-t)^2 < 8(1-q(1)), \quad (q(t)-t)^2 < 2(1-q(1)). \tag{10}$$

We say that a strategy $\psi=\psi_\nu$ is *asymptotically optimal in the principal term* if $q(1)\to 1$, as $\nu\to\infty$, i.e. $\mathbb{E}L_\nu(1)\sim\sqrt{2\nu}$; in that case (10) imply the uniform convergence of the moments

$$\sup_{t\in[0,1]}|p(t)-t|\to 0,\ \sup_{t\in[0,1]}|q(t)-t|\to 0.$$

It follows from (1) that under the optimal strategy

$$1-q(1)\sim\frac{\log\nu}{12\sqrt{2\nu}},\quad \nu\to\infty. \tag{11}$$

This relation can be called a *two-term asymptotic optimality.* Whenever this holds, the general bounds (10) imply that both $\sup_{t\in[0,1]}|p(t)-t|$ and $\sup_{t\in[0,1]}|q(t)-t|$ can be estimated as $O(\sqrt{\log\nu}/\nu^{1/4})$.

## 2.2 The stationary strategy

We call the strategy with control $\psi(t,x)=\sqrt{2/\nu}$ *stationary.* Although not feasible, the stationary strategy is an important benchmark. Clearly, $L$ is a Poisson counting process with intensity $\mathbb{E}L(1)=\sqrt{2\nu}$. Taking general constant control $\psi(t,x)=\sqrt{c/\nu}$ with some $c>0$ will yield a strategy outputting the mean length $\sqrt{\{c\wedge(2/c)\}\nu}$, which is maximal for $c=2$. In fact, a much stronger optimality property holds: the stationary strategy achieves the maximum expected length over the class of strategies that satisfy the *mean-value* constraint $\mathbb{E}X(1)\le 1$, see [1, 4, 10, 11] for proof and generalisations. This gives the well-known upper bound mentioned in the Introduction, because each feasible strategy meets the mean-value constraint.

It is seen from (4) that $X$ is a compound Poisson process

$$X(t)=\sqrt{\frac{2}{\nu}}\sum_{i=0}^{L(t)}U_i\,,$$

where $U_1,U_2,\dots$ are independent of $L$, uniformly distributed on $[0,1]$. Straightforward calculation of moments using Wald's identities yields

$$\mathbb{E}X(t)=t,\quad \mathrm{Var}X(t)=\frac{2^{3/2}t}{3\sqrt{\nu}},\quad \mathrm{Cov}(X(t),L(t))=t.$$

Since $(X,L)$ has independent increments, a functional limit in the Skorohod topology on $D[0,1]$ follows easily from the multidimensional invariance principle:

$$(\widetilde{X},\widetilde{L})\Rightarrow(W_1,W_2),\quad \text{as } \nu\to\infty,$$

where $\Rightarrow$ denotes weak convergence, and the limit process $\boldsymbol{W} := (W_1, W_2)$ is a two-dimensional Brownian motion with zero drift and covariance matrix

$$\mathbb{E}\{\boldsymbol{W}(t)^T \boldsymbol{W}(t)\} = t\,\boldsymbol{\Sigma}, \quad \text{where } \boldsymbol{\Sigma} := \begin{pmatrix} \frac{2\sqrt{2}}{3} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{12}$$

So, marginally, $W_1$ and $W_2$ are centred Brownian motions with diffusion coefficients and correlation, respectively,

$$\sigma_1 := \frac{2^{3/4}}{\sqrt{3}}, \quad \sigma_2 := \frac{1}{2^{1/4}}, \quad \rho := \frac{\sqrt{3}}{2}. \tag{13}$$

Notably, $\rho = \sigma_2/\sigma_1$, which implies that the process $\boldsymbol{W}$ satisfies the identity $2W_2 - W_1 \overset{d}{=} W_1$, which has a pre-limit analogue $2\widetilde{L} - \widetilde{X} \overset{d}{=} \widetilde{X}$. The identity can be explained by symmetry of the uniform distribution about $1/2$, which allows us to write

$$X(t) \overset{d}{=} \sqrt{\frac{2}{\nu}} \sum_{i=0}^{L(t)} (1 - U_i) = \sqrt{\frac{2}{\nu}}\, L(t) - X(t)\,.$$

The martingale (6) just coincides with the naturally centred $L$.

