Bounded-Depth Frege Complexity of Tseitin Formulas for All Graphs

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Abstract

We prove that there is a constant K such that *Tseitin* formulas for an undirected graph G requires proofs of size $2^{\mathrm{tw}(G)^{\Omega(1/d)}}$ in depth-d Frege systems for $d < \frac{K \log n}{\log \log n}$, where $\mathrm{tw}(G)$ is the treewidth of G. This extends Håstad recent lower bound for the grid graph to any graph. Furthermore, we prove tightness of our bound up to a multiplicative constant in the top exponent. Namely, we show that if a Tseitin formula for a graph G has size s, then for all large enough d, it has a depth-d Frege proof of size $2^{\mathrm{tw}(G)^{\mathcal{O}(1/d)}}\mathrm{poly}(s)$. Through this result we settle the question posed by M. Alekhnovich and A. Razborov of showing that the class of Tseitin formulas is quasi-automatizable for resolution.

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1 Introduction

Propositional proof complexity is motivated by the result of Cook and Reckhow [12] saying that if there is a propositional proof system in which any unsatisfiable formula F has a short proof of unsatisfiability (of size polynomial in the size of F), then NP = coNP. In the last 30 years the complexity of proofs was investigated for several proof systems with the aim of finding concrete evidence, and eventually a proof, that for all proof systems there is a propositional formula which is not efficiently provable, i.e. requires super-polynomial proof

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size. The approach followed to prove such lower bounds was essentially borrowed from circuit complexity. Lines in a proof are Boolean formulas and we can define different proof system according to the circuit complexity of such formulas. For example resolution, a well-known refutational system for CNFs, corresponds to a system where formulas are of depth 1. In circuit complexity we keep on trying to strength lower bounds to computationally more powerful class of circuits. In proof complexity we follow the analogous approach: to strength lower bounds to systems working on formulas computationally more powerful. The hope is that techniques used to prove lower bounds for classes of Boolean circuits could be lifted to work with proof systems operating with formulas in the same circuit class. At present however we are far from such ideal situation and in fact, in terms of circuit classes, lower bounds for proof systems are well below those for Boolean circuits.

The complexity of proofs in resolution is largely studied. The first lower bound for (a restriction of) resolution was given by Tseitin in [35]. To obtain his result Tseitin introduced a class of formulas (nowadays known as *Tseitin formulas*) encoding a generalisation of the principle that the sum of the degrees of all vertices in a graph is an even number. A Tseitin formula $\mathsf{T}(G,f)$ is defined for every undirected graph $G(V,E)$ and a charging function $f : V \to \{0,1\}$. We introduce a propositional variable for every edge of G so that $\mathsf{T}(G,f)$ is a CNF representation of a linear system over the field GF(2) that for every vertex $v \in V$ states that the sum of all edges incident to v equals $f(v)$. Tseitin formulas, usually defined on graphs with good expansion properties, are among the main examples we could prove lower bounds for in different proof systems. For unrestricted resolution it was Urquhart in [36] and later Ben-Sasson and Wigderson [8] who proved exponential lower bounds for Tseitin formulas over constant-degree expander graphs. Another example of an important principle largely studied in proof complexity is the Pigeonhole principle, PHP_n. Haken [19], Beame and Pitassi [4] and Ben-Sasson Wigderson [8] proved exponential resolution lower bounds for CNF encoding of the negation of PHP_n, which were later generalized and improved in several other works [13, 30, 32, 33, 9, 24].

Bounded-depth Frege extends resolution since the formulas in the line of proofs are computable by AC^0 circuits, i.e. constant-depth circuits with unbounded fan-in gates. The importance of understanding the complexity of proofs in bounded-depth-Frege systems was due at least to two reasons: (1) for general Frege systems, where formulas have no restrictions, i.e. are of depth $\mathcal{O}(\log n)$, Buss in [10] proved that the Pigeonhole principle can be proved in polynomial size, hence obtaining an exponential separation with resolution. (2) Lower bounds for AC^0-circuits were known [21, 15] and hence we could hope for applying lower bound techniques for AC^0 to lower bounds to bounded-depth Frege. Studying the complexity of proofs in bounded-depth Frege is of the utmost importance since it is a frontier proof system, i.e. one of the strongest propositional proof systems with known significant lower bounds at the moment. Any advance is then a step towards proving lower bounds for $\mathrm{AC}^0[2]$-Frege, i.e. a bounded-depth Frege admitting also formulas with parity gates, which are unknown at the moment, though we know since a long time exponential lower bounds for $\mathrm{AC}^0[2]$ circuits [34, 31, 25]. In this work we contribute to the complexity of proofs in bounded-depth Frege proving new lower bounds for Tseitin formulas.

Ajtai in [1] was the first to prove a lower bound in bounded-depth Frege. He showed that a proof of PHP_n must have a super-polynomial size. His result was later followed by several results simplifying his technique [5] and improving the lower bound [27, 26] showing that any polynomial-size Frege proof of PHP_n must have depth $\Omega(\log \log n)$. The proof complexity of Tseitin formulas in bounded-depth Frege was first considered by Urquhart and Fu in [37], a work where they simplified and adapted the lower bound for the PHP_n to the case of Tseitin formulas over a complete graph. Ben-Sasson in [7], proved exponential lower bounds for the Tseitin formulas over constant-degree expander graphs using a new reduction from the pigeonhole principle [37]. All these lower bounds are adaptation of the technique of [27, 26], hence vanish when the depth of formulas in the proof is more than $\log\log n$. In a very recent major breakthrough [28] showed that Tseitin formulas over a 3-expander graph of n nodes requires super-polynomial bounded-depth Frege proofs at depth $\mathcal{O}(\sqrt{\log n})$. Their result was later improved to depth up to $\frac{C\log n}{\log\log n}$ by Håstad in [22] but for Tseitin formulas defined only on the 2-dimensional grid, where C is a positive constant.

Proofs of $\mathsf{T}(G,f)$ were studied in terms of the treewidth of G, $\mathrm{tw}(G)$, for resolution [2, 17] and for OBDD proof systems [18]. We use Håstad result to prove tight bounds on the complexity of proofs in bounded-depth Frege of $\mathsf{T}(G,f)$ *over any graph* G in terms of the treewidth of G. Our main result is the following theorem:

▶ **Theorem 1.** *There is a constant K such that for any graph G over n nodes and for all $d \leq K\frac{\log n}{\log\log n}$, every depth-$d$ Frege proof of $\neg\mathsf{T}(G,f)$ has size at least $2^{\mathrm{tw}(G)^{\Omega(1/d)}}$. Furthermore, for all large enough d there exist depth-d Frege proofs of $\neg\mathsf{T}(G,f)$ of size $2^{\mathrm{tw}(G)^{\mathcal{O}(1/d)}}\mathrm{poly}(|\mathsf{T}(G,f)|)$.*

A class of unsatisfiable CNFs F_n is (quasi-)automatizable in a proof systems S, if there exists a deterministic algorithm that, given F in F_n returns a proof in S in time which is (quasi-)polynomial in $|F|+|\tau_F|$, where $|\tau_F|$ is the size of shortest proof of F in S. Theorem 1, together with the results from [17, 20, 2, 3] implies that *for any graph* G, the class of Tseitin formulas is *quasi-automatizable* in all systems between treelike resolution and constant-depth Frege. This answers the problem of [2] of extending to all graphs the quasi-automazibablity of $\mathsf{T}(G,f)$ in resolution, known only for graphs with bounded cyclicity [2].

