


Partial Complementation of Graphs


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Abstract

A *partial complement* of the graph G is a graph obtained from G by complementing all the edges in one of its induced subgraphs. We study the following algorithmic question: for a given graph G and graph class \mathcal{G} , is there a partial complement of G which is in \mathcal{G} ? We show that this problem can be solved in polynomial time for various choices of the graphs class \mathcal{G} , such as bipartite, degenerate, or cographs. We complement these results by proving that the problem is NP-complete when \mathcal{G} is the class of r -regular graphs.

2012 ACM Subject Classification Mathematics of computing → Graph algorithms, Theory of computation → Graph algorithms analysis

Keywords and phrases Partial complementation, graph editing, graph classes

Digital Object Identifier 10.4230/LIPIcs.SWAT.2018.21

Related Version A full version of the paper is available at <http://arxiv.org/abs/1804.10920>

Acknowledgements We thank Saket Saurabh for helpful discussions, and also a great thanks to the anonymous reviewers who provided valuable feedback.

1 Introduction

One of the most important questions in graph theory concerns the efficiency of recognition of a graph class \mathcal{G} . For example, how fast we can decide whether a graph is chordal, 2-connected,

¹ Supported by the Research Council of Norway via the projects “CLASSIS” and “MULTIVAL”.

² Supported by the Research Council of Norway via the project “CLASSIS”.

³ Supported by the Research Council of Norway via the project “MULTIVAL”.

⁴ Supported by project “DEMOGRAPH” (ANR-16-CE40-0028).



triangle-free, of bounded treewidth, bipartite, 3-colorable, or excludes some fixed graph as a minor? In particular, the recent developments in parameterized algorithms are driven by the problems of recognizing of graph classes which do not differ up to a “small disturbance” from graph classes recognizable in polynomial time. The amount of disturbance is quantified in “atomic” operations required for modifying an input graph into the “well-behaving” graph class \mathcal{G} . The standard operations could be edge/vertex deletions, additions or edge contractions. Many problems in graph algorithms fall into this graph modification category: is it possible to add at most k edges to make a graph 2-edge connected or to make it chordal? Or is it possible to delete at most k vertices such that the resulting graph has no edges or contains no cycles?

A rich subclass of modification problems concerns edge editing problems. Here the “atomic” operation is the change of adjacency, i. e. for a pair of vertices u, v , we can either add an edge uv or delete the edge uv . For example, the CLUSTER EDITING problem asks to transform an input graph into a cluster graph, that is a disjoint union of cliques, by flipping at most k adjacency relations.

Besides the basic edge editing, it is natural to consider problems where the set of removed and added edges should satisfy some structural constraints. In particular, such problems were considered for *complementation* problems. Recall that the *complement* of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G . Seidel (see [19, 20, 21]) introduced the operation that is now known as the *Seidel switch*. For a vertex v of a graph G , this operation complements the adjacencies of v , that is, it removes the edges incident to v and makes v adjacent to the non-neighbors of v in G . Respectively, for a set of vertices U , the Seidel switching, that is, the consecutive switching for the vertices of U , complements the adjacencies between U and its complement $V(G) \setminus U$. The study of the algorithmic question whether it is possible to obtain a graph from a given graph class by the Seidel switch was initiated by Ehrenfeucht et al. [7]. Further results were established in [11, 12, 13, 16, 15]. Another important operation of this type is the *local complementation*. For a vertex v of a graph G , the *local complementation of G at v* is the graph obtained from G by replacing $G[N(v)]$ by its complement. This operation plays crucial role in the definition of *vertex-minors* [17] and was investigated in this contest (see, e.g. [6, 18]). See also [2, 14] for some algorithmic results concerning local complementations.

In this paper we study the *partial complement* of a graph, which was introduced by Kamiński, Lozin, and Milanič in [14] in their study of the clique-width of a graph. A *partial complement* of a graph G is a graph obtained from G by complementing all the edges of one of its induced subgraphs. More formally, for a graph G and $S \subseteq V(G)$, we define $G \oplus S$ as the graph with the vertex set $V(G)$ whose edge set is defined as follows: a pair of distinct vertices u, v is an edge of $G \oplus S$ if and only if one of the following holds:

- $uv \in E(G) \wedge (u \notin S \vee v \notin S)$, or
- $uv \notin E(G) \wedge u \in S \wedge v \in S$.

Thus when the set S consists only of two vertices $\{u, v\}$, then the operation changes the adjacency between u and v , and for a larger set S , $G \oplus S$ changes the adjacency relations for all pairs of vertices of S .

We say that a graph H is a partial complement of the graph G if H is isomorphic to $G \oplus S$ for some $S \subseteq V(G)$. For a graph class \mathcal{G} and a graph G , we say that there is a *partial complement of G to \mathcal{G}* if for some $S \subseteq V(G)$, we have $G \oplus S \in \mathcal{G}$. We denote by $\mathcal{G}^{(1)}$ the class of graphs such that its members can be partially complemented to \mathcal{G} .

Let \mathcal{G} be a graph class. We consider the following generic algorithmic problem.