## 2.3 Self-similar asymptotically optimal strategies

We call strategy *self-similar* if the control $\psi = \psi_\nu$ has the form

$$\psi(t, x) := (1 - x)\delta(\nu(1 - t)(1 - x)), \quad (t, x) \in [0, 1]^2. \tag{14}$$

for some function $\delta : \mathbb{R}_+ \to [0, 1]$. Note that such a strategy is feasible and $\psi_\nu(0, 0) = \delta(\nu)$. The rationale behind this definition is the following. Assuming $x$ to be the running maximum at time $t$, the remaining part of the chain should be selected from the north-east rectangle spanned on $(t, x)$ and $(1, 1)$, and by the optimality principle the subsequence selected from the rectangle should have maximal expected length. Mapping the rectangle onto $[0, 1]^2$ it is readily seen that the subproblem is an independent replica of the original problem of optimal selection from the unit square with intensity parameter $\nu(1 - t)(1 - x)$. The martingale (6) assumes the form

$$M(t) = L(t) + F(\nu(1 - t)(1 - X(t)) - F(\nu), \tag{15}$$

where the *value function* $F$, for given control, depends on one variable

$$F(\nu) := \mathbb{E}L_\nu(1), \quad F(0) = 0.$$

**Assumption.** From this point on we assume that *the selection strategy is self-similar as defined by* (14)*, with function* $\delta$ *having asymptotics*

$$\delta(\nu) = \sqrt{2/\nu} + O\left(\nu^{-1}\right), \quad \nu \to \infty. \tag{16}$$

The assumption is central and deserves comments. Whenever $\nu(1 - x)(1 - t)$ is large, (16) implies asymptotics of the control

$$\psi(t, x) \sim \sqrt{\frac{2(1 - x)}{\nu(1 - t)}}, \tag{17}$$

which shows that near the diagonal $x = t$ the acceptance window is about the same as for the stationary strategy. Away from the diagonal the acceptance window is close to that for the stationary strategy adjusted to the rectangle north-east of $(t, x)$.

It is known [11] that the optimal strategy satisfies the asymptotic expansion

$$\delta^*(\nu) \sim \sqrt{2/\nu} - (3\nu)^{-1} + O(\nu^{-3/2}). \tag{18}$$

A minor adjustment of Theorem 6 in [11] shows that if we assume, more generally, the relation $\delta(\nu) \sim \sqrt{2/\nu} + \beta/\nu$ with some parameter $\beta \in \mathbb{R}$, then asymptotic expansions of the moments (1), (2) are still valid, with only constant terms depending on $\beta$. Using a sandwich argument based on Lemma 1, it can be further shown that under the assumption (16) expansions of the moments hold but with constant terms being replaced by some $O(1)$ remainders. In particular, condition (16) ensures the two-term asymptotic optimality (11), equivalent to the asymptotic expansion of the value function,

$$F(\nu) = \sqrt{2\nu} - \frac{1}{12}\log(\nu + 1) + O(1). \tag{19}$$

We stress that the logarithmic term here (as well as in the counterpart of the variance formula (2)) is not affected by the remainder in (16), rather appears due to the self-similar adjustment of the (feasible version of) stationary strategy, as incorporated in (17).

Approximation (17) is not useful when $t$ or $x$ are too close to 1, so that $\nu(1 - t)(1 - x)$ varies within $O(1)$. To embrace the full range of the variables, for the sequel we choose $\beta > 1$ large enough to meet the bounds

$$\left|\psi(t, x) - \sqrt{\frac{2(1 - x)}{\nu(1 - t)}}\right| \quad < \quad \frac{\beta}{\nu(1 - t)}, \quad \text{for} \quad (t, x) \in [0, 1) \times [0, 1). \tag{20}$$

This will be employed along with the bound

$$\psi(t, x) \quad < \quad \frac{1}{\nu(1 - t)}, \quad \text{for} \quad 1 - x < \frac{1}{\nu(1 - t)} \tag{21}$$

which follows by feasibility.