Using a result in [3, 23] we can also prove that the size of proofs of $\mathsf{T}(G,f)$ in proof systems between tree-like resolution and bounded-depth Frege are quasi-polynomially correlated, i.e. *if $\mathsf{T}(G,f)$ has a proof of size S in bounded-depth Frege, then it has a proof of size at most $2^{\mathrm{poly}(\log(S+|\mathsf{T}(G,f)|))}$ in treelike resolution and vice versa.* This result provides evidence to the conjecture of Urquhart that the shortest resolution proofs of $\mathsf{T}(G,f)$ are regular. Finally other consequences of Theorem 1 are: (1) It gives polynomial size Frege proofs of $\mathsf{T}(G,f)$ of depth $\log(\mathrm{tw}(G))$. (2) It improves the lower bounds of [7, 28] since expanders have treewidth $\Omega(n)$ and on such graphs our lower bound is $2^{n^{\Omega(1/d)}}$, which works for larger d than [7, 28]; (3) Induces a strict depth-hierarchy for the proof complexities of Tseitin formulas over an infinite sequence of graphs G_n.

Overview of the proof technique

In Theorem 17 we prove the lower bound from Theorem 1. The proof is based on the improvement of the Excluded Grid Theorem by Robertson and Seymour recently obtained by Chuzhoy [11]: an arbitrary graph G contains as a minor a $r\times r$ grid, where $r=\Omega\left(\mathrm{tw}(G)^{1/37}\right)$. More precisely we use the corollary of this result (see Corollary 8) stating that any graph G has a *wall* of size r as a *topological minor* (i.e. can be obtained from G by several removing of vertices, edges and *suppressions*, see Fig. 1 and Fig. 3). Our proof consists of two parts: at first, we show that if H is a topological minor of G, then any bounded-depth Frege proof of a Tseitin formula $\mathsf{T}(G,f)$ can be transformed to a proof of a $\mathsf{T}(H,f')$, with constant increase in depth and polynomial increase in size. And then we prove a lower bound on the size of depth-d Frege proof of Tseitin formulas based on walls. In this proof we use the lower bound for grid graphs proved by Håstad [22].

In Theorem 18 we prove the upper bound from Theorem 1. We consider the *compact representation* of linear functions $\mathbb{F}_2^n \to \mathbb{F}_2$ on variables $x_1, x_2, \dots, x_n$ by propositional formulas of depth d and of size $2^{n^{\mathcal{O}(1/d)}}$. We show that for linear functions f and g if the equations $f(x) = a$ and $g(x) = b$ are given in our representation, then there is a derivation of $(h + g)(x) = a + b$ of depth d and of size $2^{n^{\mathcal{O}(1/d)}}$. We also show that if a linear equation is represented in CNF, then it is possible to infer its compact representation with depth d and size $2^{n^{\mathcal{O}(1/d)}}$. Since a Tseitin formula is an unsatisfiable system of linear equations written in CNF, hence it is possible to prove a Tseitin formula in size $2^{m^{\mathcal{O}(1/d)}}$ and depth d, where m is the number of edges in G. However we wish to have the treewidth of G instead of m. We consider a tree-partition of a graph G, the vertices of G are split into bags and there exists a tree such that bags are nodes of this tree and if two vertices of G are connected, then they are either in one bag or in adjacent bags. It is known that there is a tree partition where the size of bags are at most $\mathcal{O}(\mathrm{tw}(G)\Delta(G))$ [38]. Since the number of edges touching a given bag is $\mathcal{O}(\mathrm{tw}(G)\Delta(G)^2)$ we can use the compact representation to take care of the equations involving the parity of sum of adjacent bags with proofs growing in terms of the treewidth of G.

Organization

The paper is divided into four sections. After the Preliminary section, we have Section 3 for the lower bound (Theorem 17), Section 4 for the upper bound (Theorem 18). Proofs omitted due to space constraints may be found in [16].

2 Preliminaries

2.1 Formulas and restrictions

We consider propositional formulas over binary $\vee$ and $\wedge$, unary $\neg$ and Boolean constants $\mathbf{0}, \mathbf{1}$. We represent formulas as rooted trees such that internal vertices are labeled with connectives and leaves are labeled with propositional variables or Boolean constants. The *depth* of a formula is the maximal number of alternations of types of connectives over all the paths from the root to a leaf plus one.

We assume that disjunctions with unbounded fanin are represented via binary disjunctions. By default, we mean that $\bigvee_{i=1}^n x_i$ is right-associative; i.e., denotes $(\dots(x_1 \vee x_2) \vee \dots) \vee x_{n-1}) \vee x_n$; we also assume the same for $\bigwedge$.

We denote by $\mathsf{vars}(F)$ the set of variables of a formula F. A *partial assignment* α for a formula F is mapping from $\mathsf{vars}(F) \to \{0, 1, *\}$, where $\alpha(x) = *$ if x is unassigned. We denote by $\mathsf{dom}(\alpha) = \alpha^{-1}(\{0, 1\})$ the set of variables in F which α assigns a Boolean value.

2.2 Pudlák-Buss games

We use the game interpretation of Frege proofs introduced by Pudlák and Buss [29]. Let us define a game with two players Pavel and Sam. The game starts with initial conditions of the form $\varphi_1 = a_1, \dots, \varphi_k = a_k$, where $\varphi_1, \varphi_2, \dots, \varphi_k$ are propositional formulas and $a_1, a_2, \dots, a_k \in \{0, 1\}$ such that $\bigwedge_{i=1}^k (\varphi_i = a_i)$ is identically false. Sam claims that he knows an assignment of variables that satisfies $\bigwedge_{i=1}^k (\varphi_i = a_i)$, the goal of Pavel is to convict Sam. At each his move Pavel asks Sam the value of a propositional formula and Sam gives an answer. The game stops when Pavel convicts Sam, namely Pavel finds an immediate contradiction among initial conditions and Sam's answers. An immediate contradiction with a Boolean connective $\circ$ of arity t is a set of $(t+1)$ formulas $\alpha_1, \ldots, \alpha_t$ and $\circ(\alpha_1, \ldots, \alpha_t)$ with claimed values $a_1, \ldots, a_t$ and b such that $\circ(a_1, \ldots, a_k) \neq b$. In particular, $\mathbf{0}$ with claimed value 1 is an immediate contradiction.

A strategy of Pavel is a function that maps initial conditions and the history of a game to a propositional formula (request). A winning strategy is a strategy that allows Pavel to convict Sam for any behaviour of Sam. A winning strategy of Pavel can be represented as a binary tree whose nodes are labeled with Pavel's requests and edges correspond to Sam's answers. A leaf of the tree corresponds to an immediate contradiction among initial conditions and equalities corresponding to the path from the root to this leaf.

A Pudlák-Buss game derivation of a formula ψ from formulas $\varphi_1, \varphi_2, \ldots, \varphi_s$ is a tree of a Pavel's winning strategy in a game with initial conditions $\varphi_1 = 1, \varphi_2 = 1, \ldots, \varphi_s = 1, \psi = 0$. In that follows by derivations we always mean Pudlák-Buss game derivations. We are interested in the two complexity parameters of derivations: 1) the size of a derivation S that equals the total size of formula ψ and all formulas that are used as labels of nodes; 2) the depth of a derivation d is the maximum depth of ψ and formulas that are used as labels of nodes. We use the notation $\varphi_1, \ldots, \varphi_s \vdash_d \psi$ for a derivation of ψ from $\varphi_1, \varphi_2, \ldots, \varphi_s$ of depth at most d. A derivation of φ is a derivation of φ from the empty set of formulas.

