Geodesic Obstacle Representation of Graphs

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Abstract

An *obstacle representation* of a graph is a mapping of the vertices onto points in the plane and a set of connected regions of the plane (called *obstacles*) such that the straight-line segment connecting the points corresponding to two vertices does not intersect any obstacles if and only if the vertices are adjacent in the graph. The obstacle representation and its *plane* variant (in which the resulting representation is a plane straight-line embedding of the graph) have been extensively studied with the main objective of minimizing the number of obstacles. Recently, Biedl and Mehrabi [5] studied *non-blocking grid obstacle representations* of graphs in which the vertices of the graph are mapped onto points in the plane while the straight-line segments representing the adjacency between the vertices is replaced by the L_1 (Manhattan) shortest paths in the plane that avoid obstacles.

In this paper, we introduce the notion of *geodesic obstacle representations* of graphs with the main goal of providing a generalized model, which comes naturally when viewing line segments as shortest paths in the Euclidean plane. To this end, we extend the definition of obstacle representation by allowing *some* obstacles-avoiding shortest path between the corresponding points in the underlying metric space whenever the vertices are adjacent in the graph. We consider both *general* and *plane* variants of geodesic obstacle representations (in a similar sense to obstacle representations) under any polyhedral distance function in $\mathbb{R}^d$ as well as shortest path distances in graphs. Our results generalize and unify the notions of obstacle representations, plane obstacle representations and grid obstacle representations, leading to a number of questions on such representations.

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1 Introduction

An obstacle representation of an (undirected simple) graph G is pair (φ, S) where $\varphi : V(G) \to \mathbb{R}^2$ maps vertices of G to distinct points in $\mathbb{R}^2$ and S is a set of connected subsets of $\mathbb{R}^2$ with the property that, for every $u, w \in V(G)$, $uw \in E(G)$ if and only if the line segment with endpoints $\varphi(u)$ and $\varphi(w)$ is disjoint from $\cup S$. The elements of S are called *obstacles*. It is easy to see that every graph G has an obstacle representation: obtain a straight-line drawing of G by taking any φ that does not map three vertices of G onto a single line, and let S be the set of the open faces in the resulting arrangement of line segments.

Since every graph has an obstacle representation, this defines a natural graph parameter called the *obstacle number*, $\text{obs}(G) = \min\{|S| : (\varphi, S) \text{ is an obstacle representation of } G\}$. Since their introduction by Alpert et al. [2], obstacle numbers have been studied extensively with the main goal of bounding the obstacle numbers of various classes of graphs (see e.g. [3, 8, 11, 12, 14, 16] and the references therein).

For planar graphs, there is also a natural notion of a *plane obstacle representation* (φ, S) which is an obstacle representation in which φ defines a plane straight-line embedding of G. This leads to *plane obstacle number*: $\text{plane-obs}(G) = \min\{|S| : (\varphi, S) \text{ is a plane obstacle representation of } G\}$. Using Euler's formula, it is not hard to see that the plane obstacle number of any n-vertex planar graph is $O(n)$: let φ define any plane drawing of G with no three vertices collinear and take S to be the set of open faces in this drawing. Since an n-vertex planar graph has at most $2n - 4$ faces, this implies $\text{plane-obs}(G) \le 2n - 4$.

Recently, Biedl and Mehrabi [5] studied *non-blocking grid obstacle representations* of graphs, consisting of the pair (φ, S) as before in which φ maps the vertices of the graph to points in the plane and S is a set of obstacles, but the adjacency in the graph is represented by replacing straight-line segments with L_1 shortest paths in the plane. That is, for every $u, w \in V(G)$, $uw \in E(G)$ if and only if some L_1 shortest path from $\varphi(u)$ to $\varphi(w)$ is disjoint from $\cup S$; see Figure 1 for an illustration of these obstacle representations.

Geodesic obstacle representation. In this paper, we generalize the notions of obstacle representations [2], plane obstacle representations, and grid obstacle representations [5] by introducing *geodesic obstacle representations* of graphs. This natural generalization of obstacle representations comes from viewing line segments as shortest paths in the Euclidean plane. An obstacle representation (φ, S) has the property that $uw \in E(G)$ if and only if the shortest path from $\varphi(u)$ to $\varphi(w)$ does not intersect $\cup S$. The Euclidean distance is a very special case because the shortest path between any two points p and q is unique. To

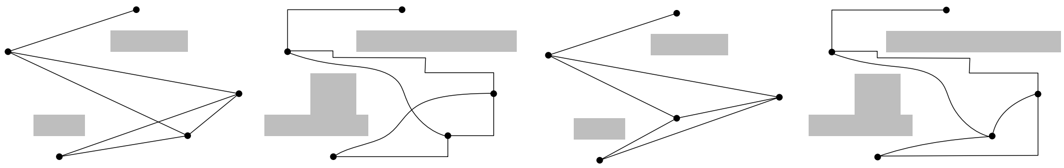

Figure 1 Four different obstacle representations of the same graph G: (a) an obstacle representation, (b) a geodesic obstacle representation under L_1 distance, (c) a plane obstacle representation, and (d) a non-crossing geodesic obstacle representation under L_1 distance.

accommodate other distance measures, we extend the definition of obstacle representation by saying that $uw \in E(G)$ if and only if *some* shortest path from $\varphi(u)$ to $\varphi(w)$ does not intersect $\cup S$. In this way, we can obtain many generalizations of obstacle representations by changing the underlying distance measure. For example, with the L_1 distance measure, every xy-monotone path is a shortest path. Therefore, if (φ, S) is an obstacle representation under L_1, then $uw \in E(G)$ if and only if there is some xy-monotone path from u to w that avoids $\cup S$. Analogous to plane obstacle representations, we can define *non-crossing* geodesic obstacle representations in which φ defines a plane embedding of graph G. Under the L_1 metric, this non-crossing version is equivalent to non-blocking grid obstacle representations as defined by Biedl and Mehrabi [5].