## 3 Generators

A major technical difficulty in showing the convergence in $D[0, 1]$ is the singularity of (17) at $t = 1$. This will be handled in two steps. First, we bound the time variable away from $t = 1$ and show the convergence of the generators on a sufficiently big space of test functions. Then we will apply domination arguments to bound fluctuations near the right endpoint, thus justifying convergence on the full $[0, 1]$.

The processes we consider are not time-homogeneous, therefore by computing generators we include the time variable in the state vector. From (4), the generator of the jump process $(X, L)$ is

$$\mathcal{L}_\nu f(t, x, \ell) = f_t(t, x, \ell) + \nu \int_0^{\psi(t,x)} \{f(t, x + u, \ell + 1) - f(t, x, \ell)\} \mathrm{d}u.$$

For the processes centered by $t$ we should include $-f_x - f_\ell$ in the generator. Then, with the change of variables

$$x \to x\nu^{-1/4} + t, \;\; \ell \to (\ell\nu^{-1/4} + t)\sqrt{2\nu}, \;\;\; \widetilde{\psi}(t, x) := \nu^{1/4}\psi(t, x\nu^{-1/4} + t)$$

we arrive at the generator of $(\widetilde{X}, \widetilde{L})$

$$\widetilde{\mathcal{L}}_\nu f = f_t - \nu^{1/4}(f_x + f_\ell) + \nu^{3/4} \int_0^{\widetilde{\psi}(t,x)} \{f(t, x+u, \ell+v) - f(t,x,\ell)\}\mathrm{d}u, \tag{22}$$

where we abbreviate $f = f(t, x, \ell)$ etc., and

$$v := (4\nu)^{-1/4} \tag{23}$$

We extend $\widetilde{\mathcal{L}}_\nu f$ by 0 outside the reachable range of $(\widetilde{X}, \widetilde{L})$. Note that the range of $\widetilde{X}(t)$ lies within the bounds

$$-t\nu^{1/4} \le x \le (1-t)\nu^{1/4}.$$

We fix $h \in (0,1)$ and focus on $t \in [0, 1-h]$, so achieving uniformly in this range

$$\widetilde{\psi}(t,x) = O(\nu^{-1/4}), \tag{24}$$

and for $k \ge 1$

$$\widetilde{\psi}^k(t,x) = \left(2 - \frac{2x}{\nu^{1/4}(1-t)}\right)^{k/2} \nu^{-k/4} + O(\nu^{-(k+2)/4}), \text{ for } x \le (1-t)\nu^{1/4} - \frac{1}{\nu^{3/4}(1-t)} \tag{25}$$

as dictated by the bounds (20), (21).

Now let $\mathcal{D}$ be the space of vanishing at infinity functions $f \in C_0^3([0,1] \times \mathbb{R}^2)$ which satisfy a rapid decrease property

$$\sup |x^k f_\bullet(t,x,\ell)| < \infty,$$

where $f_\bullet$ is any derivative of $f$ of the first or second order and $k > 0$. Set

$$D^>_{h,\nu} := \{(t,x,\ell) : t \in [0, 1-h],\ |x| > \nu^{1/16}\}, \quad D^<_{h,\nu} := \{(t,x,\ell) : t \in [0,1-h],\ |x| \le \nu^{1/16}\}.$$

We shall be using that for $f \in \mathcal{D}$

$$\lim_{\nu \to \infty} \sup_{D^>_{h,\nu}} |\nu^k f_\bullet(x)| = 0. \tag{26}$$

The integrand in (22) expands as

$$f(t, x+u, \ell+v) - f(t,x,\ell) = f_x u + f_\ell v + \frac{1}{2} f_{xx}\, u^2 + f_{x\ell}\, uv + \frac{1}{2} f_{\ell\ell}\, v^2 + R,$$

where the remainder can be estimated as

$$|R| \le c \sum_{i=0}^{3} u^i v^{3-i},$$

with constant $c$ chosen bigger than the maximum absolute value of any third derivative of $f$. Hence for the integrated remainder we have a uniform estimate

$$\nu^{3/4} \left| \int_0^{\widetilde{\psi}} R du \right| \le \nu^{3/4} c \sum_{i=1}^{4} \widetilde{\psi}^i v^{4-i} = O(\nu^{-1/4}),$$

using (24), (23).