PARTIAL COMPLEMENT TO \mathcal{G} (PC \mathcal{G})

Input: A simple undirected graph G .

Question: Is there a partial complement of G to \mathcal{G} ?

In other words, how difficult is it to recognize the class $\mathcal{G}^{(1)}$? In this paper we show that there are many well-known graph classes \mathcal{G} such that $\mathcal{G}^{(1)}$ is recognizable in polynomial time. We show that

- PARTIAL COMPLEMENT TO \mathcal{G} is solvable in time $\mathcal{O}(f(n) \cdot n^4 + n^6)$ when \mathcal{G} is a triangle-free graph class recognizable in time $f(n)$. For example, this implies that when \mathcal{G} is the class of bipartite graphs, the class $\mathcal{G}^{(1)}$ is recognizable in polynomial time.
- PARTIAL COMPLEMENT TO \mathcal{G} is solvable in time $f(n) \cdot n^{\mathcal{O}(1)}$ when \mathcal{G} is a d -degenerate graph class recognizable in time $f(n)$. Thus when \mathcal{G} is the class of planar graphs, class of cubic graphs, class of graph of bounded treewidth, or class of H -minor free graphs, then the class $\mathcal{G}^{(1)}$ is recognizable in polynomial time.
- PARTIAL COMPLEMENT TO \mathcal{G} is solvable in polynomial time when \mathcal{G} is a class of bounded clique-width expressible in monadic second-order logic (with no edge set quantification). In particular, if \mathcal{G} is the class of P_4 -free graphs (cographs), then $\mathcal{G}^{(1)}$ is recognizable in polynomial time.
- PARTIAL COMPLEMENT TO \mathcal{G} is solvable in polynomial time when \mathcal{G} can be described by a 2×2 M -partition matrix. Therefore $\mathcal{G}^{(1)}$ is recognizable in polynomial time when \mathcal{G} is the class of split graphs, as they can be described by such a matrix.

Nevertheless, there are cases when the problem is NP-hard. In particular, we prove that this holds when \mathcal{G} is the class of r -regular graphs.

2 Partial complementation to triangle-free graph classes

A triangle is a complete graph on three vertices. Many graph classes does not allow the triangle as a subgraph, for instance trees, forests, or graphs with large girth. In this paper we show that partial complementation to triangle-free graphs can be decided in polynomial time.

More precisely, we show that if a graph class \mathcal{G} can be recognized in polynomial time and it is triangle-free, then we can also solve PARTIAL COMPLEMENT TO \mathcal{G} in polynomial time.

Our algorithm is constructive, and returns a *solution* $S \subseteq V(G)$, that is a set S such that $G \oplus S$ is in \mathcal{G} . We say that a solution *hits* an edge uv (or a non-edge \overline{uv}), if both u and v are contained in S .

Our algorithm considers each of the following cases.

- (i) There is a solution S of size at most two.
- (ii) There is a solution S containing two vertices that are non-adjacent in G .
- (iii) There is a solution S such that it form a clique of size at least 3 in G .
- (iv) G is a no-instance.

Case (i) can be resolved in polynomial time by brute-force, and thus we start from analyzing the structure of a solution in Case (ii). We need the following observation.

► **Observation 1.** *Let \mathcal{G} be a class of triangle-free graphs and let G be an instance of PARTIAL COMPLEMENT TO \mathcal{G} , where $S \subseteq V(G)$ is a valid solution. Then*

- a) $G[S]$ does not contain an independent set of size 3, and
- b) for every triangle $\{u, v, w\} \subseteq V(G)$, at least two vertices are in S .

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Because all non-edges between vertices in $G[S]$ become edges in $G \oplus S$ and vice versa, whereas all (non-) edges with an endpoint outside S remain untouched, we see that the observation holds.

Let us recall that a graph G is a *split graph* if its vertex set can be partitioned into $V(G) = C \cup I$, where C is a clique and I is an independent set. Let us note that the vertex set of a split graph can have several *split partitions*, i.e. partitions into a clique and independent set. However, the number of split partitions of an n -vertex split graphs is at most n . The analysis of Case (ii) is based on the following lemma.

► **Lemma 2.** *Let \mathcal{G} be a class of triangle-free graphs and let G be an instance of PARTIAL COMPLEMENT TO \mathcal{G} . Let $S \subseteq V(G)$ be a valid solution which is not a clique, and let $u, v \in S$ be distinct vertices such that $uv \notin E(G)$. Then*

- a) *the entire solution S is a subset of the union of the closed neighborhoods of u and v , that is $S \subseteq N_G[u] \cup N_G[v]$;*
- b) *every common neighbor of u and v must be contained in the solution S , that is $N_G(u) \cap N_G(v) \subseteq S$;*
- c) *the graph $G[N(u) \setminus N(v)]$ is a split graph. Moreover, $(N(u) \setminus N(v)) \cap S$ is a clique and $(N(u) \setminus N(v)) \setminus S$ is an independent set.*

Proof. We will prove each point separately, and in order.

