Spectral Aspects of Symmetric Matrix Signings

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Abstract

The spectra of signed matrices have played a fundamental role in social sciences, graph theory, and control theory. In this work, we investigate the computational problems of *finding* symmetric signings of matrices with natural spectral properties. Our results are the following:

1. We characterize matrices that have an invertible signing: a symmetric matrix has an invertible symmetric signing *if and only if* the support graph of the matrix contains a perfect 2-matching. Further, we present an efficient algorithm to search for an invertible symmetric signing.
2. We use the above-mentioned characterization to give an algorithm to find a minimum increase in the support of a given symmetric matrix so that it has an invertible symmetric signing.
3. We show NP-completeness of the following problems: verifying whether a given matrix has a symmetric signing that is singular or has bounded eigenvalues. However, we also illustrate that the complexity could differ substantially for input matrices that are adjacency matrices of graphs.

We use combinatorial techniques in addition to classic results from matching theory.

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1 Introduction

The spectra of several graph-related matrices such as the adjacency and the Laplacian matrices have become fundamental objects of study in computer science. In this work, we undertake a systematic and comprehensive investigation of the spectrum and the invertibility of *symmetric signings* of matrices. We study natural spectral properties of symmetric signed matrices and address the computational problems of finding and verifying the existence of symmetric signings with these spectral properties.

For a real-valued symmetric $n \times n$ matrix M and a $\{\pm 1\}$-valued $n \times n$ matrix s – which we refer to as a *signing* – we define the *signed matrix* $M(s)$ to be the matrix obtained by taking entry-wise products of M and s. Signed adjacency matrices (respectively, Laplacians) correspond to signed matrices $M(s)$ where M is the adjacency matrix (respectively, Laplacian) of a graph. We say that s is a *symmetric* signing if s is a symmetric matrix and an *off-diagonal* signing if all the diagonal entries of s are $+1$. In this work we consider the following computational problems:

BoundedEvalueSigning: Given a real symmetric matrix M and a real number λ, verify if there exists an off-diagonal symmetric signing s such that the largest eigenvalue $\lambda_{\max}(M(s))$ is at most λ.

IncludeSigning: Given a real symmetric matrix M and a real number λ, verify if there exists an off-diagonal symmetric signing s such that $M(s)$ has λ as an eigenvalue.

AvoidSigning: Given a real symmetric matrix M and a real number λ, verify if there exists a symmetric signing s such that $M(s)$ does not have λ as an eigenvalue.

It suffices to focus on instances where λ is 0. Indeed, solving an instance of one of the above problems on input (M, λ) corresponds exactly to solving the same problem on input $(M - \lambda I, 0)$. Hence, we focus our attention on the corresponding specialized problems:

NsdSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is negative semi-definite.

SingularSigning: Given a real symmetric matrix M, verify if there exists an off-diagonal symmetric signing s such that $M(s)$ is singular.

InvertibleSigning: Given a real symmetric matrix M, verify if there exists a symmetric signing s such that $M(s)$ is invertible (that is, non-singular).

1.1 Motivations

Spectra of Signed Matrices and Expanders. Let G be a connected d-regular graph on n vertices and let $d = \lambda_0 > \lambda_1 \geq \cdots \geq \lambda_{n-1}$ be the eigenvalues of its adjacency matrix. Then G is a Ramanujan expander if $\max_{|\lambda_i|<d} |\lambda_i| \leq 2\sqrt{d-1}$. Efficient construction of Ramanujan expanders of arbitrary degrees remains an important open problem.[1] A combinatorial approach to this problem, initiated by Friedman [9], is to obtain larger Ramanujan graphs from smaller ones while preserving the degree. A 2-*lift* H of G is obtained as follows: Introduce two copies of each vertex u of G, say u_1 and u_2, as the vertices of H and for each edge $\{u, v\}$ in G, introduce either $\{u_1, v_2\}, \{u_2, v_1\}$ or $\{u_1, v_1\}, \{u_2, v_2\}$ as edges of H. There is a bijection between 2-lifts and symmetric signed adjacency matrices of G. Furthermore, the eigenvalues of the adjacency matrix of a 2-lift H are given by the union of the eigenvalues of the adjacency matrix of the base graph G (also called the "old" eigenvalues) and the signed adjacency matrix of G that corresponds to the 2-lift. (the "new" eigenvalues).

Marcus, Spielman, and Srivastava [16] showed that every d-regular bipartite graph has a 2-lift whose new eigenvalues are bounded in absolute value by $2\sqrt{d-1}$. However, this result [16] is not constructive and their work raises the question of whether there is an efficient

[1] While efficient construction of bipartite Ramanujan *multi-graphs* of all degrees is known [5], it still remains open to efficiently construct bipartite Ramanujan *simple* graphs of all degrees.

algorithm to find a symmetric signing that minimizes the largest eigenvalue. This motivates investigating BoundedEvalueSigning which is the decision variant of the computational problem. More precisely, it motivates investigating BoundedEvalueSigning when the input matrix is an adjacency matrix.

It is also natural to investigate the complexity of several related problems. As we will see in the next section, BoundedEvalueSigning is NP-hard for arbitrary symmetric matrices. The reduction which shows BoundedEvalueSigning is NP-hard suggests a close relationship with AvoidSigning which is also NP-hard. Hoping to make progress on BoundedEvalueSigning for adjacency matrices, we investigate AvoidSigning for adjacency matrices. IncludeSigning arises naturally as the complement of AvoidSigning.

