# Precongruences and Parametrized Coinduction for Logics for Behavioral Equivalence

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**Abstract**

We present a new proof system for equality of terms which present elements of the final coalgebra of a finitary set functor. This is most important when the functor is finitary, and we improve on logical systems which have already been proposed in several papers. Our contributions here are (1) a new logical rule which makes for proofs which are somewhat easier to find, and (2) a soundness/completeness theorem which works for all finitary functors, in particular removing a weak pullback preservation requirement that had been used previously. Our work is based on properties of precongruence relations and also on a new parametrized coinduction principle.

**1998 ACM Subject Classification** F.4.1 Mathematical Logic

**Keywords and phrases** precongruence, kernel bisimulation, finitary functor, coalgebra, behavioural equivalence

**Digital Object Identifier** 10.4230/LIPIcs.CALCO.2017.23

## 1 Introduction

The goal of this paper to construct a sound and complete logic for behavioral equivalence for finitary set functors. That is, we aim to construct a logical system whose assertions include equations between variables. Given a finitary set functor $F$ and a finite coalgebra $(X, f : X \to FX)$ we would like it to be the case that $\vdash x = y$ if and only if $x$ and $y$ have the same image in the final $F$-coalgebra. We would like the proof system to have the expected properties: for example, proofs should be finite and should be easily checkable. Moreover, even to propose a syntax which pertains to a finitary set functor $F$, we prefer to work with a presentation of $F$ as a quotient of a signature functor, say $H$ (see [5]).

We build on a line of work in this area. [15] presented a sound and complete logic for terms which denote elements of the final coalgebra of a variant of finite power set functor. This was greatly extended in [17] by moving to arbitrary finitary functors preserving weak pullbacks. Indeed the results in that paper heavily depended on preservation of weak pullbacks, as the soundness and completeness depended on properties of Aczel-Mendler bisimulations. Our first achievement is to remove the restriction to functors which preserve weak pullbacks. Working with *precongruences* (from [1]) obviates this restriction. What makes this work is the fact the final coalgebra of a finitary functor $F$ may be described as the quotient of the final coalgebra for its related signature functor $H$ by the greatest precongruence.

Our second achievement is a new proof rule in the logic which allows precongruences to be built up in phases. This replaces a rule in [17] which required a bisimulation to be

* This work was partially supported by a grant from the Simons Foundation (#245591 to Lawrence Moss).


7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017).
Editors: Filippo Bonchi and Barbara König; Article No. 23; pp. 23:1–23:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

"guessed all at once." The soundness of this new principle relies on properties of greatest fixed points of *parametrized precongruence relations*, a variant of precongruences, and is therefore quite different than the earlier approach in [17].

*Related logical systems.* The logical system in this paper is related to many others. Moschovakis's $FLR_0$, as presented in [11] provided logical systems for equality of recursive terms. But the semantics was presented either in terms of least fixed points on complete partial orders, or more generally in settings with a categorical fixed point operator (essentially iteration theories). [14] showed that the interpretation into final coalgebras was complete for these logics, but the only functors considered were signature functors on Set. The main work in that paper concerned constructing the semantics and checking the soundness of the rules, because $FLR_0$ allows terms like "$x$ `where` $x = x$". [15] went beyond signature functors by considering a variant the finite power set functor. The syntax had term formers for sets: if $t_1$, ..., $t_n$ are terms, then $\{t_1, \ldots, t_n\}$ is a term. The logical rules then incorporated a version of the Axiom of Extensionality from set theory, and it was not clear how to extend this to a wider class of functors. This was done for finitary set functors in [17], using the representation of such functors as quotients of signature functors. Indeed the kernel equivalences used in the representation were directly incorporated into the inference rules. But as mentioned above, the properties of the system depended on the preservation of weak pullbacks. As in [17], the main work is in getting a usable syntax and in proving the soundness of the logic; proving the completeness is (somewhat surprisingly) an easier task.

The expression calculus of Bonsangue et al. in [8] and [7] or Milius' related work in the setting of vector spaces [13] are further related work. What we do in this paper goes in a different direction.

**Outline.** In Section 2, we recall basic definitions of coalgebras and kernel bisimulations as well as facts on finitary set functors and on coinduction principles. In Section 3, we recall the notion of a precongruence relation. We develop this notion from scratch, recalling results from [1, 9] that we shall use. We then specialize this notion to finitary functors and prove that the final coalgebra of a finitary functor is the quotient of the final coalgebra for its related signature functor by the greatest precongruence. In Section 4, we propose our logic for behavioral equivalence for coalgebras of finitary set functors. We use our results on precongruences to prove the soundness and completeness and completeness of our logic.