- a) Assume for the sake of contradiction that the solution S contains a vertex $w \notin N_G[u] \cup N_G[v]$. But then $\{u, v, w\}$ is an independent set in G , which contradicts item a) of Observation 1.
- b) Assume for the sake of contradiction that the solution S does not contain a vertex $w \in N_G(u) \cap N_G(v)$. Then the edges uw and vw will both be present in $G \oplus S$, as well as the edge uv . Together, these forms a triangle.
- c) We first claim that the solution S is a vertex cover for $G[N(u) \setminus N(v)]$. If it was not, then there would exist an edge u_1u_2 of $G[N(u) \setminus N(v)]$ such that both endpoints $u_1, u_2 \notin S$, yet u_1, u_2 would form a triangle with u in $G \oplus S$, which would be a contradiction. Hence $(N(u) \setminus N(v)) \setminus S$ is an independent set. Secondly, we claim that $(N(u) \setminus N(v)) \cap S$ forms a clique. If not, then there would exist $u_1, u_2 \in (N(u) \setminus N(v)) \cap S$ which are nonadjacent. In this case $\{u_1, u_2, v\}$ is an independent set, which contradicts item a) of Observation 1. Taken together, these claims imply the last item of the lemma. ◀

We now move on to examine the structure of a solution for the third case, when there exists a solution which is a clique of size at least three.

► **Lemma 3.** *Let \mathcal{G} be a class of triangle-free graphs and let G be an instance of PARTIAL COMPLEMENT TO \mathcal{G} . Let $S \subseteq V(G)$ be a solution such that $|S| \geq 3$ and $G[S]$ is a clique. Let $u, v \in S$ be distinct. Then*

- a) *the solution S is contained in their common neighborhood, that is $S \subseteq N_G[u] \cap N_G[v]$, and*
- b) *the graph $G[N_G[u] \cap N_G[v]]$ is a split graph where $(N_G[u] \cap N_G[v]) \setminus S$ is an independent set.*

Proof. We prove each point separately, and in order.

- a) Assume for the sake of contradiction that the solution S contains a vertex w which is not in the neighborhood of both u and v . This contradicts that S is a clique.
- b) We claim that S is a vertex cover of $G[N_G[u] \cap N_G[v]]$. Because S is also a clique, the statement of the lemma will then follow immediately. Assume for the sake of contradiction that S is not a vertex cover. Then there exist an uncovered edge w_1w_2 , where $w_1, w_2 \in N_G[u] \cap N_G[v]$, and also $w_1, w_2 \notin S$. Since $\{u, w_1, w_2\}$ form a triangle,

we have by b) of Observation 1 that at least two of these vertices are in S . That is a contradiction, so our claim holds. \blacktriangleleft

We now have everything in place to present the algorithm.

► **Algorithm 4** (PARTIAL COMPLEMENT TO \mathcal{G} where \mathcal{G} is triangle-free).

Input: An instance G of PC \mathcal{G} where \mathcal{G} is a triangle-free graph class recognizable in time $f(n)$ for some function f .

Output: A set $S \subseteq V(G)$ such that $G \oplus S$ is in \mathcal{G} , or a correct report that no such set exists.

1. By brute force, check if there is a solution of size at most 2. If yes, return this solution.
2. For every non-edge \overline{uv} of G :
 - a. If either $G[N_G(u) \setminus N_G(v)]$ or $G[N_G(u) \setminus N_G(v)]$ is not a split graph, skip this iteration and try the next non-edge.
 - b. Let (I_u, C_u) and (I_v, C_v) denote a split partition of $G[N_G(u) \setminus N_G(v)]$ and $G[N_G(v) \setminus N_G(u)]$ respectively. For each pair of split partitions $(I_u, C_u), (I_v, C_v)$:
 - i. Construct solution candidate $S' := \{u, v\} \cup (N_G(u) \cap N_G(v)) \cup C_u \cup C_v$
 - ii. If $G \oplus S'$ is a member of \mathcal{G} , return S'
3. Find a triangle $\{x, y, z\}$ of G
4. For each edge in the triangle $uv \in \{xy, xz, yz\}$:
 - a. If $G[N_G(u) \cap N_G(v)]$ is not a split graph, skip this iteration and try the next edge.
 - b. For each possible split partition (I, C) of $G[N_G(u) \cap N_G(v)]$:
 - i. Construct solution candidate $S' := \{u, v\} \cup C$
 - ii. If $G \oplus S'$ is a member of \mathcal{G} , return S'
5. Return ‘NONE’

► **Theorem 5.** *Let \mathcal{G} be a class of triangle-free graphs such that deciding whether an n -vertex graph is in \mathcal{G} is solvable in time $f(n)$ for some function f . Then PARTIAL COMPLEMENT TO \mathcal{G} is solvable in time $\mathcal{O}(n^6 + n^4 \cdot f(n))$.*

Proof. We will prove that Algorithm 4 is correct, and that its running time is $\mathcal{O}(n^4 \cdot (n^2 + f(n)))$. We begin by proving correctness. Step 1 is trivially correct. After Step 1 we can assume that any valid solution has size at least three, and we have handled Case (i) when there exists a solution of size at most two. We have the three cases left to consider: (ii) There exists a solution which hits a non-edge, (iii) there is a solution S such that in $G \oplus S$ vertices of S form a clique of size at least 3, and (iv) no solution exists.