▶ **Lemma 2.** *Assume that there is a derivation $\varphi_1, \ldots, \varphi_k \vdash_{d_1} \psi_1$ of size S_1 and also there is a derivation $\varphi_1, \ldots, \varphi_k, \psi_1 \vdash_{d_2} \psi_2$ of size S_2, then there is a derivation $\varphi_1, \ldots, \varphi_k \vdash_{\max\{d_1,d_2\}} \psi_2$ of size $S_1 + S_2$.*

Proof. Let us create the new tree with the root labelled with ψ_1 such that edge form the root labelled with 0 goes to the root of the first derivation and edge labelled with 1 goes to the root of the second derivation. ◀

▶ **Lemma 3.** *1. If a formula φ has a Frege derivation of size S and depth d, then φ has a Pudlák-Buss game derivation of size $\mathcal{O}(S^2)$ and depth d. 2. If φ has a Pudlák-Buss game derivation of size S and depth d, then φ has a Frege derivation of size $\mathcal{O}(S^3)$ and depth $d + \mathcal{O}(1)$.*

▶ **Lemma 4.** *Let ψ_1 and ψ_2 be two formulas of depth at most d such that $|\mathsf{vars}(\psi_1) \cup \mathsf{vars}(\psi_2)| = k$ and ψ_1 semantically implies ψ_2. Then there exists a derivation $\psi_1 \vdash_d \psi_2$ of size at most $2^k \left(|\psi_1|^2 + |\psi_2|^2\right)$.*

A shortcut contradiction for the disjunction is a situation where Pavel asks Sam formulas $\bigvee_{i=1}^{k} \alpha_i$ and α_j for $j \in [k]$ and gets the answers 0 and 1 respectively. Similarly a shortcut contradiction for the conjunction is a situation where Pavel asks Sam formulas $\bigwedge_{i=1}^{k} \alpha_i$ and α_j for $j \in [k]$ and gets the answers 1 and 0. *An ordinary derivation* is a derivation which does not use shortcut contradictions.

▶ **Lemma 5.** *Consider a derivation of size S and of depth d that uses shortcut contradictions in leaves. Then there is an ordinary derivation of size at most S^3 and of depth d.*

Tseitin Formulas. Let $G(V, E)$ be an undirected graph and $v \in V$. We denote by $E(v)$ the set of edges in E incident with v and by $N(v)$ the set of neighbours $u \in V$ of v, i.e. the u such that $(u, v) \in E(v)$.

A vertex-charging for $G(V, E)$ is a mapping $f : V \longrightarrow \{0, 1\}$. We say that f is an odd-charging of G if $\sum_{v \in V} f(v) \equiv 1 \mod 2$. The *Tseitin formulas* defined on G using variables $x_e, e \in E$ are the formulas: $\mathsf{T}(G, f) := \bigwedge_{v \in V} \mathsf{Par}(v)$, where $\mathsf{Par}(v)$ is a CNF formula representing $\bigoplus_{e \in E(v)} x_e = f(v)$.

▶ **Lemma 6** ([36]). $\mathsf{T}(G, f)$ *is unsatisfiable if and only if there is a connected component U of G such that the restriction of f on U is odd-charging.*

In this work we will work with the tautological form of Tseitin formulas in the form of $\neg\,\mathsf{T}(G, f)$.

2.3 Grids, Walls, Minors, Topological Minors and Treewidth

We consider 4 structural operations on undirected graphs $G = (V, E)$ possibly with parallel edges, but without loops. We follow [6, 14].

- *edge removal* of $e \in E$. It produces the graph $[G \backslash e] = (V, E \backslash \{e\})$.
- *vertex removal* of $v \in V$. It produces the graph $[G \backslash v] = (V \backslash \{v\}, E \backslash E(v))$, where $E(v)$ is the set of edges in E incident with $v \in V$.
- *edge contraction* of $e = (uv) \in E$. Is the replacement of u and v with a single vertex such that edges incident to the new vertex are the edges other than e that were incident with u or v. The resulting graph $G \star e$ has one edge less than G.
- *vertex suppression* of a vertex v in G of degree 2. Let u and w be v's neighbours in G. The suppression of v is obtained by deleting v along with two edges (uv) and (wv) and adding a new edge (wu) (possibly parallel to an existing one). The resulting graph $[G \backslash_s v]$ has one vertex less than G. See Figure 1.

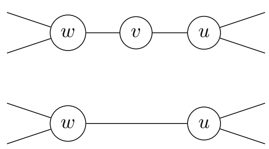

Figure 1 Suppression of v from G.

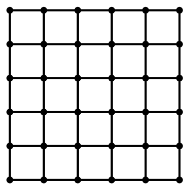

Figure 2 The grid $\mathcal{H}_{5,5}$.

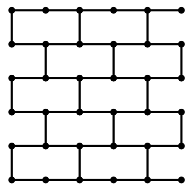

Figure 3 The wall $\mathcal{W}_5$.

A graph H is a *minor* of G if H can be obtained from G by a sequences of edge and vertex removals and edge contractions. A graph H is a *topological minor* of G if H can be obtained from G by a sequence of edge removals, vertex removals and by vertex suppressions [6, 14].

The grid $\mathcal{H}_{m,n}$ is the graph of the cellular rectangle $m \times n$; it has $(m+1)(n+1)$ vertices and $n(m+1) + m(n+1)$ edges, among them $n(m+1)$ horizontal and $m(n+1)$ vertical edges. See fig. 2.

The wall $\mathcal{W}_n$ is a subgraph of $\mathcal{H}_{n,n}$ that is obtained by the removing of several vertical edges. Vertical edges of $\mathcal{H}_{n,n}$ are in n rows and we enumerate them in every row from the left to the right. In the odd rows we remove all vertical edges with even numbers and in even rows we remove all vertical edges with odd numbers. See fig. 3.

A tree decomposition of an undirected graph $G(V, E)$ is a tree $T = (V_T, E_T)$ such that every vertex $u \in V_T$ corresponds to a set $X_u \subseteq V$ and it satisfies the following properties: 1. The union of X_u for $u \in V_T$ equals V. 2. For every edge $(a, b) \in E$ there exists $u \in V_T$ such that $a, b \in X_u$. 3. If a vertex $a \in V$ is in the sets X_u and X_v for some $u, v \in V_T$, then it is also in X_w for all w on the path between u and v in T.

The width of a tree decomposition is the maximum $|X_u|$ for $u \in V_T$ minus one. *A treewidth* of a graph G is the minimal value of the width among all tree decompositions of the graph G.

Recall the following Theorem proved in [11].

▶ **Theorem 7** ([11]). *If G has a treewidth t, then it has the grid $\mathcal{H}_{r,r}$ as a minor, where $r = \Omega(t^{1/37})$.*

The following Corollary was mentioned in [6].

