Parameterized complexity of games with monotonically ordered ω-regular objectives

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Abstract

In recent years, two-player zero-sum games with multiple objectives have received a lot of interest as a model for the synthesis of complex reactive systems. In this framework, Player 1 wins if he can ensure that all objectives are satisfied against any behavior of Player 2. When this is not possible to satisfy all the objectives at once, an alternative is to use some preorder on the objectives according to which subset of objectives Player 1 wants to satisfy. For example, it is often natural to provide more significance to one objective over another, a situation that can be modelled with lexicographically ordered objectives for instance. Inspired by recent work on concurrent games with multiple ω-regular objectives by Bouyer et al., we investigate in detail turned-based games with monotonically ordered and ω-regular objectives. We study the threshold problem which asks whether player 1 can ensure a payoff greater than or equal to a given threshold w.r.t. a given monotonic preorder. As the number of objectives is usually much smaller than the size of the game graph, we provide a parametric complexity analysis and we show that our threshold problem is in FPT for all monotonic preorders and all classical types of ω-regular objectives. We also provide polynomial time algorithms for Büchi, coBüchi and explicit Muller objectives for a large subclass of monotonic preorders that includes among others the lexicographic preorder. In the particular case of lexicographic preorder, we also study the complexity of computing the values and the memory requirements of optimal strategies.

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1 Introduction

Two-player zero-sum games played on directed graphs form an adequate framework for the *synthesis of reactive systems* facing an uncontrollable environment [21]. To model properties to be enforced by the reactive system within its environment, games with Boolean objectives and games with quantitative objectives have been studied, for example games with ω-regular objectives [15] and mean-payoff games [23].

Recently, games with *multiple* objectives have received a lot of attention since in practice, a system must usually satisfy several properties. In this framework, the system wins if it can ensure that *all* objectives are satisfied no matter how the environment behaves. For instance, generalized parity games are studied in [11], multi-mean-payoff games in [22], and multidimensional games with heterogeneous ω-regular objectives in [7].

When multiple objectives are conflicting or if there does not exist a strategy that can enforce all of them at the same time, it is natural to consider trade-offs. A general framework for defining trade-offs between n (Boolean) objectives $\Omega_1, \ldots, \Omega_n$ consists in assigning to each infinite path π of the game a payoff $v \in \{0,1\}^n$ such that $v(i) = 1$ iff π satisfies Ω_i, and then to equip $\{0,1\}^n$ with a preorder $\precsim$ to define a preference between pairs of payoffs: $v \precsim v'$ whenever payoff v' is preferred to payoff v. Because the ideal situation would be to satisfy *all* the objectives together, it is natural to assume that the preorder $\precsim$ has the following *monotonicity* property: if v' is such that whenever $v(i) = 1$ then $v'(i) = 1$, then it should be the case that v' is preferred to v.

As an illustration, let us consider a game in which Player 1 strives to enforce three objectives: Ω_1, Ω_2, and Ω_3. Assume also that Player 1 has no strategy ensuring all three objectives at the same time, that is, Player 1 cannot ensure the objective $\Omega_1 \cap \Omega_2 \cap \Omega_3$. Then several options can be considered, see e.g. [6]. First, we could be interested in a strategy of Player 1 ensuring a maximal subset of the three objectives. Indeed, a strategy that enforces both Ω_1 and Ω_3 should be preferred to a strategy that enforces Ω_3 only. This preference is usually called the *subset preorder*. Now, if Ω_1 is considered more important than Ω_2 itself considered more important than Ω_3, then a strategy that ensures the most important possible objective should be considered as the most desirable. This preference is called the *maximize preorder*. Finally, we could also translate the relative importance of the different objectives into a *lexicographic preorder* on the payoffs: satisfying Ω_1 and Ω_2 would be considered as more desirable than satisfying Ω_1 and Ω_3 but not Ω_2. Those three examples are all monotonic preorders.

In this paper, we consider the following threshold problem: given a game graph G, a set of *ω-regular objectives*[3] $\Omega_1, \ldots, \Omega_n$, a monotonic preorder $\precsim$ on the set $\{0,1\}^n$ of payoffs, and a threshold μ, decide whether Player 1 has a strategy such that for all strategies of Player 2, the outcome of the game has payoff v greater than or equal to μ (for the specified preorder), i.e. $\mu \precsim v$. As the number n of objectives is typically much smaller than the size of the game graph G, it is natural to consider a parametric analysis of the complexity of the threshold problem in which the number of objectives and their size are considered to be fixed parameters of the problem. Our main results are as follows.

[3] We cover all classical ω-regular objectives: reachability, safety, Büchi, co-Büchi, parity, Rabin, Streett, explicit Muller, or Muller.

Contributions

First, we provide *fixed parameter tractable solutions* to the threshold problem for *all* monotonic preorders and for *all* classical types of ω-regular objectives. Our solutions rely on the following ingredients:

1. We show that solving the threshold problem is equivalent to *solve a game with a single objective* Ω that is a union of intersections of objectives taken among $\Omega_1, \dots, \Omega_n$ (Theorem 3). This is possible by *embedding* the monotonic preorder $\precsim$ in the subset preorder and by translating the threshold μ in preorder $\precsim$ into an antichain of thresholds in the subset preorder. A threshold in the subset preorder is naturally associated with a conjunction of objectives, and an antichain of thresholds leads to a union of such conjunctions.
2. We provide a fixed parameter tractable algorithm to solve games with a single objective Ω as described previously for all types of ω-regular objectives $\Omega_1, \dots, \Omega_n$, leading to a *fixed parameter algorithm for the threshold problem* (Theorem 4). Those results build on the recent breakthrough of Calude et al. that provides a quasipolynomial time algorithm for parity games as well as their fixed parameter tractability [9], and on the fixed parameter tractability of games with an objective defined by a *Boolean combination of Büchi objectives* (Proposition 5).

Second, we consider games with a preorder $\precsim$ having a *compact embedding*, with the main condition that the antichain of thresholds resulting from the embedding in the subset preorder is of *polynomial size*. The maximize preorder, the subset preorder, and the lexicographic preorder, given as examples above, all possess this property. For games with a compact embedding, we go *beyond fixed parameter tractability* as we are able to provide deterministic polynomial time solutions for Büchi, coBüchi, and explicit Muller objectives (Theorem 6). Polynomial time solutions are not possible for the other types of ω-regular objectives as we show that the threshold problem for the *lexicographic preorder* with reachability, safety, parity, Rabin, Streett, and Muller objectives cannot be solved in polynomial time unless P = PSPACE (Theorem 7). Finally, we present a *full picture* of the study of the lexicographic preorder for each studied objective. We give the exact complexity class of the threshold problem, show that we can obtain the values from the threshold problem (which thus yields a polynomial algorithm for Büchi, co-Büchi and Explicit Muller objectives, and an FPT algorithm for the other objectives) and provide tight memory requirements for the optimal and winning strategies (Table 2).

Related Work

In [6], Bouyer et al. investigate concurrent games with multiple objectives leading to payoffs in $\{0,1\}^n$ which are ordered using Boolean circuits. While their threshold problem is slightly more general than ours, their games being concurrent and their preorders being not necessarily monotonic, the algorithms that they provide are nondeterministic and guess witnesses whose size depends polynomially not only in the number of objectives but also in the size of the game graph. Their algorithms are sufficient to establish membership to PSPACE for all classical types of ω-regular objectives but they do not provide a basis for the parametric complexity analysis of the threshold problem. In stark contrast, we provide deterministic algorithms whose complexity only depends polynomially in the size of the game graph. Our new deterministic algorithms are thus instrumental to a finer complexity analysis that leads to fixed parameter tractability for all monotonic preorders and all ω-regular objectives. We also provide tighter lower-bounds for the important special case of lexicographic preorder, in particular for parity objectives.