Solvability Index of a Signed Matrix. The notion of *balance* of a symmetric signed matrix has been studied extensively in social sciences [14, 11, 13, 17]. A signed adjacency matrix is *balanced* if there is a partition of the vertex set such that all edges within each part are positive, and all edges in between two parts are negative (one of the parts could be empty). A number of works [3, 10, 18, 17, 19, 20] have explored the problem of minimally modifying signed graphs (or signed adjacency matrices) to convert it into a balanced graph.

In this work, we introduce a related problem regarding symmetric signed matrices: Given a symmetric matrix M, what is the smallest number of non-diagonal zero entries of M whose replacement by non-zeroes gives a symmetric matrix M' that has an invertible symmetric signing? We define this quantity to be the *solvability index*[2]. Knowing this number might be helpful in studying systems of linear equations in signed matrices that might be ill-defined, and thus do not have a (unique) solution and in minimally modifying such matrices so that the resulting linear system becomes (uniquely) solvable. We use classic graph-theoretic techniques to show that solvability index is indeed computable efficiently.

1.2 Our Results

Intriguingly, the complexity of BoundedEvalueSigning has not been studied in the literature even though it is widely believed to be a difficult problem in the graph sparsification community. We shed light on this problem by showing that it is NP-complete.

▶ **Theorem 1.** NsdSigning *and* SingularSigning *are NP-complete.*

Theorem 1 also implies that BoundedEvalueSigning and AvoidSigning are NP-complete. In contrast to SingularSigning, we show that InvertibleSigning admits an efficient algorithm. In fact, we show a stronger result: there exists an algorithm to efficiently solve the search variant of InvertibleSigning, which we denote by SearchInvertible-Signing (here the goal is to *find* an invertible signing if it exists).

▶ **Theorem 2.** *There exists a polynomial-time algorithm to solve* SearchInvertibleSigning.

Theorem 2 also implies that the search variant of IncludeSigning is solvable efficiently. Our proof of Theorem 2 leads to a structural characterization for the existence of invertible signings through the existence of *perfect* 2*-matchings* in the support graph of the matrix.

[2] Our definition of *solvability index* is similar to the notion of *frustration index* [12, 1]. The *frustration index* of a matrix M is the minimum number of non-zero off-diagonal entries of M whose deletion results in a *balanced signed graph*. Computing the frustration index of a signed graph is NP-hard [15].

We believe that this structural characterization could be of independent interest and hence, discuss it in detail in Section 1.2.1.

The hard instances generated by our proof of Theorem 1 are real symmetric matrices with non-zero diagonal entries and hence, it does not resolve the computational complexity of the problem of finding a signing of a given *graph-related* matrix (for example, the adjacency matrix) that minimizes its largest eigenvalue. Our next result provides some evidence that one might be able to design efficient algorithms to solve the NP-complete problems appearing in Theorem 1 for graph-related matrices. In particular, we show that SingularSigning and its search variant admit efficient algorithms when the input matrix corresponds to the adjacency matrix of a *bipartite* graph.

▶ **Theorem 3.** *There exists a polynomial-time algorithm to verify if the adjacency matrix A_G of a given bipartite graph G has a symmetric signing s such that $A_G(s)$ is singular; and if so, find such a signing.*

Finally, we define the *solvability index* of a real symmetric matrix M to be the smallest number of non-diagonal zero entries that need to be converted to non-zeroes so that the resulting symmetric matrix has an invertible symmetric signing. We emphasize that the support-increase operation that we consider preserves symmetry, that is, if we replace the zero entry $A[i,j]$ by α, then the zero entry $A[j,i]$ is also replaced by α. We give an efficient algorithm to find the solvability index of a given symmetric matrix M.

▶ **Theorem 4.** *There exists a polynomial-time algorithm to find the solvability index of a given real symmetric matrix.*

1.2.1 Structural Characterization for Invertible Signings

Theorem 2, in particular, implies that InvertibleSigning is solvable efficiently. In fact, our proof-technique gives an efficient characterization for the existence of an invertible signing. This characterization also leads to an alternative algorithm to solve InvertibleSigning. We believe that this characterization might be of independent interest and hence describe it here.

The *support graph* of a real symmetric $n \times n$ matrix M is an undirected graph G where the vertex set of G is $[n] := \{1, \ldots, n\}$, and the edge set of G is $\big\{\{u,v\} \mid M[u,v] \neq 0\big\}$. We note that G could have self-loops depending on the diagonal entries of M. A *perfect 2-matching* in a graph G with edge set E is an assignment $x : E \to \{0,1,2\}$ of values to the edges such that $\sum_{e \in \delta(v)} x_e = 2$ holds for every vertex v in G (where $\delta(v)$ denotes the set of edges incident to v). Equivalently, a perfect 2-matching in a graph G is a vertex-disjoint union of edges and cycles (cycles could be loop edges) in G such that each vertex is incident to at least one edge. We show the following characterization:

▶ **Theorem 5.** *Let M be a symmetric $n \times n$ matrix and let G be the support graph of M. The following are equivalent:*

- (i) *There exists a symmetric signing s such that the signed matrix $M(s)$ is invertible.*
- (ii) *The support graph G contains a perfect 2-matching.*

▶ Remark 1. The structural characterization of Theorem 5 leads to a polynomial-time algorithm to solve InvertibleSigning – it suffices to verify if the support graph of the input matrix contains a perfect 2-matching which can be done in polynomial-time.

