# Energy Mean-Payoff Games


**Véronique Bruyère**
Université de Mons (UMONS), Belgium
veronique.bruyere@umons.ac.be

**Quentin Hautem**[1]
Université de Mons (UMONS), Belgium
q.hautem@gmail.com

**Mickael Randour**[2]
Université de Mons (F.R.S.-FNRS & UMONS), Belgium
mickael.randour@gmail.com

**Jean-François Raskin**
Université libre de Bruxelles (ULB), Belgium
jraskin@ulb.ac.be



## Abstract

In this paper, we study one-player and two-player energy mean-payoff games. Energy mean-payoff games are games of infinite duration played on a finite graph with edges labeled by 2-dimensional weight vectors. The objective of the first player (the protagonist) is to satisfy an energy objective on the first dimension and a mean-payoff objective on the second dimension. We show that optimal strategies for the first player may require infinite memory while optimal strategies for the second player (the antagonist) do not require memory. In the one-player case (where only the first player has choices), the problem of deciding who is the winner can be solved in polynomial time while for the two-player case we show co-NP membership and we give effective constructions for the infinite-memory optimal strategies of the protagonist.



**2012 ACM Subject Classification** Software and its engineering → Formal methods; Theory of computation → Logic and verification; Theory of computation → Solution concepts in game theory

**Keywords and phrases** two-player zero-sum games played on graphs, energy and mean-payoff objectives, complexity study and construction of optimal strategies

**Digital Object Identifier** 10.4230/LIPIcs.CONCUR.2019.21

**Related Version** A full version of the paper is available at [14], `https://arxiv.org/pdf/1907.01359.pdf`.

**Funding** Work partially supported by the PDR project *Subgame perfection in graph games* (F.R.S.-FNRS), the grant n°F.4520.18 *ManySynth* (F.R.S.-FNRS), the ARC project *Non-Zero Sum Game Graphs: Applications to Reactive Synthesis and Beyond* (Fédération Wallonie-Bruxelles), the EOS project *Verifying Learning Artificial Intelligence Systems* (F.R.S.-FNRS & FWO), and the COST Action 16228 *GAMENET* (European Cooperation in Science and Technology).


## 1 Introduction

Graph games with $\omega$-regular objectives are a canonical mathematical model to formalize and solve the reactive synthesis problem [33]. Extensions of graph games with quantitative objectives have been considered more recently as a model where, not only the correctness,

[1] supported by a FRIA fellowship
[2] F.R.S.-FNRS Research Associate



30th International Conference on Concurrency Theory (CONCUR 2019).
Editors: Wan Fokkink and Rob van Glabbeek; Article No. 21; pp. 21:1–21:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

but also the quality of solutions for the reactive synthesis problem can be formalized and optimized. A large effort has been invested in studying games with various kinds of objectives, see e.g. [5, 12, 15, 19, 21, 22, 25, 35, 36], see also Chapter 27 of [3] and the survey [13].

Two particularly important classes of objectives are *mean-payoff* and *energy* objectives. In a mean-payoff game, the edges of the game graph are labeled with integer weights that model payoffs received by the first player (the protagonist) and paid by the second player (the antagonist) when the edge is taken. The game is played for infinitely many rounds, and the protagonist aims at maximizing the mean value of edges traversed during the game while the antagonist tries to minimize this mean value. Mean-payoff games have been studied in [25] where it is shown that memoryless optimal strategies exist for both players. As a corollary of this result, mean-payoff games can be decided in NP $\cap$ co-NP. While pseudo-polynomial time algorithms for solving mean-payoff games have been developed in [12, 36] as well as the recent pseudo-quasi-polynomial time algorithm in [24], it is a long standing open question whether or not those games can be solved in polynomial time. Energy games were defined more recently in [16]. In an energy game, edges are also labeled with integer weights that represent gains or losses of energy. In such a game, the protagonist tries to build an infinite path for which the total sum of energy in all the prefixes is bounded from below, while the antagonist has the opposite goal. Energy games can also be decided in NP $\cap$ co-NP and it is known that they are *inter-reducible* with mean-payoff games [5].

*Energy mean-payoff* games that combine an energy and a mean-payoff objectives have not been yet studied. This is the main goal of this paper. It is a challenging problem for several reasons. First, multi-dimensional *homogeneous* extensions of mean-payoff and energy games have been studied in a series of recent contributions [21, 29, 34, 35], and those works show that when going from one dimension to several, the close relationship between mean-payoff games and energy games is lost and specific new techniques need to be designed for solving those extensions. Second, *pushdown mean-payoff games* have been studied in [22] and shown to be undecidable. Decision problems for energy mean-payoff games can be reduced to decision problems of pushdown mean-payoff games, even to the subclass of pushdown mean-payoff games with a one-letter stack alphabet. Unfortunately, pushdown mean-payoff games are undecidable in general and to the best of our knowledge the one-letter stack alphabet case has not been studied.

**Main contributions.** In this paper, we prove that energy mean-payoff games are decidable. More precisely, their decision problems lie in co-NP (Theorem 7) for both cases of strict and non-strict inequality in the threshold constraint for the mean-payoff objective. To obtain this result, we first study *one-player* energy mean-payoff games and characterize precisely the game graphs in which $\mathcal{P}_1$ (the protagonist) can build an infinite path that satisfies the energy mean-payoff objective (Theorem 5 and Theorem 6). This characterization leads to polynomial time algorithms to solve the decision problems in the one-player case (Theorem 3). Then we show that in *two-player* energy mean-payoff games memoryless optimal strategies always exist for $\mathcal{P}_2$ (the antagonist) who aims at spoiling the energy mean-payoff objective of $\mathcal{P}_1$ (Proposition 8). Combined with the polynomial time algorithms for the one-player case, this result leads to co-NP membership of the decision problems. While the memoryless result for $\mathcal{P}_2$ allows us to understand how this player should play in energy mean-payoff games, it does not prescribe how $\mathcal{P}_1$ should play from winning vertices. To show how to effectively construct optimal strategies for $\mathcal{P}_1$, we consider a reduction to *4-dimensional energy games* in case of strict inequality for mean-payoff objective (Proposition 12). With the result of [29], this implies the existence of finite-memory strategies for $\mathcal{P}_1$ to play optimally and of a pseudo-polynomial time algorithm to solve those instances. For non-strict inequalities, this reduction cannot be applied as, even for the one-player case, infinite-memory strategies are sometimes necessary to play optimally. In this case, we show how we can combine an infinite number of finite-memory strategies, that are played in sequence, in order to play optimally (Proposition 13).

**Related work.** As already mentioned, multi-dimensional conjunctive extensions of mean-payoff games and multi-dimensional conjunctive extensions of energy games have been considered [18, 21, 35]. Deciding the existence of a winning strategy for $\mathcal{P}_1$ in those games is co-NP-complete. Games with any Boolean combination of mean-payoff objectives have been shown undecidable in [34]. Games with mean-payoff objectives and $\omega$-regular constraints have been studied in [20], while games with energy objectives and $\omega$-regular constraints have been studied in [17], and their multi-dimensional extensions in [2, 21, 23].

In [29], the authors have studied multi-dimensional energy games for the fixed initial credit and provided a pseudo-polynomial time algorithm to solve them when the number of dimensions is fixed. Energy games with bounds on the energy level have been studied in [26, 28]. Games with the combination of an energy objective and an average-energy objective are investigated in [6, 7]. This seemingly related class of games is actually quite different from the energy mean-payoff games studied in this paper: e.g., they are EXPSPACE-hard whereas our games are in co-NP. Infinite-state energy games are investigated in [1] where energy objectives are studied on infinite game structures, induced by one-counter automata or pushdown automata. Some work on other models dealing with energy have been studied, as battery edge systems [4] and consumption games [8]. In the latter games, minimization of running costs have also been investigated [10]. Optimizing the expected mean-payoff in energy MDP's have been studied in [11]. In [32], Kucera presents an overview of results related to games and counter automata, which are close to energy constraints.

We now discuss *mean-payoff pushdown games* [22] in more details. In those games, a stack is associated with a finite game structure, and players move from vertex to vertex while applying operations on the stack. Those operations are *push* a letter, *pop* a letter or *skip* and can be respectively represented with weights 1, $-1$ and 0. The authors show that one-player pushdown games can be solved in polynomial time, thanks to the existence of *pumpable paths*. Moreover, already in this case, $\mathcal{P}_1$ needs infinite memory to win in mean-payoff pushdown games. In the two-player setting, determining the winner is undecidable. Doing a straight reduction of one-player energy mean-payoff games to one-player mean-payoff pushdown games would lead to a pseudo-polynomial solution, whereas we show here that we can solve the former games in polynomial time. In addition, we cannot use the concept of pumpable paths to obtain those results as the construction of [22] is inherent to the behavior of the stack of mean-payoff pushdown games. Indeed, after one step, the height of the stack can only change of one unity $(+1, -1, 0)$, whereas in energy mean-payoff games, the energy level can vary from $-W$ to $+W$, for an arbitrarily large integer $W \in \mathbb{N}$.