## 2 Preliminaries

In this section, we recall definitions and basic results about coalgebras, kernel bisimulations, and finitary functors. Our setting is the category Set, and all functors are assumed to be Set-endofunctors. Section 2.3 presents some facts on another background topic, greatest fixed points of operators.

### 2.1 Coalgebras

Given a Set-endofunctor $F$, an *$F$-coalgebra* is a set $X$ together with a map $f : X \to FX$. The set is often called the *carrier* of the coalgebra, while $f$ gives its *structure*.

An *$F$-coalgebra morphism* from an $F$-coalgebra $(X, f)$ to another $F$-coalgebra $(Y, g)$ is a map $\varphi : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & FX \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle F\varphi} \\ Y & \xrightarrow{g} & FY \end{array}$$

$F$-coalgebras together with coalgebra morphisms between them form a category. When this category has a final object, we call it the *final $F$-coalgebra* and denote it by $\nu F$. Points in coalgebras which have the same image in $\nu F$ are *behaviorally equivalent* to one another.

A *kernel bisimulation* from one coalgebra $(X, f)$ to another $(Y, g)$ is a pullback of a cospan $(X, f) \to (Z, h) \leftarrow (Y, g)$. Here we are following the name from [18], but in various other sources these are called cocongruences. In case $(X, f) = (Y, g)$, such a pullback is called a *behavioral equivalence*, even if the final $F$-coalgebra does not exist.

(Aczel-Mendler) bisimulations are a related notion. An ($F$-)*bisimulation* is a relation $R \subseteq X \times Y$ such that there is an $F$-coalgebra structure on $R$, say $\rho : R \to FR$, such that the two projections $\pi_1 : R \to X$, $\pi_2 : R \to Y$, are coalgebra morphisms.

These coalgebraic notions are parametrized by the functor $F$, which is often clear from context. As usual, when this is the case, we drop the prefixed $F$ and simply say "coalgebra", "coalgebra morphism", "bisimulation", and so on.

As shown in Staton [18], Aczel-Mendler bisimulations and kernel bisimulations agree when $F$ preserves weak pullbacks. In Chapter 1 of Kurz's thesis [12], it is shown that much of universal coalgebra [16] can be recovered when $F$ does not preserve weak pullbacks by replacing Aczel-Mendler bisimulations with behavioral equivalences.

## 2.2 Finitary functors

If $\mathcal{A}$ is a category with filtered colimits, then a functor $F : \mathcal{A} \to \mathcal{B}$ is *finitary* if $F$ preserves filtered colimits. In this paper, the main category of interest is $\mathsf{Set}$. For $F : \mathsf{Set} \to \mathsf{Set}$, there are several important equivalent formulations to the notion of a finitary functor.

First: for each $X$ and each $x \in FX$, there is a finite $Y \subseteq X$ such that $x \in Fi[FY]$, where $i : Y \hookrightarrow X$ is the inclusion. This equivalence was proven by Adámek and Porst [5]. This formulation implies that many functors of interest are finitary including the finite power set functor, $F(X) = A \times X$ for all sets $A$, any signature functor (with finite arities), and the discrete distribution functor.

Even more useful for our purposes is a formulation given in terms of *presentations via signatures and relations*. We begin with a signature $\Sigma$, and thus with a functor $H_\Sigma : \mathsf{Set} \to \mathsf{Set}$. For each set $X$, $H_\Sigma(X)$ is the set of all flat terms $f(x_1, \ldots, x_k)$, where $f$ is a $k$-ary symbol in $\Sigma$, and $x_1, \ldots, x_k \in X$. For functions $\phi : X \to Y$, the functor performs a simple variable renaming: $H_\Sigma\phi : f(x_1, \ldots, x_n) \mapsto f(\phi(x_1), \ldots, \phi(x_n))$. The adjective *flat* or *basic* here refers to the fact that terms consist only of symbols applied to variables: $f$ applied to $n$ other non-variable terms is not a term in our context.