▶ Remark 2. We present a constructive proof of Theorem 5 via a generalization (see Theorem 8 in Section 2). Our proof of Theorem 5 is constructive but we are aware of a non-constructive proof using Combinatorial Nullstellensatz. This alternative non-constructive proof is available online in an earlier arXiv version of this [4].

1.3 Related Work

Skew Symmetric Matrix of Indeterminates. A square skew-symmetric matrix of indeterminates with zeroes on the diagonal is known as the *Tutte matrix* of its support graph. A well-known result by Tutte shows that the determinant polynomial of the Tutte matrix is non-zero if and only if the corresponding support graph has a perfect matching. Our result in Theorem 5 can be interpreted as a variant of Tutte's result to square *symmetric* matrices of indeterminates with zeros on the diagonal.

Cunningham and Geelen [6] extended Tutte's work along a different direction by giving a characterization of invertible submatrices of the Tutte matrix using *path-matchings*. Given a graph G with vertex set V and vertex subsets $R, L \subseteq V$, a (R, L)-*path-matching* in G is a collection of vertex-disjoint paths from R to L and edges in $G[V \setminus (R \cup L)]$. A *perfect* (R, L)-path-matching is a (R, L)-path-matching in which every vertex in G is incident to some edges of the vertex-disjoint paths. They showed that the determinant polynomial of a square submatrix of the Tutte matrix of G with column set R and row set L is non-zero if and only if there exists a perfect (R, L)-path-matching in G. The notion of *cycle-covers* that we introduce in Section 2 and our result in Theorem 8 can be interpreted as variants of Cunningham and Geelen's result to square *symmetric* matrices of indeterminates with zeros on the diagonal.

Our results in Theorems 5 and 8 go further than Cunningham and Geelen's result by not only giving similar characterizations for the determinant to be a non-zero polynomial but also by giving polynomial-time algorithms to find a point in $\{\pm 1\}^E$ at which the polynomial is non-zero.

Minimum Rank Problems. A line of work seemingly related to ours is the minimum rank problem (e.g., see [8, 7]): given a graph G, the goal is to compute the minimum rank of the weighted adjacency matrix of a graph obtained by giving non-zero real-valued weights to the edges of G. We emphasize that the allowed weights in the minimum rank problem are arbitrary and are not simply signs of the given adjacency matrix as in the case of our work. A signed variant of the minimum rank problem has also been addressed in the literature: given a sign pattern matrix S, the goal is to compute the minimum rank of a matrix with real-valued entries whose sign pattern is identical to S. Once again, we emphasize the distinction between the signed variant of the minimum rank problem and the problems studied in our work: in the signed variant of the minimum rank problem, the sign pattern is the input and the goal is to find a matrix with real-valued entries matching the input sign pattern and achieving minimum rank. In contrast, the problems studied in our work have real-valued entries as inputs and the goal is to find a symmetric sign pattern of the entries to achieve the specified spectral properties.

A year after posting our work on arXiv [4], Akbari, Ghafari, Kazemian, and Nahvi [2] also posted an article addressing InvertibleSigning[3]. They show the same characterization as Theorem 5 with a proof identical to the non-constructive proof appearing in the early arXiv version of our work [4]. We emphasize that in addition to showing the structural characterization in Theorem 5, this work resolves the search problem in Theorem 2, and

[3] Our arXiv post dated Nov, 2016: `https://arxiv.org/abs/1611.03624`; the post by Akbari, Ghafari, Kazemian, and Nahvi dated Aug, 2017: `https://arxiv.org/abs/1708.07118`.

moreover shows a much more general structural characterization in Theorem 8 with a constructive proof.

1.4 Organization

In Section 1.5, we review definitions and notations. In Section 2, we describe an efficient algorithm to find an invertible signing (Theorem 2). Due to space constraints, we refer the reader to the full version of the paper for all missing proofs [4].

1.5 Preliminaries

Unless otherwise specified, all matrices are symmetric and take values over the reals. Since all of our results are for symmetric signings, we will just use the term *signing* to refer to a symmetric signing in the rest of this work. We denote the entry-wise product of two $n \times n$ matrices M and s as $M(s)$ (even when s is not necessarily a signing).

Let S_n be the set of permutations of n elements, M be a real symmetric $n \times n$ matrix, and s be a symmetric $n \times n$ signing. Then, the *permutation expansion* of the determinant of a signed matrix $M(s)$ is given by

$$\det M(s) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{n} M(s)[i, \sigma(i)].$$

A permutation σ in S_n has a unique cycle decomposition and hence corresponds to a vertex-disjoint union of directed cycles on n vertices. Removing the orientation gives an undirected graph which is a vertex disjoint union of cycles, self-loops, and matching edges.

2 Finding Invertible Signings

In this section, we focus on invertible signings and the proof of Theorem 2. We prove a much more general statement in comparison to the one given in Theorem 5, which we believe could be of independent interest. We start by introducing the background needed to state the general version.