Integrating the Taylor polynomial yields

$$\widetilde{\mathcal{L}}_\nu f = f_t - \nu^{1/4}(f_x + f_\ell) + \nu^{3/4}\left\{ \frac{f_x\,\widetilde{\psi}^2}{2} + f_\ell\, v\widetilde{\psi} + \frac{f_{xx}\widetilde{\psi}^3}{6} + \frac{f_{x\ell}\,\widetilde{\psi}^2}{2} v + \frac{f_{\ell\ell} v^2 \widetilde{\psi}}{2} \right\} + O(\nu^{-1/4}).$$

Applying (26)

$$\lim_{\nu\to\infty} \sup_{D^{>}_{h,\nu}} |\widetilde{\mathcal{L}}_\nu f(t,x,\ell)| = 0. \tag{27}$$

Thus we focus on the range $D^{<}_{h,\nu}$, where (20) and (25) can be employed. From (20)

$$-\nu^{1/4} f_x + \nu^{3/4}\frac{1}{2} f_x \widetilde{\psi}^2 = -\frac{x}{1-t} f_x + O(\nu^{-1/4}).$$

Observing that in this range $|x\nu^{-1/4}| \le \nu^{-3/16}$ for $k>0$ we expand as

$$\widetilde{\psi}^k(t,x,\ell) = 2^{k/2}\nu^{-k/4} - \frac{2^{k/2-1}x}{1-t}\nu^{-(k+1)/4} + O(\nu^{-(k+1)/4-1/8}),$$

with the remainder estimate being uniform over $D^{<}_{h,\nu}$. The remaining calculations is a careful book-keeping using this formula and that the derivatives are uniformly bounded.

Define operator

$$\widetilde{\mathcal{L}} f := f_t - \frac{x}{1-t} f_x - \frac{x}{2(1-t)} f_\ell + \frac{{\sigma_1}^2}{2} f_{xx} + \frac{{\sigma_2}^2}{2} f_{\ell\ell} + \sigma_1\sigma_2\rho f_{x\ell},$$

with $\sigma_1, \sigma_2$, and $\rho$ given by (13).

▶ **Lemma 2.** *For $f \in \mathcal{D}$ and $h \in (0,1)$*

$$\lim_{\nu\to\infty} \sup_{(t,x,\ell)\in[0,1-h]\times\mathbb{R}^2} |\widetilde{\mathcal{L}}_\nu f(t,x,\ell) - \widetilde{\mathcal{L}} f(t,x,\ell)| = 0.$$

Operator $\widetilde{\mathcal{L}}$ is the generator of a Gaussian diffusion process which satisfies the stochastic differential equation

$$\mathrm{d}Y_1(t) = -\frac{Y_1(t)}{1-t}\,\mathrm{d}t + \mathrm{d}W_1(t), \tag{28}$$

$$\mathrm{d}Y_2(t) = -\frac{Y_1(t)}{2(1-t)}\,\mathrm{d}t + \mathrm{d}W_2(t), \tag{29}$$

with zero initial value, where $\boldsymbol{W} = (W_1, W_2)$ is the two-dimensional Brownian motion with covariance $\boldsymbol{\Sigma}$ introduced in (12).

From the equation for the first component (28), it is seen that $Y_1$ is a Brownian bridge

$$Y_1(t) = (1-t)\int_0^t \frac{\mathrm{d}W_1(s)}{1-s}, \tag{30}$$

with the covariance function $\mathrm{Cov}(Y_1(s), Y_1(t)) = \sigma_1 s(1-t)$, $0 \le s \le t \le 1$. In particular, $Y_1(1) = 0$. We shall discuss the second component later on.