Considering the L_1 metric in the plane, one can view a geodesic obstacle representation of G as a partition of the neighbours of each vertex $u \in V(G)$ into four sets based on which of the four quadrants relative to u the neighbours of u are in the representation. Consequently, if $uv, vw \in E(G)$ in such a way that v is in the same quadrant of u as w is in the quadrant of v in a representation, then we must have $uw \in E(G)$ since there is an xy-monotone path from u to w in the representation. Notice that it is now not clear whether every graph has a geodesic obstacle representation. In fact, Pach [15] showed that there exists a bipartite graph that does not admit a grid obstacle representation. Indeed, the focus of this paper is to determine, for a class $\mathcal{G}$ of graphs, whether or not every member of $\mathcal{G}$ has a geodesic obstacle representation (under some metric space). Clearly, the existence of such representations is more likely if one extends the definition of monotonicity by considering $2k$ equal-angled *cones* around each vertex (instead of $2k = 4$ quadrants), where $k > 2$ is an integer. This leads us to the general question of, informally speaking, what is the minimum integer $k > 0$ for which every member of $\mathcal{G}$ has a geodesic obstacle representation when shortest paths are defined by monotone paths relative to such $2k$ equal-angled cones around each vertex. In this paper, with this "parameter k", we study geodesic obstacle representations and its non-crossing version under polyhedral distance functions in $\mathbb{R}^d$ as well as shortest path distances in graphs. See Section 2 for a formal definition of this generalized notion of obstacle representations.

Related work. It is known that every n-vertex graph has obstacle number $O(n \log n)$ [3] and some n-vertex graphs have obstacle number $\Omega(n/(\log\log n)^2)$ [8]. For planar graphs, there exist planar graphs with obstacle number 2 (the icosahedron is an example [4]), but the best upper bound on the obstacle number of an n-vertex planar graph is $O(n)$. Recall the $O(n)$ upper bound on the plane obstacle number of any n-vertex planar graph by Euler's formula. A lower bound of $\Omega(n)$ is also not difficult: any plane drawing of the $\sqrt{n} \times \sqrt{n}$ grid $G_{\sqrt{n}\times\sqrt{n}}$ has at least $n - 2\sqrt{n}$ bounded faces. Each of these faces has at least four vertices and therefore requires at least one obstacle, so plane-obs$(G_{\sqrt{n}\times\sqrt{n}}) \geq n - 2\sqrt{n}$. Gimbel et al. [11] have nailed the leading constant by showing that every planar graph has plane obstacle number at most $n-3$, the maximum being attained by planar bipartite graphs. See [2, 3, 8, 11] and the references therein for more details of results on obstacle number and its plane version.

While the obstacle numbers have been extensively studied under the Euclidean distance as shortest path, not much is known about obstacle representations under other shortest path metrics. In fact, we are only aware of the works of Bishnu et al. [6], and Biedl and Mehrabi [5] both of which considered only a restricted version of obstacle representations. Bishnu et al. [6] showed that any n-vertex planar graph G has an obstacle representation on an $O(n^4) \times O(n^4)$ grid in the plane under L_1 metric, with the additional restriction that, for any $uw \in E(G)$, the shortest path from $\varphi(u)$ to $\varphi(w)$ also avoids $\varphi(v)$ for all $v \in V(G) \setminus \{u, w\}$ (in addition to avoiding $\cup S$). Biedl and Mehrabi [5] relaxed this "vertex blocking" constraint and were able to show that every n-vertex planar bipartite graph has a non-blocking grid obstacle representation on an $O(n) \times O(n)$ grid. They left open the problem of finding other classes of graphs for which such non-blocking grid obstacle representations exist and, in particular, whether every planar graph has such a representation.

Our results. In this paper, we prove the following results:

- For any integer $k > 1$, there is a graph with $O(k^2)$ vertices that does not have a geodesic obstacle representation with parameter k. On the other hand, every n-vertex graph has a geodesic obstacle representation with every $k \geq n$.
- For any integer $d > 1$ and any integer $k > 1$, there exists a graph that does not have a geodesic obstacle representation in $\mathbb{R}^d$ with parameter k. On the other hand, every n-vertex graph has a geodesic obstacle representation in $\mathbb{R}^3$ with $k = \lceil (1/2) \log_2 n + 2 \rceil$.
- Every planar graph of treewidth at most 2 (and hence every outerplanar graph) has a non-crossing geodesic obstacle representation with $k = 2$; i.e., a non-blocking obstacle representation.
- Not every planar 3-tree has a non-crossing geodesic obstacle representation with $k = 2$, answering the question asked by Biedl and Mehrabi [5] negatively. Moreover, not every planar 4-connected triangulation has a non-crossing geodesic obstacle representation with $k = 2$.
- Every planar 3-tree has a non-crossing geodesic obstacle representation with $k = 3$. Furthermore, every 3-connected cubic planar graph has a non-crossing geodesic obstacle representation with $k = 7$.
- Every n-vertex graph admits a non-crossing geodesic obstacle representation when taking the D-cube graph as the underlying distance metric, where $D = C \log n$ for some constant $C > 0$.

Organization. We first give some definitions and notation in Section 2. Then, we show our results for (general) geodesic obstacle representations in Section 3 and for its non-crossing version in Section 4. Finally, we give our result for graph metrics in Section 5, and conclude the paper with a discussion on open problems in Section 6.

Throughout this paper, the proofs of lemmas and theorems marked with $(*)$ are given in the full version of the paper [7] due to space constraints.