Symmetric signings of asymmetric sub-matrices. Let M be a symmetric $n \times n$ matrix. For $X, Y \subseteq [n]$ being a subset of row and column indices of the same cardinality, let $M[X, Y]$ denote the submatrix of M obtained by picking the rows in X and the columns in Y. We note that $M[X, Y]$ is a square matrix, but it may *not* be symmetric even though M is symmetric. We are interested in finding a symmetric $n \times n$ signing s so that the square submatrix $M(s)[X, Y]$ is invertible. We emphasize that for a symmetric signing s, the (possibly asymmetric) matrix $M(s)[X, Y]$ is symmetric on $X \cap Y$, that is, the $[i, j]$'th and the $[j, i]$'th entries of the matrix $M(s)[X, Y]$ are the same for every $i, j \in X \cap Y$.

Perfect 2-matchings in subgraphs. Let G be a simple undirected graph, possibly containing self-loops. Let X, Y be vertex subsets of G. We consider the subgraph $G[X \cup Y]$ induced by $X \cup Y$. An (X, Y)-*cycle-cover* is a collection of edges of the subgraph $G[X \cup Y]$ that induce a vertex-disjoint union of paths and cycles (cycles could be loop edges) in $G[X \cup Y]$ such that (1) every cycle is a subgraph of $G[X \cap Y]$, (2) every vertex of $X \cup Y$ is incident to at least one edge, and (3) every path either has one end-vertex in $X \setminus Y$, the other end-vertex in $Y \setminus X$, and all intermediate vertices in $X \cap Y$, or has both end-vertices in $X \cap Y$ with only one edge (see Figure 1 for an example). We note that (X, X)-cycle-covers correspond to perfect 2-matchings in $G[X]$ and hence, (V, V)-cycle-covers correspond to perfect 2-matchings in G. It follows that in (X, X)-cycle-covers all paths are only a single edge in G. Furthermore, the existence of an (X, Y)-cycle-cover is possible only if $|X| = |Y|$. The following lemma states that the existence of an (X, Y)-cycle-cover in a given graph can be verified efficiently.

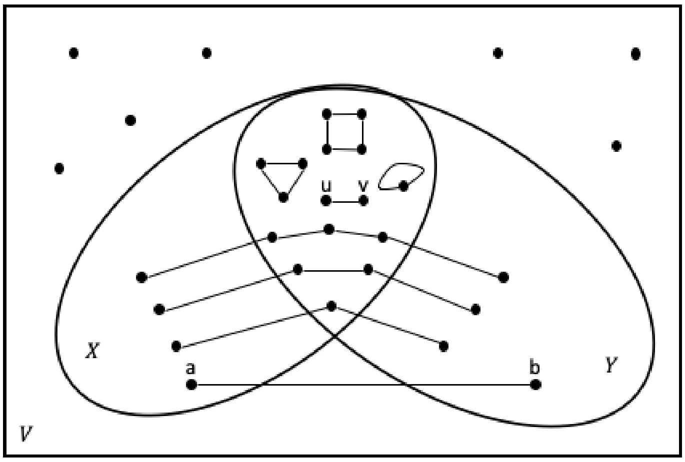

Figure 1 An (X, Y)-cycle-cover F. Furthermore, by our definitions below, the edge $\{a, b\}$ is in Paths(F) while the edge $\{u, v\}$ is in Matchings(F).

▶ **Lemma 6.** *There exists a polynomial-time algorithm that decides if there is an (X, Y)-cycle-cover in a given graph G for given vertex subsets X, Y of G.*

The key observation to prove Lemma 6 is that finding an (X, Y)-cycle-cover can be reduced to finding a perfect matching in an auxiliary bipartite graph.

Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $X, Y \subseteq [n]$ with $|X| = |Y|$ and s be a symmetric $n \times n$ matrix. Recall that we are interested in finding a symmetric $n \times n$ signing s so that the square submatrix $M(s)[X, Y]$ is invertible. We derive a convenient expression for $\det(M(s)[X, Y])$ that is based on (X, Y)-cycle-covers. For an (X, Y)-cycle-cover F, let Cycles(F), Paths(F), and Matchings(F) denote the set of cycles in F, paths in F with end-vertices in $X \setminus Y$ and $Y \setminus X$, and paths in F that are contained in $G[X \cap Y]$, respectively. Moreover, let Loops(F) and NTCs(F) denote the set of self-loops and non-trivial-cycles in F. We emphasize that Cycles(F) = Loops(F) ∪ NTCs(F). We also note that Cycles(F), Paths(F), and Matchings(F) are all vertex-disjoint from one another and if $X = Y$ then Paths$(F) = \emptyset$. We define

$$M(s)_{\text{Cycles}(F)} := \prod_{C\in\text{Cycles}(F)} \prod_{\{u,v\}\in C} M(s)[u, v],$$

$$M(s)_{\text{Paths}(F)} := \prod_{P\in\text{Paths}(F)} \prod_{\{u,v\}\in P} M(s)[u, v], \text{ and}$$

$$M(s)_{\text{Matchings}(F)} := \prod_{\{u,v\}\in\text{Matchings}(F)} M(s)[u, v]^2.$$

With this notation, we have the following claim that the determinant of $M(s)[X, Y]$ is a $\{\pm 1\}$-linear combination of terms corresponding to (X, Y)-cycle-covers in G.

