Counting Cubic Maps with Large Genus

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Abstract

We derive an asymptotic expression for the number of cubic maps on orientable surfaces when the genus is proportional to the number of vertices. Let Σ_g denote the orientable surface of genus g and $\theta = g/n \in (0, 1/2)$. Given $g, n \in \mathbb{N}$ with $g \to \infty$ and $\frac{n}{2} - g \to \infty$ as $n \to \infty$, the number $C_{n,g}$ of cubic maps on Σ_g with $2n$ vertices satisfies

$$C_{n,g} \sim (g!)^2 \, \alpha(\theta) \, \beta(\theta)^n \, \gamma(\theta)^{2g}, \quad \text{as} \quad g \to \infty,$$

where $\alpha(\theta), \beta(\theta), \gamma(\theta)$ are differentiable functions in $(0, 1/2)$. This also leads to the asymptotic number of triangulations (as the dual of cubic maps) with large genus. When g/n lies in a closed subinterval of $(0, 1/2)$, the asymptotic formula can be obtained using a local limit theorem. The saddle-point method is applied when $g/n \to 0$ or $g/n \to 1/2$.

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1 Introduction

Since the seminal work of Tutte on planar maps [19], various types of maps on surfaces have attracted much attention (see e.g. [3, 4, 11, 13]). Most of results on maps deal with the case when the genus is *constant*. When the genus is proportional to the number of vertices, edges or faces, there are only a few results, which deal with either maps with one face (also known as unicellular maps) [1, 7, 18] or triangular maps (also known as triangulations) [6].

In this paper we study cubic maps (and their dual, triangular maps) on orientable surfaces of non-constant genus. As demonstrated in [8, 15], such cubic maps form base cases in the study of sparse random graphs of non-constant genus. Furthermore, the study of random graphs of non-constant genus has only been initiated very recently [8, 16], and it is likely to prove to be the most interesting – the “evolution” of random graphs of non-constant genus depends heavily on the ratios between the genus, the number of edges, and the number of vertices, and it “transforms” from a random forest to the classical Erdős-Rény random graph.

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We let Σ_g denote the orientable surface of genus g. A map on Σ_g is a connected graph G that is *embedded* on Σ_g in such a way that each component of $\Sigma_g - G$, called *a face*, is simply connected region. A map on Σ_g will be called a map with genus g. Throughout the paper, a map is always *rooted*, meaning that an edge is distinguished together with an end vertex and a side of it.

A map is called *cubic* if all its vertices have degree 3. The dual of a cubic map is called a *triangular map* whose faces all have degree 3. Let $C_{n,g}$ be the number of cubic maps with $2n$ vertices and genus g and $T_{n,g}$ be the number of triangular maps with n vertices and genus g. Recall Euler's formula for a map with v vertices, e edges, f faces, and genus g:

$$v - e + f = 2 - 2g.$$

In addition, a triangular map with e edges and f faces satisfies $2e = 3f$, and therefore a triangular map with v vertices and genus g has exactly $2(v + 2g - 2)$ faces, which in dual corresponds to a cubic map with $2(v + 2g - 2)$ vertices and genus g. Thus, we have

$$C_{n,g} \;=\; T_{n-2g+2,g}. \tag{1}$$

A direct consequence is that there are no cubic maps on Σ_g with $2n$ vertices (and hence $C_{n,g} = 0$), if $2g > n + 1$. Therefore, throughout the paper we assume $2g \le n + 1$.

When g is *constant*, the following asymptotic formulas for $T_{n,g}$ and $C_{n,g}$ were determined by Gao [10]: as $n \to \infty$,

$$C_{n,g} \;\sim\; 3 \cdot 6^{(g-1)/2}\, t_g\, n^{5(g-1)/2}\, (12\sqrt{3})^n, \tag{2}$$

$$T_{n,g} \;\sim\; 3 \cdot (2^9 \cdot 3^7)^{(g-1)/2}\, t_g\, n^{5(g-1)/2}\, (12\sqrt{3})^n. \tag{3}$$

In fact, the constant t_g appears universally in the asymptotic formulas for various rooted maps on Σ_g [3, 4, 11, 13]. Its asymptotic expression was derived by Bender, Gao, and Richmond [5]:

$$t_g \;\sim\; \frac{10\,(3/5)^{1/2}\,\Gamma(1/5)\,\Gamma(4/5)\,\sin(\pi/5)}{2^{1/2}\,\pi^{5/2}} \left(\frac{1440\,g}{e}\right)^{-g/2}, \quad \text{as} \quad g \to \infty. \tag{4}$$

In this paper we study cubic maps on Σ_g when g is *non-constant*, particularly when $g/n \in (0, 1/2)$. We determine the asymptotic behavior of the generating function for $C_{n,g}$ (Theorem 1) and an asymptotic expression for $C_{n,g}$ (Theorems 2 and 3) as $g \to \infty$ and $n - 2g \to \infty$.