▶ **Corollary 8** ([6]). *If G a has treewidth t, then it has the wall $\mathcal{W}_r$ as a topological minor, where $r = \Omega(t^{1/37})$.*

3 The Lower Bound

3.1 Topological Minors and Tseitin Formulas

Le φ be a formula and let α be a partial assignment to variables of φ. Define $\varphi[\alpha]$ to be the formula obtained from φ substituting each variable x in the domain of α, with the constant assigned to x by α. Notice that φ and $\varphi[\alpha]$ have the same size and depth.

▶ **Lemma 9.** *Let Φ_a and Φ'_a for $a \in A$ be propositional formulas of depth at most d such that $|\mathsf{vars}(\Phi_a) \cup \mathsf{vars}(\Phi'_a)| \leq k$. Assume that for all $a \in A$, Φ_a is semantically equivalent to Φ'_a. Then $\neg \bigwedge_{a\in A} \Phi'_a \vdash_{d+\mathcal{O}(1)} \neg \bigwedge_{a \in A'} \Phi_a$ of size at most $2^k \mathrm{poly}\left(\sum_{a\in A}(|\Phi_a| + |\Phi'_a|)\right)$, where $A' = \{a \in A \mid \Phi_a$ is not identically true$\}$.*

▶ **Lemma 10** ([18]). *Let $G(V,E)$ be a connected graph and $H(V',E')$ be a connected subgraph of G with $E' \neq \emptyset$ that is obtained from G by the deletion of some vertices and edges. For every unsatisfiable Tseitin formula $\mathsf{T}(G,f)$ there exists a partial assignment α to variables x_e for $e \in E \setminus E'$ such that α does not falsify any clause of $\mathsf{T}(G,f)$.*

▶ **Lemma 11.** *Let $G(V,E)$ be a connected graph and $H(V',E')$ be a connected subgraph of G. Assume that there is a derivation $\vdash_d \neg \mathsf{T}(G,f)$ of size S. Then for some f' there is a derivation $\vdash_{d+\mathcal{O}(1)} \neg \mathsf{T}(H,f')$ of size $S + \mathrm{poly}(|\mathsf{T}(G,f)|)$.*

Proof. Let T be the game tree of $\vdash_d \neg \mathsf{T}(G,f)$. Let α be given by Lemma 10 that is defined on all variables x_e for $e \in E \setminus E'$ and does not falsify any clause of $\mathsf{T}(G,f)$. $T[\alpha]$ be the tree obtained form T applying the substitution α to all the queried formulas. Size and depth do not change, hence $T[\alpha]$ defines a derivation $\vdash_d \neg \mathsf{T}(G,f)[\alpha]$ of size S. $\neg \mathsf{T}(G,f)$ has the form $\neg \bigwedge_{v \in V} \mathsf{Par}(v)$, where $\mathsf{Par}(v)$ is a parity condition of the vertex v. Hence, $\neg \mathsf{T}(G,f)[\alpha]$ is of the form $\neg \bigwedge_i \mathsf{Par}(v)[\alpha]$. If $v \notin V'$, then α assigns values to all variables from $\mathsf{Par}(v)$, since α does not falsify $\mathsf{Par}(v)$, α satisfies $\mathsf{Par}(v)$, hence $\mathsf{Par}(v)[\alpha]$ is identically true. If $v \in V'$, then $\mathsf{Par}(v)[\alpha]$ is a parity statement depending on variables x_e, where $e \in E'$ is incident to v. Hence, for $v \in V'$, $\mathsf{Par}(v)[\alpha]$ is semantically equivalent to a parity condition of a Tseitin formula $\mathsf{T}(H,\varphi')$ for some charging φ'. Let Δ be the maximal degree of G. Then every parity condition of $\mathsf{T}(H,\varphi')$ or $\mathsf{T}(G,\varphi)$ depends on at most Δ variables. Notice that since we represent parities in CNF, $|\mathsf{T}(G,f)| \geq 2^\Delta$. By Lemma 9, there is a derivation $\neg \mathsf{T}(G,f)[\alpha] \vdash_{\mathcal{O}(1)} \neg \mathsf{T}(H,f')$ of size $\mathrm{poly}(|\mathsf{T}(G,f)|)$. The claim follows using the size S, depth d derivation of $\neg \mathsf{T}(G,f)[\alpha]$ together with Lemma 2. ◀

A 1-*substitution* for a formula φ is a partial function mapping variables of φ into its literals. After applying a 1-substitution σ to φ, the depth of the new formula $\varphi[\sigma]$ can increase by one. However 1-substitutions are closed under composition: if σ_1 maps $[y \mapsto \neg z]$ and σ_2 maps $[x \mapsto \neg y]$, then $\sigma = \sigma_1 \circ \sigma_2$ is the 1-substitution $[x \mapsto z, y \mapsto \neg z]$. We use 1-substitutions to handle in $\mathsf{T}(G,f)$ the operation of *vertex suppression* on the graph G. Let $G = (V,E)$ be a graph and $v \in V$ be a node and let $\mathsf{T}(G,f)$ be a Tseitin formula on G. Let v

be a degree-2 vertex v in G with neighbours u and w. Consider the following 1-substitution σ_v and the charge function f_v for $[G\backslash_s v]$:

$$\sigma_v = \begin{cases} [x_{vw} \mapsto x_{wu}, x_{vu} \mapsto x_{wu}] & \text{if } f(v) = 0 \\ [x_{vw} \mapsto x_{wu}, x_{vu} \mapsto \neg x_{wu}] & \text{if } f(v) = 1 \end{cases} \qquad f_v(z) = \begin{cases} f(z) & \text{if } z \in V \setminus \{u, v\} \\ f(u) + f(v) & \text{if } z = u \end{cases}$$

Let $G(V, E)$ be a graph and $f : V \to \{0, 1\}$ be a charging. Let A be a finite set. We say that a formula Ψ is a *pseudo Tseitin formula based on G and f with fake vertices in A*, and we write Ψ is $\mathsf{T}^*_A(G, f)$, if Ψ has the form $\bigwedge_{v \in V \cup A} \psi_v$, where

1. for all $v \in V$, ψ_v is a propositional formula depending on variables x_e for all edges e incident to v. And ψ_v is semantically equivalent to the parity condition $\mathsf{Par}(v)$ of $\mathsf{T}(G, f)$.
2. for all $v \in A$, ψ_v is a tautology.

▶ **Lemma 12.** *Let $G(V, E)$ be a connected constant-degree graph over n vertices. Let $[G\backslash_s v]$ be the graph obtained after the suppression of a degree-2 vertex v in G. If Ψ is $\mathsf{T}^*_A(G, f)$, then $\Psi[\sigma_v]$ is $\mathsf{T}^*_{A \cup \{v\}}([G\backslash_s v], f_v)$.*

Proof. Assume that v is linked to two vertices w and u in G. Let A be the set of fake vertices of Ψ so Ψ has the form $\bigwedge_{x \in V \cup A} \psi_x$, hence $\Psi[\sigma_v]$ is $\bigwedge_{x \in V \cup A} \psi_x[\sigma_v]$. For $x \in A$, $\psi(x)$ is a tautology, hence $\psi_x[\sigma_v]$ is also a tautology. By the definition of σ_v, $\psi_v[\sigma_v]$ is a tautology. It is not hard to verify that for $x \in V \setminus \{v\}$, $\psi_x[\sigma_v]$ is equivalent to parity condition of $\mathsf{T}([G\backslash_s v], f_v)$. Hence, $\Psi[\sigma_v]$ is $\mathsf{T}^*_{A \cup \{v\}}([G\backslash_s v], f_v)$ ◀