In the case that there exists a solution S hitting a non-edge uv , we will at some point guess this non-edge in Step 2 of the algorithm. By Lemma 2, we have that both $G[N_G(u) \setminus N_G(v)]$ and $G[N_G(v) \setminus N_G(u)]$ are split graphs, so we do not miss the solution S in Step 2a. Since we try every possible combinations of split partitions in Step 2b, we will by Lemma 2 at some point construct S' correctly such that $S' = S$.

In the case that there exist only solutions which hits exactly a clique, we first find some triangle $\{x, y, z\}$ of G . It must exist, since a solution S is a clique of size at least three. By Observation 1b, at least two vertices of the triangle must be in the S . At some point in step 4 we guess these vertices correctly. By Lemma 3b we know that $G[N_G(u) \cap N_G(v)]$ is a split graph, so we will not miss S in Step 4a. Since we try every split partition in Step 4b, we will by Lemma 3 at some point construct S' correctly such that $S' = S$.

Lastly, in the case that there is no solution, we know that there neither exists a solution of size at most two, nor a solution which hits a non-edge, nor a solution which hits a clique of size at least three. Since these three cases exhaust the possibilities, we can correctly report that there is no solution when none was found in the previous steps.

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For the runtime, we start by observing that Step 1 takes time $\mathcal{O}(n^2 \cdot f(n))$. The sub-procedure of Step 2 is performed $\mathcal{O}(n^2)$ times, where step 2a takes time $\mathcal{O}(n \log n)$. The sub-procedure of Step 2b takes time at most $\mathcal{O}(n^2 + f(n))$, and it is performed at most $\mathcal{O}(n^2)$ times. In total, Step 2 will use no longer than $\mathcal{O}(n^4 \cdot (n^2 + f(n)))$ time. Step 3 is trivially done in time $\mathcal{O}(n^3)$. The sub-procedure of Step 4 is performed at most three times. Step 4a is done in $\mathcal{O}(n \log n)$ time, and step 4b is done in $\mathcal{O}(n \cdot (n^2 + f(n)))$ time, which also becomes the asymptotic runtime of the entire step 4. The worst running time among these steps is Step 2, and as such the runtime of Algorithm 4 is $\mathcal{O}(n^4 \cdot (n^2 + f(n)))$. ◀

3 Complement to degenerate graphs

For $d > 0$, we say that a graph G is d -degenerate, if every induced (not necessarily proper) subgraph of G has a vertex of degree at most d . For example, trees are 1-degenerate, while planar graphs are 5-degenerate.

► **Theorem 6.** *Let \mathcal{G} be a class of d -degenerate graphs such that deciding whether an n -vertex graph is in \mathcal{G} is solvable in time $f(n)$ for some function f . Then PARTIAL COMPLEMENT TO \mathcal{G} is solvable in time $f(n) \cdot n^{2^{\mathcal{O}(d)}}$.*

Proof. Let G be an n -vertex graph. We are looking for a vertex subset S of G such that $G \oplus S \in \mathcal{G}$.

We start from trying all vertex subsets of G of size at most $2d$ as a candidate for S . Thus, in time $\mathcal{O}(n^{2d} \cdot f(n))$ we either find a solution or conclude that a solution, if it exists, should be of size more than $2d$.

Now we assume that $|S| > 2d$. We try all subsets of $V(G)$ of size $2d + 1$. Then if G can be complemented to \mathcal{G} , at least one of these sets, say X , is a subset of S . In total, we enumerate $\binom{n}{2d+1}$ sets.

First we consider the set Y of all vertices in $V(G) \setminus X$ with at least $d + 1$ neighbors in X . The observation here is that most vertices from Y are in S . More precisely, if more than

$$\alpha = \binom{|X|}{d+1} \cdot d + 1 = \binom{2d+1}{d+1} \cdot d + 1$$

vertices of Y are not in S , then $G \oplus S$ contains a complete bipartite graph $G_{d+1, d+1}$ as a subgraph, and hence $G \oplus S$ is not d -degenerate. Thus, we make at most $\binom{n}{\alpha}$ guesses on which subset of Y is in S .

Similarly, when we consider the set Z of all vertices from $V(G) \setminus X$ with at most d neighbors in X , we have that at most α of vertices from Z could belong to S . Since $V(G) = X \cup Y \cup Z$, if there is a solution S , it will be found in at least one from

$$\binom{n}{2d+1} \cdot \alpha^2 = n^{2^{\mathcal{O}(d)}}$$

of the guesses. Since for each set S we can check in time $f(n)$ whether $G \oplus S \in \mathcal{G}$, this concludes the proof. ◀

4 Complement to M-partition

Many graph classes can be defined by whether it is possible to partition the vertices of graphs in the class such that certain internal and external edge requirements of the parts are met. For instance, a complete bipartite graph is one which can be partitioned into two sets such

that every edge between the two sets is present (external requirement), and no edge exists within any of the partitions (internal requirements). Other examples are split graphs and k -colorable graphs. Feder et al. [8] formalized such partition properties of graph classes by making use of a symmetric matrix over $\{0, 1, \star\}$, called an M -partition.