**Structure of the paper.** In Sect. 2, we introduce the necessary notations and preliminaries to this work. In Sect. 3, we study the one-player energy mean-payoff games. In Sect. 4, we study the two-player energy mean-payoff games.

## 2 Preliminaries

In this section, we introduce energy mean-payoff games and the related decision problems studied in this paper.

**Games structures.** A *game structure* is a weighted directed graph $G = (V, V_1, V_2, E, w)$ such that $V_1, V_2$ form a partition of the finite set $V$, $V_i$ is the set of vertices controlled by player $\mathcal{P}_i$ , $i \in \{1, 2\}$, $E \subseteq V \times V$ is the set of edges such that for all $v \in V$, there exists $v' \in V$ such that $(v, v') \in E$, and $w = (w_1, w_2) : E \to \mathbb{Z}^2$ is a weight function that assigns a pair of weights $w(e) = (w_1(e), w_2(e))$ to each edge $e \in E$. In the whole paper, we denote by $|V|$ the number of vertices of $V$, by $|E|$ the number of edges of $E$, and by $||E|| \in \mathbb{N}_0$ the largest absolute value used by the weight function $w$. We say that a game structure is a *player-i game structure* when player $\mathcal{P}_i$ controls all the vertices, that is, $V_i = V$.

A *play* in $G$ from an *initial vertex* $v_0$ is an infinite sequence $\rho = \rho_0\rho_1 \ldots \rho_k \ldots$ of vertices such that $\rho_0 = v_0$ and $(\rho_k, \rho_{k+1}) \in E$ for all $k \geq 0$. A *factor* of $\rho$, denoted by $\rho[k, \ell]$, is the finite sequence $\rho_k\rho_{k+1} \ldots \rho_\ell$. When $k = 0$, we say that $\rho[0, \ell]$ is the *prefix* of length $\ell$ of $\rho$. The *suffix* $\rho_k\rho_{k+1} \ldots$ of $\rho$ is denoted by $\rho[k, \infty]$. The set of plays in $G$ is denoted by $\mathsf{Plays}(G)$ or simply $\mathsf{Plays}$. A path or a cycle is *simple* if there are no two occurrences of the same vertex (except for the first and last vertices in the cycle). A *multicycle* $\mathcal{C}$ is a multiset of simple cycles (that may or may not be connected to each other). We extend the weight function $w$ to paths (resp. cycles, multicycles) as the sum $w(\pi) = (w_1(\pi), w_2(\pi))$ of the weights of their edges. In particular, for a multicycle $\mathcal{C}$, we have $w(\mathcal{C}) = \sum_{\pi \in \mathcal{C}} w(\pi)$.

Given a path $\pi = \pi_0\pi_1 \cdots \pi_n$, we consider its *cycle decomposition* into a *multiset of simple cycles* as follows. We push successively vertices $\pi_0, \pi_1, \ldots$ onto a stack. Whenever we push a vertex $\pi_\ell$ equal to a vertex $\pi_k$ already in the stack, i.e. a simple cycle $C = \pi_k \cdots \pi_\ell$ is formed, we remove this cycle from the stack except $\pi_k$ (we remove all the vertices until reaching $\pi_k$ that we let in the stack) and add $C$ to the cycle decomposition multiset of $\pi$. The cycle decomposition of a play $\rho = \rho_0\rho_1 \ldots$ is defined similarly.

For each dimension $j \in \{1, 2\}$, the weight or *energy level* of the prefix $\rho[0, k]$ of a play $\rho$ is $w_j(\rho[0, k])$, and the *mean-payoff-inf* (resp. *mean-payoff-sup*) of $\rho$ is $\underline{\mathsf{MP}}_j(\rho) = \liminf_{k\to\infty} \frac{1}{k} \cdot w_j(\rho[0, k])$ (resp. $\overline{\mathsf{MP}}_j(\rho) = \limsup_{k\to\infty} \frac{1}{k} \cdot w_j(\rho[0, k])$). The following properties hold for both mean-payoff values. First, they are prefix-independent, that is, $\underline{\mathsf{MP}}_j(\pi\rho) = \underline{\mathsf{MP}}_j(\rho)$ and $\overline{\mathsf{MP}}_j(\pi\rho) = \overline{\mathsf{MP}}_j(\rho)$ for all finite paths $\pi$. Second for a play $\rho = \rho_0 \ldots \rho_{k-1}(\rho_k \ldots \rho_l)^\omega$ that is eventually periodic, its mean-payoff-inf and mean-payoff-sup values coincide and are both equal to the average weight of the cycle $\rho_k \ldots \rho_l\rho_k$, that is, $\frac{1}{l-k+1} \cdot w_j(\rho_k \ldots \rho_l\rho_k)$.

**Strategies.** Given a game structure $G$, a *strategy* $\sigma_i$ for player $\mathcal{P}_i$ is a function $V^* \cdot V_i \to V$ that assigns to each path $\pi v$ ending in a vertex $v \in V_i$ a vertex $v'$ such that $(v, v') \in E$. Such a strategy $\sigma_i$ is *memoryless* if it only depends on the last vertex of the path, i.e. $\sigma_i(\pi v) = \sigma_i(\pi' v)$ for all $\pi v, \pi' v \in V^* \cdot V_i$. It is a *finite-memory* strategy if it can be encoded by a deterministic *Moore machine* $\mathcal{M} = (M, m_0, \alpha_U, \alpha_N)$ where $M$ is a finite set of states (the memory of the strategy), $m_0 \in M$ is an initial memory state, $\alpha_U : M \times V \to M$ is an update function, and $\alpha_N : M \times V_i \to V$ is a next-move function. Such a machine defines a strategy $\sigma_i$ such that $\sigma_i(\pi v) = \alpha_N(\widehat{\alpha}_U(m_0, \pi), v)$ for all paths $\pi v \in V^* \cdot V_i$, where $\widehat{\alpha}_U$ extends $\alpha_U$ to paths as expected. The *memory size* of $\sigma_i$ is then the size $|M|$ of $\mathcal{M}$. In particular $\sigma_i$ is memoryless when it has memory size one.

Given a strategy $\sigma_i$ for $\mathcal{P}_i$, a play $\rho$ is *consistent* with $\sigma_i$ if for all its prefixes $\rho[0, k] \in V^* \cdot V_i$, we have $\rho_{k+1} = \sigma_i(\rho[0, k])$. A finite path $\pi$ consistent with $\sigma_i$ is defined similarly. Given a finite-memory strategy $\sigma_i$ and its Moore machine $\mathcal{M}$, we denote by $G(\sigma_i)$ the game structure obtained as the product of $G$ with $\mathcal{M}$. Notice that the set of plays from an initial vertex $v_0$ that are consistent with $\sigma_i$ is then exactly the set of plays in $G(\sigma_i)$ starting from $(v_0, m_0)$ where $m_0$ is the initial memory state of $\mathcal{M}$.

**Objectives.** Given a game structure $G$ and an initial vertex $v_0$, an *objective* for player $\mathcal{P}_1$ is a set of plays $\Omega \subseteq \mathsf{Plays}(G)$. Given a strategy $\sigma_1$ for $\mathcal{P}_1$, we say that $\sigma_1$ is *winning for* $\mathcal{P}_1$ *from* $v_0$ if all plays $\rho \in \mathsf{Plays}(G)$ from $v_0$ that are consistent with $\sigma_1$ satisfy $\rho \in \Omega$. Given a strategy $\sigma_2$ for $\mathcal{P}_2$, we say that $\sigma_2$ is *winning for* $\mathcal{P}_2$ *from* $v_0$ if all plays $\rho \in \mathsf{Plays}(G)$ from $v_0$ that are consistent with $\sigma_2$ satisfy $\rho \notin \Omega$.

We here consider the following objectives for dimension $j \in \{1, 2\}$:

- *Energy objective.* Given an *initial credit* $c_0 \in \mathbb{N}$, the objective $\mathsf{Energy}_j(c_0) = \{\rho \in \mathsf{Plays}(G) \mid \forall k \geq 0, c_0 + w_j(\rho[0, k]) \geq 0\}$ requires that the energy level remains always nonnegative in dimension $j$.
- *Mean-payoff-inf objective.* The objective $\underline{\mathsf{MP}}_j(\sim 0) = \{\rho \in \mathsf{Plays}(G) \mid \underline{\mathsf{MP}}_j(\rho) \sim 0\}$ with $\sim \in \{>, \geq\}$ requires that the mean-payoff-inf value is $\sim 0$ in dimension $j$.
- *Mean-payoff-sup objective.* The objective $\overline{\mathsf{MP}}_j(\sim 0) = \{\rho \in \mathsf{Plays}(G) \mid \overline{\mathsf{MP}}_j(\rho) \sim 0\}$ with $\sim \in \{>, \geq\}$ requires that the mean-payoff-sup value is $\sim 0$ in dimension $j$.

Notice that it is not a restriction to work with threshold 0 in mean-payoff-inf/sup objectives. Indeed arbitrary thresholds $\frac{a}{b} \in \mathbb{Q}$ can be reduced to threshold 0 by replacing the weight function $w$ of $G$ by the function $b \cdot w - a$.