▶ **Lemma 13.** *Let $G(V, E)$ be a graph and $f : V \to \{0, 1\}$ and $W = \{v_1, \ldots, v_k\}$ be degree 2 nodes in V suppressed in that order from G and $[G\backslash_s W]$ be the resulting graph. Let σ_i be the corresponding 1-substitutions and let $\sigma = \sigma_k \circ \ldots \circ \sigma_1$. There is a charging f_k of G such that if Ψ is $\mathsf{T}^*_A(G, f)$, then $\Psi[\sigma]$ is $\mathsf{T}^*_{A \cup W}([G\backslash_s W], f_k)$.*

▶ **Lemma 14.** *Let G be a connected graph on n vertices and with the maximal degree at most 3. Let H be obtained from G by several suppressions. Assume that there is a derivation of $\neg\mathsf{T}(G, f)$ of size S and depth d. Then for some charging f_k there is a derivation of $\neg\mathsf{T}(H, f_k)$ of size $\mathcal{O}(S) + \mathrm{poly}(n)$ and depth $d + \mathcal{O}(1)$.*

Proof. Assume that, in order, to get H from G we have to apply suppressions for vertices $W = \{v_1, \ldots, v_k\}$. Let σ_i be the 1-substitutions corresponding to the suppression of v_i, and let $\sigma = \sigma_k \circ \cdots \circ \sigma_1$. $\mathsf{T}(G, f)$ is $\mathsf{T}^*_\emptyset(G, f)$. Let f_k be the charging given by Lemma 13 applied to $\mathsf{T}(G, f)$ and $[G\backslash_s W] = H$. Then $\mathsf{T}(G, f)[\sigma]$ is $\mathsf{T}^*_W(H, f_k)$. We apply the 1-substitution σ to the given derivation of $\neg\mathsf{T}(G, f)$ and we get a derivation of $\neg\mathsf{T}(G, f)[\sigma]$ of size $\mathcal{O}(S)$ and depth at most $d + 1$. By Lemma 9, applied on $\mathsf{T}(G, f)[\sigma]$ and $\mathsf{T}(H, f_k)$, there is a derivation $\neg\mathsf{T}(G, f)[\sigma] \vdash_{d+\mathcal{O}(1)} \neg\mathsf{T}(H, f_k)$ of size $\mathrm{poly}(n)$. Combining the two derivations together by Lemma 2 we obtain a derivation $\vdash_{d+\mathcal{O}(1)} \mathsf{T}(H, f_k)$ of size $\mathcal{O}(s) + \mathrm{poly}(n)$. ◀

3.2 From Walls To Grids

▶ **Lemma 15.** *If there exists a derivation $\vdash_d \neg\mathsf{T}(\mathcal{W}_n, f)$ of size S, then there exists a derivation $\vdash_{d+\mathcal{O}(1)} \neg\mathsf{T}(M_n, f')$ of size $\mathcal{O}(S) + \mathrm{poly}(n)$, where M_n is a connected constant-degree graph that contains $\mathcal{H}_{n, \lfloor \frac{n-1}{2} \rfloor}$ as a subgraph.*

Proof. Consider a set I of all the horizontal edges of $\mathcal{W}_n$ that belong to odd columns (on fig. 4 and 5 edges from I are red). I is a matching, i.e. no two edges from I are incident to the same vertex. If we contract all edges from I, we get the graph M_n that for odd n

coincides with $\mathcal{H}_{n,\frac{n-1}{2}}$ and for even n coincides with a graph that is obtained from $\mathcal{H}_{n,\lfloor\frac{n}{2}\rfloor}$ by the removal of several edges from the last vertical (see fig. 4 and 5). For every $e \in I$ we denote its left vertex by u_e and the right vertex by v_e. Let E_{u_e} be the set of edges of $\mathcal{W}_n$ incident to u_e except e. Let τ_e denote a CNF formula encoding $\bigoplus_{f\in E_{u_e}} x_f = f(u_e)$.

Consider a game tree T for the derivation of the Tseitin tautology $\neg\mathsf{T}(\mathcal{W}_n, f)$ of size S and depth d. To every formula used in this tree we apply the substitution that replaces every occurrence of x_e with τ_e. We denote the resulting tree by T'.

Notice that T' is a correct game tree of a derivation $\vdash_{d+\mathcal{O}(1)} \neg F$, where F is obtained from $\mathsf{T}(\mathcal{W}_n, f)$ by the same substitution. The depth of this derivation is increased by at most a constant since in several leaves we hang a formula of constant depth; here we also use that I is a matching and thus we do not add new occurrences of variables corresponding edges from I. The size of τ_e is $\mathcal{O}(1)$, hence any formula from the derivation is increased in at most a constant factor, thus the size of the derivation defined by the tree T' is $\mathcal{O}(S)$.

We define a function f' on vertices of M_n as follows. If a vertex w of the graph M_n is obtained by merging the vertices w', w'' of the graph $\mathcal{W}_n$, then $f'(w) = (f(w')+f(w'')) \bmod 2$. If the vertex w of $\mathcal{H}_{n,\lfloor n/2\rfloor}$ is obtained from the vertex w of $\mathcal{W}_n$, then $f'(w) = f(w)$.

Now we show how to derive $\neg\mathsf{T}(M_n, f')$ from $\neg F$. $\mathsf{T}(\mathcal{W}_n, f)$ is a Tseitin formula and it has the following structure: $\bigwedge_{v\in V} \psi_v$, where V is the set of vertices of $\mathcal{W}_n$ and ψ_v is a CNF formula encoding a parity condition for the vertex v. F differs from $\mathsf{T}(\mathcal{W}_n, f)$ only in conditions corresponding to vertices that are incident to an edge from I (if n is even, then there are vertices in $\mathcal{W}_n$ that are not incident to any edge from I). Notice that F has the form $\bigwedge_{v\in V} \psi'_v$ where ψ'_v is obtained by substitution from ψ_v. Let $w = u_e$ for some $e \in I$, then the formula ψ'_w is identically true. If $w = v_e$, then the condition ψ'_w is equivalent to the parity condition of the merged vertex $\{u_e, v_e\}$ in the Tseitin formula $\mathsf{T}(M_n, f')$, but ψ'_w is not written in canonical form.

Since all degrees in M_n are at most 4, then by Lemma 9 there exists a derivation $\neg F \vdash_{d+\mathcal{O}(1)} \neg\mathsf{T}(M_n, f')$ of size $\mathrm{poly}(n)$. The claim follows by Lemma 2. ◀

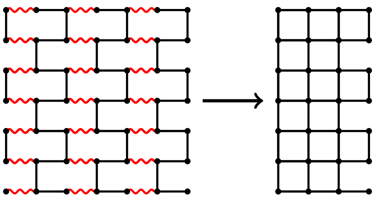

Figure 4 $\mathcal{W}_6$ is contracted to M_6.

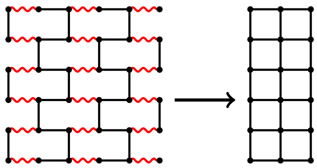

Figure 5 $\mathcal{W}_5$ is contracted to M_5.

3.3 Putting it all together

We use Håstad's Theorem from [22].