The space $\mathcal{D}$ is dense in a larger space $C_0^3([0,1-h]\times\mathbb{R}^2)$. Since the differentiability properties of functions are preserved under averaging over normally distributed translations, $\mathcal{D}$ is invariant under the semigroup of $\boldsymbol{Y}$. Thus by Watanabe's theorem (see [13], Proposition

17.9) $\mathcal{D}$ is a core of operator $\widetilde{\mathcal{L}}$. The above Lemma 2 and Theorem 17.25 from [13] now imply weak convergence

$$(\widetilde{X}_\nu, \widetilde{L}_\nu) \Rightarrow (Y_1, Y_2) \;\text{ in }\; D[0, 1-h] \tag{31}$$

for every $h \in (0,1)$. A closer inspection of the above approximation errors suggests that the quality of convergence deteriorates as $h \to 0$.

We encountered the Brownian motion $\boldsymbol{W}$ in connection with the free-endpoint stationary strategy in Section 2.2. Now we see that the variable control (17) causes a drift that forces the running maximum to timely arrive at the north-east corner of the square.

## 4 The functional limit in $D[0,1]$

The martingale problem for $\widetilde{\mathcal{L}}$ is well-posed on the complete interval, and the SDE (28) has a unique strong solution. This suggests to extend convergence (31) to the full $[0,1]$. To that end, we need to monitor the behaviour of $\widetilde{\mathcal{L}}_\nu f$ for $t$ close to 1.

Since (31) entails convergence of finite-dimensional distributions for times $t < 1$ and ensures that the modulus of continuity behaves properly over $[0, 1-h]$, to justify tightness of $\widetilde{X}_\nu$'s, and hence their convergence on $[0,1]$, it will be enough to show that

$$\lim_{h\to 0} \limsup_{\nu} \mathbb{P}\Big( \sup_{t\in[1-h,1]} |\widetilde{X}_\nu(t)| > h^{1/4}\Big) = 0. \tag{32}$$

Define $\xi_{\nu,h}$ by setting

$$\widetilde{X}_\nu(1-h) = \sigma_1 \sqrt{h(1-h)}\, \xi_{\nu,h}.$$

Since $\widetilde{X}_\nu(1-h) \xrightarrow{d} Y_1(1-h)$ the distribution of $\xi_{\nu,h}$ is close to $\mathcal{N}(0,1)$ for large $\nu$.

By self-similarity of the selection strategy, $((X_\nu(t)-t),\ t \in [1-h,1])$ has the same distribution as $(h^{-1}(X_{\nu h^2}(t)-t),\ t\in[0,1])$ with the initial value $X_{\nu h^2}(0) = \nu^{-1/4}\sigma_1\sqrt{(1-h)/h}\,\xi_{\nu,h}$, as is seen by zooming in the corner square north-east of the point $(1-h, 1-h)$ with factor $h^{-1}$. Changing variable $\nu h^2 \to \nu$, (32) translates as a compact containment condition

$$\lim_{h\to 0} \limsup_{\nu} \mathbb{P}\Big( \sup_{t\in[0,1]} |\widetilde{X}_\nu(t)| > h^{-1/4}\Big) = 0 \tag{33}$$

under the initial value $\widetilde{X}_\nu(0) = \sqrt{1-h}\,\xi_{\nu,h}$.

To verify (33) we shall squeeze the running maximum $X$ between $X^{\downarrow}$ and $X^{\uparrow}$ whose normalised versions satisfy the compact containment condition. We force the majorant and the minorant to live on the opposite sides of the diagonal. Both have independent, almost stationary increments, so that functional limits can be readily identified. For simplicity we will assume $X_\nu(0) = 0$. The general case with $X_\nu(0)$ of the order $\nu^{-1/4}$ can be handled by the same method.

### 4.1 Majorant

Define process $X^{\uparrow} = X^{\uparrow}_\nu$ as solution to

$$\mathrm{d}X^{\uparrow}(t) = \int_0^{\psi^{\uparrow}(t)} x\Pi^*(\mathrm{d}t\mathrm{d}x) + 1(X^{\uparrow}(t) = t)\mathrm{d}t,$$

$X^{\uparrow}(0) = K\nu^{-1/2}$ for some big enough $K > 0$, with control

$$\psi^{\uparrow}(t) := \sqrt{\frac{2}{\nu}} + \frac{\beta}{\nu(1-t)} 1(t \le 1 - K\nu^{-1/2})$$

not depending on $x$. Notation $1(\cdots)$ is used for indicators. The process never drops below the line $x = K\nu^{-1/2} + t$, and whenever the line is hit the path drifts along it for some time. By the construction, above the diagonal the process $X^{\uparrow}$ increases faster than $X$, and is, in fact, a majorant.