**Decision problems.** In this paper we consider the following four variants of a decision problem implying an energy objective on the first dimension and a mean-payoff objective on the second dimension. Let $\sim \in \{>, \geq\}$:

- The *energy mean-payoff decision problem* $\mathsf{E} \cap \underline{\mathsf{MP}}^{\sim 0}$ asks, given a game structure $G$ and an initial vertex $v_0$, to decide whether there exist an initial credit $c_0 \in \mathbb{N}$ and a winning strategy $\sigma_1$ for player $\mathcal{P}_1$ from $v_0$ for the objective $\Omega = \mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(\sim 0)$.
- The *energy mean-payoff decision problem* $\mathsf{E} \cap \overline{\mathsf{MP}}^{\sim 0}$ asks, given a game structure $G$ and an initial vertex $v_0$, to decide whether there exist an initial credit $c_0 \in \mathbb{N}$ and a winning strategy $\sigma_1$ for player $\mathcal{P}_1$ from $v_0$ for the objective $\Omega = \mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(\sim 0)$.

In this context, we also use the terminology of *energy mean-payoff objectives* or *energy mean-payoff games*. Let us give two illustrating examples.

▶ **Example 1.** Consider the player-1 game structure $G$ depicted in Figure 1. Consider the cycle $C = v_0 v_0 v_0 v_1 v_1 v_1 v_0$ that loops twice on $v_0$, goes to $v_1$, loops twice on $v_1$, and comes back to $v_0$. Observe that $w(C) = (w_1(C), w_2(C)) = (0, 2)$. Hence $\mathcal{P}_1$ has a winning strategy, that consists in looping forever in this cycle $C$, for all four variants of the energy mean-payoff decision problem.

▶ **Example 2.** Consider now the player-1 game structure $G$ depicted in Figure 2. It differs from the previous game structure only by the weight $(-1, 1)$ (instead of $(-1, 3)$) of the edge $(v_1, v_1)$. We claim that $\mathcal{P}_1$ has no winning strategy for any of the two decision problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{>0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{>0}$. We will explain why in Section 3. However $\mathcal{P}_1$ has a winning strategy for both problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{\geq 0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{\geq 0}$ with initial credit $c_0 = 0$. Such a strategy consists in repeatedly executing the following round, with $Z = 1$ initially, and $Z$ incremented by 1 after each round: loop $Z$ times on $v_0$, go to $v_1$, loop $Z$ times on $v_1$, and come back to $v_0$. Such a strategy with infinite memory is clearly winning for the energy objective. It is also winning for the mean-payoff-inf objective because with $Z$ increased by 1 at each round, the cost of moving from $v_0$ to $v_1$ and from $v_1$ to $v_0$ becomes negligible.

No finite-memory strategy is winning in this case. Indeed assume the contrary: there exists a winning strategy that induces a cycle $C$ in which it loops forever. This cycle necessarily uses both simple cycles $C_0 = (v_0, v_0)$ and $C_1 = (v_1, v_1)$ as the strategy is winning. As these cycles are not connected, $C$ has to also use the simple cycle $C_3 = (v_0, v_1, v_0)$. As $w(C_1) = -w(C_2)$ and $w(C_3) = (0, -2)$, it is easy to see that it is impossible that this strategy satisfies both energy and mean-payoff-inf/sup objectives simultaneously.

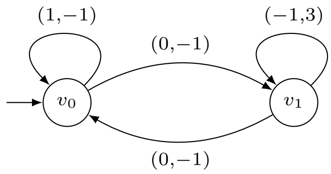


**Figure 1** Energy mean-payoff game where $\mathcal{P}_1$ wins with finite memory for problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{>0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{>0}$.

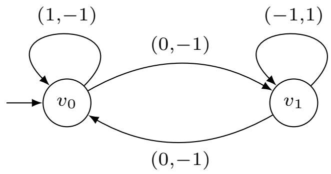


**Figure 2** Energy mean-payoff game where $\mathcal{P}_1$ needs infinite memory to win for problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{\geq 0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{\geq 0}$.

## 3 One-player setting

Within this section, we investigate player-1 game structures, that is, game structures where player $\mathcal{P}_1$ is the only one to play. In this context, $\mathcal{P}_1$ has a winning strategy for the energy mean-payoff objective for some initial credit $c_0$ if and only if there exists a play belonging to this objective. For player-1 game structures, we show that the energy mean-payoff decision problem can be solved in polynomial time for all of its four variants. However depending on the used relation $\sim \in \{>, \geq\}$ for the mean-payoff objective, memory requirements for winning strategies of $\mathcal{P}_1$ differ. We already know that $\mathcal{P}_1$ needs infinite memory in case of non-strict inequalities by Example 2. In case of strict inequalities, we show that finite-memory strategies are always sufficient for $\mathcal{P}_1$, as in Example 1. All these results will be useful in Section 4 when we will investigate the general case of two-player energy mean-payoff games.

▶ **Theorem 3.** *The energy mean-payoff decision problem for player-1 game structures can be solved in polynomial time. Moreover,*

- *for both problems* $\mathsf{E} \cap \underline{\mathsf{MP}}^{>0}$ *and* $\mathsf{E} \cap \overline{\mathsf{MP}}^{>0}$*, pseudo-polynomial-memory strategies are sufficient and necessary for* $\mathcal{P}_1$ *to win;*
- *for both problems* $\mathsf{E} \cap \underline{\mathsf{MP}}^{\geq 0}$ *and* $\mathsf{E} \cap \overline{\mathsf{MP}}^{\geq 0}$*, in general,* $\mathcal{P}_1$ *needs infinite memory to win.*

To prove Theorem 3, we characterize the existence of a winning strategy for $\mathcal{P}_1$ for some initial credit $c_0$ by the existence of a particular cycle or multicycle, that we call *good*.

▶ **Definition 4.** *Let $G$ be a game structure and $v_0$ be an initial vertex.*

- *We say that a cycle $C$ is a* good cycle *if $w_1(C) \geq 0$ and $w_2(C) > 0$. A good cycle $C$ is* reachable *if it is reachable from $v_0$.*
- *We say that a multicycle $\mathcal{C}$ is a* good multicycle *if $w_1(\mathcal{C}) \geq 0$ and $w_2(\mathcal{C}) \geq 0$. A good multicyle $\mathcal{C}$ is* reachable *if all its simple cycles are in the same connected component reachable from $v_0$.*

There exists a simple characterization of the existence of a winning strategy for $\mathcal{P}_1$ for either the objective $\mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(> 0)$ or the objective $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(> 0)$ for some initial credit $c_0$: both are equivalent to the existence of a reachable good cycle.

▶ **Theorem 5.** *Let $G$ be a player-1 game structure and $v_0$ be an initial vertex. The following assertions are equivalent.*

1. *There exist an initial credit $c_0$ and a winning strategy for $\mathcal{P}_1$ from $v_0$ for the objective* $\mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(> 0)$.
2. *There exist an initial credit $c_0$ and a winning strategy for $\mathcal{P}_1$ from $v_0$ for the objective* $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(> 0)$.
3. *There exists a reachable good cycle.*

In case of non-strict inequalities, there exists also a simple characterization: $\mathcal{P}_1$ can win for either the objective $\mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(\geq 0)$ or the objective $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(\geq 0)$ for some initial credit $c_0$ if and only if there exists a reachable good multicycle.

▶ **Theorem 6.** *Let $G$ be a player-1 game structure and $v_0$ be an initial vertex. The following assertions are equivalent.*

1. *There exist an initial credit $c_0$ and a winning strategy for $\mathcal{P}_1$ from $v_0$ for the objective* $\mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(\geq 0)$.
2. *There exist an initial credit $c_0$ and a winning strategy for $\mathcal{P}_1$ from $v_0$ for the objective* $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(\geq 0)$.
3. *There exists a reachable good multicycle.*

A similar characterization appears for multi-mean-payoff games and multi-energy games studied in [35]: when the objective is an intersection of several mean-payoff-inf objectives (resp. several energy objectives), and when he plays alone, $\mathcal{P}_1$ has a winning strategy if and only if there exists a reachable non negative multicycle (resp. a reachable non negative cycle) in the game structure. Nevertheless, the proofs of those results differ substantially from the proofs of our results.