▶ **Theorem 16** ([22]). *There is a constant $K > 0$ such that for $d \le \frac{K\log n}{\log\log n}$ any depth d derivation of $\neg\mathsf{T}(\mathcal{H}_{n,n}, f))$ has size at least $2^{n^{\Omega(1/d)}}$.*

▶ **Theorem 17.** *There exist constants $K > 0$ and $C > 0$ such that for every connected graph G of treewidth t and every $d \le \frac{K\log n}{\log\log n} - C$, any depth d derivation of $\neg\mathsf{T}(G, f))$ has size at least $2^{t^{\Omega(1/d)}}$.*

Proof. Suppose that $\neg\mathsf{T}(G,f)$ have a derivation of size S and depth d. By Corollary 8 we know that G contains the wall $\mathcal{W}_r$ as a topological minor, where $r=\Omega(t^{1/37})$. Consider a sequence of operations (edge/vertex removals and suppressions) that transform G to $\mathcal{W}_r$. Assume that removals do not follow suppressions. And let G' be a subgraph of G that is obtained from G by application of all removals (hence, $\mathcal{W}_r$ can be obtained from G' by application of several suppressions).

By Lemma 11, for some f' there is a derivation of $\neg\mathsf{T}(G',f')$ of size $\mathrm{poly}(|\mathsf{T}(G,f)|)+S$ and depth $d+\mathcal{O}(1)$. Since $\mathcal{W}_r$ can be obtained from G' by application of several suppressions, G' is connected. Suppressions can not increase the degrees, hence all degrees in G' are at most 3. By Lemma 14, for some f'' there is a derivation of $\neg\mathsf{T}(\mathcal{W}_r,f'')$ of size $\mathrm{poly}(|\mathsf{T}(G,f)|)+S$ and depth $d+\mathcal{O}(1)$. By Lemma 15, for some f''' there is a derivation of $\neg\mathsf{T}(M_r,f''')$ of size $\mathrm{poly}(|\mathsf{T}(G,f)|)+\mathcal{O}(S)$ and depth $d+\mathcal{O}(1)$, where M_r is connected constant-degree graph containing $\mathcal{H}_{\lfloor(r-1)/2\rfloor}$ as a subgraph. And finally by Lemma 11, for some f'''' there is a Frege derivation of $\mathsf{T}(\mathcal{H}_{\lfloor(r-1)/2\rfloor},f'''')$ of size $\mathrm{poly}(|\mathsf{T}(G,f)|)+\mathcal{O}(S)$ and depth $d+\mathcal{O}(1)$. Notice that S is the size of a derivation of $\neg\mathsf{T}(G,f)$, hence $S\geq|\mathsf{T}(G,f)|$. Thus, for some constants C and c there is a derivation of $\neg\mathsf{T}(\mathcal{H}_{\lfloor(r-1)/2\rfloor},f'''')$ of size S^c and depth $d+C$.

By Theorem 16, there is a constant K such that if $d+C\leq\frac{K\log n}{\log\log n}$, then $S^c\geq 2^{\lfloor(r-1)/2\rfloor^{\Omega(1/(d+C))}}$. Hence $S\geq 2^{r^{\Omega(1/d)}}$ and, thus, $S\geq 2^{t^{\Omega(1/d)}}$. ◀

4 The Upper Bound

In this section we prove the following Theorem:

▶ **Theorem 18.** *Let $G(V,E)$ be a connected undirected graph and $\mathsf{T}(G,f)$ be an unsatisfiable Tseitin formula. Then for all large enough d the formula $\neg\mathsf{T}(G,f)$ has a derivation of depth d and size $2^{\mathrm{tw}(G)^{\mathcal{O}(1/d)}}\mathrm{poly}(|\mathsf{T}(G,f)|)$.*

In order to prove Theorem 18 we define a *compact representation* of parity by depth-d formulas, then we show that we can efficiently derive the sum of $\mathbb{F}_2$-linear equations using the compact representation of parities. And then we prove Theorem 18 using a tree-partition of the graph G.

4.1 A compact representation of parity

Let $t_1,t_2,\ldots,t_d$ be natural numbers, where d is a non-negative integer. Let $U_0,U_1,\ldots,U_d$ be partitions of a finite set F. We say that a list of partitions $U=(U_0,U_1,\ldots,U_d)$ is a $(t_1,\ldots,t_d)$*-refinement* of F if the following conditions hold:

1. U_0 consists of the only element $U_{0,1}=F$.
2. For every i, U_{i+1} is a subpartition of U_i such that every element of U_i is split into t_{i+1} parts. Hence, U_i split F into m_i parts: $U_{i,1},U_{i,2},\ldots,U_{i,m_i}$, where $m_i=\prod_{j=1}^{i}t_j$.
3. All elements of U_d have cardinality at most 1.

Let U be a $(t_1,\ldots,t_d)$-refinement of a set F and let $U_{i,j}$ be one of the blocks of this refinement. Then U induces on each of the blocks U_{ij} a $(t_{i+1},\ldots,t_d)$-refinement U' which is obtained by restricting $U_i,\ldots,U_d$ to the set U_{ij}. U' is called a *sub-refinement* of U_{ij} in U.

▶ **Lemma 19.** *Let F be a set of size n and $d\geq 0$ be an integer. Let $t_1,\ldots,t_d$ be integers such that $t_1\cdot t_2\cdot\ldots\cdot t_d\geq n$. Then there exists a $(t_1,\ldots,t_d)$-refinement U of F.*

For $a\in\{0,1\}$ and natural number n we define a Boolean function $\mathrm{PARITY}_n^a:\{0,1\}^n\to\{0,1\}^n$ such that $\mathrm{PARITY}_n^a(x_1,\ldots,x_n)=1$ iff $\bigoplus_{i=1}^n x_i=a$ for all $x_1,\ldots,x_n\in\{0,1\}$.

▶ **Lemma 20.** *Let n and d be positive integers and U be a $(t_1, t_2, \dots, t_d)$-refinement of $[n]$. Then there exists a formula representing PARITY_n^b of depth at most $3d + 1$ and of size $\prod_{i=1}^d 2^{t_i+1} t_i$.*

Proof. Let us prove by backward induction on i from d to 0 that for every $j \in [\prod_{k=1}^i t_k]$, there is a formula representing $\bigoplus_{k \in U_{i,j}} x_k$ of depth $3(d-i)$ and of size $\prod_{q=i+1}^d 2^{t_q+1} t_q$. If $i = d$, then $|U_{d,j}| \le 1$, hence $\bigoplus_{k \in U_{i,j}} x_k$ is either 0 or a variable x_k and thus has size 1 and depth 0.

Assume that $i < d$. Let $\ell_1, \ell_2, \dots, \ell_{t_{i+1}}$ be such that $U_{i,j} = U_{i+1,\ell_1} \sqcup U_{i+1,\ell_2} \sqcup \dots \sqcup U_{i+1,\ell_{t_{i+1}}}$. Let for $r \in [t_{i_1}]$, β_r be a representation of $\bigoplus_{k \in U_{i+1,\ell_r}} x_k$ of size $\prod_{q=i+2}^d 2^{t_q+1} t_q$ and depth $3(d-i-1)$ that exists by the induction hypothesis. Consider a CNF-representation of $\beta_1 \oplus \dots \oplus \beta_{t_{i+1}}$: $\bigoplus_{k \in U_{i,j}} x_k = \bigwedge_{\substack{S \subseteq \{1,\dots,t_{i+1}\} \\ |S| \bmod 2 = 0}} \left(\bigvee_{s \in S} \neg \beta_s \vee \bigvee_{s \notin S} \beta_s \right)$. After the substitution of the representations of $\beta_1, \dots, \beta_{t_{i+1}}$ we obtain a formula of size at most $2^{t_{i+1}} t_{i+1} \cdot \prod_{q=i+2}^d 2^{t_q+1} t_q + 2^{t_{i+1}} t_{i+1} \le \prod_{q=i+1}^q 2^{t_q+1} t_q$ and of depth $3(d-i-1) + 3 = 3(d-i)$.