► **Definition 7** (M -partition). For a $k \times k$ matrix M , we say that a graph G belongs to the graph class \mathcal{G}_M if its vertices can be partitioned into k (possibly empty) sets X_1, X_2, \dots, X_k such that, for every $i \in [k]$, if

- $M[i, i] = 1$, then X_i is a clique and if $M[i, i] = 0$, then X_i is an independent set, and for every $i, j \in [k], i \neq j$,
- if $M[i, j] = 1$, then every vertex of X_i is adjacent to all vertices of X_j ,
- if $M[i, j] = 0$, then there is no edges between X_i and X_j .

Note that if $M[i, j] = \star$, then there is no restriction on the edges between vertices from X_i and X_j .

For example, for matrix

$$M = \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix}$$

the corresponding class of graphs is the class of bipartite graphs, while matrix

$$M = \begin{pmatrix} 0 & \star \\ \star & 1 \end{pmatrix}$$

identifies the class of split graphs.

In this section we prove the following theorem.

► **Theorem 8.** *Let $\mathcal{G} = \mathcal{G}_M$ be a graph class described by an M -partition matrix of size 2×2 . Then PARTIAL COMPLEMENT TO \mathcal{G} is solvable in polynomial time.*

In particular, Theorem 8 yields polynomial algorithms for PARTIAL COMPLEMENT TO \mathcal{G} when \mathcal{G} is the class of split graphs or (complete) bipartite graphs. The proof of our theorem is based on the following beautiful dichotomy result of Feder et al. [8] on the recognition of classes \mathcal{G}_M described by 4×4 matrices.

► **Proposition 9** ([8, Corollary 6.3]). *Suppose M is a symmetric matrix over $\{0, 1, \star\}$ of size $k = 4$. Then the recognition problem for \mathcal{G}_M is*

- NP-complete when M contains the matrix for 3-coloring or its complement, and no diagonal entry is \star .
- Polynomial time solvable otherwise.

► **Lemma 10.** *Let M be a symmetric $k \times k$ matrix giving rise to the graph class $\mathcal{G}_M = \mathcal{G}$. Then there exists a $2k \times 2k$ matrix M' such that for any input G to PARTIAL COMPLEMENT TO \mathcal{G} , it is a yes-instance if and only if G belongs to $\mathcal{G}_{M'}$.*

Proof. Given M , we construct a matrix M' in linear time. We let M' be a matrix of dimension $2k \times 2k$, where entry $M'[i, j]$ is defined as $M[\lceil \frac{i}{2} \rceil, \lceil \frac{j}{2} \rceil]$ if at least one of i, j is even, and $\neg M[\frac{i+1}{2}, \frac{j+1}{2}]$ if i, j are both odd. Here, $\neg 1 = 0$, $\neg 0 = 1$, and $\neg \star = \star$. For example, for matrix

$$M = \begin{pmatrix} 0 & \star \\ \star & 1 \end{pmatrix}$$

the above construction results in

$$M' = \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 0 & \star & \star \\ \star & \star & 0 & 1 \\ \star & \star & 1 & 1 \end{pmatrix}.$$

We prove the two directions separately.

(\implies) Assume there is a partial complementation $G \oplus S$ into \mathcal{G}_M . Let X_1, X_2, \dots, X_k be an M -partition of $G \oplus S$. We define partition $X'_1, X'_2, \dots, X'_{2k}$ of G as follows. For every vertex $v \in X_i$, $1 \leq i \leq k$, we assign v to X'_{2i-1} if $v \in S$ and to X'_{2i} otherwise.

We now show that every edge of G respects the requirements of M' . Let $uv \in E(G)$ be an edge, and let $u \in X_i$ and $v \in X_j$. If at least one vertex from $\{u, v\}$, say v is not in S , then uv is also an edge in $G \oplus S$, thus $M[i, j] \neq 0$. Since $v \notin S$, it belongs to set $v \in X'_{2j}$. Vertex u is assigned to set X'_ℓ , where ℓ is either $2i$ or $2i - 1$, depending whether u belongs to S or not. But because $2j$ is even irrespectively of ℓ , $M'[\ell, 2j] = M[i, j] \neq 0$.

Now consider the case when both $u, v \in S$. Then the edge does not persist after the partial complementation by S , and thus $M[i, j] \neq 1$. We further know that u is assigned to X'_{2i-1} and v to X'_{2j-1} . Both $2i - 1$ and $2j - 1$ are odd, and by the construction of M' , we have that $M'[2i - 1, 2j - 1] \neq 0$, and again the edge uv respects M' . An analogous argument shows that also all non-edges respect M' .

(\impliedby) Assume that there is a partition $X'_1, X'_2, \dots, X'_{2k}$ of G according to M' . Let the set S consist of all vertices in odd-indexed parts of the partition. We now show that $G \oplus S$ can be partitioned according to M . We define partition X_1, X_2, \dots, X_k by assigning each vertex $u \in X'_i$ to $X_{\lceil \frac{i}{2} \rceil}$. It remains to show that X_1, X_2, \dots, X_k is an M -partition of $G \oplus S$.