Following the notation in [5] we let $C_g(x) := \sum_{n \ge 0} C_{n,g} x^n$ denote the generating function for cubic maps on Σ_g. The parametrization given by (15) in [5]

$$x \;=\; \frac{1}{12\sqrt{3}}(1-s)\sqrt{1+2s} \quad (0 < s < 1)$$

was quite useful when the genus g is constant. However, in order to study the asymptotic behaviors of $C_g(x)$ and $C_{n,g}$ for *non-constant* genus g satisfying $g/n \in (0, 1/2)$, it turns out to be more convenient to use the following parametrization

$$x(t) := \; \frac{t}{4}(1+2t)^{-3/2}. \tag{5}$$

Note that $x(t)$ is monotonically increasing in $t \in [0, 1]$. In addition, we define functions θ, r, A, σ^2 in $t \in (0, 1)$ by

$$\theta(t) := \frac{1}{2} - \frac{3t}{4(1+2t)\sqrt{1-t}} \ln \frac{1+\sqrt{1-t}}{1-\sqrt{1-t}}, \tag{6}$$

$$r(t) := \frac{2(1+2t)\sqrt{1-t}}{3t}\theta(t), \tag{7}$$

$$\sigma^2(t) := \frac{1}{2\theta^2(t)} - \frac{2t^2-t+2}{2(1-t)^2\theta(t)}, \tag{8}$$

$$A(t) := \frac{27K}{8}(1+2t)^{-1/2}\left(\frac{t}{2(1-t)\theta(t)}\right)^{3/2}, \tag{9}$$

where $K \doteq 1.2 \times 10^{-6}$ is some positive constant.

Our first main result is the following asymptotic expression for $C_g(x)$.

▶ **Theorem 1.** *Let x be on the complex plane. Uniformly for $|x|$ in any given closed subinterval of $\big(0, 1/(12\sqrt{3})\big)$, the generating function $C_g(x)$ for cubic maps with genus g satisfies*

$$C_g(x) \;=\; C_g(x(t)) \;=\; (g!)^2\, A(t)\, r(t)^{-2g}\, (1+O(1/g)), \quad \text{as} \quad g \to \infty. \tag{10}$$

Our next main result is the following asymptotic expression for $C_{n,g}$.

▶ **Theorem 2.** *For g/n in a given closed subinterval of $(0, 1/2)$, let $\tau \in (0,1)$ be determined by $\theta(\tau) = g/n$. Then the number $C_{n,g}$ of cubic maps with $2n$ vertices and genus g satisfies*

$$C_{n,g} \;\sim\; (g!)^2 \frac{A(\tau)}{\sqrt{2\pi\, g\, \sigma^2(\tau)}}\, x(\tau)^{-n}\, r(\tau)^{-2g}, \quad \text{as} \quad g \to \infty. \tag{11}$$

Using (11) and (1) we also obtain the following asymptotic formula for the number of triangular maps (i.e. triangulations) with n vertices and genus g:

$$T_{n,g} \;\sim\; (g!)^2 \frac{A(\tau)\, x(\tau)^2}{\sqrt{2\pi\, g\, \sigma^2(\tau)}}\, x(\tau)^{-n}\, (x(\tau) r(\tau))^{-2g}, \quad \text{as} \quad g \to \infty. \tag{12}$$

The rest of the paper is organized as follows. In the next section, we provide proofs of Theorems 1 and 2. In Section 4 we extend Theorem 2 to cover the boundary cases $g/n \to 0$ or $g/n \to 1/2$ (Theorem 3). In Section 5 we compare our asymptotic result on $C_{n,g}$ with a very recent result on the asymptotic number of triangular maps by Budzinski and Louf [6]. We conclude the paper with further discussions on cubic graphs on orientable surfaces in Section 6.

2 Proof of Theorems 1 and 2

Proof of Theorem 1. We begin with the function $F_g(x)$ defined by

$$F_g(x) \;=\; 3x^3 C_g'(x) + 2x^2 C_g(x) \quad (g \geq 0). \tag{13}$$

Rewriting (13) in [5], which is derived from the Goulden-Jackson recursion for cubic maps [14], we obtain the following recursion: for $g \geq 1$,

$$\frac{1-t}{1+2t}F_g(x) + x^2C_g(x)$$
$$= 36x^4F''_{g-1}(x) + 12x^3F'_{g-1}(x) + 6x^3\delta_{g,1} + 12\sum_{h=1}^{g-1}F_h(x)F_{g-h}(x), \tag{14}$$

where $\delta_{g,1}$ is equal to 1 if $g = 1$ and 0 otherwise.

Furthermore, by definition of the functions $x(t)$ and $r(t)$ in (5)–(7) and with some computation, we obtain

$$\frac{dx}{dt} = \frac{1}{4}(1-t)(1+2t)^{-5/2}, \tag{15}$$
$$\frac{x(t)}{x'(t)} = t + \frac{3t^2}{1-t}, \tag{16}$$
$$\frac{dr}{dt} = -\frac{1}{3}t^{-2}(1-t)^{3/2}, \tag{17}$$
$$\frac{dr}{dx} = -\frac{4}{3}t^{-2}(1-t)^{1/2}(1+2t)^{5/2}, \tag{18}$$
$$\frac{d^2r}{dx^2} = \frac{8}{3}t^{-3}(1-t)^{-3/2}(1+2t)^4(4t^2-5t+4). \tag{19}$$