Due to the lack of space, we only give the main ideas of the proofs (see [14] for more details):

1. We begin by illustrating the statements of Theorems 5 and 6 with the two previous examples. Let us first come back to the game structure of Figure 1. The cycle $C$ mentioned in Example 1 is a reachable good cycle since $w(C) = (0, 2)$. By Theorem 5, it follows that $\mathcal{P}_1$ is winning for the energy mean-payoff decision problem with strict inequalities (and thus also with non-strict inequalities), as already observed in Example 1. Let us now come back to the game structure of Figure 2. Recall from Example 2 that there exists a winning strategy for $\mathcal{P}_1$ in case of non-strict inequalities, but no winning strategy in case of strict inequalities. By Theorem 6, there should exist a reachable good multicycle. Indeed, consider the multicycle $\mathcal{C} = \{C, C'\}$ with $C = (v_0, v_0)$ and $C' = (v_1, v_1)$: we have $w(\mathcal{C}) = w(C) + w(C') = (1, -1) + (-1, 1) = (0, 0)$. Moreover by Theorem 5, there is no reachable good cycle in this game.
2. By Theorems 5 and 6, solving the energy mean-payoff decision problem reduces to decide whether there exists a reachable good cycle (resp. multicycle). This can be tested in polynomial time thanks to a variant of a result in [31] that states that deciding the existence of a cycle (resp. multicycle) of weight $(0, 0)$ can be done in polynomial time. This established the polynomial complexity stated in Theorem 3.
3. Theorem 5 is proved as follows. For Implication (3) $\Rightarrow$ (1), a winning strategy for $\mathcal{P}_1$ consists in reaching the good cycle and looping in it forever. Implication (1) $\Rightarrow$ (2) is trivial since $\overline{\mathsf{MP}}_2(\rho) \geq \underline{\mathsf{MP}}_2(\rho)$ for all plays $\rho$. However, the proof of Implication (2) $\Rightarrow$ (3) is rather technical, it is thus detailed in [14].
4. The proof of Theorem 6 requires a more precise characterization by good cycles and multicycles. We show that the existence of a good cycle is equivalent to the existence of

- either one good cycle that is simple,
- or two simple cycles $C$, $C'$ with respective weight vectors $w(C) = (-x, y)$ and $w(C') = (x', -y')$ that satisfy $x, x', y \in \mathbb{N}_0$ and $y' \in \mathbb{N}$ and make an angle $< 180^o$ (see Figure 3).

We show a similar equivalence for the existence of a good multicycle with the existence of

- either a good multicycle $\mathcal{C} = \{C\}$ composed of a unique simple cycle $C$,
- or two simple cycles $C$, $C'$ as before with the difference that $w(C), w(C')$ make an angle $\leq 180^o$ (instead of $< 180^o$).

The proof of these results is based on the cycle decomposition of paths and on geometrical arguments on the weights of those simple cycles.
Let us illustrate this characterization with our two running examples. In case of Figure 1, the good cycle is characterized by the two cycles $C = (v_1, v_1)$, $C' = (v_0, v_0)$ with respective weights $(-1, 3)$, $(1, -1)$. In case of Figure 2, the good multicycle is characterized by the two cycles $C = (v_1, v_1)$, $C' = (v_0, v_0)$ with respective weights $(-1, 1)$, $(1, -1)$. Moreover one can check that there is no good cycle (the conditions given in the characterization do not hold in Figure 2).

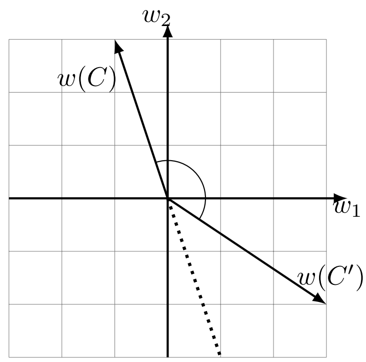


**Figure 3** Geometrical view of the characterization for good cycles.

**5.** With the characterization given previously in Item 4., in case of strict inequalities, pseudo-polynomial-memory strategies are sufficient for $\mathcal{P}_1$ to win, as stated in Theorem 3. Indeed when there exist two simple cycles $C$, $C'$ with weight vectors $w(C)$, $w(C')$ as in Figure 3, one can construct a good cycle as follows. There always exists a linear combination of vectors $w(C)$, $w(C')$, with pseudo-polynomial positive coefficients, that is $> (0, 0)$ and that balances the cost of moving from $C$ to $C'$ and from $C'$ to $C$ (this is however not possible when those vectors make an angle of $180^o$). The fact that in case of strict inequalities, pseudo-polynomial-memory strategies are necessary for $\mathcal{P}_1$ to win is proved in [14]. Notice that in case of non-strict inequalities, Theorem 3 states that $\mathcal{P}_1$ needs infinite memory to win, that we already know from Example 2.

**6.** Theorem 6 is proved as follows. Implication (1) $\Rightarrow$ (2) is immediate. For Implication (3) $\Rightarrow$ (1), with the two simple cycles $C, C'$ of the characterization given in Item 4., one can construct a winning strategy with infinite memory for $\mathcal{P}_1$ that is similar to the strategy of Example 2. To prove (2) $\Rightarrow$ (3), suppose that $\mathcal{P}_1$ is winning for the objective $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(\geq 0)$ for some $c_0$. Then he is also winning for $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(> -\epsilon)$ for all $\epsilon > 0$. We consider the game structure $G_\epsilon$ obtained from $G$ by replacing function $w_2$ by function $w_2 + \epsilon$. Hence $\mathcal{P}_1$ is now winning in this game $G_\epsilon$ for the objective $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(> 0)$, and by Theorem 5 there exists a reachable good cycle in $G_\epsilon$. Therefore, with the characterization of Item 4., for each $\epsilon$, there exist either one good simple cycle $C_\epsilon$, or two simple cycles $C_\epsilon, C'_\epsilon$ with weight vectors as in Figure 3. The main part of the proof is to extract from these sequences of cycles a reachable good multicycle thanks to the characterization of Item 4. (Notice that when $\epsilon$ converges to 0, the angle $> 180^o$ made by the vectors of Figure 3 converges to an angle $\geq 180^o$.)

## 4 Two-player setting

In this section we consider two-player energy mean-payoff games. We show that the four variants of the energy mean-payoff decision problem are in co-NP. To establish this, we show that if the answer to this problem is No, then $\mathcal{P}_2$ has a spoiling memoryless strategy $\sigma_2$ that he can use for all initial credits $c_0 \in \mathbb{N}$. In the game structure $G(\sigma_2)$, $\mathcal{P}_1$ is then the only player and we can apply the results of the previous section, in particular Theorem 3. We also show that in case of mean-payoff objectives with strict inequality, the energy mean-payoff decision problem can be reduced to the unknown initial credit problem for 4-dimensional energy games. If follows by [29] that our decision problem can be solved in pseudo-polynomial time and that finite-memory winning strategies with pseudo-polynomial size for $\mathcal{P}_1$ exist and can effectively be constructed. In case of mean-payoff objectives with non-strict inequality, we already know that infinite memory is necessary for $\mathcal{P}_1$ in player-1 energy mean-payoff games by Theorem 3. We show how to construct such strategies. The results that we establish in this section are summarized in the following theorem.

▶ **Theorem 7.** *The energy mean-payoff decision problem for two-player game structures is in co-NP. Moreover,*
- *both problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{>0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{>0}$ can be solved in pseudo-polynomial time and exponential-memory strategies are sufficient for $\mathcal{P}_1$ to win;*
- *for both problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{\geq 0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{\geq 0}$, in general, $\mathcal{P}_1$ needs infinite memory to win.*

*In all cases, winning strategies can be effectively constructed for both players.*

The proof of this result is detailed in the following sections.

### 4.1 Memoryless winning strategies for $\mathcal{P}_2$

For all four variants of mean-payoff energy objective, we here establish that $\mathcal{P}_2$ does not need any memory for his winning strategies. Therefore, thanks to Theorem 3, the energy mean-payoff decision problem can be solved in co-NP.

▶ **Proposition 8.** *Let $\sim \in \{>, \geq\}$. For all energy mean-payoff games $G$ and all initial vertices $v_0$, if the answer to the energy mean-payoff problem $\mathsf{E} \cap \underline{\mathsf{MP}}^{\sim 0}$ (resp. $\mathsf{E} \cap \overline{\mathsf{MP}}^{\sim 0}$) is No, then there exists a memoryless strategy $\sigma_2$ for $\mathcal{P}_2$ such that for all initial credits $c_0 \in \mathbb{N}$, no play $\rho$ consistent with $\sigma_2$ from $v_0$ belongs to $\Omega = \mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(\sim 0)$ (resp. to $\Omega = \mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(\sim 0)$).*

The proof of this proposition is given in [14]. Note that energy objectives are not prefix-independent objectives and as a consequence this proposition does not directly follow from the results of [30] where are given general conditions that guarantee the existence of a memoryless winning strategy for one of the players. However our proof is an adaptation of the proof technique of [9, 21, 27, 30].

Notice that from Theorems 5-6 and Proposition 8, we directly get the following corollary.

▶ **Corollary 9.** *For all energy mean-payoff games $G$ and initial vertices $v_0$, let $\sim \in \{>, \geq\}$. Then $\mathcal{P}_1$ is winning from $v_0$ for $\mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(\sim 0)$ for some initial credit $c_0$ if and only if he is winning from $v_0$ for $\mathsf{Energy}_1(c_0) \cap \overline{\mathsf{MP}}_2(\sim 0)$ for some initial credit $c_0$.*

While Proposition 8 allows us to obtain the membership in co-NP of the decision problems and to effectively construct winning memoryless strategies for $\mathcal{P}_2$, unfortunately it does not tell us how $\mathcal{P}_1$ must play from a winning vertex (when spoiling strategies do not exist for $\mathcal{P}_2$). In the following two sections we provide results that show how $\mathcal{P}_1$ needs to play in order to win energy mean-payoff games. We first show that $\mathcal{P}_1$ can win with finite memory for the case of strict inequalities, and then we provide infinite-memory winning strategies for the case of non-strict inequalities. For the later case, we already know that infinite memory is necessary even player-1 game structures (see Theorem 3).