Therefore we have constructed a representation of PARITY_n^1 of the needed size and depth. The representation of PARITY_n^0 could be constructed as $\neg\varphi$ where φ is the obtained representation of PARITY_n^1. ◀

We call the representation of PARITY_n^a obtained by Lemma 20 *the compact representation* of PARITY_n^a with respect to a $(t_1, \dots, t_d)$-refinement U.

Let us define for $S \subseteq [n]$ and for $a \in \{0, 1\}$, $\mathrm{PARITY}_{n,S}^a(x_1, \dots, x_n) = (\neg a) \oplus \bigoplus_{i \in S} x_i$. We define a compact representation of $\mathrm{PARITY}_{n,S}^a$ with respect to a $(t_1, \dots, t_d)$-refinement U as the result of substitutions $x_j := \mathbf{0}$ for all $j \notin S$ to the compact representation of PARITY_n^a with respect to U. We denote the compact representation of $\mathrm{PARITY}_{n,S}^a(x_1, x_2, \dots, x_n)$ w.r.t. U by $\Phi^a(S, U)$.

▶ **Lemma 21.** *Let U be a $(t_1, \dots, t_d)$-refinement of $[n]$ and U' be a sub-refinement of U_{ij} in U. Then for every $S \subseteq U_{ij}$ there exists a derivation $\Phi^a(S, U') \vdash_{3d+\mathcal{O}(1)} \Phi^a(S, U)$ of size at most $4|\Phi^a(S, U)|^3$.*

4.2 Summation of linear equations

Let $S \triangle T$ be the symmetric difference of sets S and T i.e. $S \triangle T = (S \cup T) \setminus (S \cap T)$.

▶ **Lemma 22.** *Let U be a $(t_1, \dots, t_d)$-refinement of $[n]$. Let $S_1, S_2, \dots, S_k \subseteq [n]$ and $a_1, \dots, a_k \in \{0, 1\}$. Then there exists a constant c such that:*

1. *There exists a derivation $\Phi^{a_1}(S_1, U), \Phi^{a_2}(S_2, U), \dots, \Phi^{a_k}(S_k, U) \vdash_{3d+\mathcal{O}(1)} \Phi^{a_1 \oplus \dots \oplus a_k}(S_1 \triangle \dots \triangle S_k, U)$ of size at most $c \cdot k \cdot |\Phi^1(\varnothing, U)|^6$.*
2. *If $\bigwedge_{i \in [k]} \left(\bigoplus_{j \in S_i} x_j = a_i \right)$ is unsatisfiable then there exists a derivation $\Phi^{a_1}(S_1, U), \Phi^{a_2}(S_2, U), \dots, \Phi^{a_k}(S_k, U) \vdash_{3d+\mathcal{O}(1)} \mathbf{0}$ of size at most $c \cdot k \cdot |\Phi^1(\varnothing, U)|^6$.*

4.3 Tree-partition width

Let $G(V, E)$ be an undirected graph and $S_1, \dots, S_m$ be a partition of V. $S_1, \dots, S_m$ is a tree-partition of G if there exists a tree $T([m], E_T)$ such that every edge e of G connects either two vertices from the same part S_i or connects a vertex from S_i and a vertex from S_j, where i and j are adjacent in T, i.e. $(i, j) \in E_T$. A *width* of a tree-partition $S_1, S_2, \dots, S_m$ is the size of the largest set S_i for $i \in [m]$. A *tree-partition width* of a graph G is the smallest width among all tree-partitions of G. We denote the tree-partition width of G by $\mathrm{tpw}(G)$.

If we add a new vertex in the middle of every edge (i, j) of the tree T and put the set $S_i \cup S_j$ on it, we will get a tree decomposition of G, hence $\mathrm{tw}(G) \le 2\mathrm{tpw}(G) - 1$.

The following theorem shows an inequality in the other direction.

▶ **Theorem 23** ([38]). *If* $\mathrm{tw}(G) \ge 1$, *then* $\mathrm{tpw}(G) \le 10\Delta(G)\mathrm{tw}(G)$, *where* $\Delta(G)$ *is the maximum degree of* G.

So, $\mathrm{tw}(G)$ and $\mathrm{tpw}(G)$ coincide up to a multiplicative constants for constant degree graphs.

▶ **Theorem 24.** *Let* $G(V, E)$ *be a connected graph and let a Tseitin formula* $\mathsf{T}(G, f)$ *be unsatisfiable. Then there exists a derivation* $\vdash_{3d+\mathcal{O}(1)} \mathsf{T}(G, f)$ *of size at most* $\mathrm{poly}(|\mathsf{T}(G, f)|) \cdot 2^{(\mathrm{tpw}(G)\Delta(G))^{\mathcal{O}(1/d)}}$, *where* $\Delta(G)$ *is the maximum degree of* G.

Proof. Let $S_1, \ldots, S_m$ be a tree-partition of G with width $\mathrm{tpw}(G)$ and let $T([m], E_T)$ be the corresponding tree. W.l.o.g. we assume that T is a rooted tree with the root m; for all $i \in [m-1]$, $p(i)$ denotes its parent and for all $i \in [m]$, $\mathrm{s}(i)$ denotes the set of direct successors of i . W.l.o.g. we assume that $p(i) > i$ for all $i \in [m-1]$.

Since $\mathsf{T}(G, f)$ is unsatisfiable and G is connected, $\bigoplus_{v\in V} f(v) = \bigoplus_{i\in[m]} \bigoplus_{v\in S_i} f(v) = 1$. We consider the sum $\bigoplus_{i\in[m]} \bigoplus_{e\in E(S_i, V\setminus S_i)} x_e$. Since each x_e occurs in the sum exactly twice, the sum (modulo 2) is 0 for all values of x_e. Then for each assignment to $\{x_e\}_{e\in E}$ there exists i_0 such that $\bigoplus_{v\in S_{i_0}} f(v) \ne \bigoplus_{e\in E(S_{i_0}, V\setminus S_{i_0})} x_e$. The first part of Pavel's strategy is to find such i_0.

Pavel will request parity of the sum of all edges between S_i and S_j for all $(i, j) \in E_T$. In order to represent these formulas in a compact way we now define m different $(t_1, \ldots, t_d)$-refinements $W^1, \ldots, W^m$; for every i, W^i is a refinement of the set $E\left(S_i, \bigcup_{j\in \mathrm{s}(i)} S_j\right)$ of all edges connecting a vertex from S_i with a vertex from $\bigcup_{j\in s(i)} S_j$. We construct appropriate refinements W^i later.

Pavel asks Sam the values of $\bigoplus_{e\in E(S_i, S_{p(i)})} x_e$ represented as $\Phi^1\left(E\left(S_i, S_{p(i)}\right), W^{p(i)}\right)$ for $i \in [m-1]$ in the increasing order until he finds i_0 such that $\bigoplus_{e\in E(S_{i_0}, V\setminus S_{i_0})} x_e \ne \bigoplus_{v\in S_{i_0}} f(v)$.