The particular class of games with multiple Büchi objectives ordered with the maximize preorder has been considered in [2]. The interested reader will find in that paper clear practical motivations for considering multiple objectives and ordering them. The lexicographic ordering of objectives has also been considered in the context of quantitative games: lexicographic mean-payoff games in [5], some special cases of lexicographic quantitative games in [8, 16], and lexicographically ordered energy objectives in [12].

In [1] and [19], the authors investigate partially (or totally) ordered specifications expressed in LTL. None of their complexity results leads to the results of this paper since the complexity is de facto much higher with objectives expressed in LTL. Moreover no FPT result is provided in those references.

Structure of the paper

In Section 2, we present all the useful notions about games with monotonically ordered ω-regular objectives. In Section 3, we show that solving the threshold problem is equivalent to solve a game with a single objective that is a union of intersections of objectives (Theorem 3), and we establish the main result of this paper: the fixed parameter complexity of the threshold problem (Theorem 4). Section 4 is devoted to games with a compact embedding and in particular to the threshold problem for lexicographic games. The last section is dedicated to the study of computing the values and memory requirements of optimal strategies in the case of lexicographic games (Table 2). Full paper is available on arXiv.[4]

2 Monotonically ordered ω-regular games

We consider zero-sum turn-based games played by two players, $\mathcal{P}_1$ and $\mathcal{P}_2$, on a finite directed graph. Given *several objectives*, we associate with each play of this game a vector of bits called *payoff*, the components of which indicate the objectives that are satisfied. The set of all payoffs being equipped with a *preorder*, $\mathcal{P}_1$ wants to ensure a payoff greater than or equal to a given threshold against any behavior of $\mathcal{P}_2$. In this section we give all the useful notions and the studied problem.

Preorders

Given some non-empty set P, a *preorder* over P is a binary relation $\precsim \subseteq P \times P$ that is reflexive and transitive. The *equivalence relation* $\sim$ associated with $\precsim$ is defined such that $x \sim y$ if and only if $x \precsim y$ and $y \precsim x$. The *strict partial order* $\prec$ associated with $\precsim$ is then defined such that $x \prec y$ if and only if $x \precsim y$ and $x \not\sim y$. A preorder $\precsim$ is *total* if $x \precsim y$ or $y \precsim x$ for all $x, y \in P$. A set $S \subseteq P$ is *upper-closed* if for all $x \in S$, $y \in P$, if $x \precsim y$, then $y \in S$. An *antichain* is a set $S \subseteq P$ of pairwise incomparable elements, that is, for all $x, y \in S$, if $x \neq y$, then $x \not\precsim y$ and $y \not\precsim x$.

Game structures and strategies

A *game structure* is a tuple $G = (V_1, V_2, E)$ where (V, E) is a finite directed graph, with $V = V_1 \cup V_2$ the set of vertices and $E \subseteq V \times V$ the set of edges such that for each $v \in V$, there exists $(v, v') \in E$ for some $v' \in V$ (no deadlock), and (V_1, V_2) forms a partition of V such that V_i is the set of vertices controlled by player $\mathcal{P}_i$ with $i \in \{1, 2\}$.

[4] See `https://arxiv.org/abs/1707.05968`.

A *play* of G is an infinite sequence of vertices $\pi = v_0 v_1 \ldots \in V^\omega$ such that $(v_k, v_{k+1}) \in E$ for all $k \in \mathbb{N}$. We denote by $\mathsf{Plays}(G)$ the set of plays in G. *Histories* of G are finite sequences $\rho = v_0 \ldots v_k \in V^+$ defined in the same way. Given a play $\pi = v_0 v_1 \ldots$, the set $\mathsf{Occ}(\pi)$ denotes the set of vertices that occur in π, and the set $\mathsf{Inf}(\pi)$ denotes the set of vertices visited infinitely often along π, i.e., $\mathsf{Occ}(\pi) = \{v \in V \mid \exists k \geq 0, v_k = v\}$ and $\mathsf{Inf}(\pi) = \{v \in V \mid \forall k \geq 0, \exists l \geq k,\ v_l = v\}$. Given a set $\Omega \subseteq V^\omega$, we denote by $\overline{\Omega}$ the set $V^\omega \setminus \Omega$.

A *strategy* σ_i for $\mathcal{P}_i$ is a function $\sigma_i \colon V^* V_i \to V$ assigning to each history $\rho v \in V^* V_i$ a vertex $v' = \sigma_i(\rho v)$ such that $(v, v') \in E$. It is *memoryless* if $\sigma_i(\rho v) = \sigma_i(\rho' v)$ for all histories $\rho v, \rho' v$ ending with the same vertex v, that is, if σ_i is a function $\sigma_i \colon V_i \to V$. It is *finite-memory* if $\sigma_i(\rho v)$ only needs finite memory of the history ρv (recorded by a Moore machine). The *size* of σ_i is the size of its Moore machine.

The set of all strategies of $\mathcal{P}_i$ is denoted by Σ_i. Given a strategy σ_i of $\mathcal{P}_i$, a play $\pi = v_0 v_1 \ldots$ of G is *consistent* with σ_i if $v_{k+1} = \sigma_i(v_0 \ldots v_k)$ for all $k \in \mathbb{N}$ such that $v_k \in V_i$. Given an *initial vertex* v_0, and a strategy σ_i of each player $\mathcal{P}_i$, we have a unique play consistent with both strategies σ_1, σ_2, called *outcome* and denoted by $\mathsf{Out}(v_0, \sigma_1, \sigma_2)$.

Single objectives and ordered objectives

An *objective for* $\mathcal{P}_1$ is a set of plays $\Omega \subseteq \mathsf{Plays}(G)$. A *game* (G, Ω) is composed of a game structure G and an objective Ω. A play π is *winning* for $\mathcal{P}_1$ if $\pi \in \Omega$, and losing otherwise. As the studied games are zero-sum, $\mathcal{P}_2$ has the opposite objective $\overline{\Omega}$, meaning that a play π is winning for $\mathcal{P}_1$ if and only if it is losing for $\mathcal{P}_2$. Given a game (G, Ω) and an initial vertex v_0, a strategy σ_1 for $\mathcal{P}_1$ is *winning from* v_0 if $\mathsf{Out}(v_0, \sigma_1, \sigma_2) \in \Omega$ for all strategies σ_2 of $\mathcal{P}_2$. Vertex v_0 is thus called *winning* for $\mathcal{P}_1$. We also say that $\mathcal{P}_1$ is winning from v_0 or that he can *ensure* Ω *from* v_0. Similarly the winning vertices of $\mathcal{P}_2$ are those from which $\mathcal{P}_2$ can ensure his objective $\overline{\Omega}$.

A game (G, Ω) is *determined* if each of its vertices is either winning for $\mathcal{P}_1$ or winning for $\mathcal{P}_2$. Martin's theorem [20] states that all games with Borel objectives are determined. The problem of *solving a game* (G, Ω) means to decide, given an initial vertex v_0, whether $\mathcal{P}_1$ is winning from v_0 (or dually whether $\mathcal{P}_2$ is winning from v_0 when the game is determined).

Instead of a *single* objective Ω, one can consider *several* objectives $\Omega_1, \ldots, \Omega_n$ that are *ordered* with respect to a preorder $\precsim$ over $\{0,1\}^n$ in the following way. We first define the payoff of a play as a vector[5] of bits the components of which indicate the objectives that are satisfied. Formally, given n objectives $\Omega_1, \ldots, \Omega_n \subseteq \mathsf{Plays}(G)$, the *payoff* function $\mathsf{Payoff} \colon \mathsf{Plays}(G) \to \{0,1\}^n$ assigns a vector of bits to each play $\pi \in \mathsf{Plays}(G)$, where for all $k \in \{1, \ldots, n\}$, $\mathsf{Payoff}_k(\pi) = 1$ if $\pi \in \Omega_k$ and 0 otherwise.