## 4.2 Strategies for $\mathcal{P}_1$: case of strict inequalities

In case of strict inequalities, our solution is based on a reduction to multi-dimensional energy games [18] for which we know how to construct strategies for $\mathcal{P}_1$.

### 4.2.1 Multi-dimensional energy games

We need to recall the concept of *$d$-dimensional energy games*, with $d \in \mathbb{N}_0$. Those games are played on $d$-dimensional game structure $G = (V, V_1, V_2, E, w)$ where the weight function $w : E \to \mathbb{Z}^d$ assigns a $d$-tuple (instead of a pair) of weights $w(e)$ to each edge $e \in E$. The *unknown initial credit problem* asks, given a $d$-dimensional game structure and an initial vertex $v_0$, to decide whether there exists an initial credit $c_0 = (c_{0,1}, \ldots, c_{0,d}) \in \mathbb{N}^d$ and a winning strategy for $\mathcal{P}_1$ for the objective $\Omega = \cap_{j=1}^{d} \mathsf{Energy}_j(c_{0,j})$. When $d = 1$ and the answer to this problem is Yes, we denote by $c(v_0) \in \mathbb{N}$ the minimum initial credit for which $\mathcal{P}_1$ has a winning strategy from $v_0$. The complexity of this problem has been first studied in [18, 21, 35] and then in [29] for a fixed number of dimensions.

▶ **Theorem 10** ([18, 21, 29, 35]). *The unknown initial credit problem for $d$-dimensional energy games can be solved in pseudo-polynomial time, that is in time $(|V| \cdot ||E||)^{\mathcal{O}(d^4)}$. If the answer to this problem is*

- Yes, *then exponential-memory strategies are sufficient and necessary for player $\mathcal{P}_1$ to win,*
- No, *then $\mathcal{P}_2$ has a spoiling memoryless strategy $\sigma_2$ that he can use for all initial credits $c_0 \in \mathbb{N}^d$.*

We recall the next useful lemma.

▶ **Lemma 11** ([17]). *Let $G$ be a 1-dimensional energy game and $v_0$ be an initial vertex. For all plays $\rho$ consistent with a winning strategy $\sigma_1$ for $\mathcal{P}_1$, if the initial credit is $c(v_0) + \Delta$ for $\Delta \geq 0$, then the energy level at all positions of $\rho$ where a state $v$ occurs is at least $c(v) + \Delta$.*

The next proposition shows that we can reduce energy mean-payoff games with strict inequality constraints to energy games with 4 dimensions.

▶ **Proposition 12.** *The problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{>0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{>0}$ for energy mean-payoff games are both polynomially reducible to the unknown initial credit problem for 4-dimensional energy games. Moreover, for the energy game $G'$ constructed from the given $G$, we have $||E'|| = ||E||$ and $|V'|, |E'|$ are linear in $|V|, |E|$, and from a finite-memory winning strategy $\sigma_1'$ of $\mathcal{P}_1$ in $G'$, we can derive a finite-memory winning strategy $\sigma_1$ of $\mathcal{P}_1$ in $G$ such that the memory size of $\sigma_1$ is upper bounded by the memory size of $\sigma_1'$.*

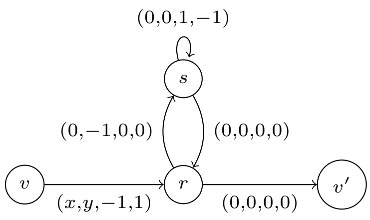


**Figure 4** Construction of a 4-dimensional energy game.

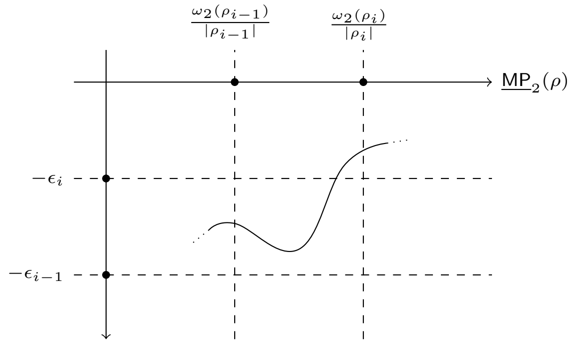


**Figure 5** $\rho$ satisfies $\underline{\mathsf{MP}}_2(\rho) \geq 0$.

**Proof.** We first explain the reduction and then we give the main intuitions that justify the correctness (see [14] for the formal detailed proof). Given an energy mean-payoff game structure $G = (V, V_1, V_2, E, w)$ with $w : E \to \mathbb{Z}^2$, we construct a 4-dimensional energy game $G' = (V', V_1', V_2', E', w')$ with $w' : E' \to \mathbb{Z}^4$ as follows. Each edge $e = (v, v') \in E$ labeled by $w(e) = (x, y)$ is replaced by the following gadget composed of:

- five edges $(v, r), (r, s), (s, s), (s, r)$, and $(r, v')$ where $r, s$ are two new vertices,
- such that $w'(v, r) = (x, y, -1, 1)$, $w'(r, s) = (0, -1, 0, 0)$, $w'(s, s) = (0, 0, 1, -1)$, $w'(s, r) = (0, 0, 0, 0)$, and $w'(r, v') = (0, 0, 0, 0)$.

This is illustrated in Figure 4. The set $V_2'$ is equal to $V_2$, and $V_1'$ is composed of all vertices of $V_1$ and the $2 \cdot |E|$ new vertices (two for each edge of $G$). By construction, we have $||E'|| = ||E||$ and $|V'|, |E'|$ are linear in $|V|, |E|$. Now, we show that if $\mathcal{P}_i$ wins in $G'$ then $\mathcal{P}_i$ wins in $G$, for $i \in \{1, 2\}$.

First, assume that $\mathcal{P}_1$ wins in $G'$ with a strategy $\sigma_1'$. By Theorem 10, we can assume that $\sigma_1'$ is a finite-memory strategy. Let us construct a corresponding strategy $\sigma_1$ in $G$. The graph $G'$ is a structural copy of $G$ where each edge is replaced by the gadget of Figure 4. So to each path $\pi'$ of $G'$ that ends in a vertex of $G$ (so not in a $s$ or $r$ vertex), there is a corresponding path $\pi$ in $G$ obtained by removing all the new vertices introduced by the gadget. We construct the strategy $\sigma_1$ as follows: the edge taken by $\sigma_i$ after history $\pi$ is simply the edge that $\sigma_i'$ enters in $G'$ after history $\pi'$. Let us show that $\sigma_1$ is winning in $G$. Remember that all dimensions in $G'$ are interpreted as energy dimensions. The first dimension which models energy in $G$ is unchanged by the gadget as all weights are 0 for the first dimension in the newly introduced edges. The second dimension, which corresponds to the mean-payoff dimension in $G$, is transformed into an energy dimension. We make a few remarks. If $\mathcal{P}_1$ decides to go from $v$ to $r$ and then directly to $v'$, the effect on the energy accumulated on the second dimension is the same as in $G$. Nevertheless because the third dimension is affected negatively by all edges with the exception of the self loops on vertices of type $s$, it is clear that $\mathcal{P}_1$ needs to take periodically the edges from $r$-vertices to $s$-vertices and loop in $s$ in order to recharge the energy on the third dimension. So, the intuition behind our construction is simple: in $G'$, $\mathcal{P}_1$ can play as in $G$ but he needs to recharge periodically dimension three by looping on $s$. Also, let us note that $\mathcal{P}_1$ always needs to leave the gadget composed of the $s$ and $r$ vertices as otherwise the fourth dimension would go arbitrary low and so this would violate the corresponding energy objectives. Finally, we note that second dimension is decreased when $\mathcal{P}_1$ takes the edge from $r$ to $s$. So, in order to satisfy the energy objective in $G'$ for dimension two, $\mathcal{P}_1$ needs to accumulate unbounded reward on that dimension in the other edges (and so in the corresponding edges in $G$). As by hypothesis the strategy $\sigma'_1$ is finite-memory, this implies that mean-payoff accumulated on dimension two will be strictly positive when playing the corresponding strategy $\sigma_1$ in $G$. This in turn implies that $\sigma_1$ is winning in $G$.

Assume now that $\sigma'_2$ is a winning strategy for $\mathcal{P}_2$ in $G'$. It can be supposed to be memoryless by Theorem 10. As $V_2 = V'_2$, we can interprete $\sigma'_2$ in $G$ thus leading to a player-1 game structure $G(\sigma'_2)$. Similar arguments as done before together with Theorem 5 show that $\sigma'_2$ is winning for $\mathcal{P}_2$ in $G$. ◀

## 4.3 Strategies for $\mathcal{P}_1$: case of non-strict inequalities

By Theorem 6, we know that infinite memory may be necessary for $\mathcal{P}_1$ to win in case of non-strict inequalities. The reduction to multi-dimensional energy games of previous section is thus not applicable for this case. Instead, we show how we can effectively construct a winning strategy for $\mathcal{P}_1$ by combining an infinite number of finite-memory strategies.