In terms of the new parameter t, the expression for $C_1(x)$ found in [5] becomes

$$C_1(t) = \frac{t(1+2t)}{4(1-t)^2}. \tag{20}$$

It follows from (5) and (13) that

$$F_1(t) = \frac{t^3(t+5)}{64(1+2t)(1-t)^4}. \tag{21}$$

To derive the asymptotic expression (10), we write

$$C_g(x) = (g!)^2r(t)^{-2g}A_g(t),\ A_g(t) = A(t) + a_1(t)g^{-1} + a_2(t)g^{-2} + \cdots,$$

and substitute it into (14). We note

$$\sum_{h=2}^{g-2}\frac{(h!(g-h)!)^2}{(g!)^2} = \sum_{h=2}^{g-2}\left(\binom{g}{h}\right)^{-2} = O\left(g^{-4}\right),$$
$$C'_g(x) = (g!)^2r(t)^{-2g}\left(A'_g(t) - \frac{2gr'(t)A_g(t)}{r(t)}\right)\frac{1}{x'(t)}.$$

Divide both sides of (14) by $(g!)^2$ and expand the resulting expressions in powers of g. Both sides become Laurent series in g with highest power equal to 1. Comparing the coefficients of g and using (13) and (5), we obtain (with the help of computer algebra system *Maple*)

$$r'(t) = -\frac{1}{3}t^{-2}(1-t)^{3/2},$$

which is (17). Observing $\lim_{t\to 1} r(t) = 0$ (see (31) in Section 4), we obtain (7).

Next we compare the coefficients of g^0 in Laurent series to obtain

$$\frac{A'(t)}{A(t)} = \frac{11-2t}{4(1-t)(1+2t)} + \frac{(1-t)^{3/2}}{2t^2r(t)}. \tag{22}$$

Integrating both sides, we obtain (9) for some constant K. The approximate value of K is obtained in Section 3. ◀

Proof of Theorem 2. Define functions $u(t)$ and $\mu(t)$ in $t \in (0,1)$ by

$$u(t) := -2\ln r(t), \tag{23}$$

$$\mu(t) := \frac{x(t)}{x'(t)}\frac{du}{dt}. \tag{24}$$

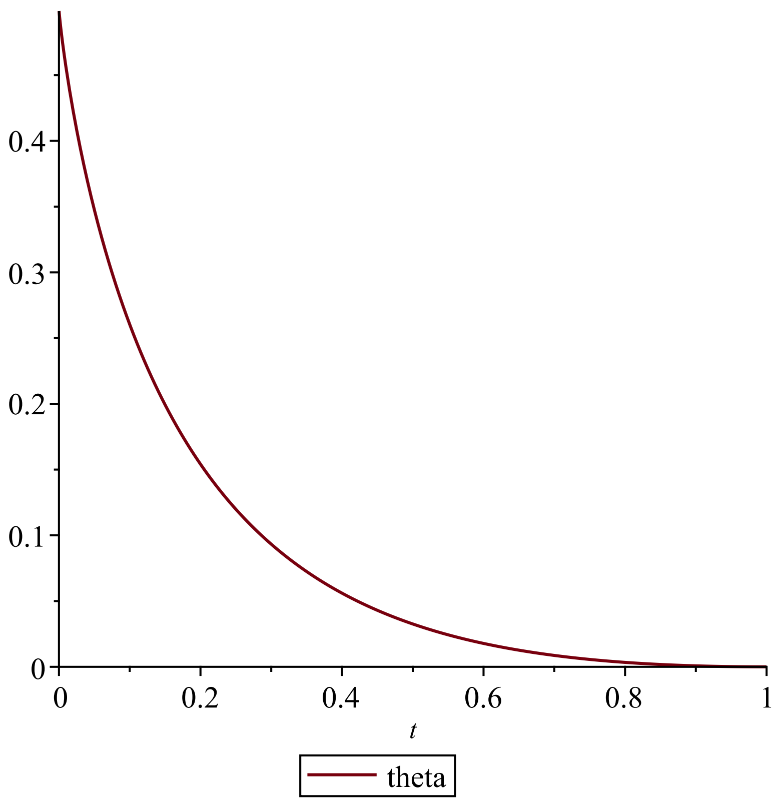

Figure 1 The plot of $\theta(t)$.

With some algebra (and with help of Maple), we find $\theta(t) = 1/\mu(t)$ and $\sigma^2(t) = \frac{x(t)}{x'(t)}\frac{d\mu}{dt}$ are as in (6) and (8), respectively (see Figures 1–2). We note that $\sigma^2(t)$ is positive for $t \in (0,1)$.