Given the preorder $\precsim$ over $\{0,1\}^n$, $\mathcal{P}_1$ prefers a play π to a play π' whenever $\mathsf{Payoff}(\pi') \precsim \mathsf{Payoff}(\pi)$. We call *ordered game* the tuple $(G, \Omega_1, \ldots, \Omega_n, \precsim)$, the payoff function of which is defined w.r.t. the objectives $\Omega_1, \ldots, \Omega_n$ and its values are ordered with $\precsim$. In this context, we are interested in the following problem.

▶ **Problem 1.** The *threshold problem* for ordered games $(G, \Omega_1, \ldots, \Omega_n, \precsim)$ asks, given a threshold $\mu \in \{0,1\}^n$ and an initial vertex $v_0 \in V$, to decide whether $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) has a strategy to ensure the objective $\Omega = \{\pi \in \mathsf{Plays}(G) \mid \mathsf{Payoff}(\pi) \succsim \mu\}$ from v_0 (resp. $\overline{\Omega} = \{\pi \in \mathsf{Plays}(G) \mid \mathsf{Payoff}(\pi) \not\succsim \mu\}$).[6]

[5] Note that in the sequel, we often manipulate equivalently vectors in $\{0,1\}^n$ and sequences of n bits.

[6] Note that when $n = 1$ and $\precsim$ is the usual order $\leq$ over $\{0,1\}$, we recover the notion of single objective with the threshold $\mu = 1$.

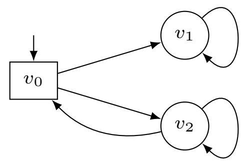

Figure 1 A simple lexicographic game.

In case $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) has such a winning strategy, we also say that he can *ensure* (resp. *avoid*) *a payoff* $\succsim \mu$.

Classical examples of preorders are the following ones [6]. Let $x, y \in \{0,1\}^n$.

- *Counting*: $x \precsim y$ if and only if $|\{j \mid x_j = 1\}| \leq |\{j \mid y_j = 1\}|$. The aim of $\mathcal{P}_1$ is to maximize the number of satisfied objectives.
- *Subset*: $x \precsim y$ if and only if $\{j \mid x_j = 1\} \subseteq \{j \mid y_j = 1\}$. The aim of $\mathcal{P}_1$ is to maximize the subset of satisfied objectives with respect to the inclusion.
- *Maximise*: $x \precsim y$ if and only if $\max\{j \mid x_j = 1\} \leq \max\{j \mid y_j = 1\}$. The aim of $\mathcal{P}_1$ is to maximize the higher index of the satisfied objectives.
- *Lexicographic*: $x \precsim y$ if and only if either $x = y$ or $\exists j \in \{1, \ldots, n\}$ such that $x_j < y_j$ and $\forall k \in \{1, \ldots, j-1\}$, $x_k = y_k$. The objectives are ranked according to their importance. The aim of $\mathcal{P}_1$ is to maximise the payoff with respect to the induced lexicographic order.

In this article, we *focus on monotonic preorders*. A preorder $\precsim$ is *monotonic* if it is compatible with the subset preorder, i.e. if $\{i \mid x_i = 1\} \subseteq \{i \mid y_i = 1\}$ implies $x \precsim y$. Hence a preorder is monotonic if satisfying more objectives never results in a lower payoff value. This is a *natural property* shared by all the examples of preorders given previously.

▶ **Example 2.** Consider the game structure G depicted on Figure 1, where circle vertices belong to $\mathcal{P}_1$ and square vertices belong to $\mathcal{P}_2$. We consider the ordered game $(G, \Omega_1, \Omega_2, \precsim)$ with $\Omega_i = \{\pi \in \mathsf{Plays}(G) \mid v_i \in \mathsf{Inf}(\pi)\}$ for $i = 1, 2$ and the lexicographic preorder $\precsim$. Therefore the function Payoff assigns value 1 to each play π on the first (resp. second) bit if and only if π visits infinitely often vertex v_1 (resp. v_2). In this ordered game, $\mathcal{P}_1$ has a strategy to ensure a payoff $\succsim 01$ from v_0. Indeed, consider the memoryless strategy σ_1 that loops in v_1 and in v_2. Then, from v_0, $\mathcal{P}_2$ decides to go either to v_1 leading to the payoff 10, or to v_2 leading to the payoff 01. As $10 \succsim 01$, this shows that any play π consistent with σ_1 satisfies $\mathsf{Payoff}(\pi) \succsim 01$. Notice that while $\mathcal{P}_1$ can ensure a payoff $\succsim 01$ from v_0, he has no strategy to enforce the single objective Ω_1 and similarly no strategy to enforce Ω_2.

Homogeneous ω-regular objectives

In this article, given a monotonically ordered game $(G, \Omega_1, \ldots, \Omega_n, \precsim)$, we want to study the threshold problem described in Problem 1 for *homogeneous ω-regular objectives*, in the sense that all the objectives $\Omega_1, \ldots, \Omega_n$ are of the same type, and taken in the following list of well-known ω-regular objectives.

Given a game structure $G = (V_1, V_2, E)$ and a subset U of V called *target set*:

- The *reachability objective* asks to visit a vertex of U at least once, i.e. $\mathsf{Reach}(U) = \{\pi \in \mathsf{Plays}(G) \mid \mathsf{Occ}(\pi) \cap U \neq \emptyset\}$.
- The *safety objective* asks to always stay in the set U, i.e. $\mathsf{Safe}(U) = \{\pi \in \mathsf{Plays}(G) \mid \mathsf{Occ}(\pi) \subseteq U\}$.

- The *Büchi objective* asks to visit infinitely often a vertex of U, i.e. $\mathsf{Buchi}(U) = \{\pi \in \mathsf{Plays}(G) \mid \mathsf{Inf}(\pi) \cap U \neq \emptyset\}$.
- The *co-Büchi objective* asks to eventually always stay in the set U, i.e. $\mathsf{CoBuchi}(U) = \{\pi \in \mathsf{Plays}(G) \mid \mathsf{Inf}(\pi) \subseteq U\}$.

Given a *family* $\mathcal{F} = (F_i)_{i=1}^k$ of sets $F_i \subseteq V$, and a family of *pairs* $((E_i, F_i)_{i=1}^k)$, with $E_i, F_i \subseteq V$:

- The *explicit Muller objective* asks that the set of vertices seen infinitely often is one among the sets of $\mathcal{F}$, i.e. $\mathsf{ExplMuller}(\mathcal{F}) = \{\pi \in \mathsf{Plays}(G) \mid \exists i \in \{1, \dots, k\}, \mathsf{Inf}(\pi) = F_i\}$.
- The *Rabin objective* asks that there exists a pair (E_i, F_i) such that a vertex of F_i is visited infinitely often while no vertex of E_i is visited infinitely often, i.e. $\mathsf{Rabin}((E_i, F_i)_{i=1}^k) = \{\pi \in \mathsf{Plays}(G) \mid \exists i \in \{1, \dots, k\}, \mathsf{Inf}(\pi) \cap E_i = \emptyset \text{ and } \mathsf{Inf}(\pi) \cap F_i \neq \emptyset\}$.
- The *Streett objective* asks that for each pair (E_i, F_i), a vertex of E_i is visited infinitely often or no vertex of F_i is visited infinitely often, i.e. $\mathsf{Streett}((E_i, F_i)_{i=1}^k) = \{\pi \in \mathsf{Plays}(G) \mid \forall i \in \{1, \dots, k\}, \mathsf{Inf}(\pi) \cap E_i \neq \emptyset \text{ or } \mathsf{Inf}(\pi) \cap F_i = \emptyset\}$.