▶ **Proposition 13.** *For both problems $\mathsf{E} \cap \underline{\mathsf{MP}}^{\geq 0}$ and $\mathsf{E} \cap \overline{\mathsf{MP}}^{\geq 0}$, if $\mathcal{P}_1$ is winning from an initial vertex $v_0$, then one can effectively construct a strategy for him to win from $v_0$. This strategy requires infinite memory.*

**Proof.** Remember by Corollary 9 that $\mathcal{P}_1$ is winning from $v_0$ for the objective $\mathsf{Energy}(c_0) \cap \underline{\mathsf{MP}}(\geq 0)$ for some $c_0$ if and only if he is winning from $v_0$ for the objective $\mathsf{Energy}(c_0) \cap \overline{\mathsf{MP}}(\geq 0)$ for some $c_0$. Here, we show how to construct a winning strategy for $\mathcal{P}_1$ for the mean-payoff-inf case only. Indeed such a winning strategy is also winning for the mean-payoff-sup case.

We first note that if $\mathcal{P}_1$ is winning from a vertex $v$ for the objective $\Omega(c_0) = \mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(\geq 0)$, then he is also winning from $v$ for the objective $\Omega_i(c_0) = \mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(> -\epsilon_i)$ for all $\epsilon_i = \frac{1}{2^i}$, $i \in \mathbb{N}_0$. Let $\mathsf{Win}$ be the set of vertices $v$ from which $\mathcal{P}_1$ is winning for $\Omega(c_0)$ for some $c_0$. In particular $v_0 \in \mathsf{Win}$ by hypothesis. From now on, we assume that the vertices not in $\mathsf{Win}$ are removed from $V$ leading to a game structure that we still denote by $G$. This can be done as a winning strategy for $\mathcal{P}_1$ will never enter those vertices.

For all vertices $v \in \mathsf{Win}$, we denote by $c(v) \in \mathbb{N}$ the minimum initial credit from which $\mathcal{P}_1$ is winning for $\Omega(c(v))$ from $v$. Similarly for all $i \in \mathbb{N}_0$, we denote by $c_i(v) \in \mathbb{N}$ the minimum initial credit from which he is winning for $\Omega_i(c_i(v))$ from $v$ and by $\sigma_i^v$ such a winning strategy for $\mathcal{P}_1$. Recall by Proposition 12 that all strategies $\sigma_i^v$ can be supposed to be finite-memory and to have memory size bounded by $M_i^v$. The game structure $G(\sigma_i^v)$ induced by $\sigma_i^v$ has a number of vertices equal to

$$N_i^v = |\mathsf{Win}| \cdot M_i^v \tag{1}$$

Also, we have that $c_1(v) \leq c_2(v) \leq c_3(v) \leq \ldots \leq c(v)$. Moreover as these initial credits are integers,

$$\exists k_v, \forall i \geq k_v : \quad c_i(v) = c_{k_v}(v). \tag{2}$$

Let us define

$$\kappa = \max_{v \in \mathsf{Win}} k_v \text{ and } \gamma = \max\{c_{i+1}(v) - c_i(v) \mid v \in \mathsf{Win}, i \in \mathbb{N}_0\}. \tag{3}$$

These constants will be useful later for the energy objective.

**An effective winning strategy for $\mathcal{P}_1$.** Let us define a strategy $\tau_1$ for $\mathcal{P}_1$ from $v_0$ that will be proved to be winning for $\mathcal{P}_1$. A play $\rho$ consistent with $\tau_i$ is the limit of a sequence of prefixes $\rho_i$ of increasing length constructed in the following way:

1. Initialize $i = 1$ and $\rho_0 = v_0$;
2. Assume that a prefix $\rho_{i-1}$ has been constructed so far and that its last vertex is $v_{i-1}$. Apply, starting from $v_{i-1}$, the strategy $\sigma_i^{v_{i-1}}$ (against $\mathcal{P}_2$) until the produced path $\pi_i$ consistent with $\sigma_i^{v_{i-1}}$ and the path $\rho_i$ equal to the concatenation $\rho_{i-1}$ with $\pi_i$ satisfy
$$w_2(\rho_i) > N_{i+1}^{v_i} \cdot ||E|| - |\rho_i| \cdot \epsilon_i. \tag{4}$$
3. Increment $i$ by 1 and goto 2.

Notice that in (4), we require for $w_2(\rho_i)$ more than $w_2(\rho_i) > -|\rho_i| \cdot \epsilon_i$. Indeed the latter inequality would be enough to guarantee that the mean-payoff-sup value of $\rho$ satisfies $\overline{\mathsf{MP}}(\rho) \geq 0$ but we will explain later that we need (4) to guarantee $\underline{\mathsf{MP}}(\rho) \geq 0$.

For the correctness of the given construction, we need to prove that for each $i \in \mathbb{N}_0$, there exists a path $\rho_i$ satisfying (4). This is a consequence of point $(ii)$ of the next lemma.

▶ **Lemma 14.** *As each $\sigma_i^v$ is a finite-memory strategy from $v$ winning for $\mathsf{Energy}_1(c_0) \cap \underline{\mathsf{MP}}_2(> -\epsilon_i)$,*

**(i)** *for all plays $\pi$ consistent with $\sigma_i^v$ from $v$, for all $k \in \mathbb{N}$, we have $w_2(\pi[0,k]) > -N_i^v \cdot ||E|| - k \cdot \epsilon_i$, and*

**(ii)** *for all $K \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that for all plays $\pi$ consistent with $\sigma_i^v$ from $v$, we have $w_2(\pi[0,k]) > K - k \cdot \epsilon_i$.*

**Proof.** Let us come back to the game structure $G(\sigma_i^v)$ with $N_i^v$ vertices (by (1)). As $\sigma_i^v$ is winning for the objective $\underline{\mathsf{MP}}_2(> -\epsilon_i)$, all reachable cycles $C$ in $G(\sigma_i^v)$ have a average weight

$$\frac{w_2(C)}{|C|} > -\epsilon_i. \tag{5}$$

Moreover as the weight $w_2(C)$ is an integer, $w_2(C) \geq -|C| \cdot \epsilon_i + t_C$, for some $t_C > 0$. Let $t = \min\{t_C \mid C \text{ reachable cycle in } G(\sigma_i^v)\}$. This tells us that one unit $t > 0$ of weight is accumulated each time a cycle is closed in $G(\sigma_i^v)$:

$$w_2(C) \geq -|C| \cdot \epsilon_i + t. \tag{6}$$

Let us prove $(i)$. Consider a play $\pi$ consistent with $\sigma_i^v$ from $v$, i.e., an infinite path in $G(\sigma_i^v)$. Let $k \in \mathbb{N}$ and let us reason on the cycle decomposition of $\pi[0,k]$. First, as the acyclic part of this decomposition has a length bounded by $N_i^v$, its weight is bounded below by $-N_i^v \cdot ||E||$. Second, let $\ell$ be the total length of the cycles $C$ of the cyclic decomposition of $\pi[0,k]$. As all cycles $C$ in $G(\sigma_i^v)$ satisfy (5), we conclude that the total weight of this cyclic part of $\pi[0,k]$ is bounded below by $-\ell \cdot \epsilon_i$. Finally, as $\ell \leq k$, we obtain the claimed lower bound of $(i)$, that is, $w_2(\pi[0,k]) > -N_i^v \cdot ||E|| - k \cdot \epsilon_i$.

Let us now prove $(ii)$. We simply repeat the arguments given for $(i)$ by using (6) instead of (5). If $\alpha$ cycles are closed during the cycle decomposition of $\pi[0,k]$, we then get $w_2(\pi[0,k]) \geq \alpha \cdot t - N_i^v \cdot ||E|| - k \cdot \epsilon_i$ instead of the inequality of $(i)$. So, given $K \in \mathbb{N}$, take $k \in \mathbb{N}$ such that $\alpha$ is large enough to get an accumulated positive weight $\alpha \cdot t$ such that $\alpha \cdot t - N_i^v \cdot ||E|| > K$. This establishes $(ii)$. ◀

Let us prove that $\tau_1$ is a winning strategy (with infinite memory) from $v_0$ for the objective $\Omega(d_0)$ with the initial credit

$$d_0 = \kappa \cdot \gamma + c_1(v_0) \tag{7}$$

with the constants of (3). Let $\rho$ be a play consistent with $\tau_1$ from $v_0$, i.e., $\rho$ is the limit of a sequence of prefixes $\rho_i$ as described previously in the definition of $\tau_i$. Remember that each $\rho_i$, $i \in \mathbb{N}_0$, is the concatenation of $\rho_{i-1}$ and $\pi_i$ such that $\pi_i$ is consistent with $\sigma_i^{v_{i-1}}$ from $v_{i-1}$.

**Mean-payoff-inf objective.** We begin by showing that $\rho$ satisfies $\underline{\mathsf{MP}}_2(\rho) \geq 0$. To achieve this goal, it is enough to show that for all $i \in \mathbb{N}_0$, the average weight never falls below $-\epsilon_{i-1}$ during the construction of $\rho_i$ (i.e. the construction of $\pi_i$), and this average weight is above $-\epsilon_i$ at the end of the construction of $\rho_i$ (see Figure 5).