▶ **Lemma 3.** *By coupling via* (4), $X^{\uparrow} \ge X$ *a.s.*

Let

$$S(t) := \int_0^t \int_0^{\psi^{\uparrow}(t)} x\Pi^*(\mathrm{d}s\mathrm{d}x) - t.$$

This is a process with independent increments, which we can split in two independent components

$$S(t) = \left(\int_0^t \int_0^{\sqrt{2/\nu}} x\Pi^*(\mathrm{d}s\mathrm{d}x) - t\right) + \int_0^t \int_{\sqrt{2/\nu}}^{\psi^{\uparrow}(t)} x\Pi^*(\mathrm{d}s\mathrm{d}x).$$

The mean value of the second part is estimated as

$$\frac{2\nu}{\sqrt{\nu}} \int_0^{1-K/\sqrt{\nu}} \frac{\beta}{\nu(1-t)} \mathrm{d}t = O\left(\frac{\log \nu}{\sqrt{\nu}}\right),$$

and the first is a compensated compound Poisson process. Thus $\nu^{1/4} S \Rightarrow W_1$ as $\nu \to \infty$.

Processes akin to $(X^{\uparrow}(t) - t,\ t \in [0,1])$ are common in applied probability [3, 5]. In particular, by the interpretation as the content of a single-server M/G/1 queue, the positive increments present jobs that arrive by Poisson process and are measured in terms of the demand on the service time. The downward drift occurs due to the unit processing rate when the server is busy. Borrowing a useful identity,

$$X^{\uparrow}(t) - t = S(t) - \inf_{u \in [0,t]} S(u),$$

we conclude on the weak convergence $(\nu^{1/4}(X^{\uparrow}(t) - t),\ t \in [0,1]) \Rightarrow |W_1|$ to a reflected Brownian motion.

## 4.2 Minorant

$$\psi^{\downarrow}(t,x) = \begin{cases} \left(\sqrt{\frac{2}{\nu}} - \frac{\beta}{\nu(1-t)}\right) \wedge (t-x), & \text{for} \quad 0 \le t \le 1 - K/\sqrt{\nu}, \\ 0, \quad \text{for} \quad 1 - K/\sqrt{\nu} < t \le 1. \end{cases}$$

where $K$ is sufficiently large. We can regard this as a suboptimal strategy that never selects marks $x > t$. Starting at state 0, the running maximum process stays below the diagonal throughout, and gets frozen at $t = 1 - K/\sqrt{\nu}$. A counterpart of Lemma 3, $X^{\downarrow} < X$ a.s., is readily checked.

Switching general $\beta > 0$ to $\beta = 0$ impacts $\mathbb{E}X^{\downarrow}(t)$ by $O(\nu^{-1/2} \log \nu)$ uniformly in $t \in [0,1]$. Indeed, the jumps are bounded by $2/\sqrt{\nu}$, and the expected number of jumps increases by $O(\log \nu)$.

Assuming $\beta = 0$, the process $(X^{\downarrow}(t) - t,\ t \in [0, 1 - K\nu^{-1/2}])$ is a compensated compound Poisson process on the negative halfline, with reflection at 0. We have therefore

$$(\nu^{1/4}(X^{\downarrow}(t) - t),\ t \in [0,1]) \Rightarrow -|W_1|.$$

A rigorous proof can be obtained by inspecting convergence of the generator acting on the functions $f \in \mathcal{D}$ with $f_x(t,0) = 0$.

## 4.3 The length process near termination

Having established weak convergence of $\widetilde{X}$, the fluctuations of $\widetilde{L}$ near $t = 1$ are estimated by verifying that

$$\lim_{h\to 0} \limsup_{\nu} \mathbb{P}( \sup_{t\in[1-h,1]} |\widetilde{L}(t) - \widetilde{L}(1-h)| > \epsilon) = 0. \tag{34}$$

This is done with the help of analysis of the martingale $M$.