In order to apply a generalized version (Theorem 4 in [12]) of the local limit theorem in [2, Theorem 3], we need to verify the technical condition that

$$|r(x)| > r(|x|), \quad \text{for } |x| \in \left(0, 1/(12\sqrt{3})\right) \text{ and } x \neq |x|. \tag{25}$$

We first check (by Maple) that

$$r(t) = \frac{1}{3t} + \frac{1}{2} - \ln 2 - \frac{1}{2}\ln\frac{1}{t} - \frac{1}{8}t - \frac{1}{96}t^2 - \frac{1}{384}t^3 - \frac{1}{1024}t^4 - \cdots,$$

where all the positive powers of t have negative coefficients. Applying the Lagrange inversion formula to (5), we see that $t(x)$ is a power series in x such that $[x^n]t(x)$ are all positive for all $n \geq 1$. Also the radius of convergence of $t(x)$ is $1/(12\sqrt{3}) \doteq 0.048$. This implies that $|t(x)| < t(|x|)$ for all $x \neq |x|$ with $|x| \in (0, 1/(12\sqrt{3}))$, which leads to (25). Figure 3 shows the plots of $|r(\rho e^{i\phi})|$ for $\rho \in \{0.01, 0.02, 0.03, 0.04\}$ and $0 \leq \phi \leq \pi$.

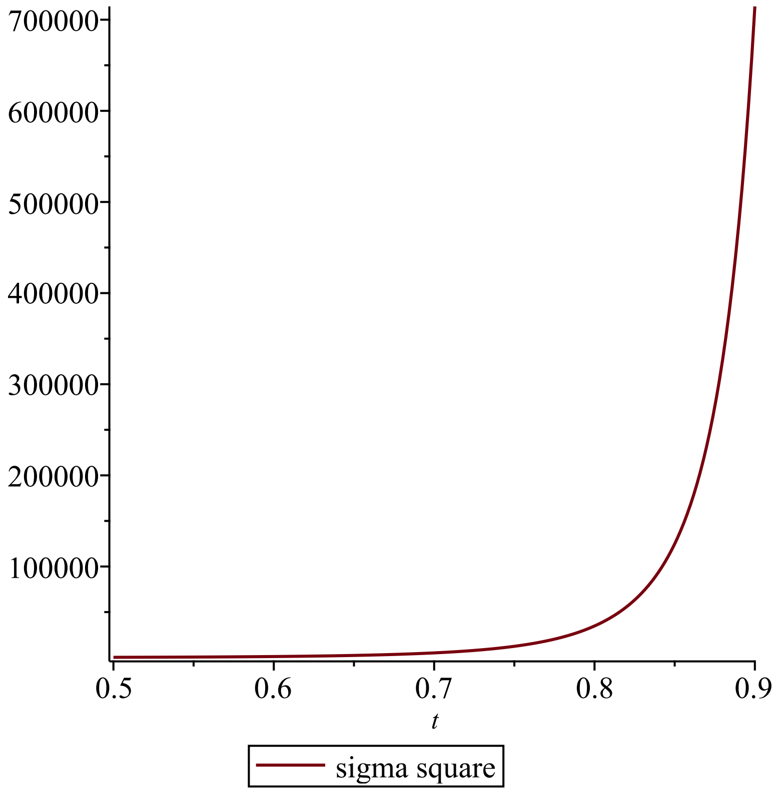

Figure 2 The plot of $\sigma^2(t)$.

Applying [12, Theorem 4] and using (23)–(24), we obtain

$$C_{n,g} \sim (g!)^2 A(\tau) r(\tau)^{-2g} x(\tau)^{-n} \frac{1}{\sqrt{2\pi\, g\, \sigma^2(\tau)}}.$$

This completes the proof of Theorem 2. ◀

3 Estimate the value of K

Our approach in the previous section does not give any information about the constant K that appeared in $A(t)$ – see (9). We may compare the exact values of $C_{n,g}$ and its asymptotic values given by (2) to obtain numerical estimation of K.

Define

$$B_{n,g} := \frac{3n+2}{(g!)^2} C_{n,g} \quad \text{for} \quad n \ge 1, g \ge 0.$$

It follows from the Goulden-Jackson recursion [5, (8)] that

$$\begin{aligned} B_{-1,0} &= 1/2, \\ B_{0,0} &= 2, \\ B_{-1,g} &= B_{0,g} = 0 \quad \text{for} \quad g \ge 1, \end{aligned}$$