2 Notation and Preliminaries

Let (X, δ) be a metric space. A *curve* over X is a function $f : [0, 1] \to X$. We call $f(0)$ and $f(1)$ the *endpoints* of the curve f and define the *image* of f as $I(f) = \{f(t) : 0 \leq t \leq 1\}$. A curve f is a *geodesic* if, for every $0 \leq t \leq 1$, $\delta(f(0), f(t)) + \delta(f(t), f(1)) = \delta(f(0), f(1))$. A *path space* is a triple $(X, \delta, \mathcal{C})$, where (X, δ) is a metric space and $\mathcal{C}$ is a set of curves over X that has the following closure property: if the curve f is in $\mathcal{C}$ then, for every $0 \le t \le 1$, $\mathcal{C}$ also contains the curves $g(x) = f(x \cdot t)$ and $h(x) = f(t + x \cdot (1 - t))$. A path space $(X, \delta, \mathcal{C})$ is *connected* if, for every distinct pair $u, w \in X$, there is some path in $\mathcal{C}$ with endpoints u and w. For a path space $P = (X, \delta, \mathcal{C})$ and a subset $R \subset X$, we denote the subspace induced by R as $P[R] = (R, \delta, \{f \in \mathcal{C} : I(f) \subseteq R\})$. The subspace that *avoids* R is defined as $P \setminus R = P[X \setminus R]$. Moreover, any curve in $P \setminus R$ is called an *R-avoiding curve.* With these definitions in hand, we are ready to define a generalization of obstacle representations.

▶ **Definition 1.** An $(X, \delta, \mathcal{C})$*-obstacle representation* of a graph G is a pair (φ, S) where $\varphi : V(G) \to X$ is a one-to-one mapping and S is a set of connected subspaces of $(X, \delta, \mathcal{C})$ with the property that, for every $u, w \in V(G)$, $uw \in E(G)$ if and only if $\mathcal{C}$ contains a $\cup S$-avoiding geodesic with endpoints $\varphi(u)$ and $\varphi(w)$.

Notice that it is now not clear whether every graph has an $(X, \delta, \mathcal{C})$-obstacle representation. Indeed, the focus of this paper is to determine, for a class $\mathcal{G}$ of graphs and a particular path space $(X, \delta, \mathcal{C})$, whether or not every member of $\mathcal{G}$ has an $(X, \delta, \mathcal{C})$-obstacle representation. This is closely related to certain types of embeddings of G into X. An *embedding* (φ, c) of a graph G into $(X, \delta, \mathcal{C})$ consists of a one-to-one mapping $\varphi : V(G) \to X$ and a function $c : E(G) \to \mathcal{C}$ such that, for each $uw \in E(G)$, the endpoints of $c(uw)$ correspond to $\varphi(u)$ and $\varphi(w)$. The embedding is *geodesic* if $c(uw)$ is a geodesic for every $uw \in E(G)$. Moreover, the embedding (φ, c) is *non-crossing* if $c(uw)$ is disjoint from $c(xz)$, for every $uw, xz \in E(G)$ with $\{u, w\} \cap \{x, z\} = \emptyset$. Observe that given an $(X, \delta, \mathcal{C})$-obstacle representation (φ, S) of G, for each $uw \in E(G)$, we can choose some $\cup S$-avoiding geodesic $c(uw) \in \mathcal{C}$ with endpoints $\varphi(u)$ and $\varphi(w)$. Then, the pair (φ, c) gives a geodesic embedding of G into X. If we can choose c such that (φ, c) is also non-crossing, then we say that the representation (φ, S) is *non-crossing.*

Distance functions. In this paper, we focus on the $(X, \delta, \mathcal{C})$-obstacle representation using polyhedral distance functions in $\mathbb{R}^d$. For a set $N = \{v_0, \ldots, v_{t-1}\}$ of vectors in $\mathbb{R}^d$, we define the *polyhedral distance function*

$$\delta_N(p, q) = \min \left\{ \sum_{i=0}^{t-1} |a_i| : q - p = \sum_{i=0}^{t-1} a_i v_i \right\} .$$

Every such distance function defines a centrally symmetric polyhedron $P_N = \{x \in \mathbb{R}^d : \delta_N(\mathbf{0}, x) \le 1\}$. The facets of P_N determine the geodesics. For a (closed) facet F of P_N, we denote the *cone* C_F as the union of all rays originating at the origin and containing a point on F (this is the conical hull of F). For a point $x \in \mathbb{R}^d$, the *F-sector* of x is $Q_F^N(x) = C_F + x$. For a facet of F of P_N, we say that a curve f is *δ_N-monotone in direction F* if, for all $0 \le a \le b \le 1$, $f(b) \in Q_F^N(f(a))$. We say that a curve is *δ_N-monotone* if it is δ_N-monotone in direction F for some facet F of P_N. Observe that a curve f is a geodesic for δ_N if and only if f is δ_N-monotone.

▶ **Observation 2.** *If uw and xz are curves that are each δ_N-monotone in direction F and $uw \cap xz$ contains at least one point p, then $\delta_k(u, z) = \delta_k(u, p) + \delta_k(p, z)$ and $\delta_k(x, w) = \delta_k(x, p) + \delta_k(p, w)$.*

When $X = \mathbb{R}^d$, we let $\mathcal{C}_d$ denote the set of curves over $\mathbb{R}^d$. For the sake of compactness, when $X = \mathbb{R}^d$, we denote the $(\mathbb{R}^d, \delta_N, \mathcal{C}_d)$-obstacle representation by *δ_N-obstacle representation.* For the plane case $d = 2$, we define, for each integer $k \ge 2 \in \mathbb{N}$, the *regular distance* *function* $\delta_k = \delta_{N_k}$, where $N_k = \{(\cos(i\pi/k), \sin(i\pi/k)) : i \in \{0, \dots, 2k-1\}\}$. In this case, the associated polygon P_N is a regular $2k$-gon. Moreover, we use δ_k-*obstacle representation* as shorthand for $(\mathbb{R}^2, \delta_k, \mathcal{C}_2)$-obstacle representation. Moreover, for a point in $\mathbb{R}^2$, we denote the i-sector of x by $Q_i^k(x)$, for $i \in \{0, \dots, 2k-1\}$.