# 5 Main result

By the domination argument, tightness of $(\widetilde{X}_\nu, \widetilde{L}_\nu)$ follows on the whole $[0,1]$, and we arrive at our main result.

▶ **Theorem 4.** *The normalised running maximum and the length process* (3) *driven by a control satisfying* (14) *and* (16) *(in particular, under the optimal online selection strategy) converge weakly in the Skorokhod space* $D[0,1]$*,*

$$(\widetilde{X}_\nu, \widetilde{L}_\nu) \Rightarrow (Y_1, Y_2), \quad \text{as} \quad \nu \to \infty,$$

*where the limit bivariate process is a Gaussian diffusion defined by the equations* (28), (29) *with zero initial conditions.*

We observed already that $Y_1$ is the Brownian bridge (30) and from (29)

$$Y_2(t) = \frac{Y_1(t)}{2} - \frac{W_1(t)}{2} + W_2(t),$$

so splitting the martingale part in independent components, we get, explicitly,

$$Y_2(t) = \int_0^t \frac{(1-t)}{2(1-s)} \mathrm{d}W_1(s) + \frac{1}{4} W_1(t) + \left( W_2(t) - \frac{3}{4} W_1(t) \right), \tag{35}$$

which is a sum of a Brownian motion, derived Brownian bridge and another independent Brownian motion.

To find the covariance structure, it is convenient to resort to matrix calculations. We may write the solution to (28), (29) as

$$\boldsymbol{Y}(t)^T = e^{a(t)} \int_0^t e^{-a(u)} \mathrm{d}\boldsymbol{W}(u)^T,$$

where

$$a(t) := A \int_0^t \frac{1}{1-u} \mathrm{d}u = A \log(1-t), \qquad A := \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 0 \end{pmatrix},$$

which yields by the Itó isometry

$$\mathbb{E}\{\boldsymbol{Y}(s)^T\boldsymbol{Y}(t)\} = \int_0^t e^{a(s)-a(u)}\boldsymbol{\Sigma} e^{(a(t)-a(u))^T}\mathrm{d}u, \qquad 0 \le s \le t \le 1.$$

Since $A$ is idempotent matrix, the exponents are readily calculated as

$$e^{a(t)} = \begin{pmatrix} 1-t & 0 \\ -\frac{t}{2} & 1 \end{pmatrix}, \qquad e^{-a(t)} = \begin{pmatrix} \frac{1}{1-t} & 0 \\ \frac{t}{2(1-t)} & 1 \end{pmatrix},$$

and we arrive at the cross-covariance matrix

$$\mathbb{E}\{\boldsymbol{Y}(s)^T\boldsymbol{Y}(t)\} \quad = \begin{pmatrix} \frac{2\sqrt{2}\,s(1-t)}{3} & \frac{2s(1-t)-(1-s)\log(1-s)}{3\sqrt{2}} \\ \frac{(1-t)(2s-\log(1-s))}{3\sqrt{2}} & \frac{2s(2-t)-(2-s-t)\log(1-s))}{6\sqrt{2}} \end{pmatrix},$$

where $0 \le s \le t \le 1$.

The limit length process $Y_2$ is *not* Markovian, since its covariance function does not satisfy the factorisation criterion (see [14], p. 148). The sum of two first terms in (35) is non-Markovian too.

## 6 Diffusion approximation in the bin-packing problem

We turn to a version of the bin-packing problem. Suppose that i.i.d., positive marks arrive by a Poisson process of intensity $\nu$ on $[0,1]$, and that the marks are sampled from a density satisfying $f(y) \sim A\alpha y^{\alpha-1}$ as $y \to 0$, with $\alpha, A > 0$. The stochastic optimisation task is to maximise the expected number of online selections under the constraint that their total does not exceed given $c > 0$.