Given a *coloring* function $p \colon V \to \{0, \dots, d\}$ that associates with each vertex a color, and $\mathcal{F} = (F_i)_{i=1}^k$ a family of subsets F_i of $p(V)$:

- The *parity objective* asks that the minimum color seen infinitely often is even, i.e. $\mathsf{Parity}(p) = \{\pi \in \mathsf{Plays}(G) \mid \min_{v \in \mathsf{Inf}(\pi)} p(v) \text{ is even}\}$.
- The *Muller objective* asks that the set of colors seen infinitely often is one among the sets of $\mathcal{F}$, i.e. $\mathsf{Muller}(p, \mathcal{F}) = \{\pi \in \mathsf{Plays}(G) \mid \exists i \in \{1, \dots, k\}, p(\mathsf{Inf}(\pi)) = F_i\}$.

In the sequel, we make the *assumption* that the considered preorders are monotonic, and by *ordered game*, we always mean monotonically ordered games. When the objectives of an ordered game are of kind X, we speak of an *ordered X game*, or of a *$\precsim$ X game* if we want to specify the used preorder $\precsim$. As already mentioned, when $n = 1$, an ordered game (with $\precsim$ equal to $\leq$) resumes to a game (G, Ω) with a single objective Ω, that is traditionally called an Ω game. For instance, an ordered game $(G, \Omega_1, \dots, \Omega_n, \precsim)$ where $\Omega_1, \dots, \Omega_n$ are reachability objectives and $\precsim$ is the lexicographic preorder is called a lexicographic reachability game, and when $n = 1$ (G, Ω_1) is called a reachability game.

Note that given an ordered game with n non-homogeneous ω-regular objectives Ω_i, we can always construct a new equivalent ordered parity game, since each objective Ω_i can be translated into a parity objective [15].

Monotonic preorders embedded in the subset preorder

We here show that solving the threshold problem for an ordered game $(G, \Omega_1, \dots, \Omega_n, \precsim)$ is equivalent to solving a game (G, Ω) with a single objective Ω equal to the union of intersections of objectives taken in $\{\Omega_1, \dots, \Omega_n\}$. The arguments are the following ones. (1) We consider the set $\{0, 1\}^n$ of payoffs ordered with $\precsim$ as well as ordered with the subset preorder $\subseteq$ (see the example of Figure 2 where $\precsim$ is the lexicographic preorder). To any payoff $\nu \in \{0, 1\}^n$, we associate the set $\delta_\nu = \{i \in \{1, \dots, n\} \mid \nu_i = 1\}$ containing all indices i such that objective Ω_i is satisfied. (2) Consider the set of payoffs $\nu \succsim \mu$ embedded in the set $\{0, 1\}^n$ ordered with $\subseteq$. By monotonicity of $\precsim$, we obtain an upper-closed set S that can be represented by the antichain of its *minimal elements* (with respect to $\subseteq$), that we denote by $\mathsf{M}(\mu)$. (3) $\mathcal{P}_1$ can ensure a payoff $\succsim \mu$ if and only if he has a strategy such that any consistent outcome π has a payoff $\nu^* \supseteq \nu$ for some $\nu \in \mathsf{M}(\mu)$, equivalently such that π satisfies (at least) the conjunction of the objectives Ω_i such that $\nu_i = 1$. (4) The objective Ω of $\mathcal{P}_1$ is thus a disjunction (over $\nu \in \mathsf{M}(\mu)$) of conjunctions (over $i \in \delta_\nu$) of objectives Ω_i. This statement is formulated in the next theorem (see again Figure 2).

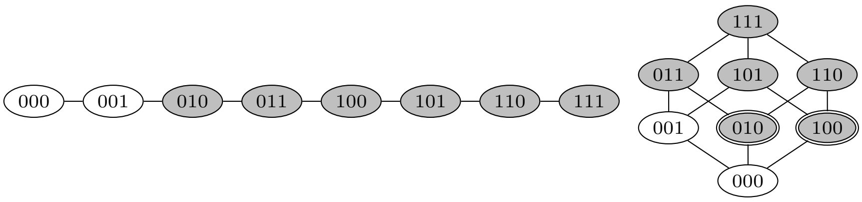

Figure 2 Gray nodes represent the set of payoffs $\nu \succsim \mu = 010$ for the lexicographic preorder and its embedding for the subset preorder. The elements of $\mathsf{M}(\mu) = \{010, 100\}$ are doubly circled.

▶ **Theorem 3.** *Let $(G, \Omega_1, \ldots, \Omega_n, \precsim)$ be an ordered game, $\mu \in \{0,1\}^n$ be some threshold, and v_0 be an initial vertex. Then, $\mathcal{P}_1$ can ensure a payoff $\succsim \mu$ from v_0 in $(G, \Omega_1, \ldots, \Omega_n, \precsim)$ if and only if $\mathcal{P}_1$ has a winning strategy from v_0 in the game (G, Ω) with the objective $\Omega = \cup_{\nu \in \mathsf{M}(\mu)} \cap_{i \in \delta_\nu} \Omega_i$.*

We end this section by giving some additional notations and terminology. Thanks to Theorem 3, we will prove in Section 3 that the threshold problem is fixed parameter tractable. The proof of this result uses two sizes depending on the number n of objectives:

- the *size* $s(n)$ of $\mathsf{M}(\mu)$. It is upper bounded by 2^n (an antichain of maximum size in the subset preorder over $\{0,1\}^n$ is of exponential size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$).
- the *size* $s'(n)$ defined as follows. In case of Büchi objectives Ω_i, we need to rewrite the objective $\cup_{\nu \in \mathsf{M}(\mu)} \cap_{i \in \delta_\nu} \Omega_i$ in conjunctive normal form $\cap_k \cup_l \Omega'_{k,l}$ with $\Omega'_{k,l} \in \{\Omega_1, \ldots, \Omega_n\}$. We denote by $s'(n)$ the size of this conjunction. It is bounded by 2^{2^n}.

In Section 4 we will show that, for several objectives, we can go beyond fixed parameter tractability by providing polynomial time algorithms when the sizes $s(n)$ and $s'(n)$ are polynomial in n. An ordered game $(G, \Omega_1, \ldots, \Omega_n, \precsim)$ is said to have a *compact embedding* (in the subset preorder) if both sizes $s(n)$ and $s'(n)$ are polynomial in n. We will also show that lexicographic games have a compact embedding.

3 Fixed parameter complexity of ordered ω-regular games

Parameterized complexity

A *parameterized language* L is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet, the second component being the parameter of the language. It is called *fixed parameter tractable* (FPT) if there is an algorithm that determines whether $(x,t) \in L$ in $f(t) \cdot |x|^c$ time, where c is a constant independent of the parameter t and f is a computable function depending on t only. We also say that L belongs to (the class) FPT. Intuitively, a language is FPT if there is an algorithm running in polynomial time w.r.t the input size times some computable function on the parameter. In this framework, we do not rely on classical polynomial reductions but rather use so called FPT-reductions. An FPT-*reduction* between two parameterized languages $L \subseteq \Sigma^* \times \mathbb{N}$ and $L' \subseteq \Sigma'^* \times \mathbb{N}$ is a function $R : L \to L'$ such that

- $(x,t) \in L$ if and only if $(x', t') = R(x,t) \in L'$,
- R is computable by an algorithm that takes $f(t) \cdot |x|^c$ time where c is a constant, and
- $t' \leq g(t)$ for some computable function g.

Moreover, if L' is in FPT, then L is also in FPT. We refer the interested reader to [13] for more details on parameterized complexity.

Our main result states that the threshold problem is in FPT for all the ordered games of this article. Parameterized complexities are given in Table 1.