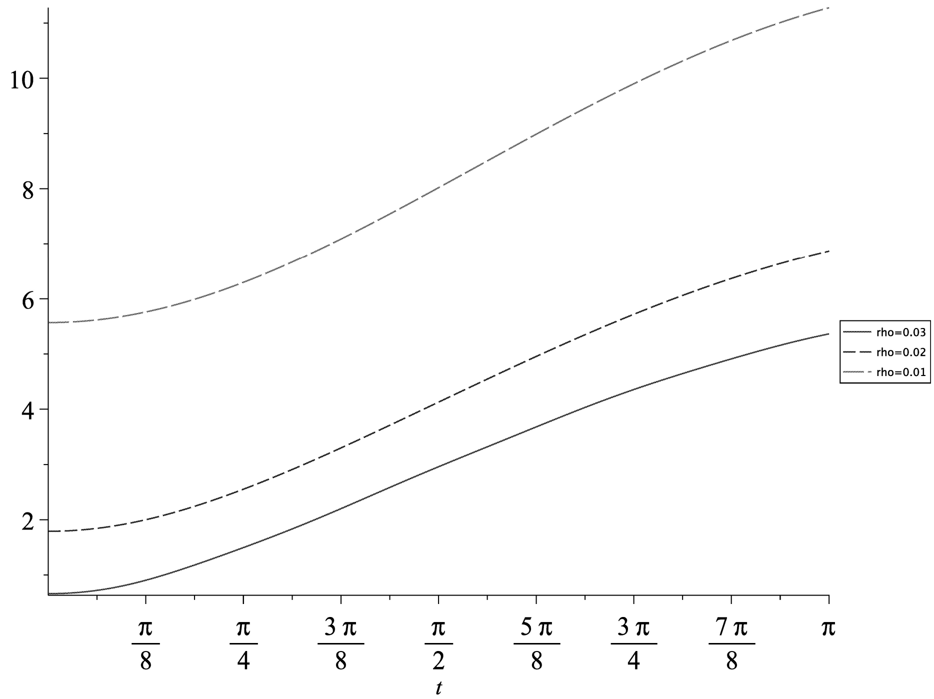

Figure 3 The plots of $|r(\rho e^{i\phi})|$.

and for $n \geq 1, g \geq 0$,

$$B_{n,g} = \frac{4(3n+2)}{n+1}\left(\frac{n(3n-2)}{g^2}B_{n-2,g-1} + \sum_{i=-1}^{n-1}\sum_{h=0}^{g}\frac{1}{\binom{g}{h}^2}B_{i,h}B_{n-2-i,g-h}\right), \tag{26}$$

where $B_{n-2,g-1}/g^2$ is understood to be 0 when $g = 0$.

Using (9) and (11) we obtain

$$\ln K \doteq \ln B_{n,g} - \frac{1}{2}\ln g + g\left(\frac{\ln x}{\theta} + 2\ln r\right) + \frac{1}{2}\ln\left(2\pi\sigma^2\right) - \ln\frac{3}{\theta} - \left(\ln\frac{27}{8} - \frac{1}{2}\ln(1+2t) + \frac{3}{2}\ln\frac{t}{2(1-t)\theta}\right).$$

We used $\theta = 1/3$ and calculated $B_{3g,g}$ for $1 \leq g \leq 150$ using (26). We then obtain

$$t \doteq 0.0569135164, x \doteq 0.0121039967, r \doteq 4.223432731, \sigma^2 \doteq 1.212044822, K \doteq 1.2 \times 10^{-6}.$$

4 Extend to the boundary

In this section we extend Theorem 2 to cover the ranges of g satisfying $g/n \to 0$ or $g/n \to 1/2$. More specifically, we shall apply the saddle-point method to prove

▶ **Theorem 3.** *Assume the same notation as in Theorem 2. Then* (11) *and* (12) *hold when*

$$g \to \infty \quad \textit{and} \quad \frac{n}{2} - g \to \infty \quad \textit{as } n \to \infty. \tag{27}$$

Proof Sketch. To study the asymptotic behaviours of $C_{n,g}$ when $\theta = g/n$ is near 0 or 1/2, we need to find the asymptotic expansions of relevant functions as $t \to 0$ or $t \to 1$. With the help of *Maple* and using (5)–(9), we obtain the following asymptotic expansions.

$$x(t) = \begin{cases} \frac{t}{4} - \frac{3t^2}{4} + O\left(t^3\right), & t \to 0, \\ \frac{1}{12\sqrt{3}}\left(1 - \frac{1}{6}(1-t)^2 - \frac{5}{27}(1-t)^3 + O\left((1-t)^4\right)\right), & t \to 1, \end{cases} \tag{28}$$

$$\frac{x(t)}{x'(t)} = \begin{cases} t + 3t^2 \sum_{k\geq 0} t^k, & t \to 0, \\ 3(1-t)^{-1} - 5 + 2(1-t), & t \to 1, \end{cases} \tag{29}$$

$$\theta(t) = \begin{cases} \frac{1}{2} + \frac{3t}{4}\ln\frac{t}{4} + O\left(t^2 \ln t\right), & t \to 0, \\ \frac{1}{15}(1-t)^2 + \frac{23}{315}(1-t)^3 + O\left((1-t)^4\right), & t \to 1, \end{cases} \tag{30}$$

$$r(t) = \begin{cases} \frac{1}{3t} + \frac{1}{2}\ln\frac{et}{4} + O(t), & t \to 0, \\ \frac{2}{15}(1-t)^{5/2} + \frac{4}{21}(1-t)^{7/2} + O\left((1-t)^{9/2}\right), & t \to 1. \end{cases} \tag{31}$$

$$\mu(t) = \begin{cases} 2 - 3t\ln t + (6\ln 2)t + O\left((t\ln t)^2\right), & t \to 0, \\ 15(1-t)^{-2} + O\left((1-t)^{-1}\right), & t \to 1, \end{cases} \tag{32}$$

$$\sigma^2(t) = \begin{cases} -3t\ln t + O(t), & t \to 0, \\ 90(1-t)^{-4} + O\left((1-t)^{-3}\right), & t \to 1, \end{cases} \tag{33}$$

$$M_3(t) := \frac{x(t)}{x'(t)}\frac{d\sigma^2}{dt} = \begin{cases} -3t\ln t + O(t), & t \to 0, \\ 1080(1-t)^{-6} + O\left((1-t)^{-5}\right), & t \to 1, \end{cases} \tag{34}$$

$$A(t) = \begin{cases} \frac{27K}{8}t^{3/2} + O\left(t^{5/2}\ln t\right), & t \to 0, \\ \frac{405\sqrt{10}K}{32}(1-t)^{-9/2} + O\left((1-t)^{-7/2}\right), & t \to 1. \end{cases} \tag{35}$$