In addition to polyhedral distance functions, we consider obstacle representations under *graph distance*. For a graph H, we denote the set of neighbours of a vertex u in H by $N_H(u)$ and the degree of u by $\deg_H(u)$. Moreover, let δ_H denote the graph distance and let $\mathcal{C}_H$ be the set of curves that define paths in H. Then, we call a $(H, \delta_H, \mathcal{C}_H)$-obstacle representation an H-obstacle representation. If we consider the infinite square grid H_4 (resp., the infinite triangular grid H_6), for instance, then it is not difficult to argue that a graph G has a non-crossing δ_2-obstacle representation (resp., non-crossing δ_3-obstacle representation) if and only if G has a non-crossing H_4-obstacle representation (resp., non-crossing H_6-obstacle representation). In general, for any integer $D > 1$, define the D-cube graph Q_D to be the graph with vertex set $V(Q_D) = \{0,1\}^D$ and that contains the edge uw if and only if u and w differ in exactly one coordinate.

3 General Representations

In this section, we show our results for the general representations. We first consider the special case of $\mathbb{R}^2$ and will then discuss our results for higher dimensions. We start by the following result.

▶ **Theorem 3.** *For any $\epsilon > 0$, there exists a graph G with $n = n(\epsilon)$ vertices such that G has no δ_k-obstacle representation for any $k < n^{1-\epsilon}$.*

Proof. For some constant $c > 0$ and all sufficiently large n, there exists a graph G with n vertices and $cn^{2-2/r}$ edges and that contains no $K_{r,r}$ as subgraph [1]. Let (φ, S) be a δ_k-obstacle representation of G and let (φ, c) be an embedding of G obtained by taking, for each $uw \in E(G)$, $c(uw)$ to be some shortest $\cup S$-avoiding path from $\varphi(u)$ to $\varphi(w)$. From this point on we identify the vertices of G with the points they are embedded to and the edges of G with the curves they are embedded to.

By definition each edge $uw \in E(G)$ is k-monotone. Since P_N has at most $2k$ facets and each edge is monotone in at least two of these directions, this means that it has some facet F such that G contains $E(G)/k$ edges that are monotone in direction F. Consider the graph G' consisting of only these edges and the embedding φ of G'. Observe that if two edges uw and xy of G' intersect at some point p, then (after appropriate relabelling), this implies that there is a $\cup S$-avoiding geodesic from u to x as well as from w to y. Therefore, $ux, uw \in E(G')$.

Therefore, if G' contains an r-tuple of pairwise crossing edges, then G' contains a $K_{r,r}$ subgraph. Now, observe that the edges of G' are monotone in some direction and (after an appropriate rotation) we can assume that they are x-monotone. We call this an x-*monotone embedding*. Valtr [17] has shown that for every fixed r, there exists a constant $C = C(r)$ such that any x-monotone embedding of any n-vertex graph with more than $Cn \log n$ edges contains a set of r pairwise crossing edges. In our case, this means that G contains a $K_{r,r}$ subgraph if $(cn^{2-2/r})/k \geq Cn \log n$, which gives a contradiction when $k \leq cn^{1-2/r}/C \log n$. The result then follows by choosing any $r > 2/\epsilon$. ◀

As $k \to \infty$, δ_k becomes the usual Euclidean distance function and δ_k-obstacle representations are just the usual obstacle representations, which we know every graph has. Thus, for every $n \in \mathbb{N}$, there is a threshold value $k(n)$ such that every n-vertex graph has a $\delta_{k(n)}$-obstacle representation. Theorem 3 shows that $k(n) \in \Omega(n^{1-\epsilon})$ and the following theorem shows that $k(n) \in O(n)$.

▶ **Theorem 4** (∗). *Every n-vertex graph G has a δ_k-obstacle representation for $k = \lceil n/2 \rceil$.*

Higher dimensions. The proof of Theorem 3 makes critical use of the fact that obstacle representations live in the plane so that any sufficiently dense (sub)graph has a k-tuple of pairwise crossing edges. An obvious question, then, is whether every graph has a δ_N-obstacle representation in $\mathbb{R}^3$ (i.e., an $(\mathbb{R}^3, \delta_N, \mathcal{C}_3)$-obstacle representation), where δ_N is some polyhedral distance function. The following theorem shows that the answer to this question is no.

▶ **Theorem 5.** *Let δ_N be a polyhedral distance function over $\mathbb{R}^d$ whose corresponding polyhedron P_N has $2k$ facets, for $k \in o(\log n)$. Then, there exists an n-vertex graph G that has no δ_N-obstacle representation.*

Proof. Let G be an n-vertex graph with no clique and no independent set of size larger than $2 \log n$. The existence of such graphs was shown by Erdős and Renyi [10]. Suppose, for the sake of contradiction, that G has some δ_N-obstacle representation (φ, S). Let $\prec$ denote lexicographic order over points in $\mathbb{R}^d$.

We will k-colour the $\binom{n}{2}$ pairs of vertices of G where the colours are facets of P_N. A pair (u, w) with $u \prec w$ is coloured with a facet F of P_N such that $w \in Q_F^N(u)$. If more than one such facet exists, we choose one arbitrarily. For each $i \in \{1, \dots, k\}$, let $\prec_i$ denote the partial order obtained by restricting the total order $\prec$ to the pairs of vertices in G with colour i. We claim that for at least one i, $\prec_i$ contains a chain $v_1 \prec_i \cdots \prec_i v_r$ of size $r \geq n^{1/k}$. To see why this is so, observe that, by Dilworth's Theorem, if $\prec_k$ does not contain a chain of length $n^{1/k}$, then it contains an antichain A_k of size $n^{1-1/k}$. Now, proceed inductively on $\prec_1, \dots, \prec_{k-1}$ and A_k, observing that every pair in A_k is coloured with $\{1, \dots, k-1\}$.