At the moment when Sam has answered the value of $\Phi^1\left(E\left(S_i, S_{p(i)}\right), W^{p(i)}\right)$ the values of $\bigoplus_{e\in E(S_i, S_j)} x_e$ for each j such that $(i, j) \in E_T$ are all determined, thus, the value of $\bigoplus_{e\in E(S_i, V\setminus S_i)} x_e$ is determined. If $\bigoplus_{e\in E(S_i, V\setminus S_i)} x_e \ne \bigoplus_{v\in S_i} f(v)$ Pavel proceeds to the next part of his strategy. Otherwise he continues to ask Sam similar questions corresponding to the vertices with larger indices.

Now we describe the strategy of Pavel in case if he finds i_0. We are going to describe this case in terms of derivation using Lemma 2 multiple times. Consider a linear system that consists of the equation $\bigoplus_{e\in E(S_{i_0}, V\setminus S_{i_0})} x_e = 1 \oplus \bigoplus_{v\in S_{i_0}} f(v)$ and all parity conditions of $\mathsf{T}(G, f)$ of the vertices from S_{i_0}. This linear system is unsatisfiable. We are going to use Lemma 22. In order to do it we need to derive the representations of these linear equations w.r.t. some refinement Q of a superset of $E(S_{i_0}, V)$.

Let for $i \in [m]$, U^i be a $(t_1, t_2, \ldots, t_d)$-refinement of the set $E(S_i)$ of all edges connecting two vertices from S_i (we construct these refinements in the end of the proof together with the refinements W^i). Let us define a $(3, t_1, \ldots, t_d)$-refinement Q as a union of $(t_1, \ldots, t_d)$-refinements $W^{i_0}, W^{p(i_0)}$ and U^{i_0} such that $Q_1 = \{E(S_{i_0}, \bigcup_{j\in \mathrm{s}(i_0)} S_j)$, $E(S_{p(i_0)}, \bigcup_{j\in \mathrm{s}(p(i_0))} S_j), E(S_{i_0})\}$ and for every $j \in \{2, 3, \ldots, d+1\}$, Q_j is the union of $W^{i_0}_{j-1}, W^{p(i_0)}_{j-1}$ and $U^{i_0}_{j-1}$.

Let a_j be Sam's answer to the question $\bigoplus_{e\in E(S_{i_0}, S_j)} x_e$ for each j that is a neighbour of i_0 in T, hence we may assume that $\Phi^{a_j}(E(S_{i_0}, S_j), W^{i_0})$ for $j \in \mathrm{s}(i_0)$ and $\Phi^{a_{p(i_0)}}(E(S_{i_0}, E_{p(i_0)}),$

$W^{p(i_0)}$) are already derived. By Lemma 21, we derive $\Phi^{a_j}(E(S_{i_0}, S_j), Q)$ from $\Phi^{a_j}(E(S_{i_0}, S_j), W^{i_0})$ for $j \in \mathrm{s}(i_0)$ and $\Phi^{a_{p(i_0)}}(E(S_{i_0}, S_{p(i_0)}), Q)$ from $\Phi^{a_{p(i_0)}}(E(S_{i_0}, S_{p(i_0)}), W^{p(i_0)})$, where a_j are Sam's answers to the corresponding questions.

By the first part of Lemma 22 we derive $\Phi^{1 \oplus \left(\bigoplus_{v \in S_{i_0}} f(v)\right)}(E(S_{i_0}, V \setminus S_{i_0}), Q)$ from the set of formulas $\{\Phi^{a_j}(E(S_{i_0}, S_j), Q) \mid (i_0, j) \in E_T\}$. We assume that the parity conditions of the vertices of G in $\mathsf{T}(G, f)$ represented as CNF are asked at the beginning of the game i.e. for each $v \in V$ we know that the CNF representation of $\bigoplus_{u:(u,v) \in E} x_e$ is true (if any clause of $\mathsf{T}(G, f)$ is false Pavel queries all subformulas of $\mathsf{T}(G, f)$ except subformulas of the clauses and gets an immediate contradiction, if any of the parity conditions is false it yields an immediate contradiction with the corresponding subset of clauses). Thus, by Lemma 4 we derive the representations of parity conditions of the vertices from S_{i_0} w.r.t. Q. Since the corresponding linear system is unsatisfiable, using the second part of Lemma 22 we get a contradiction.

▷ Claim 25. The size of the described game tree is at most $m \cdot 2^{3\Delta(G)} \Delta^2(G) \mathrm{tpw}(G) 2^{\mathcal{O}\left(\sum_{i=1}^{d} t_i\right)}$.

Let us choose $t_i = (\Delta(G)\mathrm{tpw}(G))^{2/d}$ for all $i \in [d]$. Since $|S_i| \le \mathrm{tpw}(G)$, $|E(S_i)| + \left|E\left(S_i, \bigcup_{j \in \mathrm{s}(i)} S_j\right)\right| \le \Delta(G)\mathrm{tpw}(G)$. Hence, the condition $\prod_{i=1}^{d} t_i \ge \Delta(G)\mathrm{tpw}(G) \ge |E(S_i)| + \left|E\left(S_i, \bigcup_{j \in \mathrm{s}(i)} S_j\right)\right|$ holds and, thus, for all $i \in [m]$ the refinements U^i, W^i exist by Lemma 19. If we substitute choosen values in the bound from Clam 25, we get the upper bound $m \cdot 2^{\mathcal{O}\left(3\Delta(G) + d(\Delta(G)\mathrm{tpw}(G))^{2/d}\right)} = \mathrm{poly}(|\mathsf{T}(G, f)|) \cdot 2^{(\Delta(G)\mathrm{tpw}(G))^{\mathcal{O}(1/d)}}$. ◀

Now we are ready to prove Theorem 18.

Proof of Theorem 18. Theorem 24 and Theorem 23 imply that there exists a constant c and a derivation $\vdash_{3d+\mathcal{O}(1)} \neg \mathsf{T}(G, f)$ of size at most $\mathrm{poly}(|\mathsf{T}(G, f)|) 2^{(10\Delta^2(G)\mathrm{tw}(G))^{c/d}}$. If $\mathrm{tw}(G) > \Delta(G)$ then we can rewrite our upper bound on the size as $\mathrm{poly}(|\mathsf{T}(G, f)|) 2^{(10\mathrm{tw}(G))^{3c/d}}$. If $\mathrm{tw}(G) > 1$ then it is $\mathrm{poly}(|\mathsf{T}(G, f)|) 2^{(\mathrm{tw}(G))^{\mathcal{O}(1/d)}}$. If $\mathrm{tw}(G) = 1$ then it is simply $\mathrm{poly}(|\mathsf{T}(G, f)|)$. Otherwise if $\mathrm{tw}(G) \le \Delta(G)$ we can rewrite the upper bound as $\mathrm{poly}(|\mathsf{T}(G, f)|) \cdot 2^{(10\Delta(G))^{3c/d}} = \mathrm{poly}(|\mathsf{T}(G, f)|)$ if $3c/d \le 1$. Thus, for $d \ge 3c$ the upper bound is $\mathrm{poly}(|\mathsf{T}(G, f)|) \cdot 2^{\mathrm{tw}(G)^{3c/d}}$. Therefore, for the both cases we have the needed upper bound. ◀

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