Let us show that such a property is a consequence of Lemma 14 and inequality (4) satisfied by $\rho_i$. First by (4), the average weight of $\rho_i$ satisfies $\frac{w_2(\rho_i)}{|\rho_i|} > -\epsilon_i$. Second, consider any prefix $\pi[0,k]$ of $\pi_i$ and the corresponding prefix $\rho[0,k']$ of $\rho_i$ such that $k' = k + |\rho_{i-1}|$. Then by point $(i)$ of Lemma 14, we have $w_2(\pi[0,k]) > -N_i^v \cdot ||E|| - k \cdot \epsilon_i$, and by (4) applied to $\rho_{i-1}$, we have $w_2(\rho_{i-1}) > N_i^{v_i} \cdot ||E|| - |\rho_{i-1}| \cdot \epsilon_{i-1}$. Therefore we get

$$\begin{aligned} w_2(\rho[0,k']) &= w_2(\rho_{i-1}) + w_2(\pi[0,k]) \\ &> (N_i^{v_i} \cdot ||E|| - |\rho_{i-1}| \cdot \epsilon_{i-1}) + (-N_i^v \cdot ||E|| - k \cdot \epsilon_i) \\ &> -|\rho[0,k']| \cdot \epsilon_{i-1} \end{aligned}$$

Hence, as announced, the average weight of the prefix $\rho[0,k']$ of $\rho_i$ is above $-\epsilon_{i-1}$.

**Energy objective.** It remains to explain why the energy objective is also satisfied by $\rho$ with the initial credit $d_0$ defined in (7). Recall from the definition of $\tau_1$ that $\rho$ is the limit of a sequence of prefixes $\rho_i$ such that each $\rho_i$ is the concatenation of $\rho_{i-1}$ and $\pi_i$. Recall also that $c_i(v) \in \mathbb{N}$ is the minimum initial credit for which $\sigma_i^v$ is winning from $v$.

By construction, $\pi_1$ is consistent with $\sigma_1^{v_0}$ with the initial credit $d_0 = c_1(v_0) + \Delta_1$, where $\Delta_1 = \kappa \cdot \gamma$. Hence the energy level of $\rho_1 = \pi_1$ never drops below zero and it is at least equal to $c_1(v_1) + \Delta_1$ in the last vertex $v_1$ of $\rho_1$ by Lemma 11. Similarly $\pi_2$ is consistent with $\sigma_2^{v_1}$ with the initial credit $c_1(v_1) + \Delta_1 = c_2(v_1) + \Delta_2$, where $\Delta_2 = \kappa \cdot \gamma - (c_2(v_1) - c_1(v_1))$. Hence the energy level of $\rho_2$ never drops below zero and it is at least equal to $c_2(v_2) + \Delta_2$ in the last vertex $v_2$ of $\rho_2$ by Lemma 11. This argument can be repeated for all $i \in \mathbb{N}_0$: the energy level of $\rho_i$ never drops below zero and it is at least equal to $c_i(v_i) + \Delta_i$, with $\Delta_i = \kappa \cdot \gamma - \sum_{j=1}^{i-1}(c_{j+1}(v_j) - c_j(v_j))$. Notice that we always have $\Delta_i \geq 0$ by (2) and by definition of $\kappa$ and $\gamma$ (see (3)). Therefore the energy level of $\rho$ never drops belows zero.

This proves that $\tau_1$ is a winning strategy for the objective $\mathsf{Energy}_1(d_0) \cap \underline{\mathsf{MP}}_2(\geq 0)$ and thus conclude the proof. ◀

## 4.4 Proof of Theorem 7

We conclude this section with the proof of Theorem 7.

**Proof of Theorem 7.** We establish the three assertions of the theorem as follows.

We first prove that the energy mean-payoff decision problems for two-player games $G$ are in co-NP for the four variants. This result is obtained as follows. By Proposition 8, memoryless strategies are sufficient for $\mathcal{P}_2$ to win, for all four variants. Hence, the following is an algorithm in co-NP: guess a memoryless strategy $\sigma_2$ for $\mathcal{P}_2$, and in the resulting one-player game $G(\sigma_2)$, verify in polynomial time whether $\mathcal{P}_1$ is winning thanks to Theorem 3.

Second, we consider the two variants with strict inequalities. By Proposition 12, there exists a polynomial reduction of the energy mean-payoff decision problem to the unknown initial credit problem for 4-dimensional energy games. By Theorem 10, it follows that the energy mean-payoff decision problem can be solved in pseudo-polynomial time and that exponential-memory strategies are sufficient for $\mathcal{P}_1$ to win.

Finally, we consider the last two variants with non-strict inequalities. In Proposition 13, we have shown how we can effectively construct a winning strategy for $\mathcal{P}_1$ in this case. ◀

## References

1 Parosh Aziz Abdulla, Mohamed Faouzi Atig, Piotr Hofman, Richard Mayr, K. Narayan Kumar, and Patrick Totzke. Infinite-state energy games. In Thomas A. Henzinger and Dale Miller, editors, *Joint Meeting of the Twenty-Third EACSL Annual Conference on Computer Science Logic (CSL) and the Twenty-Ninth Annual ACM/IEEE Symposium on Logic in Computer Science (LICS), CSL-LICS '14, Vienna, Austria, July 14 - 18, 2014*, pages 7:1–7:10. ACM, 2014. doi:10.1145/2603088.2603100.

2 Parosh Aziz Abdulla, Richard Mayr, Arnaud Sangnier, and Jeremy Sproston. Solving Parity Games on Integer Vectors. In Pedro R. D'Argenio and Hernán C. Melgratti, editors, *CONCUR 2013 - Concurrency Theory - 24th International Conference, CONCUR 2013, Buenos Aires, Argentina, August 27-30, 2013, Proceedings*, volume 8052 of *Lecture Notes in Computer Science*, pages 106–120. Springer, 2013. doi:10.1007/978-3-642-40184-8_9.

3 Roderick Bloem, Krishnendu Chatterjee, and Barbara Jobstmann. Graph Games and Reactive Synthesis. In Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, editors, *Handbook of Model Checking*, pages 921–962. Springer, 2018.

4 Udi Boker, Thomas A. Henzinger, and Arjun Radhakrishna. Battery transition systems. In Suresh Jagannathan and Peter Sewell, editors, *The 41st Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '14, San Diego, CA, USA, January 20-21, 2014*, pages 595–606. ACM, 2014. doi:10.1145/2535838.2535875.

5 Patricia Bouyer, Ulrich Fahrenberg, Kim Guldstrand Larsen, Nicolas Markey, and Jirí Srba. Infinite Runs in Weighted Timed Automata with Energy Constraints. In Franck Cassez and Claude Jard, editors, *Formal Modeling and Analysis of Timed Systems, 6th International Conference, FORMATS 2008, Saint Malo, France, September 15-17, 2008, Proceedings*, volume 5215 of *Lecture Notes in Computer Science*, pages 33–47. Springer, 2008. doi:10.1007/978-3-540-85778-5_4.

6 Patricia Bouyer, Piotr Hofman, Nicolas Markey, Mickael Randour, and Martin Zimmermann. Bounding Average-Energy Games. In Javier Esparza and Andrzej S. Murawski, editors, *Foundations of Software Science and Computation Structures - 20th International Conference, FOSSACS 2017, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2017, Uppsala, Sweden, April 22-29, 2017, Proceedings*, volume 10203 of *Lecture Notes in Computer Science*, pages 179–195, 2017. doi:10.1007/978-3-662-54458-7_11.

7 Patricia Bouyer, Nicolas Markey, Mickael Randour, Kim G. Larsen, and Simon Laursen. Average-energy games. *Acta Inf.*, 55(2):91–127, 2018. doi:10.1007/s00236-016-0274-1.

8 Tomás Brázdil, Krishnendu Chatterjee, Antonín Kucera, and Petr Novotný. Efficient Controller Synthesis for Consumption Games with Multiple Resource Types. In P. Madhusudan and Sanjit A. Seshia, editors, *Computer Aided Verification - 24th International Conference, CAV 2012, Berkeley, CA, USA, July 7-13, 2012, Proceedings*, volume 7358 of *Lecture Notes in Computer Science*, pages 23–38. Springer, 2012. doi:10.1007/978-3-642-31424-7_8.

9 Tomás Brázdil, Petr Jancar, and Antonín Kucera. Reachability Games on Extended Vector Addition Systems with States. *CoRR*, abs/1002.2557, 2010. arXiv:1002.2557.

10 Tomás Brázdil, David Klaska, Antonín Kucera, and Petr Novotný. Minimizing Running Costs in Consumption Systems. In Armin Biere and Roderick Bloem, editors, *Computer Aided Verification - 26th International Conference, CAV 2014, Held as Part of the Vienna Summer of Logic, VSL 2014, Vienna, Austria, July 18-22, 2014, Proceedings*, volume 8559 of *Lecture Notes in Computer Science*, pages 457–472. Springer, 2014. doi:10.1007/978-3-319-08867-9_30.