Next, consider the relation $\prec_i'$ over $v_1, \dots, v_r$ in which $v_a \prec_i v_b$ if and only if $1 \leq a < b \leq r$ and $v_a v_b \in E(G)$. Observe that $\prec_i'$ is a partial order over $\{v_1, \dots, v_r\}$. Therefore, by Dilworth's Theorem, it contains a chain of size at least $\sqrt{r}$ or it contains an antichain of size at least $\sqrt{r}$. A chain corresponds to a clique in G and an antichain corresponds to an independent set in G. This contradicts our choice of G when $\sqrt{r} > 2 \log n$, which is true for all $k \in o(\log n)$ and all sufficiently large n. ◀

Theorem 5 shows that, for some n-vertex graphs G, any δ_N-obstacle representation of G must use a distance function δ_N with $k = \Omega(\log n)$ facets. Our next result shows that, even in $\mathbb{R}^3$, a polyhedral distance function with $k = O(\log n)$ facets is indeed sufficient.

▶ **Theorem 6** (∗). *Let δ_N be any polyhedral distance function in $\mathbb{R}^d$, where $d \geq 3$, for which the polyhedron P_N has at least $2 \log_2 n$ facets. Then, every n-vertex graph G has a δ_N-obstacle representation.*

If we take t generic unit vectors in $\mathbb{R}^3$, then the polyhedral distance function determined by these vectors defines a polyhedron having $2t$ vertices and $4t-8$ triangular faces. Theorem 6 therefore implies that a polyhedral distance function determined by $t \geq (1/2) \log_2 n + 2$ unit vectors is sufficient to allow a obstacle representation of any n-vertex graph.

In constant dimensions $d > 3$, there exists sets of t vectors in $\mathbb{R}^d$ defining polytopes with $\Theta(t^{\lfloor d/2 \rfloor})$ facets. Therefore, in $\mathbb{R}^d$, every n-vertex graph has a δ_N-obstacle representation with $|N| \in O(\sqrt[\lfloor d/2 \rfloor]{\log n})$ vectors.

4 Non-Crossing Representations

In this section, we consider non-crossing δ_k-obstacle representations. The following lemma shows that these representations are equivalent to plane δ_k-obstacle embeddings.

▶ **Lemma 7** (∗). *A graph G has a non-crossing δ_k-obstacle representation if and only if G has a non-crossing δ_k-obstacle embedding.*

Lemma 7 allows us to focus our effort on studying the existence (or not) of plane δ_k-obstacle embeddings. We begin with non-crossing δ_k-obstacle embeddings of small treewidth graphs.

Treewidth. A *k-tree* is any graph that can be obtained in the following manner: we begin with a clique on $k+1$ vertices and then we repeatedly select a subset of the vertices that form a k-clique K and add a new vertex adjacent to every element in K. The class of k-trees is exactly the set of edge-maximal graphs of treewidth k. A graph G is called a *partial k-tree* if it is a subgraph of some k-tree. The class of partial k-trees is exactly the class of graphs of treewidth at most k. We will make use of the following lemma, due to Dujmović and Wood [9] in some recursive embeddings.

▶ **Lemma 8** (Dujmović and Wood [9]). *Every k-tree is either a clique on $k+1$ vertices or it contains a non-empty independent set S and a vertex $u \notin S$, such that (i) $G \setminus S$ is a k-tree, (ii) $\deg_{G\setminus S}(u) = k$, and (iii) every element in S is adjacent to u and $k-1$ elements of $N_{G\setminus S}(u)$.*

4.1 δ_2-Obstacle Representations

In this section, we focus on plane δ_2-obstacle embeddings. Recall that these are equivalent to the non-blocking planar grid obstacle representation studied by Biedl and Mehrabi [5]. We begin with the positive result that all graphs of treewidth at most 2 (i.e., partial 2-trees) have plane δ_2-obstacle embeddings.

▶ **Theorem 9.** *Every partial 2-tree has a plane straight-line δ_2-obstacle embedding.*

Proof. Let G be a partial 2-tree. We can, without loss of generality, assume that G is connected. If $|V(G)| < 4$, then the result is trivial, so we can assume $|V(G)| \geq 4$. We now proceed by induction on $|V(G)|$.

Let $T = T(G)$ be a 2-tree with vertex set $V(G)$ and that contains G. Apply Lemma 8 to find the vertex set S and the vertex u. Let x and y be the neighbours of u in $T \setminus S$. Now, apply induction to find a plane straight-line δ_2-obstacle embedding of the graph G' whose vertex set is $V(G') = V(G) \setminus S$ and whose edge set is $E(G') = E(G \setminus S) \cup \{ux, uy\}$. Denote by S_x (resp., S_y) the neighbours of x (resp., y) that belong to S.