Let $u \in X_i, v \in X_j$. Suppose first that $uv \in E(G \oplus S)$. If at least one of u, v is not in S , we assume without loss of generality that $v \notin S$. Then $uv \in E(G)$ and $v \in X'_{2j}$. For vertex $u \in X'_\ell$, irrespectively, whether ℓ is $2i$ or $2i - 1$, we have that $M'[\ell, 2j] = M[i, j] \neq 0$. But then $M[i, j] \neq 0$. Otherwise we have $u, v \in S$. Then uv is a non-edge in G , and thus $M'[2i - 1, 2j - 1] \neq 1$. But by the construction of M' , we have that $M[i, j] \neq 0$, and there is no violation of M . An analogous argument shows that if u and v are not adjacent in $G \oplus S$, it holds that $M[i, j] \neq 1$. Thus X_1, X_2, \dots, X_k is an M -partition of $G \oplus S$, which concludes the proof. \blacktriangleleft

Now we are ready to prove Theorem 8.

Proof of Theorem 8. For a given matrix M , we use Lemma 10 to construct a matrix M' . Let us note that by the construction of matrix M' , for every 2×2 matrix M we have that matrix M' has at most two 1's and at most two 0's along the diagonal. Then by Proposition 9, the recognition of whether G admits M' -partition is in P. Thus by Lemma 10, PARTIAL COMPLEMENT TO \mathcal{G} is solvable in polynomial time \blacktriangleleft

5 Partial complementation to graph classes of bounded clique-width

We show that PARTIAL COMPLEMENT TO \mathcal{G} can be solved in polynomial time when \mathcal{G} has bounded clique-width and can be expressed by an MSO₁ property. We refer to the book [3] for the basic definitions. We will use the following result of Hliněný and Oum [10].

► Proposition 11 ([10]). *There is an algorithm that for every integer k and graph G in time $O(|V(G)|^3)$ either computes a $(2^{k+1} - 1)$ expression for a graph G or correctly concludes that the clique-width of G is more than k .*

Note that the algorithm of Hliněný and Oum only approximates the clique-width but does not provide an algorithm to construct an optimal k -expression tree for a graph G of clique-width at most k . But this approximation is usually sufficient for algorithmic purposes.

Courcelle, Makowsky and Rotics [4] proved that every graph property that can be expressed in MSO_1 can be recognized in linear time for graphs of bounded clique-width when given a k -expression.

► **Proposition 12** ([4, Theorem 4]). *Let \mathcal{G} be some class of graphs of clique-width at most k such that for each graph $G \in \mathcal{G}$, a corresponding k -expression can be found in $\mathcal{O}(f(n, m))$ time. Then every MSO_1 property on \mathcal{G} can be recognized in time $\mathcal{O}(f(n, m) + n)$.*

The nice property of graphs with bounded clique-width is that their partial complementation is also bounded. In particular, Kamiński, Lozin, and Milanič in [14] observed that if G is a graph of clique-width k , then any partial complementation of G is of clique-width at most $g(k)$ for some computable function g . For completeness, we provide a more accurate upper bound whose proof is omitted in this extended abstract.

► **Lemma 13.** *Let G be a graph, $S \subseteq V(G)$. Then $\text{CWD}(G \oplus S) \leq 3\text{CWD}(G)$.*

► **Lemma 14.** *Let φ be an MSO_1 property describing the graph class \mathcal{G} . Then there exists an MSO_1 property ϕ describing the graph class $\mathcal{G}^{(1)}$ of size $|\phi| \in \mathcal{O}(|\varphi|)$.*

Proof. We will construct ϕ from φ in the following way: We start by prepending $\exists S \subseteq V(G)$. Then for each assessment of the existence of an edge in φ , say $uv \in E(G)$, replace that term with $((u \notin S \vee v \notin S) \wedge uv \in E(G)) \vee (u \in S \wedge v \in S \wedge uv \notin E(G))$. Symmetrically, for each assessment of the non-existence of an edge $uv \notin E(G)$, replace that term with $((u \notin S \vee v \notin S) \wedge uv \notin E(G)) \vee (u \in S \wedge v \in S \wedge uv \in E(G))$.

We observe that if φ is satisfiable for some graph G , then for every $S \subseteq V(G)$, the partial complementation $G \oplus S$ will yield a satisfying assignment to ϕ . Conversely, if ϕ is satisfiable for a graph G , then there exist some S such that φ is satisfied for $G \oplus S$. For the size, we note that each existence check for edges blows up by a constant factor. ◀

We are ready to prove the main result of this section.