A *flat* $\Sigma$-*equation* is a pair $(\ell, r)$ of elements of $H_\Sigma(X)$. An *instance* of a $\Sigma$-equation is an image of each side of an equation under a renaming: $(H_\Sigma\phi(\ell), H_\Sigma\phi(r))$. Consider a fixed set of $\Sigma$-equations $E$ with variables from $S$. (In our examples $S$ is usually a finite set, though we allow it to be infinite.) The instances of $E$ in $H_\Sigma(X)$ is the following:

$$\equiv_X \triangleq \{(H_\Sigma\phi(\ell), H_\Sigma\phi(r)) : \phi : S \to X, (\ell, r) \in E\}$$

For any $X$, the instances of $E$ in $H_\Sigma(X)$ is an equivalence relation on $H_\Sigma(X)$. The assignment $X \to H_\Sigma/{\equiv_X}$ extends to a functor which we call $H_\Sigma/E$. Moreover, there is a natural transformation $\epsilon : H_\Sigma \to H_\Sigma/E$. All components of $\epsilon$ are surjective. We say that $H_\Sigma/E$ is *presented by a finitary signature and a set of basic equations.*

▶ **Example 1.** Let $\mathcal{P}_3(X)$ be the set of subsets of $X$ of size $< 3$. $\mathcal{P}_3$ is a functor in the obvious way, as a subfunctor of the covariant power set. Here is a presentation of $\mathcal{P}_3$: consider a signature with two symbols: $n$ and $*$, with arities 0 and 2 respectively. (We write $*$ with infix notation.) Thus, $H_\Sigma(X)$ consists of $n$ and all terms $x * y$ for $x, y \in X$. We take $S = \{s, t\}$, and just one basic equation: $s * t = t * s$. For all $X$, the equivalence relation $\equiv_X$ identifies $x * y$ with $y * x$. The set of equivalence classes $H_\Sigma(X)/\equiv_X$ corresponds with $\mathcal{P}_3(X)$: $[n]$ corresponds with $\emptyset$, and $[x * y]$ with $\{x, y\}$. The correspondence shows that $\mathcal{P}_3$ is naturally isomorphic to $H_\Sigma/E$.

▶ **Proposition 2** ([2], 3.12)**.** *A functor $F : \mathit{Set} \to \mathit{Set}$ is finitary if and only if $F$ is naturally isomorphic to a functor presented by a finitary signature and a set of basic equations.*

## 2.3 A coinduction principle for monotone operators

Our final preliminary section deals with a coinduction principle that we shall use in the soundness proof in Section 4.2. We present the work in a fairly general setting, because it will probably be easier to read and adapt this way.

Let $L$ be a complete meet-semilattice. That is, $L$ is a poset with greatest lower bounds $\bigwedge S$ of all sets $S$. Taking $S = \emptyset$ shows that $L$ has a top element, 1. We also are interested in monotone operators $\Phi : L \to L$. By the Knaster-Tarski Theorem, $\Phi$ has a greatest post-fixed point which we write as $\Phi^*$. To obtain $\Phi$, we define iterates $\Phi^\alpha$ by transfinite recursion on the ordinal $\alpha$: $\Phi^0 = 1$, $\Phi^{\alpha+1} = \Phi(\Phi^\alpha)$, and for limit $\lambda$, $\Phi^\lambda = \bigwedge_{\alpha<\lambda} \Phi^\alpha$. Using the fact that $L$ is a set, there is an ordinal $\alpha$ such that $\Phi^{\alpha+1} = \Phi^\alpha$. We write $\Phi^\alpha$ as $\Phi^*$. $\Phi^*$ is a fixed point of $\Phi$. Moreover, if $p \leq \Phi(p)$, then $p \leq \Phi^*$.

We are frequently interested in principles which allow us to establish relationships between greatest fixed points of related operators. For any $r \in L$, we define a new operator $\Phi_r : L \to L$ by $\Phi_r(x) = \Phi(x) \vee r$. Each $\Phi_r$ is monotone, and so has a greatest fixed point, which we denote $\Phi^*_r$ as an abbreviation of $(\Phi_r)^*$. We point out $r \leq \Phi^*_r$ for all $r$.