Now, observe that, since u has degree 2 in G' and the edges ux and uy are in G', this embedding does not contain any monotone path of the form uxw or uyw for any $w \in V(G) \setminus \{u, x, y\}$. Therefore, if we place the vertices in S sufficiently close to u, we will not create any monotone path of the form ayw or axw for any $a \in S$ and any $w \in V(G) \setminus \{u, x, y\}$. What remains is to show how to place the elements of S in order to avoid unwanted monotone paths of the form uay, uax, or aub for any $a, b \in S$. There are three cases to consider:

1. $x \in Q_i^2(u)$ and $y \in Q_{i+2}^2(u)$ for some $i \in \{0, \dots, 3\}$. W.l.o.g., assume that $Q_{i+3}^2(u)$ does not intersect the segment xy. Then, we can embed the elements of S in $Q_{i+3}^2(u)$ without creating any new monotone paths; see Figure 2(a).
2. $x, y \in Q_i^2(u)$ for some $i \in \{0, \dots, 3\}$. There are two subcases:
   (i) At least one of ux or uy is in $E(G)$. Suppose $ux \in E(G)$. Then we embed S_x in $Q_i^2(u)$ and embed S_y in $Q_{i+3}^2(u)$; see Figure 2(b). The only monotone paths this creates are of the form uax with $a \in S_x$, which is acceptable since $ux \in E(G)$.

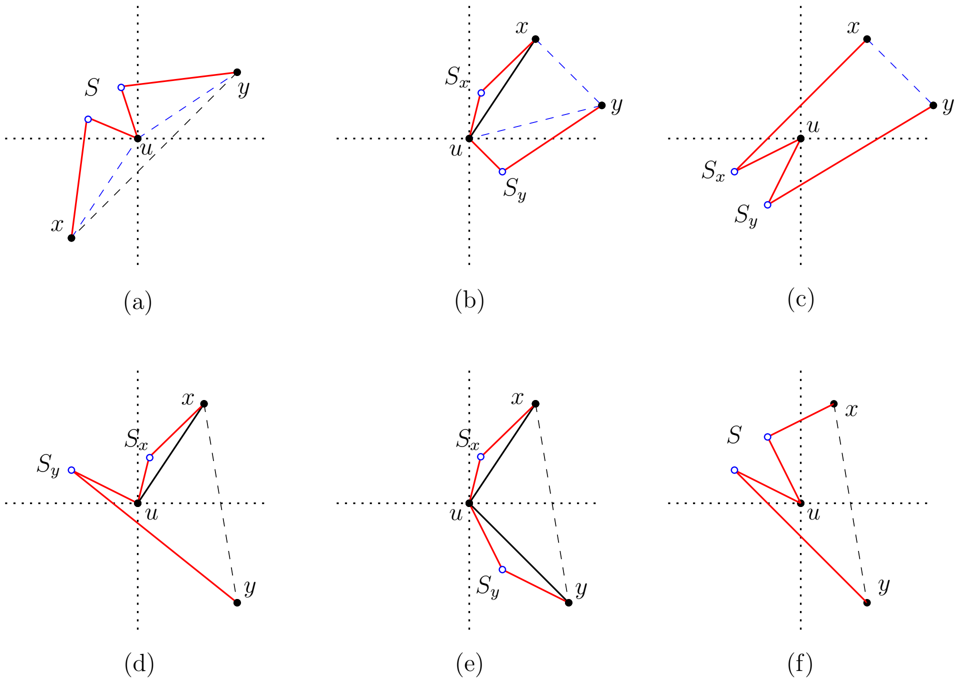

Figure 2 An illustration in supporting the proof of Theorem 9.

- **(ii)** Neither ux nor uy is in $E(G)$. In this case, we embed all of S in $Q^2_{i+2}(u)$ (see Figure 2(c)). This does not create any new monotone paths.

3. $x \in Q^2_i(u)$ and $y \in Q^2_{i+3}(u)$ for some $i \in \{0, \ldots, 3\}$. We have three subcases to consider:

- **(i)** $|\{ux, uy\} \cap E(G)| = 1$. In this case, assume $ux \in E(G)$. Then, we embed the vertices of S_x in $Q^2_i(u)$ and we embed the vertices of S_y in $Q^2_{i+1}(u)$. See Figure 2(d). The only monotone paths this creates are of the form uax with $a \in S_x$, which is acceptable since $ux \in E(G)$.
- **(ii)** $|\{ux, uy\} \cap E(G)| = 2$. In this case, we embed the vertices of S_x in $Q^2_i(u)$ and we embed the vertices of S_y in $Q^2_{i+3}(u)$ (see Figure 2(e)). The only monotone paths this creates are of the form uax with $a \in S_x$ and uby with $b \in S_y$, which is acceptable since $ux, uy \in E(G)$.
- **(iii)** $|\{ux, uy\} \cap E(G)| = 0$. In this case, we embed all of S into Q^2_{i+1} (see Figure 2(f)). This does not create any new monotone paths.

This completes the proof of the theorem. ◀

In the full version of the paper [7], we show that not every planar 3-tree admits a non-crossing δ_2-obstacle embedding.

▶ **Theorem 10** (∗). *There exists a planar 3-tree that does not have a non-crossing δ_2-obstacle embedding.*

We further prove that even 4-connectivity does not help to guarantee the existence of non-crossing δ_2-obstacle embeddings. To this end, we show that a 4-connected triangulation having a plane δ_2-obstacle representation must have a constrained 4-colouring in the sense that, for the neighbours of a vertex, which colours and in what order are they allowed to be assigned to them. The following theorem then follows by finding a 4-connected triangulation that does not admit such a constrained 4-colouring (see the full version of the paper [7]).

▶ **Theorem 11** (∗). *There exists a 4-connected triangulation G with maximum degree 7 that has no plane δ_2-obstacle embedding.*

4.2 Higher-k δ_k-Obstacle Representations

In this section, we consider the non-crossing embeddings for $k > 2$. We show that planar 3-trees have plane δ_3-obstacles embeddings and that all 3-connected cubic planar graphs have plane δ_7-obstacles embeddings. We start by planar 3-trees.