A more careful analysis of (10) gives

$$C_g(x) \;=\; (g!)^2 A(t)\, r(t)^{-2g}\left(1 + O\left(1/g\right)\right),$$

where the O-term is uniform for $0 < t < 1$. In fact, we have (with the help of Maple)

$$A_g(t) = A(t)\left(1 - \frac{3}{8}\left(2 + \frac{20t^2 - 42t + 31}{1+2t}\frac{t\,r(t)}{(1-t)^{5/2}}\right)g^{-1} + O(g^{-2})\right). \tag{36}$$

Using (31), we see that the coefficient of g^{-1} in (36) is bounded for $t \in (0,1)$.

The Cauchy integration formula and the standard saddle-point method give

$$\begin{aligned} [x^n]C_g(x) &= \frac{1}{2\pi i}\oint_{|x|=x(\tau)} C_g(x)x^{-n-1}dx \\ &\sim \frac{(g!)^2}{2\pi}x(\tau)^{-n}\int_{|\phi|\leq\pi} A(\tau e^{i\phi})\exp\left(gu(\tau e^{i\phi})\right)e^{-in\phi}d\phi \\ &\sim \frac{(g!)^2}{2\pi}A(\tau)x(\tau)^{-n}r(\tau)^{-2g}\int_{|\phi|\leq\delta}\exp\left(-\frac{g\sigma^2(\tau)\phi^2}{2} + O\left(gM_3(\tau)\delta^3\right)\right)d\phi, \end{aligned}$$

where τ is determined by the saddle-point equation $\theta(\tau) = g/n$, $M_3(\tau)$ is given in (34), and δ satisfies

$$g\sigma^2(\tau)\delta^2 \to \infty \quad\text{and}\quad gM_3(\tau)\delta^3 \to 0.$$

It follows from (33) and (34) that this condition is satisfied, provided that

$$gt\ln(1/t) \to \infty \quad\text{as}\quad t \to 0. \tag{37}$$

Using (30), we see that (37) is equivalent to

$$g\left(\frac{1}{2} - \frac{g}{n}\right) \to \infty \quad\text{as}\quad \frac{g}{n} \to \frac{1}{2}, \quad \textit{i.e.} \quad \frac{n}{2} - g \to \infty \quad\text{as}\quad \frac{g}{n} \to \frac{1}{2}.$$

This completes the proof of Theorem 3. ◀

When the order of g/n or $n-2g$ is known, the asymptotic expression of $C_{n,g}$ can be simplified using the asymptotic expansions (29)–(36). For example, we have the following corollary to Theorem 3.

▶ **Corollary 4.** *Let $\theta = g/n$. Suppose $g \to \infty$ and $g = o\left(n^{1/2}\right)$. Then*

$$C_{n,g} \sim \frac{9K}{32\,(15)^{1/4}\sqrt{2\pi g}}\theta^{-5/4}\,(g!)^2\left(12\sqrt{3}\right)^n \exp\left(\left(\frac{5}{2}\ln\frac{e}{15\theta}+2\ln\frac{15}{2}-\frac{5}{63}(15\theta)^{1/2}\right)g\right).$$

Proof. When $g = o(n^{1/2})$, we have the following expansions

$$\begin{aligned}
1-t &= (15\theta)^{1/2}\left(1-\frac{23}{42}(15\theta)^{1/2}+O(\theta)\right),\\
A &\sim \frac{405K\sqrt{10}}{32}(15\theta)^{-9//4},\\
\sigma &\sim 3\sqrt{10}(15\theta)^{-1},\\
\ln x &= \ln\frac{1}{12\sqrt{3}}-\frac{5}{2}\theta-\frac{5}{27}(15\theta)^{3/2}+O\left(\theta^2\right),\\
\ln r &= \ln 2+\frac{1}{4}\ln 15+\frac{5}{4}\ln\theta+\frac{10}{7}(15\theta)^{1/2}+O(\theta),\\
r^{-2g}x^{-n} &\sim (12\sqrt{3})^n\exp\left(g\left(\frac{5}{2}-\ln 4-\frac{1}{2}\ln 15-\frac{5}{2}\ln\theta-\frac{5}{63}(15\theta)^{1/2}\right)\right).
\end{aligned}$$

Now the result follows from Theorem 3. ◀

The following result from [6] is an immediate consequence of Theorem 2.