▶ **Lemma 7** (**(X, Y)-cycle-cover expansion**)**.** *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric $n \times n$ matrix, $X, Y \subseteq [n]$ with $|X| = |Y|$, and s be a symmetric $n \times n$ matrix. Let G be the support graph of M and $\mathcal{F}$ be the set of all (X, Y)-cycle-covers in G. Then, there exists $\lambda_F \in \{\pm 1\}$ for all $F \in \mathcal{F}$ such that*

$$\det(M(s)[X,Y]) = \sum_{F\in\mathcal{F}} \lambda_F \cdot 2^{|NTCs(F)|} \cdot M(s)_{Cycles(F)} \cdot M(s)_{Paths(F)} \cdot M(s)_{Matchings(F)}.$$

Moreover, if there are $F_1, F_2 \in \mathcal{F}$ such that Cycles(F_1) = Cycles(F_2) and Paths(F_1) = Paths(F_2) then $\lambda_{F_1} = \lambda_{F_2}$.

Proof. For simplicity, we denote $M' = M[X, Y]$. Let $k := |X|$ and let S_k denote the set of permutations on k elements. Then, by the permutation expansion of the determinant, we have

$$\det(M'(s)) = \sum_{\sigma\in S_k} \operatorname{sgn}(\sigma) \cdot \prod_{i=1}^{k} s[i, \sigma(i)] \cdot M'[i, \sigma(i)].$$

We recall that $\operatorname{sgn}(\sigma) \in \{\pm 1\}$. Moreover, if $\sigma_1, \sigma_2 \in S_k$ such that σ_1 and σ_2 have the same cycle structure then $\operatorname{sgn}(\sigma_1) = \operatorname{sgn}(\sigma_2)$. Now, we note that there is a one-to-one correspondence between S_k and bijections from X to Y. So, we may view $\sigma \in S_k$ as a bijection $\sigma' : X \to Y$. Now, consider the graph $H_{\sigma'}$ on vertex set $X \cup Y$ and edge set $F_{\sigma'} := \{\{u, v\} \mid \sigma'(u) = v\}$. Since σ' is a bijection, it follows that $F_{\sigma'}$ is an (X, Y)-cycle-cover in the complete graph on vertex set $X \cup Y$. Moreover, since each non-trivial-cycle in an (X, Y)-cycle-cover can take one of two orientations in any corresponding permutation, there are $2^{|\mathrm{NTCs}(F)|}$ distinct permutations which map to each (X, Y)-cycle-cover F. Hence,

$$\begin{aligned}\prod_{i=1}^{n} s[i, \sigma(i)] \cdot M'[i, \sigma(i)] &= \prod_{u\in X} s[u, \sigma(u)] \cdot M[u, \sigma'(u)] \\ &= M(s)_{\mathrm{Cycles}(F_{\sigma'})} \cdot M(s)_{\mathrm{Paths}(F_{\sigma'})} \cdot M(s)_{\mathrm{Matchings}(F_{\sigma'})}.\end{aligned}$$

The above-term is non-zero only if $F_{\sigma'}$ is an (X, Y)-cycle-cover in the support graph of G. Furthermore, if $F_1, F_2 \in \mathcal{F}$ such that Cycles(F_1) = Cycles(F_2) and Paths(F_1) = Paths(F_2) then $\lambda_{F_1} = \lambda_{F_2}$ since the corresponding permutations would have the same cycle structure. ◀

To prove Theorems 5 and 2, we show the following theorem which gives a generalized structural characterization: it characterizes the existence of invertible symmetric signings for (potentially asymmetric) submatrices of symmetric matrices.

▶ **Theorem 8.** *Let M be a real symmetric $n \times n$ matrix with support graph G and $X, Y \subseteq [n]$ with $|X| = |Y|$. The following are equivalent:*

(i) *There exists an (X, Y)-cycle-cover in G.*

(ii) *There exists a symmetric signing s such that $M(s)[X, Y]$ is invertible.*

Moreover, there exists a polynomial-time algorithm that takes a real symmetric $n \times n$ matrix M and $X, Y \subseteq [n]$ as input and verifies if there exists a symmetric signing s such that $M(s)[X, Y]$ is invertible and if so, find such a signing.

Notation. Let M be a real symmetric $n \times n$ matrix with support graph G. Let A and B be vertex subsets of G. We define $E[A, B]$ to be the set of edges with one end-vertex in A and the other end-vertex in B. We use $E[A]$ to denote $E[A, A]$. Let e be an edge in G corresponding to the non-zero entry $M[u, v]$ $(= M[v, u])$. We define $M^{\overline{e}}$ as the matrix obtained by setting $M[u,v]$ and $M[v,u]$ to 0. For a signing s and row and column indices $u,v \in [n]$, we can obtain another signing s' such that $s'[u,v] := -s[u,v]$, $s'[v,u] := -s[v,u]$ and $s'[i,j] := s[i,j]$ for every entry $(i,j) \in [n] \times [n] \setminus \{(u,v),(v,u)\}$. We call this operation as s' *obtained from* s *by flipping on* $\{u,v\}$.

Proof of Theorem 8. We first present a constructive proof of the characterization. We will then use the proof to design the algorithm.