▶ **Lemma 3** (Parametrized Coinduction Principle)**.** *If $s \leq \Phi(\Phi^*_{r\vee s})$, then $\Phi^*_{r\vee s} \leq \Phi^*_r$.*

**Proof.** We calculate: $\Phi^*_{r\vee s} = \Phi(\Phi^*_{r\vee s}) \vee (r \vee s) = \Phi(\Phi^*_{r\vee s}) \vee r = \Phi_r(\Phi^*_{r\vee s})$. Since $\Phi^*_r$ is the greatest fixed point of $\Phi_r$, we know $\Phi^*_{r\vee s} \leq \Phi^*_r$. ◀

We point out that $\Phi^*_r \leq \Phi^*_s$ for $r \leq s$ by montonicity arguments. One could therefore read this lemma as establishing $\Phi^*_{r\vee s} = \Phi^*_r$, provided $s$ satisfies the condition given.

There is an obvious connection to be drawn to the work of Hur et. al. [10]. In this work, there is also a constellation of related greatest fixed points, each determined by an element in the lattice. They define $G_\Phi(r)$ to be the greatest fixed point of $\Phi'_r(x) \triangleq \Phi(x \vee r)$ and work with these $G_\Phi(r)$. Our $\Phi_r$ and their $\Phi'_r$ differ very slightly, and this subtle difference leads to different properties. For example, we point out again $r \leq \Phi^*_r$ for all $r$, but $r \not\leq G_\Phi(r)$. Our parametrized coinduction principle above is our analog to their accumulation theorem.

# 3 Precongruence relations

In this section, we recall the notion of a precongruence relation from [1]. We mention their basic properties and their connections to congruences. Then we begin specializing these results towards the context of finitary functors. The theme of this section is that the greatest precongruence relation on a coalgebra $X$ is a greatest fixed point of a particular operator on the set of all relations on $X$. It turns out to be fruitful to study this operator, and we have set the stage for this in Section 2.3 just above. We also analyze the greatest fixed points of a family of related operators. These turn out to be useful in the logic presented in the final section.

▶ Notation. If $R$ is a relation on $X$, we denote its equivalence closure by $e(R)$. Given any relation $R$ the canonical function $x \mapsto [x]_{e(R)}$ sending an element of $X$ to its equivalence class in $X/e(R)$ is denoted $q_R$.

▶ **Definition 4.** Let $F : \mathsf{Set} \to \mathsf{Set}$ be a $\mathsf{Set}$ endofunctor and $(X, f : X \to FX)$ be an $F$-coalgebra. A *$F$-precongruence* (relation) on this coalgebra is a relation $R$ on $X$ such that $Fq_R(f(x)) = Fq_R(f(y))$ for all $(x, y) \in R$. In other words, $R \subseteq ker(Fq_R \circ f)$.

A *precongruence equivalence* is a precongruence relation which is also an equivalence relation.

▶ **Example 5.** We use $\mathcal{P}_3$ from Example 1. Let $X = \{x, y\}$, and consider the coalgebra structure $d$ given by $d(x) = \{x\}$ and $d(y) = \{x, y\}$. We quickly verify $R = \{(x, y)\}$ is a precongruence. First note $[x]_{e(R)} = \{x, y\} = [y]_{e(R)}$. Then it is clear that

$$\mathcal{P}_3 q_R(d(x)) = \{[x]\} = \{[x], [y]\} = \mathcal{P}_3 q_R(d(y))$$

Aczel and Mendler [1] also introduced the related notion of a *congruence*.

▶ **Definition 6.** Given a coalgebra $(X, d : X \to FX)$, a *congruence* on this coalgebra is an equivalence relation $R$ on $X$ such that $X/R$ carries a coalgebra structure $b : X/R \to F(X/R)$ and the quotient map $q_R : X \to X/R$ is a coalgebra morphism.

## 3.1 Basic properties of precongruence relations

In this section, we recall important properties of precongruences. We intend to leave the impression that precongruences and precongruence equivalences have much in common with bisimulations and bisimulation equivalences. However, an important difference is that the characterization of precongruence equivalence as kernels of coalgebra morphisms does **not** have any dependency on properties of the functor in question, unlike bisimulation equivalences. This is a key ingredient in being able to remove the weak pullback requirement from the scope of the logic presented in [17].