Table 1 Fixed parameter tractability of ordered games $(G, \Omega_1, \ldots, \Omega_n, \precsim)$: for $i \in \{1, \ldots, n\}$, k_i/d_i denotes the number of pairs/colors of each Rabin/Streett/Muller objective Ω_i. Sizes $s(n)$ and $s'(n)$ are resp. upper bounded by 2^n and 2^{2^n}. For $j \in \{1, 2, 3\}$, $M_j = 2^{m_j}$ where $m_1 = \sum_{i=1}^{n} 2 \cdot k_i$, $m_2 = m_3 = \sum_{i=1}^{n} d_i$, and $N_1 = s(n) \cdot \sum_{i=1}^{n} 2 \cdot k_i$, $N_2 = s(n) \cdot \sum_{i=1}^{n} \frac{d_i^2}{2}$, $N_3 = s(n) \cdot \sum_{i=1}^{n} 2^{d_i} \cdot d_i$.

Objectives	Parameters	Threshold problem
Reachability, Safety	n	$O(s(n) \cdot n + 2^n \cdot (\lvert V\rvert + \lvert E\rvert))$
Büchi	n	$O(s(n) \cdot n + s'(n) \cdot \lvert V\rvert^2)$
co-Büchi	n	$O(s(n) \cdot n + s(n) \cdot \lvert V\rvert^2)$
Explicit Muller	n	$O(s(n) \cdot n + (s(n) \cdot \max_i \lvert\mathcal{F}_i\rvert)^3 \cdot \lvert V\rvert^2 \cdot \lvert E\rvert^2)$
Rabin, Streett	$n, k_1, \ldots, k_n$	$O((2^{M_1} \cdot N_1 + M_1^{M_1}) \cdot \lvert V\rvert^5)$
Parity	$n, d_1, \ldots, d_n$	$O((2^{M_2} \cdot N_2 + M_2^{M_2}) \cdot \lvert V\rvert^5)$
Muller	$n, d_1, \ldots, d_n$	$O((2^{M_3} \cdot N_3 + M_3^{M_3}) \cdot \lvert V\rvert^5)$

▶ **Theorem 4.** *The threshold problem is in* FPT *for ordered reachability, safety, Büchi, co-Büchi, explicit Muller, Rabin, Streett, parity, and Muller games.*

The proof of this theorem needs to introduce additional kinds of games (G, Ω) with a single ω-regular objective Ω, like the Boolean Büchi games. It also needs to show that solving the latter games is in FPT.

Parameterized complexity of Boolean Büchi games

Let G be a game structure and $U_1, \ldots, U_m$ be m target sets. Let ϕ be a Boolean formula over variables $x_1, \ldots, x_m$. We say that a play π satisfies $(\phi, U_1, \ldots, U_m)$ if the truth assignment $\{x_i = 1$ if and only if $\mathsf{Inf}(\pi) \cap U_i \neq \emptyset$, and $x_i = 0$ otherwise$\}$ satisfies ϕ. An objective Ω is a *Boolean combination of Büchi objectives*, or shortly a *Boolean Büchi* objective, if $\Omega = \{\pi \in \mathsf{Plays}(G) \mid \pi$ satisfies $(\phi, U_1, \ldots, U_m)\}$. It is denoted by $\mathsf{BooleanBuchi}(\phi, U_1, \ldots, U_m)$.

All operators $\vee$, $\wedge$, $\neg$ are allowed in Boolean Büchi objectives. However we denote by $|\phi|$ the *size* of ϕ equal to the number of disjunctions and conjunctions inside ϕ, and we say that the Boolean Büchi objective $\mathsf{BooleanBuchi}(\phi, U_1, \ldots, U_m)$ is *of size* $|\phi|$ *and with* m *variables*. The definition of $|\phi|$ is not the classical one that usually counts the number of operators $\vee, \wedge, \neg$ and variables. This is not a restriction since one can transform any Boolean formula ϕ into one such that negations only apply on variables.

We need to introduce some other kinds of ω-regular objectives with Boolean combinations of objectives that are limited to

- intersections of objectives: like a *generalized reachability* objective or a *generalized Büchi* objective denoted respectively by $\mathsf{GenReach}(U_1, \ldots, U_m)$ and $\mathsf{GenBuchi}(U_1, \ldots, U_m)$,
- unions of intersections (UI) of objectives: like a *UI reachability* objective, a *UI safety* objective, or a *UI Büchi* objective.

▶ **Proposition 5.** *Solving Boolean Büchi games* (G, Ω) *is in* FPT*, with an algorithm in* $O(2^M \cdot |\phi| + (M^M \cdot |V|)^5)$ *time with* $M = 2^m$ *such that* m *is the number of variables of* ϕ *in the Boolean Büchi objective* Ω*.*

Proof. Let us show the existence of an FPT-reduction from Boolean Büchi games to Muller games. For this purpose, consider a Boolean Büchi game (G, Ω) with the objective $\Omega = \mathsf{BooleanBuchi}(\phi, U_1, \ldots, U_m)$, where ϕ is a Boolean formula over variables $x_1, \ldots, x_m$, and m is seen as a parameter. We build an adequate Muller game $(G, \mathsf{Muller}(p, \mathcal{F}))$ on the same game structure and parameterized by the number of colors. The coloring function p and the family $\mathcal{F}$ are constructed as follows.

To any vertex $v \in V$, we associate a color $p(v) = \mu$ which is a subset of $\{1, \dots, m\}$ in the following way: $i \in \mu$ if and only if $v \in U_i$.[7] Intuitively, we keep track for all i, whether a vertex belongs to U_i or not. The total number M of colors is thus equal to 2^m. One can notice that $(*)$ a play π visits a vertex $v \in U_i$ if and only if π visits a color μ that contains i.

To any subset F of $p(V)$, we associate the truth assignment $\chi(F) \in \{0,1\}^m$ of variables $x_1, \dots, x_m$ such that for all i, $\chi(F)_i = 1$ if there exists $\mu \in F$ such that $i \in \mu$, and 0 otherwise. The idea (by $(*)$) is that the set F of colors visited infinitely often by a play π corresponds to the set $\mathsf{Inf}(\pi)$ of vertices visited infinitely often such that $\chi(F)_i = 1$ if and only if $\mathsf{Inf}(\pi) \cap U_i \neq \emptyset$. We then define $\mathcal{F} = \{F \subseteq p(V) \mid \chi(F) \models \phi\}$, that is, $\mathcal{F}$ corresponds to the set of all truth assignments satisfying ϕ.

In this way we have the desired FPT-reduction: first, parameter $M = 2^m$ only depends on parameter m. Second, we have that $\mathcal{P}_1$ is winning in $(G, \mathsf{BooleanBuchi}(\phi, U_1, \dots, U_m))$ from an initial vertex v_0 if and only if he is winning in $(G, \mathsf{Muller}(p, \mathcal{F}))$ from v_0. Indeed, a play π satisfies $(\phi, U_1, \dots, U_m)$ if and only if the truth assignment ($x_i = 1$ if and only $\mathsf{Inf}(\pi) \cap U_i \neq \emptyset$, and $x_i = 0$ otherwise) satisfies ϕ. This is equivalent to have that $F = p(\mathsf{Inf}(\pi))$ belongs to $\mathcal{F}$ (by definition of $\chi(F)$), that is, π belongs to $\mathsf{Muller}(p, \mathcal{F})$. Third, the construction of the Muller game is in $O(2^{2^m} \cdot |\phi|)$ time since it requires $O(|V| + |E|)$ time for the game structure, $O(m \cdot |V|)$ time for the coloring function p, and $O(2^{2^m} \cdot |\phi|)$ time for the family $\mathcal{F}$.