Lemma 7 immediately shows that (ii) implies (i): If we have a symmetric signing s such that $M(s)[X,Y]$ is invertible, then at least one of the terms in the (X,Y)-cycle-cover expansion of $\det(M(s)[X,Y])$ is non-zero. Hence, there exists an (X,Y)-cycle-cover in G.

We show that (i) implies (ii). Suppose not. Among the counterexamples, consider the ones with $|X|$ minimum and among these, pick a matrix M with minimum number of non-zero entries. Without loss of generality, let M be an $n \times n$ matrix with support graph G and let $X, Y \subseteq [n]$ with $|X| = |Y|$. Since we chose a counterexample, we have that

(A) there exists an (X,Y)-cycle-cover in G, but

(B) there is no symmetric signing s such that $M(s)[X,Y]$ is invertible.

We will arrive at a contradiction by showing that a signing s satisfying (ii) exists. We begin with the following claim about the counterexample.

▷ Claim 9. $E[X \setminus Y, Y] = \emptyset$ and $E[Y \setminus X, X] = \emptyset$.

Proof. Suppose there exists an edge $e \in E[X \setminus Y, Y]$. Let $e := \{u,v\}$ with $u \in X \setminus Y$ and $v \in Y$. Then there exists $\alpha \in \{\pm 1\}$ such that the determinant of $M(s)[X,Y]$ can be expressed as a linear function of $s[u,v]$:

$$\det(M(s)[X,Y]) = \alpha \cdot s[u,v] \cdot M[u,v] \cdot \det(M(s)[X-u, Y-v]) + \det(M^{\overline{e}}(s)[X,Y]). \quad (1)$$

Case 1: Suppose there exists an (X,Y)-cycle-cover F containing e. We observe that $F - e$ is an $(X-u, Y-v)$-cycle-cover in G. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M(s)[X-u, Y-v]) \neq 0$. Since $\det(M(s)[X,Y])$ is a linear function of $s[u,v]$, it follows that $\det(M(s)[X,Y]) \neq 0$ or $\det(M(s')[X,Y]) \neq 0$ where s' is a signing obtained from s by flipping on $\{u,v\}$. Hence, we have a contradiction to assumption B about the counterexample.

Case 2: Suppose that every (X,Y)-cycle-cover in G does not contain e. Then there is no $(X-u, Y-v)$-cycle-cover in G. Since (ii) implies (i), it follows that $\det(M(s)[X-u, Y-v]) = 0$ for every symmetric signing s. Let F be an (X,Y)-cycle-cover in G (as promised to exist by A). Then F is an (X,Y)-cycle-cover in $G-e$. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M^{\overline{e}}(s)[X,Y]) \neq 0$. By (1), we observe that $\det(M(s)[X,Y]) \neq 0$. Thus, the symmetric signing s is a contradiction to assumption B about the counterexample.

Hence, $E[X \setminus Y, Y] = \emptyset$. Similarly $E[Y \setminus X, X] = \emptyset$. ◁

Now, if $X \setminus Y \neq \emptyset$ and there is no edge $e \in E[X \setminus Y, Y]$, then there is no (X,Y)-cycle-cover in G, a contradiction to assumption A about the counterexample. Hence, $X \setminus Y = \emptyset$. Similarly, $Y \setminus X = \emptyset$. Thus, we have $X = Y$ in the counterexample. We next show that the counterexample cannot have any self-loop edges.

▷ Claim 10. There are no self-loop edges in $E[X]$.

Proof. Suppose there exists a self-loop edge in $E[X]$. Let $e = \{u, u\}$ for some $u \in X$. Then, we again have that $\det(M(s)[X, Y])$ is a linear function of $s[u, u]$:

$$\det(M(s)[X, X]) = s[u, u] \cdot M[u, u] \cdot \det(M(s)[X - u, X - u]) + \det(M^{\overline{e}}(s)[X, X]). \quad (2)$$

We arrive at a contradiction by proceeding similar to the proof of the previous claim. We avoid restating the proof in the interests of brevity. ◁

By Claim 10, the counterexample has no self-loop edges in $E[X]$. Our next claim strengthens this further by showing that the counterexample has no (X, Y)-cycle-cover with cycle edges.

▷ Claim 11. Every (X, X)-cycle-cover in G has no cycles.

Proof. Suppose there exists an (X, X)-cycle-cover F in G with a cycle C induced by F. Let $e = \{u, v\}$ be an edge in the cycle. By Claim 10, we know that $u \neq v$. We observe that $\det(M(s)[X, X])$ is a quadratic function of $s[u, v]$, i.e., there exists $\alpha \in \{\pm 1\}$ such that the determinant of $M(s)[X, X]$ can be expressed as

$$\begin{aligned} \det(M(s)[X, X]) = & -s[u, v]^2 \cdot M[u, v]^2 \cdot \det(M(s)[X - \{u, v\}, X - \{u, v\}]) \\ & + 2\alpha \cdot s[u, v] \cdot M[u, v] \cdot \det(M^{\overline{e}}(s)[X - u, X - v]) \\ & + \det(M^{\overline{e}}(s)[X, X]). \end{aligned} \quad (3)$$