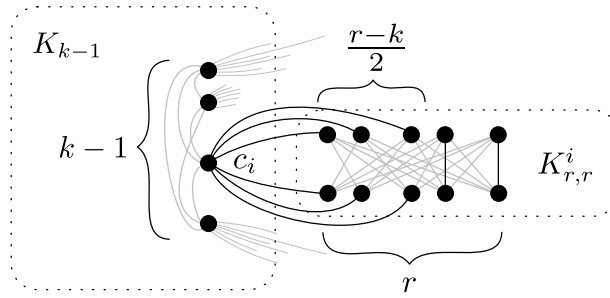
► **Theorem 15.** *Let \mathcal{G} be a graph class expressible in MSO_1 which has bounded clique-width. Then $\text{PARTIAL COMPLEMENT TO } \mathcal{G}$ is solvable in polynomial time.*

Proof. Let φ be the MSO_1 formula which describes \mathcal{G} , and let G be an n -vertex input graph. We apply Proposition 11 for G and in time $\mathcal{O}(n^3)$ either obtain a $(2^{3k+1} - 1)$ expression for G or conclude that the clique-width of G is more than $3k$. In the latter case, by Lemma 13, G cannot be partially complemented to \mathcal{G} .

We then obtain an MSO_1 formula ϕ from Lemma 14, and apply Proposition 12, which works in time $f(k, \phi) \cdot n$ for some function f . In total, the runtime of the algorithm is $f(k, \phi) \cdot n + n^3$. ◀

We remark that if clique-width expression is provided along with the input graphs, and \mathcal{G} can be expressed in MSO_1 , then there is a linear time algorithm for $\text{PARTIAL COMPLEMENT TO } \mathcal{G}$. This follows directly from Lemma 14 and Proposition 12.

Theorem 15 implies that for every class of graphs \mathcal{G} of bounded clique-width characterized by a finite set of finite forbidden induced subgraphs, e.g. P_4 -free graphs (also known as cographs) or classes of graphs discussed in [1], the $\text{PARTIAL COMPLEMENT TO } \mathcal{G}$ problem is solvable in polynomial time. However, Theorem 15 does not imply that $\text{PARTIAL COMPLEMENT TO } \mathcal{G}$ is solvable in polynomial time for \mathcal{G} being of the class of graphs having



■ **Figure 1** The graph $\text{GDG}_{k,r}$ is built of k parts, namely a clique K_{k-1} , and $k-1$ complete bipartite graphs $K_{r,r}^1, \dots, K_{r,r}^{k-1}$ with some rewiring.

clique-width at most k . This is because such a class \mathcal{G} cannot be described by MSO_1 . Interestingly, for the related class \mathcal{G} of graphs of bounded *rank-width* (see [5] for the definition) at most k , the result of Oum and Courcelle [6] combined with Theorem 15 implies that $\text{PARTIAL COMPLEMENT TO } \mathcal{G}$ is solvable in polynomial time.

6 Hardness of partial complementation to r -regular graphs

Let us remind that a graph G is r -regular if all its vertices are of degree r . We consider the following restricted version of $\text{PARTIAL COMPLEMENT TO } \mathcal{G}$.

PARTIAL COMPLEMENT TO r -REGULAR (PCrR)
Input: A simple undirected graph G , a positive integer r .
Question: Does there exist a vertex set $S \subseteq V(G)$ such that $G \oplus S$ is r -regular?

In this section, we show that $\text{PARTIAL COMPLEMENT TO } r\text{-REGULAR}$ is NP-complete by a reduction from $\text{CLIQUE IN } r\text{-REGULAR GRAPH}$.

CLIQUE IN r -REGULAR GRAPH (KrR)
Input: A simple undirected graph G which is r -regular, a positive integer k .
Question: Does G contain a clique on k vertices?

We will need the following well-known proposition.

► **Proposition 16** ([9]). $\text{CLIQUE IN } r\text{-REGULAR GRAPH}$ is NP-complete.

► **Theorem 17.** $\text{PARTIAL COMPLEMENT TO } r\text{-REGULAR}$ is NP-complete.

Proof. We begin by defining a gadget which we will use in the reduction. For integers $r > k$ such that $r - k$ is even, we build the graph $\text{GDG}_{k,r}$ as follows. Initially, we let $\text{GDG}_{k,r}$ consist of one clique on $k - 1$ vertices, as well as $k - 1$ distinct copies of $K_{r,r}$. These are all the vertices of the gadget, which is a total of $(k - 1) + 2r \cdot (k - 1)$ vertices. We denote the vertices of the clique c_1, c_2, \dots, c_{k-1} , and we let the complete bipartite graphs be denoted by $K_{r,r}^1, K_{r,r}^2, \dots, K_{r,r}^{k-1}$. For a bipartite graph $K_{r,r}^i$, let the vertices of the two parts be denoted by $a_1^i, a_2^i, \dots, a_{\frac{r-k}{2}}^i$ and $b_1^i, b_2^i, \dots, b_{\frac{r-k}{2}}^i$ respectively.

We will now do some rewiring of the edges to complete the construction of $\text{GDG}_{k,r}$. Recall that $r - k$ is even and positive. For each vertex c_i of the clique, add one edge from c_i to each of $a_1^i, a_2^i, \dots, a_{\frac{r-k}{2}}^i$. Similarly, add an edge from c_i to each of $b_1^i, b_2^i, \dots, b_{\frac{r-k}{2}}^i$. Now remove

the edges $a_1^i b_1^i, a_2^i b_2^i, \dots, a_{\frac{r-k}{2}}^i b_{\frac{r-k}{2}}^i$. Once this is done for every $i \in [k-1]$, the construction is complete. See Figure 1.