From this FPT-reduction and as solving Muller games is in $O((d^d \cdot |V|)^5))$ time where d is the number of colors [9], we have an algorithm solving the Boolean Büchi game in $O(2^M \cdot |\phi| + (M^M \cdot |V|)^5)$ time, where $M = 2^m$. ◀

Proof of FPT membership for ordered games

Thanks to Theorem 3, we provide a proof of Theorem 4 with the parameterized complexities given in Table 1.

Proof of Theorem 4. By Theorem 3, solving the threshold problem for an ordered game $(G, \Omega_1, \dots, \Omega_n, \precsim)$ is equivalent to solving a classical game (G, Ω) with $\Omega = \cup_{\nu \in \mathsf{M}(\mu)} \cap_{i \in \delta_\nu} \Omega_i$. We have $|\mathsf{M}(\mu)| = s(n)$ and $|\delta_\nu| \leq n \; \forall \nu \in \mathsf{M}(\mu)$. Recall that $s(n) \leq 2^n$ and $s'(n) \leq 2^{2^n}$.

We first show that the threshold problem for ordered reachability, safety, Büchi, co-Büchi, and explicit Muller games is in FPT with parameter n. The reduction provided in Theorem 3 is an FPT-reduction as the number of disjunctions/conjunctions in Ω only depends on n. Moreover the construction of the game (G, Ω) is in $O(|V| + |E| + s(n) \cdot n)$ time. In the following items we describe a second FPT-reduction to add to the first one. The sum of the complexities of both FPT-reductions leads to the complexities given in Table 1, rows 2-5.

- If each Ω_i is a reachability (resp. safety) objective, then (G, Ω) is a UI reachability (resp. safety) game that can be reduced to a reachability (resp. safety) game over a game structure of size $2^n \cdot |V|$ [14].[8] The latter is solved in $O(2^n \cdot (|V| + |E|))$ time.
- If Ω is a union of intersections of Büchi objectives, then it can be rewritten as the intersection of unions of Büchi objectives which is a generalized Büchi objective with at most $s'(n)$ target sets. The latter game is solved in $O(s'(n) \cdot |V|^2)$ time by [10]. The

[7] Our definition of color requires μ to be an integer. It suffices to associate with v a vector $\mu^v \in \{0,1\}^m$ such that $\mu^v_i = 1$ if $v \in U_i$ and 0 otherwise, and to define the coloring function $p \colon V \to \{0, \dots, 2^m - 1\}$ that associates with each vertex v the color $p(v)$ such that its binary encoding is equal to μ^v.

[8] This result does not appear explicitly in [14] but can be easily adapted to the case of UI reachability (resp. safety) objectives.

union of intersections of co-Büchi objectives is the complementary of a generalized Büchi objective with at most $s(n)$ target sets, leading to an algorithm in $O(s(n) \cdot |V|^2)$ time.

- If each Ω_i is an explicit Muller objective $\mathsf{ExplMuller}(\mathcal{F}_i)$ then Ω is also an explicit Muller objective. Indeed the intersection (resp. union) of explicit Muller objectives is an explicit Muller objective such that $\cap_i \mathsf{ExplMuller}(\mathcal{F}_i) = \mathsf{ExplMuller}(\mathcal{F})$ with $\mathcal{F} = \cap_i \mathcal{F}_i$ (resp. $\cup_i \mathsf{ExplMuller}(\mathcal{F}_i) = \mathsf{ExplMuller}(\mathcal{F})$ with $\mathcal{F} = \cup_i \mathcal{F}_i$). Thus Ω can be here rewritten as $\mathsf{ExplMuller}(\mathcal{F})$ for some set $\mathcal{F}$ such that $|\mathcal{F}| \leq \sum_{\nu \in \mathsf{M}(\mu)} \min_{j \in \delta_\nu} |\mathcal{F}_j|$. The latter game is solved in $O(|\mathcal{F}| \cdot (|V| \cdot |E| + |\mathcal{F}|)^2) = O((s(n) \cdot \max_i |\mathcal{F}_i|)^3 \cdot |V|^2 \cdot |E|^2)$ time by [17].

We now show that the threshold problem for ordered parity, Rabin, Streett, and Muller games is in FPT thanks to Proposition 5.

- Let us show that the threshold problem for ordered parity games is in FPT with parameters $n, d_1, \ldots, d_n$. If each Ω_i is a parity objective with d_i colors, then each Ω_i is a Boolean Büchi objective of size at most $\frac{d_i^2}{2}$ and using d_i variables. Indeed, as a play is winning for Ω_i if and only there exists an even priority seen infinitely often along the play and no lower priority seen infinitely often. Therefore, Ω is a Boolean Büchi objective Ω' of size $|\phi| \leq s(n) \cdot \sum_{i=1}^{n} \frac{d_i^2}{2}$, and with $m = \sum_{i=1}^{n} d_i$ variables as $\cup_{\nu \in \mathsf{M}(\mu)} \{\Omega_i \mid i \in \delta_\nu\} \subseteq \{\Omega_1, \ldots, \Omega_n\}$. We thus have an FPT-reduction to the game (G, Ω') depending on the parameters $n, d_1, \ldots, d_n$ and with an algorithm in $O(|V|+|E|+|\phi|)$ time. By Proposition 5, solving the game (G, Ω') is in FPT with an algorithm in $O(2^M \cdot |\phi| + (M^M \cdot |V|)^5)$ time with $M = 2^m$. Thus the threshold problem is in FPT with parameters $n, d_1, \ldots, d_n$, with an overall algorithm in $O((2^M \cdot N + M^M) \cdot |V|^5)$ time where $N = 2^n \cdot \sum_{i=1}^{n} \frac{d_i^2}{2}$.
- The arguments are similar for ordered Rabin, Streett, and Muller games. The only differences are the upper bound on size $|\phi|$ and the number m of variables of the related formula ϕ. ◀

4 Ordered games with a compact embedding

In the previous section, we have shown that solving the threshold problem for ordered ω-regular games is in FPT. This result depends on sizes $s(n)$ and $s'(n)$ which vary with the number n of objectives. In this section, we study ordered games with a compact embedding, that is, such that these sizes are polynomial in n.

Beyond fixed parameter tractability

While the threshold problem is in FPT for ordered Büchi, co-Büchi, and explicit Muller games, it becomes polynomial as soon as their preorder has a compact embedding. This is a direct consequence of Table 1, rows 2-4.

▶ **Theorem 6.** *The threshold problem is solved in polynomial time for ordered Büchi, co-Büchi, and explicit Muller games with a compact embedding.*

One can easily prove that ordered games using the subset or the maximize preorder have a compact embedding. We will later prove that this also holds for the lexicographic preorder. Nevertheless it is not the case for the counting preorder. Indeed solving the threshold problem for counting Büchi games is co-NP-complete [6].

Recall that solving the threshold problem for ordered Büchi games reduces to solving some UI Büchi game (by Theorem 3). Whereas solving the latter games is coNP-complete [4], solving the threshold problem for ordered Büchi games is only polynomial when they have a compact embedding (see Theorem 6).

There is no hope to extend Theorem 6 to the other ω-regular objectives studied in this article, unless P = PSPACE. Indeed, we have PSPACE-hardness of the threshold problem for the following lexicographic games.

▶ **Theorem 7.** *(1) Lexicographic games have a compact embedding and (2) the threshold problem is* PSPACE*-hard for lexicographic reachability, safety, Rabin, Streett, parity, and Muller games.*

The rest of this section is devoted to the proof of Theorem 7.