▶ **Theorem 12.** *Every planar 3-tree has a plane δ_3-obstacle embedding.*

Proof Sketch. Here, we sketch the proof; see the full version of the paper [7] for the complete proof. The proof is by induction on $n = |V(G)|$ in which our inductive hypothesis is that every n vertex planar 3-tree has a plane δ_3-obstacle embedding in which the neighbours of each vertex u occupy at least 3 of the sectors $Q_0^3(u), \ldots, Q_5^3(u)$. The key to our proof is the result of Dujmovic and Wood [9] when specialized to planar 3-trees, which says that every planar 3-tree is either K_4 or has a vertex u and an independent set S ($|S| \leq 3$) such that $G \setminus S$ is a 3-tree, u has degree 3 in $G \setminus S$ with neighbours x, y and z, and every vertex r in S forms a clique with exactly one of uxy, uyz or uzx.

By applying this result and recursing on $G \setminus S$ (when $n > 4$), we obtain a plane δ_3-obstacle embedding of $G \setminus S$. By our induction hypothesis, there are two cases depending on the locations of x, y and z with respect to u. In both cases, the elements of S are placed close enough to u that we do not create any new δ_3-monotone paths involving vertices other than those in $\{u, x, y, z\} \cup S$. Since $\{u, x, y, z\}$ form a clique, we only need to worry about (possibly) creating a new δ_3-monotone path involving at least one vertex of S. ◀

We next show that every 3-connected cubic planar graph has a plane δ_7-obstacle embedding. The algorithm contrustructs a δ_7-obstacle embedding by adding one vertex per time according to a canonical ordering of the graph [13], and at each step it maintains a set of geometric invariants which guarantee its correctness. The key ingredients are the fact that each new vertex v to be inserted has exactly two neighbors in the already constructed representation, together with the existence of a set of edges whose removal disconnects the representation in two parts, each containing one of the two neighbors of v. A sufficient stretching of these edges allows for a suitable placement for vertex v. See the full version of the paper for details [7].

▶ **Theorem 13** (∗). *Every 3-connected cubic plane graph has a plane δ_7-obstacle embedding.*

5 Graph Metrics

In this section, we consider the problem under graph distances. Recall the graph D-cube, Q_D whose vertex set is $V(Q_D) = \{0,1\}^D$ and that contains the edge uw if and only u and w differ in exactly one coordinate. It is not hard to see that every n vertex graph has a Q_n-obstacle representation: Each vertex of G is assigned a coordinate with a single 1 bit. Then, for any two vertices u and w there are exactly two shortest paths in Q_n joining them and they each have length 2. One path goes through the intermediate vertex $\mathbf{0} = (0, \ldots, 0)$ and the other goes through $u + w$. Therefore, by placing an obstacle at $\mathbf{0}$ and at each $u + w$ for which $uw \notin E(G)$, we obtain a Q_D-obstacle representation of G. The following theorem shows we can do this with much fewer coordinates.

▶ **Theorem 14.** *There exists a constant $C > 0$ such that, for $D = C \log n$, every n-vertex graph has a non-crossing Q_D-obstacle representation.*

Proof. Consider the following embedding (φ, c) of G into Q_D: For each $u \in V(G)$, $\varphi(u)$ is a random element of $\{0,1\}^D$. We use the notation u_i to denote the ith coordinate of u. Let $\prec$ denote lexicographic order on D-tuples. For each edge $uw \in E(G)$ with $u \prec w$, we take $c(uw)$ to be the *greedy* path that visits, for $i = 0, \dots, D$, the vertex $uw_i = (w_1, \dots, w_i, u_{i+1}, \dots, u_D)$. Thus $uw_0, \dots, uw_D$ is a sequence of vertices that—after removing duplicates—is a shortest path, in Q_D, from u to w. Note that there is an asymmetry here that we should be careful of, so for $u \prec w$, we define $wu_i = (w_1, \dots, w_{D-i}, u_{D-i+1}, \dots, u_D) = uw_{D-i}$. Here are some observations about the embedding (φ, c):

1. All vertex distances are close to $D/2$: The distance between any two vertices is a binomial$(D, 1/2)$ random variable. Therefore, by Chernoff's bounds, for any constant $\epsilon > 0$ and for any vertex pair $u \neq w$, $\Pr\{|\delta_{Q_D}(u,w) - D/2| > \epsilon(D/2)\} \le n^{-\Omega(C)}$. By the union bound, the probability that there exists any pair of vertices $u \neq w$ with $|\delta_{Q_D}(u,w) - D/2| > \epsilon(D/2)$ is also $n^{-\Omega(C)}$.
2. The embedding is non-crossing: For any four distinct vertices $u \prec w$ and $x \prec y$, and any $i, j \in \{0, \dots, D\}$, the vertices uw_i and xy_j are independent random D-bit strings. Therefore, $\Pr\{\delta_{Q_D}(uw_i, xy_j) \le 1\} = (D+1)/2^D$. By the union bound, the probability that there exists any four vertices u, w, x, y and any pair of indices i, j for which $\delta_{Q_D}(uw_i, xy_j) \le 1$ is at most $n^4(D+1)^3/2^D = n^{-\Omega(C)}$.
3. No geodesic passes close to a vertex except its endpoints: Let u, w, and x be distinct vertices and $r \in \{0, \dots, D\}$ be an integer. Then, the probability that there exists any geodesic with endpoints u and w that contains a vertex z with $\delta_{Q_D}(z, x) \le r$ is at most $n^{-\Omega(C)}$. To see why this is so, suppose that such a geodesic, C, contains a vertex z such that $\delta_{Q_D}(z, x) \le r$. Then, at least one of the following events occurs:
   (a) $\delta_{Q_D}(u, w) \ge (1+\epsilon)D/2$;
   (b) $\delta_{Q_d}(u, x) \le (1+\epsilon)D/4 + r$; or
   (c) $\delta_{Q_d}(w, x) \le (1+\epsilon)D/4 + r$.
   Point 1 above establishes that the probability of the first event is $n^{-\Omega(C)}$ and that, for $r \le (1-3\epsilon)D/4$, the probability of each of the other two events is $n^{-\Omega(C)}$. Applying the union bound over all 3 events, and over all $\binom{n}{3}$ choices of u, w, and x then shows that the probability that there is any triple u, w, x such that any geodesic from u to w passes within distance $(1-3\epsilon)D/4$ of x is $n^{-\Omega(C)}$.
4. Paths diverge quickly: Let $xu, xw \in E(G)$, be two edges of G with the common endpoint x and let $r \in \{0, \dots, D\}$. We want to show that the directed paths xu and xw diverge quickly. There are three cases to consider:
   a. $x \prec u$ and $x \prec w$. In this case $xu_r = xw_r$ if and only if $u_1, \dots, u_r = w_1, \dots, w_r$, so $\Pr\{xu_r = xw_r\} = 2^{-r}$.
   b. $x \prec u$ and $w \prec x$. In this case, we consider $xu_r = u_1, \dots, u_r, x_{r+1}, \dots, x_D$ and $xw_r = wx_{D-r} = x_1, \dots, x_{D-r}, w_{D-r+1}, \dots, w_D$. For any choice of i, these two strings have independent bits in at least r locations, so $\Pr\{xu_r = xw_r\} \le 2^{-r}$.
   c. $u \prec x$ and $w \prec x$. In this case $xu_r = ux_{D-r} = x_1, \dots, x_{D-r}, u_{D-r+1}, \dots, u_D$ and $xw_r = wx_{D-r} = x_1, \dots, x_{D-r}, w_{D-i+1}, \dots, w_D$. So $\Pr\{xu_r = xw_r\} = 2^{-r}$.