Let $Z_\nu(t), N_\nu(t)$, $t \in [0,1]$ denote the sum and the number of selected marks at time $t$ under the optimal selection policy, which has a control function $\psi(t,z) = (c-z)\,\delta(\nu(1-t)(c-z)^\alpha)$, where

$$\delta(\nu) = \frac{\gamma_1}{\nu^{1/(\alpha+1)}} + O(\nu^{-2/(\alpha+1)}), \quad \nu \to \infty, \ \gamma_1 = \left(\frac{(\alpha+1)}{A\,\alpha}\right)^{1/(\alpha+1)},$$

see [9]. It was shown in [9] that the mean number of selections has asymptotics $u(\nu) \sim \gamma_2\,\nu^{1/(\alpha+1)}$, $\nu \to \infty$, with

$$\gamma_2 = \left(\frac{c(\alpha+1)}{A\,\alpha}\right)^{1/(\alpha+1)}.$$

This suggests the normalisations

$$\widetilde{Z}_\nu(t) := \nu^{1/(2(\alpha+1))}\,(Z_\nu(t) - ct)\,, \quad \widetilde{N}_\nu(t) := \nu^{1/(2(\alpha+1))}\left(\frac{N_\nu(t)}{\gamma_2 \nu^{1/(\alpha+1)}} - t\right).$$

The infinitesimal generator of $(t, \widetilde{Z}_\nu(t), \widetilde{N}_\nu(t))$ is

$$\mathcal{L}_\nu f(t,z,n) = f_t - \nu^{1/(\alpha+1)}(cf_z + f_n) + \nu^{1-1/(2(\alpha+1))}\int_0^{\widetilde{\psi}(t,z)} \left( f\left(t, z+y, n+\frac{1}{\gamma_2\nu^{1/(2(\alpha+1))}}\right) - f(t,z,n)\right) f(y)\mathrm{d}y,$$

where

$$\widetilde{\psi}(t,z) = \nu^{1/(2(\alpha+1))}\,\psi(t, z\nu^{-1/(2(\alpha+1))} + tc).$$

A fairly long computation yields the asymptotics, as $\nu \to \infty$,

$$\mathcal{L}_\nu f(t,z,n) \sim f_t - \frac{z}{1-t} f_z - \frac{\alpha z}{c(\alpha+1)(1-t)} f_n + \frac{{\sigma_3}^2}{2} f_{zz} + \frac{{\sigma_4}^2}{2} f_{nn} + \rho_0 \sigma_3 \sigma_4 f_{zn},$$

where

$$\sigma_3 = \frac{(\alpha+1)^{(\alpha+2)/(2(\alpha+1))}\sqrt{c}}{\sqrt{\alpha+2}}, \ \sigma_4 = \frac{1}{A^{1/(2(\alpha+1))}(\alpha+1)^{\alpha/(2(\alpha+1))}},$$

and

$$\rho_0 = \frac{\alpha^{\alpha/(\alpha+1)}\sqrt{\alpha+2}}{c^{(\alpha-1)/(2(\alpha+1))}A^{\alpha/(2(\alpha+1))}(\alpha+1)^{2\alpha/(\alpha+1)}}.$$

Using this we were able to show the functional convergence

$$(\widetilde{Z}_\nu, \widetilde{N}_\nu) \Rightarrow (Y_3, Y_4), \quad \text{as} \ \ \nu \to \infty,$$

in $D[0, 1-h]$ for every $h \in (0,1)$, where the limit process $(Y_3(t), Y_3(t))$ is a Gaussian diffusion satisfying the SDE

$$\mathrm{d}Y_3(t) = -\frac{Y_3(t)}{1-t}\,\mathrm{d}t\, + \mathrm{d}W_3(t), \quad \mathrm{d}Y_4(t) = -\frac{\alpha Y_3(t)}{c(\alpha+1)(1-t)}\,\mathrm{d}t\, + \mathrm{d}W_4(t)$$

with zero initial conditions. Here, $(W_3, W_4)$ is a two-dimensional centred Brownian motion with the covariance matrix $\mathbf{\Sigma_0} = \begin{pmatrix} {\sigma_3}^2 & \rho_0\sigma_3\sigma_4 \\ \rho_0\sigma_3\sigma_4 & {\sigma_4}^2 \end{pmatrix}$.

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