We observe the following property of vertices a_j^i, b_j^i , and c_i of $\text{GDG}_{k,r}$.

► **Observation 18.** *For every $i \in [k-1]$ and $j \in [r]$, it holds that the degrees of a_j^i and b_j^i in $\text{GDG}_{k,r}$ are both exactly r , whereas the degree of c_i is $r-1$.*

We are now ready to prove that **CLIQUE IN r -REGULAR GRAPH** is many-one reducible to **PARTIAL COMPLEMENT TO r -REGULAR**.

► **Algorithm 19** (Reduction KrR to PCrR).

Input: An instance (G, k) of KrR .

Output: An instance (G', r) of PCrR such that it is a yes-instance if and only if (G, k) is a yes-instance of KrR .

1. If $k < 7$ or $k \geq r$, solve the instance of KrR by brute force. If it is a yes-instance, return a trivial yes-instance to PCrR , if it is a no-instance, return a trivial no-instance to PCrR .
2. If $r-k$ is odd, modify G by taking two copies of G which are joined by a perfect matching between corresponding vertices. Then r increase by one, whereas k remains the same.
3. Construct the graph G' by taking the disjoint union of G and the gadget $\text{GDG}_{k,r}$. Here, r denotes the regularity of G after step 2 is performed. Return (G', r) .

Let $n = |V(G)|$. We observe that the number of vertices in the returned instance is at most $2n + (k-1) + 2r \cdot (k-1)$, which is $\mathcal{O}(n^2)$. The running time of the algorithm is $\mathcal{O}(n^7)$ and thus is polynomial.

The correction of the reduction follows from the following two lemmata.

► **Lemma 20.** *Let (G, k) be the input of Algorithm 19, and let (G', r) be the returned result. If (G, k) is a yes-instance to **CLIQUE IN r -REGULAR GRAPH**, then (G', r) is a yes-instance of **PARTIAL COMPLEMENT TO r -REGULAR**.*

Proof. Let $C \subseteq V(G)$ be a clique of size k in G . If the clique is found in step 1, then (G', r) is a trivial yes-instance, so the claim holds. Thus, we can assume that the graph G' was constructed in step 3. If G was altered in step 2, we let C be the clique in one of the two copies that was created. Let $S \subseteq V(G')$ consist of the vertices of C as well as the vertices of the clique K_{k-1} of the gadget $\text{GDG}_{k,r}$. We claim that S is a valid solution to (G', r) .

We show that $G' \oplus S$ is r -regular. Any vertex not in S will have the same number of neighbors as it had in G' . Since the only vertices that weren't originally of degree r were those in the clique K_{k-1} , all vertices outside S also have degree r in $G' \oplus S$. What remains is to examine the degrees of vertices of C and of K_{k-1} .

Let c_i be a vertex of K_{k-1} in G' . Then c_i lost its $k-2$ neighbors from K_{k-1} , gained k neighbors from C , and kept $r-k$ neighbors in $K_{r,r}^i$. We see that its new neighborhood has size $k+r-k=r$.

Let $u \in C$ be a vertex of the clique from G . Then u lost $k-1$ neighbors from C , gained $k-1$ neighbors from K_{k-1} , and kept $r-(k-1)$ neighbors from $G-C$. In total, u will have $r-(k-1)+(k-1)=r$ neighbors in $G' \oplus S$. Since every vertex of $G' \oplus S$ has degree r , it is r -regular, and thus (G', r) is a yes-instance. ◀

► **Lemma 21.** *Let (G, k) be the input of Algorithm 19, and let (G', r) be the returned result. If (G', r) is a yes-instance to **PARTIAL COMPLEMENT TO r -REGULAR**, then (G, k) is a yes-instance of **CLIQUE IN r -REGULAR GRAPH**.*

Proof of Lemma 21 is omitted due to space constraints. Lemmata 20 and 21 together with Proposition 16 conclude the proof of NP-hardness. Membership in NP is trivial, so NP-completeness holds. ◀

We remark that if r is a constant not given with the input, the problem becomes polynomial time solvable by Theorem 6.

7 Conclusion and open problems

In this paper we initiated the study of PARTIAL COMPLEMENT TO \mathcal{G} . Many interesting questions remain open. In particular, what is the complexity of the problem when \mathcal{G} is

- the class of chordal graphs,
- the class of interval graphs,
- the class of graph excluding a path P_5 as an induced subgraph,
- the class graphs with max degree $\leq r$, or
- the class of graphs with min degree $\geq r$

More broadly, it is also interesting to see what happens as we allow more than one partial complementation; how quickly can we recognize the class $\mathcal{G}^{(k)}$ for some class \mathcal{G} ? It will also be interesting to investigate what happens if we combine partial complementation with other graph modifications, such as the Seidel switch.

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