▶ **Corollary 5.** *Let $g, n \to \infty$ such that $g/n \to \theta_0 \in (0, 1/2)$. Let t_0 be determined by $\theta(t_0) = \theta_0$ and $x_0 = x(t_0)$, where $x(t)$ and $\theta(t)$ are defined by (6) and (5). Then we have*

$$\frac{C_{n+1,g}}{C_{n,g}} \to \frac{1}{x_0}, \quad \textit{as} \quad n\to\infty.$$

Proof. Let t_1 be determined by $\theta(t_1) = g/(n+1)$. Since

$$\frac{g}{n+1} = \theta_0 - \frac{\theta_0}{n} + O\left(\frac{\theta_0}{n^2}\right),$$

and the function $\theta(t)$ is differentiable and has nonzero derivative in $(0, 1)$, we have

$$t_1 = t_0 + O\left(\frac{1}{n}\right).$$

Hence

$$x(t_1)\to x(t_0),\ r(t_1)\to r(t_0),\ A(t_1)\to A(t_0),\ \sigma^2(t_1)\to\sigma^2(t_0) \quad \text{as} \quad n\to\infty.$$

Writing $f(t) := \ln x(t) + 2\theta_0 \ln r(t)$ and applying Theorem 2, we obtain

$$\begin{aligned}
\frac{C_{n+1,g}}{C_{n,g}} &\sim \frac{1}{x(t_0)}\left(\frac{x(t_1)}{x(t_0)}\right)^{-n}\left(\frac{r(t_1)}{r(t_0)}\right)^{-2g}\\
&= \frac{1}{x(t_0)}\exp\left(-n\left(f(t_1)-f(t_0)\right)\right)\\
&= \frac{1}{x(t_0)}\exp\left(-n\left(f'(t_0)(t_1-t_0)+O\left((t_1-t_0)^2\right)\right)\right).
\end{aligned}$$

Using (23) and (24) and noting $\theta_0 = 1/\mu(t_0)$, we obtain

$$f'(t_0) = \frac{x'(t_0)}{x(t_0)} + 2\theta_0 \frac{r'(t_0)}{r(t_0)} = 0.$$

Thus

$$\frac{C_{n+1,g}}{C_{n,g}} \sim \frac{1}{x(t_0)} \exp\left(-O\left(n/n^2\right)\right) \to \frac{1}{x_0},$$

as desired. ◀

5 Cross-check with a result on triangulations in [6]

In [6] Budzinski and Louf resolved a conjecture of Benjamini and Curien on the local limits of uniform random triangulations whose genus is proportional to the number of faces. As a consequence, they derived an asymptotic formula for the number of triangulations up to sub-exponential factors.

Using notations in [6], let $\tau(n,g)$ denote the number of triangulations (i.e. triangular maps) with $2n$ faces and genus g, which of course is equal to the number $C_{n,g}$ to cubic maps (as dual) with $2n$ vertices and genus g. For any $\lambda \in (0, 1/(12\sqrt{3})]$ let $h \in (0, 1/4]$ be such that

$$\lambda = \frac{h}{(1+8h)^{3/2}} \quad \text{and} \quad \psi(\lambda) = \frac{h \ln\left(1+\sqrt{1-4h}\right)/(1-\sqrt{1-4h})\big)}{(1+8h)\sqrt{1-4h}}.$$

For any $\vartheta \in [0, 1/2)$ let $\lambda = \lambda(\vartheta)$ be the unique solution of the equation

$$\psi(\lambda) = \frac{1-2\vartheta}{6}.$$

In [6, Theorem 3] it was shown that for $g = g(n)$ satisfying $0 \le g \le \frac{n+1}{2}$ and $g/n \to \vartheta \in [0, 1/2]$, we have

$$\tau(n,g) \;=\; n^{2g} \exp(f(\vartheta)\, n + o(n)), \quad \text{as} \quad n \to \infty, \tag{38}$$

where $f(0) = \log(12\sqrt{3})$, $f(1/2) = \log(6/e)$, and

$$f(\vartheta) \;=\; 2\vartheta \ln(12\vartheta/e) - (1-2\vartheta)\int_{2\vartheta}^{1} \ln \lambda(\vartheta/z)dz, \quad \text{for} \quad \vartheta \in (0, 1/2),$$

in which we have corrected the factor $-(1-2\vartheta)$ in front of the integral – see (3) in [6, Theorem 3] for comparison.

In order to compare (38) with our result (11), we note the following relations between parameters:

$$t = 4h, \quad x = \lambda, \quad \mu = \frac{1}{\vartheta}.$$

Using Stirling's formula (up to the sub-exponential factor), we may rewrite our asymptotic expression of $C_{n,g}$ in (11) as

$$C_{n,g} \;\approx\; n^{2g} \exp(q(t)\, n), \quad \text{as} \quad n \to \infty, \tag{39}$$

where

$$q(t) \;=\; 2(-\ln r + \ln \theta - 1)\theta - \ln x. \tag{40}$$

We note that, as $t \to 0$, $\theta \to 1/2$ and consequently

$$q(t) \;\to\; -\ln(rx) - \ln(2e) \;\to\; \ln(6/e) \;=\; f(1/2).$$

As $t \to 1$, we have $\theta \to 0$ and

$$q(t) \;\to\; 2\theta \ln(\theta/r) - \ln x \;\to\; \ln 12\sqrt{3} \;=\; f(0).$$

These two values match with those in (38).