11 Tomás Brázdil, Antonín Kucera, and Petr Novotný. Optimizing the Expected Mean Payoff in Energy Markov Decision Processes. In Cyrille Artho, Axel Legay, and Doron Peled, editors, *Automated Technology for Verification and Analysis - 14th International Symposium, ATVA 2016, Chiba, Japan, October 17-20, 2016, Proceedings*, volume 9938 of *Lecture Notes in Computer Science*, pages 32–49, 2016. doi:10.1007/978-3-319-46520-3_3.

12 Lubos Brim, Jakub Chaloupka, Laurent Doyen, Raffaella Gentilini, and Jean-François Raskin. Faster algorithms for mean-payoff games. *Formal Methods in System Design*, 38(2):97–118, 2011. doi:10.1007/s10703-010-0105-x.

13 Véronique Bruyère. Computer Aided Synthesis: A Game-Theoretic Approach. In Émilie Charlier, Julien Leroy, and Michel Rigo, editors, *Developments in Language Theory - 21st International Conference, DLT 2017, Liège, Belgium, August 7-11, 2017, Proceedings*, volume 10396 of *Lecture Notes in Computer Science*, pages 3–35. Springer, 2017. doi:10.1007/978-3-319-62809-7_1.

14 Véronique Bruyère, Quentin Hautem, Mickael Randour, and Jean-François Raskin. Energy mean-payoff games. *CoRR*, abs/1907.01359, 2019. arXiv:1907.01359.

15 Véronique Bruyère, Quentin Hautem, and Jean-François Raskin. On the Complexity of Heterogeneous Multidimensional Games. In Josée Desharnais and Radha Jagadeesan, editors, *27th International Conference on Concurrency Theory, CONCUR 2016, August 23-26, 2016, Québec City, Canada*, volume 59 of *LIPIcs*, pages 11:1–11:15. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016. doi:10.4230/LIPIcs.CONCUR.2016.11.

16 Arindam Chakrabarti, Luca de Alfaro, Thomas A. Henzinger, and Mariëlle Stoelinga. Resource Interfaces. In Rajeev Alur and Insup Lee, editors, *Embedded Software, Third International Conference, EMSOFT 2003, Philadelphia, PA, USA, October 13-15, 2003, Proceedings*, volume 2855 of *Lecture Notes in Computer Science*, pages 117–133. Springer, 2003. doi:10.1007/978-3-540-45212-6_9.

17 Krishnendu Chatterjee and Laurent Doyen. Energy parity games. *Theor. Comput. Sci.*, 458:49–60, 2012. doi:10.1016/j.tcs.2012.07.038.

18 Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, and Jean-François Raskin. Generalized Mean-payoff and Energy Games. In Kamal Lodaya and Meena Mahajan, editors, *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science, FSTTCS 2010, December 15-18, 2010, Chennai, India*, volume 8 of *LIPIcs*, pages 505–516. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2010. doi:10.4230/LIPIcs.FSTTCS.2010.505.

19 Krishnendu Chatterjee, Laurent Doyen, Mickael Randour, and Jean-François Raskin. Looking at mean-payoff and total-payoff through windows. *Inf. Comput.*, 242:25–52, 2015. doi:10.1016/j.ic.2015.03.010.

20 Krishnendu Chatterjee, Thomas A. Henzinger, and Marcin Jurdzinski. Mean-Payoff Parity Games. In *20th IEEE Symposium on Logic in Computer Science (LICS 2005), 26-29 June 2005, Chicago, IL, USA, Proceedings*, pages 178–187. IEEE Computer Society, 2005. doi:10.1109/LICS.2005.26.

21 Krishnendu Chatterjee, Mickael Randour, and Jean-François Raskin. Strategy synthesis for multi-dimensional quantitative objectives. *Acta Inf.*, 51(3-4):129–163, 2014. doi:10.1007/s00236-013-0182-6.

22 Krishnendu Chatterjee and Yaron Velner. The Complexity of Mean-Payoff Pushdown Games. *J. ACM*, 64(5):34:1–34:49, 2017. doi:10.1145/3121408.

23 Thomas Colcombet, Marcin Jurdzinski, Ranko Lazic, and Sylvain Schmitz. Perfect half space games. In *LICS Proceedings*, pages 1–11. IEEE Computer Society, 2017. doi:10.1109/LICS.2017.8005105.

24 Laure Daviaud, Marcin Jurdzinski, and Ranko Lazic. A pseudo-quasi-polynomial algorithm for mean-payoff parity games. In Anuj Dawar and Erich Grädel, editors, *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*, pages 325–334. ACM, 2018. doi:10.1145/3209108.3209162.

**25** A. Ehrenfeucht and J. Mycielski. Positional strategies for mean payoff games. *International Journal of Game Theory*, 8:109–113, 1979. doi:10.1007/BF01768705.

**26** Uli Fahrenberg, Line Juhl, Kim G. Larsen, and Jirí Srba. Energy Games in Multiweighted Automata. In Antonio Cerone and Pekka Pihlajasaari, editors, *Theoretical Aspects of Computing - ICTAC 2011 - 8th International Colloquium, Johannesburg, South Africa, August 31 - September 2, 2011, Proceedings*, volume 6916 of *Lecture Notes in Computer Science*, pages 95–115. Springer, 2011. doi:10.1007/978-3-642-23283-1_9.

**27** Hugo Gimbert and Wieslaw Zielonka. Games Where You Can Play Optimally Without Any Memory. In Martín Abadi and Luca de Alfaro, editors, *CONCUR 2005 - Concurrency Theory, 16th International Conference, CONCUR 2005, San Francisco, CA, USA, August 23-26, 2005, Proceedings*, volume 3653 of *Lecture Notes in Computer Science*, pages 428–442. Springer, 2005. doi:10.1007/11539452_33.

**28** Line Juhl, Kim Guldstrand Larsen, and Jean-François Raskin. Optimal Bounds for Multiweighted and Parametrised Energy Games. In Zhiming Liu, Jim Woodcock, and Huibiao Zhu, editors, *Theories of Programming and Formal Methods - Essays Dedicated to Jifeng He on the Occasion of His 70th Birthday*, volume 8051 of *Lecture Notes in Computer Science*, pages 244–255. Springer, 2013. doi:10.1007/978-3-642-39698-4_15.

**29** Marcin Jurdzinski, Ranko Lazic, and Sylvain Schmitz. Fixed-Dimensional Energy Games are in Pseudo-Polynomial Time. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, *Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II*, volume 9135 of *Lecture Notes in Computer Science*, pages 260–272. Springer, 2015. doi:10.1007/978-3-662-47666-6_21.

**30** Eryk Kopczynski and Damian Niwinski. A simple indeterminate infinite game. In Vasco Brattka, Hannes Diener, and Dieter Spreen, editors, *Logic, Computation, Hierarchies*, volume 4 of *Ontos Mathematical Logic*, pages 205–212. De Gruyter, 2014. doi:10.1515/9781614518044.205.

**31** S. Rao Kosaraju and Gregory F. Sullivan. Detecting Cycles in Dynamic Graphs in Polynomial Time (Preliminary Version). In Janos Simon, editor, *Proceedings of the 20th Annual ACM Symposium on Theory of Computing, May 2-4, 1988, Chicago, Illinois, USA*, pages 398–406. ACM, 1988. doi:10.1145/62212.62251.

**32** Antonín Kucera. Playing Games with Counter Automata. In Alain Finkel, Jérôme Leroux, and Igor Potapov, editors, *Reachability Problems - 6th International Workshop, RP 2012, Bordeaux, France, September 17-19, 2012, Proceedings*, volume 7550 of *Lecture Notes in Computer Science*, pages 29–41. Springer, 2012. doi:10.1007/978-3-642-33512-9_4.

**33** Amir Pnueli and Roni Rosner. On the Synthesis of an Asynchronous Reactive Module. In Giorgio Ausiello, Mariangiola Dezani-Ciancaglini, and Simona Ronchi Della Rocca, editors, *Automata, Languages and Programming, 16th International Colloquium, ICALP89, Stresa, Italy, July 11-15, 1989, Proceedings*, volume 372 of *Lecture Notes in Computer Science*, pages 652–671. Springer, 1989. doi:10.1007/BFb0035790.

**34** Yaron Velner. Robust Multidimensional Mean-Payoff Games are Undecidable. In Andrew M. Pitts, editor, *Foundations of Software Science and Computation Structures - 18th International Conference, FoSSaCS 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11-18, 2015, Proceedings*, volume 9034 of *Lecture Notes in Computer Science*, pages 312–327. Springer, 2015. doi:10.1007/978-3-662-46678-0_20.

**35** Yaron Velner, Krishnendu Chatterjee, Laurent Doyen, Thomas A. Henzinger, Alexander Moshe Rabinovich, and Jean-François Raskin. The complexity of multi-mean-payoff and multi-energy games. *Inf. Comput.*, 241:177–196, 2015. doi:10.1016/j.ic.2015.03.001.

**36** U. Zwick and M. Paterson. The Complexity of Mean Payoff Games on Graphs. *Theoretical Computer Science*, 158(1-2):343–359, 1996.