Lexicographic games

We now focus on the lexicographic preorder $\precsim$. Let us first provide several useful terminology and comments on this preorder. Recall that the lexicographic preorder is monotonic. It is also total, hence $x \sim y$ if and only if $x = y$, and $x \prec y$ if and only if $\neg(y \precsim x)$. Given a vector $x \in \{0,1\}^n$, we denote by $\overline{x}$ the *complement* of x, i.e. $\overline{x}_i = 1 - x_i$, for all $i \in \{1,\dots,n\}$. We denote by $x - 1$ the *predecessor* of $x \neq 0^n$, that is, the greatest vector which is strictly smaller than x. We define the *successor* $x + 1$ of x similarly. In the sequel, as the threshold problem is trivial for $x = 0^n$, we do not consider this threshold. By abuse of notation, we keep writing $x \in \{0,1\}^n$ without mentioning that $x \neq 0^n$. We denote by $\mathsf{Last}_1(x)$ the last index i of x such that $x_i = 1$, i.e. $\mathsf{Last}_1(x) = \max\{i \in \{1,\dots,n\} \mid x_i = 1\}$. Note that $\mathcal{P}_1$ can ensure a payoff $\succsim x \neq 0^n$ if and only if he can ensure a payoff $\succ x - 1$, and when $\mathcal{P}_2$ can avoid a payoff $\succsim x$, we rather say that $\mathcal{P}_2$ can *ensure* a payoff $\prec x$.

We now prove that the lexicographic games have a compact embedding (Part (1) of Theorem 7): we first show that $s(n)$ is polynomial in Proposition 8, and we then show that $s'(n)$ is also polynomial in Proposition 10.

▶ **Proposition 8.** *Let $x \in \{0,1\}^n$. Then the set $\mathsf{M}(x)$ is equal to $\{x\} \cup \{y^j \in \{0,1\}^n \mid x_j = 0 \wedge j < \mathsf{Last}_1(x)\}$, where for all $j \in \{1,\dots,\mathsf{Last}_1(x) - 1\}$, we define the vector $y^j \in \{0,1\}^n$ as equal to $x_1 \dots x_{j-1}10^{n-j}$ (x and y^j share the same (possibly empty) prefix $x_1 \dots x_{j-1}$). Moreover, $s(n) = |\mathsf{M}(x)| \leq n$.*

▶ **Example 9.** Consider the vector $x = 0010100$ such that $\mathsf{Last}_1(x) = 5$. Then, the set $\mathsf{M}(x)$ is equal to $\{x\} \cup \{1000000, 0100000, 0011000\}$.

Proof of Proposition 8. We recall that $\mathsf{M}(x)$ is the set of minimal elements (with respect to the subset preorder $\subseteq$) of the set of payoffs $y \succsim x$ embedded in the set $\{0,1\}^n$ ordered with $\subseteq$. Let us show both inclusions between $\mathsf{M}(x)$ and $M = \{x\} \cup \{y^j \in \{0,1\}^n \mid x_j = 0 \wedge j < \mathsf{Last}_1(x)\}$.

Let $y \in \mathsf{M}(x)$. If $y = x$, then trivially $y \in M$. Otherwise, assume $y \succ x$ and let j be the first index such that $y_j = 1$ and $x_j = 0$. Note that $x_1 \dots x_{j-1} = y_1 \dots y_{j-1}$ since $y \succ x$. We associate with y the vector $y^j = y_1 \dots y_{j-1}10^{n-j}$. Note that $y^j \succ x$. By minimality of y and by construction of y^j, we obtain $y = y^j$ showing that $y \in M$.

For the second inclusion, as the lexicographic preorder is monotonic, we have $x \in \mathsf{M}(x)$. Now, consider some $y^j \in M$ such that $x_j = 0$ and $j < \mathsf{Last}_1(x)$. Let us show that y^j belongs to $\mathsf{M}(x)$, that is, $y^j \succsim x$ and there is no $y \succsim x$, $y \neq y^j$, such that $y \subset y^j$ (i.e. $\{i \mid y_i = 1\} \subset \{i \mid y^j_i = 1\}$). First, we clearly have $y^j \succsim x$ since $y^j = x_1 \dots x_{j-1}10^{n-j}$ and $x_j = 0$. Towards a contradiction, assume now that there exists some $y \succsim x$, $y \neq y^j$, such that $y \subset y^j$. Let i be the first index such that $y_i = 0$ and $y^j_i = 1$. As $y \subset y^j$, we have $i \leq j$. If $i < j$, then y has $x_1 \dots x_{i-1}0$ as prefix, $y^j_i = x_i = 1$, showing that $y \prec x$ in contradiction with $y \succsim x$. If $i = j$, then $y = x_1 \dots x_{j-1}0^{n-j+1}$, and again $y \prec x$ since $j < \mathsf{Last}_1(x)$ by construction of y^j. ◀

▶ **Proposition 10.** *Let $(G, \Omega_1, \dots, \Omega_n, \precsim)$ be a lexicographic Büchi game and $\mu \in \{0,1\}^n$. Then, $\Omega = \cup_{\nu \in \mathsf{M}(\mu)} \cap_{i \in \delta_\nu} \Omega_i$ can be rewritten in conjunctive normal form with a conjunction of size $s'(n) \leq n$.*

Proof. The proof uses the property that given a lexicographic game $(G, \Omega_1, \dots, \Omega_n, \precsim)$ and a threshold $\mu \in \{0,1\}^n$, $\mathcal{P}_1$ can ensure a payoff $\succsim \mu$ in $(G, \Omega_1, \dots, \Omega_n, \precsim)$ if and only if $\mathcal{P}_1$ can ensure a payoff $\precsim \overline{\mu}$ in the lexicographic game $(G, \overline{\Omega_1}, \dots, \overline{\Omega_n}, \precsim)$. By Theorem 3 and Martin's theorem [20], equivalently, $\mathcal{P}_2$ cannot satisfy the objective $\cup_{\nu \in \mathsf{M}(\overline{\mu}+1)} \cap_{i \in \delta_\nu} \overline{\Omega_i}$. This is equivalent to say that $\mathcal{P}_1$ can satisfy the complement of the latter objective, that is, the objective $\cap_{\nu \in \mathsf{M}(\overline{\mu}+1)} \cup_{i \in \delta_\nu} \Omega_i$. We have $|\mathsf{M}(\overline{\mu}+1)| \leq n$ by Proposition 8. ◀

We finally prove Part (2) of Theorem 7.

Proof of Theorem 7, Part (2). Let us study the complexity lower bounds.