Furthermore, $F - e$ is an $(X - u, X - v)$-cycle-cover in G. Since we have a smallest counterexample, it follows that there exists a symmetric signing s such that $\det(M^{\overline{e}}(s)[X - u, X - v]) \neq 0$. We now define the quadratic function

$$\begin{aligned} f(x) := & -x^2 \cdot M[u, v]^2 \cdot \det(M(s)[X - \{u, v\}, X - \{u, v\}]) \\ & + 2\alpha x \cdot M[u, v] \cdot \det(M^{\overline{e}}(s)[X - u, X - v]) + \det(M^{\overline{e}}(s)[X, X]), \end{aligned}$$

and consider the roots of the quadratic equation $f(x) = 0$. Since $\det(M^{\overline{e}}(s)[X - u, X - v]) \neq 0$, the sum of the roots of this quadratic equation is non-zero. Since the real roots of a quadratic function are symmetric about the extreme point of the parabola defined by the function (i.e., symmetric about $\arg\min f(x)$), there exists $x \in \{\pm 1\}$ that is *not* a root of $f(x)$. Hence, either $\det(M(s)[X, Y]) \neq 0$ or $\det(M(s')[X, Y]) \neq 0$ where s' is a signing obtained from s by flipping on $\{u, v\}$. Thus, either s or s' contradict assumption B about the counterexample. ◁

By Claims 9 and 10, the counterexample has $X = Y$ with no loop edges in $E[X]$. Furthermore, by Claim 11, every (X, X)-cycle-cover in G has no cycles. By definition of (X, X)-cycle-covers, it follows that each (X, X)-cycle-cover in G corresponds to a perfect matching in $G[X]$. Let N be an (X, X)-cycle-cover in G.

▷ Claim 12. N is the unique (X, X)-cycle-cover in G.

Proof. Let e be an arbitrary edge in N. Suppose there exists an (X, X)-cycle-cover N' in $G - e$. Then, Claims 10 and 11 imply that N' is also a perfect matching in $G[X]$. We consider $N'' := N \cup N'$. Since N and N' are perfect matchings in $G[X]$, the set of edges N'' induces a vertex-disjoint union of edges and cycles of even length in $G[X]$. Hence, N'' is an (X, X)-cycle-cover in G. Furthermore, since $e \in N \setminus N'$, it follows that N'' contains at least one cycle. This contradicts Claim 11. Thus, every edge $e \in N$ belongs to every (X, X)-cycle-cover in G. Consequently, N is the unique (X, X)-cycle-cover in G. ◁

Since N is the unique (X, X)-cycle-cover in G, by Lemma 7, we have that

$$\det(M(s)[X, X]) = (-1)^{|N|} \prod_{\{u,v\}\in N} M(s)[u, v]^2$$

which is non-zero for every signing s. Thus, there exists a symmetric signing s such that $\det(M(s)[X, X]) \neq 0$, a contradiction to assumption B about the counterexample. This completes the proof of the characterization. We note that the above proof of the characterization is constructive and immediately leads to the algorithm FindSigning(M, X, Y) in Algorithm 1.

Algorithm 1 The algorithm FindSigning(M, X, Y).

FindSigning(M, X, Y):

Input: $M \in \mathbb{R}^{n\times n}$ with support graph G, and $X, Y \subseteq [n]$ satisfying $|X| = |Y|$.

Output: A symmetric signing $s \in \{\pm 1\}^{n\times n}$ such that $M(s)[X, Y]$ is invertible if such a signing exists.

1. If $|X| = |Y| \leq 1$, then brute-force search for a symmetric signing s for which $\det(M(s))[X, Y] \neq 0$:
   1.1. If such a signing exists, return s.
   1.2. Else return “No Invertible Signing”.
2. If there exists no (X, Y)-cycle-cover then return “No Invertible Signing”.
3. If $E[X \setminus Y, Y] \cup E[Y \setminus X, X] \neq \emptyset$:
   3.1. Pick $e := \{u, v\} \in E[X \setminus Y, Y]$ such that $u \in X \setminus Y$ and $v \in Y$.
   3.2. If there is no $(X - u, Y - v)$-cycle-cover in G:
      3.2.1 $s \leftarrow$ FindSigning$(M^{\overline{e}}, X, Y)$.
   3.3. Else: *(when there is an $(X - u, Y - v)$-cycle-cover in G)*
      3.3.1 $s \leftarrow$ FindSigning$(M, X - u, Y - v)$.
   3.4. If $M(s)[X, Y]$ is invertible then return s.
   3.5. Else return s' obtained from s by flipping on $\{u, v\}$.
4. Else: *(when sets X and Y are identical)*
   4.1. If there exists an (X, Y)-cycle-cover in G with a cycle edge $\{u, v\}$:
      4.1.1. $s \leftarrow$ FindSigning$(M, X - u, Y - v)$.
      4.1.2. If $M(s)[X, Y]$ is invertible then return s.
      4.1.3. Else return s' obtained from s by flipping on $\{u, v\}$.
   4.2. Else: *(when all (X, Y)-cycle-covers are perfect matchings in $G[X]$)*
      4.2.1 Return **1** *(the all-positive signing)*.

We now describe an efficient implementation of the non-trivial steps in FindSigning. In Step 1, the algorithm performs a brute-force search. We note that the search needs to be conducted only for the entries $s[u, v]$ where $u, v \in X \cup Y$ since $\det(M(s)[X, Y])$ is independent of the remaining entries of the signing s. Since $|X \cup Y| \leq 2$, the search can be conducted in constant time by picking an arbitrary sign for the remaining entries.