   If we choose $r = \alpha \log n$, then this probability is at most $n^{-\Omega(\alpha)}$. Again, the union bound shows that the probability that there is any u, w, or x such that $xu_r = xw_r$ is at most $n^{-\Omega(\alpha)}$.

In the following, we choose C sufficiently large and $\alpha < (1/4 - \epsilon)C$ also sufficiently large so that with probability greater than 0, we obtain an embedding for which all four of preceding properties hold. Therefore, there exists some embedding (φ, c) such that 1. for all $u, w \in V(G)$, $|\delta_{Q_D}(u,w) - D/2| \le \epsilon D/2$; 2. for all $uw, xy \in E(G)$ with $\{u, w\} \cap \{x, y\} = \emptyset$,

$\delta_{Q_D}(c(uw), c(xy)) > 1$; 3. for all $uw \in E(G)$ and $x \in V(G) \setminus \{u, w\}$, $\delta_{Q_D}(c(uw), x) \geq (1-\epsilon)D/4$; and 4. for all $xu, xw \in E(G)$ and all $r \geq \alpha \log n$, $xu_r \neq xw_r$.

To obtain a Q_D-obstacle representation (φ, S) we take S to contain all the vertices not used in any path of the embedding (φ, c). To verify that this is indeed a non-crossing Q_D-obstacle representation, we need only verify that, for any $u, w \in V(G)$ with $uw \notin E(G)$, $\delta_{Q_D \setminus S}(u, w) > \delta_{Q_D}(u, w)$. This is implied by the following inequality, which relates distances in G to those in $Q_D \setminus S$:

$$\delta_{Q_D \setminus S}(u, w) \geq \delta_G(u, w)(1-\epsilon)D/2 - (\delta_G(u, w) - 1)2\alpha \log n \ . \qquad (1)$$

Note (1) is sufficient since, if $uw \notin E(G)$, then $\delta_G(u, w) \geq 2$ and (1) implies $\delta_{Q_D \setminus S}(u, w) \geq (1-\epsilon)D - 2\alpha \log n = ((1-\epsilon)C - 2\alpha) \log n > (1+\epsilon)D/2$, which contradicts Property 1. Thus, all that remains is to establish (1). To do this, consider any path P from u to w in $Q_D \setminus S$. Since the only vertices in $Q_D \setminus S$ are those that are used by some embedded edge of G, the path P consists of a sequence of subpaths $P_0, \ldots, P_k$ where each P_i is a subpath of $c(x_i y_i)$ for some edge $x_i y_i \in E(G)$. Note that Property 3 implies that $x_0 = u$ and that $x_k = w$. Furthermore, Properties 2 and 3 imply that $x_i = y_{i-1}$ for each $i \in \{1, \ldots, k\}$. Therefore, $x_0, \ldots, x_k$ is a path in G from u to w, so $k \geq \delta_G(u, w)$. Finally, Property 4 implies that, for each $i \in \{1, \ldots, k-1\}$, the portion of $c(x_i, x_{i+1})$ not used by P_i has length at most $2\alpha \log n$. Thus, the length of P is at least $k(1-\epsilon)D/2 - 2(k-1)\alpha \log n$, as required. ◀

It is worth noting that Theorem 14 is closely related to Theorem 6. Indeed, before perturbing it, the point set X used in the proof of Theorem 6 is a projection of the vertices of Q_D with $D = \lceil \log_2 n \rceil$ onto $\mathbb{R}^3$. In Theorem 6 we then perturb X to obtain a non-crossing embedding. In the proof of Theorem 14 we have to be more careful to avoid crossings.

6 Conclusion

In this paper, we introduced the geodesic obstacle representation of graphs, providing a unified generalization of obstacle representations and grid obstacle representations. Our work leaves several problems open. As perhaps the main question, does every planar graph admit a non-crossing δ_k-obstacle representation for some constant k? It would be also interesting to extend the classes of graphs for which non-crossing δ_k-obstacle representations exist for small values of k. For graph metrics, given two graphs G and H, is it NP-hard to decide if G has an H-obstacle representation?

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