- The PSPACE-hardness of the threshold problem for lexicographic reachability (resp. safety) games is obtained thanks to a polynomial reduction from solving generalized reachability games which is PSPACE-complete [14]. Let (G, Ω) be a generalized reachability game with $\Omega = \mathsf{GenReach}(U_1, \dots, U_n)$. Let $(G, \Omega_1, \dots, \Omega_n, \precsim)$ be the lexicographic reachability (resp. safety) game with $\Omega_i = \mathsf{Reach}(U_i)$ (resp. $\Omega_i = \mathsf{Safe}(V \setminus U_i)$) $\forall i$.
  - Reachability: We have that $\mathcal{P}_1$ is winning in (G, Ω) from v_0 if and only if $\mathcal{P}_1$ can ensure a payoff $\succsim \mu = 1^n$ from v_0 in the lexicographic reachability game $(G, \Omega_1, \dots, \Omega_n, \precsim)$.
  - Safety: We claim that $\mathcal{P}_1$ is winning in (G, Ω) from v_0 if and only if $\mathcal{P}_1$ can ensure a payoff $\succsim \mu = 0^{n-1}1$ from v_0 in the lexicographic safety game. This follows from the determinacy of generalized reachability games, and from the fact that $\mathcal{P}_1$ can ensure a payoff $\succsim \mu$ from v_0 in the lexicographic safety game if and only if $\mathcal{P}_2$ is losing in the generalized reachability game (G, Ω) from v_0.
- The hardness of the threshold problem for lexicographic parity games is obtained thanks to a polynomial reduction from solving games (G, Ω) the objective Ω of which is a union of a Rabin objective and a Streett objective, which is known to be PSPACE-complete [3]. Let $\Omega = \mathsf{Rabin}((E_i, F_i)_{i=1}^{n_1}) \cup \mathsf{Streett}((E_i, F_i)_{i=n_1+1}^{n})$. As any Rabin (resp. Streett) objective is the union (resp. intersection) of parity objectives [11], we can rewrite Ω as $\Omega = \cup_{i=1}^{n_1}(\mathsf{Parity}(p_i)) \cup (\cap_{i=n_1+1}^{n} \mathsf{Parity}(p_i))$, where all p_i are coloring functions. Let $(G, \Omega_1, \dots, \Omega_n, \precsim)$ be the lexicographic parity game where $\Omega_i = \mathsf{Parity}(p_i)$ for all i. We claim that $\mathcal{P}_1$ is winning in the game (G, Ω) from v_0 if and only if $\mathcal{P}_1$ can ensure a payoff $\succsim \mu$ from v_0 in the lexicographic parity game $(G, \Omega_1, \dots, \Omega_n, \precsim)$ where $\mu = 0^{n_1}1^{n-n_1}$. Indeed, if a play π satisfies $\mathsf{Payoff}(\pi) \succsim \mu$ then either $\mathsf{Payoff}(\pi) = \mu$ in which case $\pi \in \cap_{i=n_1+1}^{n} \mathsf{Parity}(p_i)$, i.e. π satisfies the Streett objective, or $\mathsf{Payoff}(\pi) \succ \mu$ in which case there exists $1 \in \{1, \dots, n_1\}$ such that $\pi \in \mathsf{Parity}(p_i)$, i.e. π satisfies the Rabin objective. Conversely, if a play π satisfies the Streett or the Rabin objective then $\mathsf{Payoff}(\pi) \succsim \mu$ since $\mathsf{Payoff}(\pi) \succsim \mu$ (resp. $\succ \mu$) as soon as π satisfies the Streett (resp. Rabin) objective.
- As parity objectives are a special case of Rabin (Streett) objectives, the lower bound follows (from the previous item) for both lexicographic Rabin and Streett games.
- Lexicographic Muller games with $n = 1$ and $\mu = 1$ are a special case of Muller games and solving the latter games is PSPACE-complete [18].

This completes the proof. ◀

5 Values and optimal strategies for lexicographic games

In this section, we first recall the notion of values and optimal strategies. We then show how to compute the values in lexicographic games, and what are the memory requirements for the related optimal strategies. This yields a *full picture* of the study of lexicographic games, see

Table 2 Overview of the results for lexicographic games with ω-regular objectives.

<table>
<tr><th></th><th>Threshold problem</th><th>Value</th><th>$\mathcal{P}_1$ memory</th><th>$\mathcal{P}_2$ memory</th></tr>
<tr><td>Büchi</td><td rowspan="3">P-complete</td><td rowspan="3">polynomial</td><td>linear</td><td>memoryless</td></tr>
<tr><td>Co-Büchi</td><td>memoryless</td><td>linear</td></tr>
<tr><td>Explicit Muller</td><td colspan="2">exponential</td></tr>
<tr><td>Reachability, safety</td><td rowspan="4">PSPACE-complete and FPT</td><td rowspan="4">exponential and FPT</td><td rowspan="4" colspan="2">exponential</td></tr>
<tr><td>Parity</td></tr>
<tr><td>Streett, Rabin</td></tr>
<tr><td>Muller</td></tr>
</table>

Table 2. In this table, the second column indicates the complexity of the threshold problem (Theorems 4, 7 and PSPACE upper bounds follow from results of [6]), the third one indicates the complexity of computing the values and the remaining columns summarize the memory requirements of winning and optimal strategies (Theorem 12 hereafter).

Values and optimal strategies

In a lexicographic game, one can define the best reward that $\mathcal{P}_1$ can ensure from a given vertex, that is, the highest threshold μ for which $\mathcal{P}_1$ can ensure a payoff $\succsim \mu$. Dually, we can also define the worst reward that $\mathcal{P}_2$ can ensure.

In the following definition, the infimum and supremum functions are applied with $\precsim$.

▶ **Definition 11.** Given a lexicographic game $(G, \Omega_1, \dots, \Omega_n, \precsim)$, for every vertex $v \in V$, the *upper value* $\overline{\mathsf{Val}}(v)$ and the *lower value* $\underline{\mathsf{Val}}(v)$ are defined as:

$$\overline{\mathsf{Val}}(v) = \inf_{\sigma_2 \in \Sigma_2} \sup_{\sigma_1 \in \Sigma_1} \mathsf{Payoff}(\mathsf{Out}(v, \sigma_1, \sigma_2)) \text{ and } \underline{\mathsf{Val}}(v) = \sup_{\sigma_1 \in \Sigma_1} \inf_{\sigma_2 \in \Sigma_2} \mathsf{Payoff}(\mathsf{Out}(v, \sigma_1, \sigma_2)).$$

The lexicographic game $(G, \Omega_1, \dots, \Omega_n, \precsim)$ is *value-determined* if $\underline{\mathsf{Val}}(v) = \overline{\mathsf{Val}}(v)\ \forall v \in V$. In this case, we write $\mathsf{Val}(v) = \overline{\mathsf{Val}}(v) = \underline{\mathsf{Val}}(v)$ and we call $\mathsf{Val}(v)$ the *value* of v. Note that the inequality $\underline{\mathsf{Val}}(v) \precsim \overline{\mathsf{Val}}(v)$ always holds. If $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) can ensure a payoff $\succsim \underline{\mathsf{Val}}(v)$ (resp. $\precsim \overline{\mathsf{Val}}(v)$) from v, his related winning strategy σ_1^* (resp. σ_2^*) is called *optimal from* v.

Notice that for all lexicographic games such that the objectives $\Omega_1, \dots, \Omega_n$ are Borel sets, we have that these games are value-determined and have optimal strategies by Theorem 3 and Martin's theorem [20]. In the following theorem, we go further by giving time complexities and memory sizes of the optimal strategies.

▶ **Theorem 12.** *(1) The value of each vertex in lexicographic Büchi, co-Büchi, and explicit Muller games can be computed with a polynomial time algorithm, and with an exponential time and an* FPT *algorithm for lexicographic reachability, safety, parity, Rabin, Streett, and Muller games.*

(2) The following assertions hold for both winning strategies of the threshold problem and optimal strategies. Linear memory strategies are necessary and sufficient for $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) while memoryless strategies are sufficient for $\mathcal{P}_2$ (resp. $\mathcal{P}_1$) in lexicographic Büchi (resp. co-Büchi) games. Exponential memory strategies are both necessary and sufficient for both players in lexicographic reachability, safety, explicit Muller, parity, Rabin, Streett, and Muller games.

We only give a sketch of the proof.

First, for the considered lexicographic games, the values can be obtained by solving n times well-chosen threshold problems. Therefore, results of Part (1) of Theorem 12 follows from the second column of Table 2. In addition to give the exact value of a vertex, this procedure also shows that optimal strategies correspond to winning strategies for specific threshold problems. Therefore, we just have to analyze memory requirements of winning strategies for the threshold problem in lexicographic games to obtain those of optimal strategies. Upper bounds on memory sizes of winning strategies are obtained by analyzing the several reductions done in the proof of Theorem 4 in the case of a preorder with a compact embedding. Lower bounds for lexicographic Büchi and co-Büchi games are obtained thanks to a reduction from generalized Büchi games, and for the other lexicographic games thanks to the reductions proposed in the proof of Theorem 7, Part (2). This yields results of Part (2) of Theorem 12.

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