6 Discussions: cubic graphs on orientable surfaces

Graphs that are closely related to cubic maps on Σ_g are cubic graphs with genus at most g, which play a crucial role in the study of phase transitions in sparse random graphs on orientable surfaces, as it was shown in [8, 15]. Let $\tilde{H}_{n,g}$ denote the number of vertex-labeled cubic graphs with $2n$ vertices and genus at most g and let $H_{n,g} = \tilde{H}_{n,g}/(2n)!$. In [8, 9], it was shown that if g is constant, then

$$H_{n,g} \;\sim\; c_g\, n^{5(g-1)/2-1}\gamma^n,$$

where γ does not depend on g and is the same constant as planar case (i.e. when $g = 0$), and if $g \le \frac{n+1}{2}$, then

$$a_g\, n^{2g} \;\le\; H_{n,g} \;\le\; b_g\, g^{-4g}\, n^{6g}. \tag{41}$$

Note that if $g > \frac{n+1}{2}$, then $H_{n,g}$ is equal to the total number of cubic graphs with $2n$ vertices (without restriction on the genus). Therefore, we have

$$H_{n,g} \;\sim\; e^{-2}\,\frac{(6n-1)!!}{(3!)^{2n}} \;=\; e^{-2}\,\frac{(6n)!}{(3n)!\,2^{3n}\,(3!)^{2n}} \;\sim\; e^{-2}\left(\frac{6}{e^3}\right)^n n^{3n}. \tag{42}$$

So far, an asymptotic expression for $H_{n,g}$ when $g/n \in (0, 1/2]$ is not known.

▶ **Problem 1.** *Derive an asymptotic expression for $H_{n,g}$ when $1 \ll g \le \frac{n+1}{2}$.*

As it turned out, it is quite difficult to resolve Problem 1. Let us first compare asymptotic behaviors of $H_{n,g}$ and $C_{n,g}$, particularly when $g/n \to 1/2$. If the asymptotic formula (11) of $C_{n,g}$ *would* hold also for $g/n \to 1/2$, then

$$C_{n,g} \;\approx\; n^n, \quad \textit{as} \quad g/n \to 1/2. \tag{43}$$

The upper bound in (41) indicates that the super-exponential factor of $H_{n,g}$ could be

$$H_{n,g} \;\approx\; n^n, \quad \text{as} \quad g/n \to 1/2, \tag{44}$$

which matches with (43). Note however that (42) suggests that the super-exponential factor of $H_{n,g}$ might be

$$H_{n,g} \;\approx\; n^{3n}, \quad \text{as} \quad g/n \to 1/2, \tag{45}$$

which is substantially larger than (43).

It would be interesting to check whether or not the asymptotic formula of $C_{n,g}$ in Theorem 2 holds even for $g/n \to 1/2$.

▶ **Problem 2.** *Determine asymptotic behavior of $C_{n,g}$ when $g/n \to 1/2$.*

Another natural, but challenging task in view of (44) and (45) is the following.

▶ **Problem 3.** *Does there exist a threshold function $t^* = t^*(n) = o(n)$ such that*

$$H_{n,g} \;\approx\; \begin{cases} n^n, & \text{as} \quad g - n/2 = o(t^*), \\ n^{h(c)\,n}, & \text{as} \quad g - n/2 = c\,t^* \quad (\text{for} \quad c \in \mathbb{R}), \\ n^{3n}, & \text{as} \quad g - n/2 = \omega(t^*), \end{cases}$$

for some function $h : \mathbb{R} \to [1,3]$ satisfying $h(c) \to 1$ as $c \to -\infty$ and $h(c) \to 3$ as $c \to \infty$?

As very recent results on sparse random graphs with large genus [8] revealed, the most interesting unknown case is when the genus is linear in the number of vertices.

▶ **Problem 4.** *Derive an asymptotic expression for $H_{n,g}$ when $g/n \in (0, 1/2]$.*

To this end, we may apply the following steps (analogous ideas were successfully utilized in [9] when g is constant).

(S1) We first derive asymptotic formula for 2-connected cubic maps (equivalently, loopless triangular maps) of genus $g = \theta n$. This will be done as follows.
- Show that triangular maps with a non-contractible loop are negligible by cutting through such a loop and bounding the number of such maps by triangular maps of genus $g - 1$.
- For contractible loops, we can apply the usual composition technique to derive equations of generating functions relating loopless triangular maps and all triangular maps.

(S2) Similarly, we derive formulas for 3-connected cubic maps (equivalently, triangular maps without loops or multiple edges).

(S3) In order to go from 3-connected cubic *maps* on Σ_g to 3-connected cubic *graphs* on Σ_g, we apply Robertson-Vitray uniqueness embedding result or Thomassen's LEW result (see e.g. [17]). We need to show that almost all such cubic maps have representativity larger than $2g+3$ (equivalently, all non-contractible cycles have length greater than $2g+3$).

(S4) Finally, in order to go from 3-connected cubic graphs to cubic graphs with lower connectivity we apply standard connectivity-decomposition arguments.

Note that (S3) can be quite challenging since the genus g is linear in n, and we only have g^2 to play with the error term.

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