Lemma 6 implies that Steps 2 and 3.2 can be implemented to run in polynomial time. We recall that any cycle edge in an (X, X)-cycle-cover must be a cycle edge in some perfect 2-matching in $G[X]$. Claim 13 shows that Step 4.1 can be implemented to run in polynomial time. Finally, the recursive algorithm terminates in polynomial time since each recursive call reduces either $|X \cup Y|$ or the number of non-zero entries in M. ◀

▷ Claim 13. There is a polynomial-time algorithm that given a graph, finds an edge that belongs to a cycle in some perfect 2-matching of the graph or decides that no such edge exists.

Algorithm 2 The algorithm FindCycleEdge(G).

FindCycleEdge(G):
Input: A graph G with vertex set V.
Output: An edge e that is a cycle edge in some perfect 2-matching in G if one exists.

1. If there exists no perfect 2-matching in G then return "No edge".
2. Let F be a perfect 2-matching in G.
3. If F contains a cycle C then return any edge in C.
4. For $e \in F$:
    4.1. Let N_e be a perfect 2-matching in $G - e$ if one exists.
    4.2. If N_e exists and has a cycle C then return any edge in C.
5. If Step 4 finds N_e for some $e \in F$, then return e.
6. Else return "No edge".

Proof. To prove the claim we consider the algorithm FindCycleEdge(G) in Algorithm 2. If at any point we find a perfect 2-matching with a cycle then we return an edge from it. Hence, it only remains to show the correctness of Steps 5 and 6. Let F be a perfect 2-matching with no cycle edge. Suppose there exists a perfect 2-matching N_e for some edge e with no cycle edge. Then N_e and F are both perfect matchings in G. It follows that $N_e \cup F$ will be a perfect 2-matching where e is in a cycle and hence Step 5 is correct to return e. Now suppose that for all e there is no perfect 2-matching N_e. It follows that G has one unique perfect 2-matching F that is a perfect matching and hence Step 6 correctly returns that no cycle edge exists.

Using the algorithm from Lemma 6 we can perform Steps 1, 2, and 4.1 in polynomial time. Thus, FindCycleEdge(G) runs in polynomial time. ◁

References

1 R. Abelson and M. Rosenberg. Symbolic psycho-logic: A model of attitudinal cognition. *Behavioral Science*, 3:1–13, 1958.
2 S. Akbari, A. Ghafari, K. Kazemian, and M. Nahvi. Some Criteria for a Signed Graph to Have Full Rank. ArXiv. `arXiv:1708.07118`.
3 J. Akiyama, D. Avis, V. Chvátal, and H. Era. Balancing signed graphs. *Discrete Appl. Math.*, 3:227–233, 1981.
4 C. Carlson, K. Chandrasekaran, H. Chang, and A. Kolla. Invertibility and Largest Eigenvalue of Symmetric Matrix Signings. *arXiv e-prints*, November 2016. `arXiv:1611.03624`.
5 M. Cohen. Ramanujan Graphs in Polynomial Time. In *Proceedings of the 57th Annual IEEE Symposium on Foundations of Computer Science*, pages 276–281, 2016.
6 W. Cunningham and J. Geelen. The Optimal Path-Matching Problem. *Combinatorica*, 17(3):315–337, 1997. `doi:10.1007/BF01215915`.
7 S. Fallat and L. Hogben. Variants on the minimum rank problem: A survey II. ArXiv. `arXiv:1102.5142`.
8 S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications*, 426(2):558–582, 2007.
9 J. Friedman. Relative expanders or weakly relatively Ramanujan graphs. *Duke Math. J.*, 118(1):19–35, 2003.
10 P. Hansen. Labelling algorithms for balance in signed graphs. *Problémes Combinatoires et Théorie des Graphes*, pages 215–217, 1978.
11 F. Harary. On the notion of balance of a signed graph. *Michigan Math. J.*, 2(2):143–146, 1953.
12 F. Harary. On the measurement of structural balance. *Behavioral Science*, 4(4):316–323, 1959.
13 F. Harary and J. Kabell. A simple algorithm to detect balance in signed graphs. *Mathematical Social Sciences*, 1(1):131–136, 1980.

14 F. Heider. Attitudes and Cognitive Organization. *J. Psych.*, 21:107–112, 1946.

15 F. Hüffner, N. Betzler, and R. Niedermeier. Optimal edge deletions for signed graph balancing. In *Proceedings of the 6th international conference on Experimental algorithms*, pages 297–310, 2007.

16 A. Marcus, D. Spielman, and N. Srivastava. Interlacing Families I: Ramanujan Graphs of All Degrees. *Annals of Mathematics*, 182(1):307–325, 2015.

17 K. Osamu and S. Iwai. Studies on the balancing, the minimal balancing, and the minimum balancing processes for social groups with planar and nonplanar graph structures. *J. Math. Psychology*, 18(2):140–176, 1978.

18 V. Sivaraman. *Some topics concerning graphs, signed graphs and matroids.* Ph.D. dissertation, Ohio State University, 2012.

19 T. Zaslavsky. The geometry of root systems and signed graphs. *Amer. Math. Monthly*, 2:88–105, 1981.

20 T. Zaslavsky. Signed graphs. *Discrete Appl. Math.*, 1:47–74, 1982.