▶ **Proposition 7.** *Suppose $(X, d)$ is an $F$-coalgebra. The following hold:*

1. *The diagonal $\Delta_X$ is a precongruence equivalence.*
2. *Arbitrary unions of precongruence relations are precongruence relations.*
3. *If a relation is a precongruence, so is its inverse.*
4. *If two relations containing the diagonal are precongruences, so is their composition.*
5. *A relation is a precongruence if and only if its equivalence closure is a precongruence.*

**Proof.** 1 and 3 follow from the definition of a precongruence.

2. Suppose $R_i$ is a collection of precongruence relations. Let us write $S$ for the union $\bigcup R_i$. Suppose $(x, y) \in S$. Then $(x, y) \in R_k$ for some $k$, and since $e(R_k) \subseteq e(S)$, there is a morphism $q$ so that $q_S = q \circ q_{R_k}$. Since $R_k$ is a precongruence, we know $Fq_{R_k}(d(x)) = Fq_{R_k}(d(y))$. By applying $Fq$ to both sides, we find that $S$ is a precongruence.

4. If $R$ and $S$ contain the diagonal, then their composition $R \cdot S$ contains both $R$ and $S$. Suppose $(x, z) \in R \cdot S$, so there exists $y \in X$ such that $(x, y) \in R$ and $(y, z) \in S$. Since $R$ and $S$ are precongruences, we know that $Fq_R(d(x)) = Fq_R(d(y))$ and $Fq_S(d(y)) = Fq_S(d(z))$. This implies $Fq_{R \cdot S}(d(x)) = Fq_{R \cdot S}(d(y))$ and $Fq_{R \cdot S}(d(y)) = Fq_{R \cdot S}(d(z))$, as desired.

5. The above properties can be combined to show that if a relation is a precongruence, its equivalence closure is as well. If the equivalence closure of a relation is a precongruence, a fortiori that relation is a precongruence as well. ◀

Proposition 7 implies the following result, fundamental to the rest of our work.

▶ **Theorem 8** ([1]). *For every $F$-coalgebra $(X, f)$, there is a largest precongruence $R$ on $X$. $R$ is an equivalence relation.*

**Proof.** Here is a sketch; we shall see a different proof of it in Section 3.4. Precongruence relations being closed under unions also implies that there is a greatest precongruence on any given coalgebra. This greatest precongruence must also be an equivalence relation by the fifth property above. ◀

▶ Remark. In contrast to congruences, precongruences are not closed under intersections. To see this, define an $Id$-coalgebra on $X = \{x, y, z\}$ by $d(x) = y$, $d(y) = z$ and $d(z) = x$. Then the two relations $R = \{(x, y), (y, z)\}$ and $S = \{(x, y), (z, y)\}$ are both precongruences, but their intersection is not.

Next we recall results about precongruence equivalences. A particularly important result for our purposes is that they coincide with kernels of coalgebra morphisms.

▶ **Theorem 9** ([1], Lemma 5.1). *Let $(X, d)$ be an $F$-coalgebra and let $R \subseteq X \times X$. The following are equivalent:*
1. *$R$ is a congruence on $X$.*
2. *$R$ is the kernel of a coalgebra morphism $\varphi$ with domain $(X, d)$.*
3. *$R$ is a precongruence equivalence on $X$.*

We can use these results to establish a relationship between precongruences and bisimulations:

▶ **Corollary 10.** *Suppose that $(X, d)$ is an $F$-coalgebra. Then $R$ is a precongruence relation on $X$ if and only if $e(R)$ is a congruence. If $F$ moreover preserves weak pullbacks, then $R$ is a precongruence relation on $X$ if and only if $e(R)$ is a bisimulation.*

**Proof.** We know $R$ is a precongruence if and only if $e(R)$ is a precongruence. By Theorem 9, $e(R)$ is a precongruence if and only if it is the kernel of an $F$-coalgebra morphism. When $F$ preserves weak pullbacks, kernels of coalgebra morphisms exactly coincide with bisimulation equivalences, see [16, Proposition 5.7, 5.8]. ◀

In particular, the behavioral equivalence relation has to be a precongruence equivalence.

▶ **Proposition 11.** *Let $F$ be a Set-endofunctor and suppose the final coalgebra of $F$, $\nu F$, exists. If $(X, d)$ an $F$-coalgebra, then the kernel of the final coalgebra map $h_X : X \to \nu F$ is the greatest precongruence relation on $X$.*

**Proof.** Let $R$ be the greatest precongruence relation on $X$. By Theorem 9, $ker(h_X)$ is a precongruence equivalence and hence $ker(h_X) \subseteq R$.

Also by Theorem 9, we know $q_R : X \to X/R$ is an $F$-coalgebra morphism. Since $X/R$ is an $F$-coalgebra it also has a final morphism into $\nu F$, which we denote $h_{X/R}$. By finality, $h_X = h_{X/R} \circ q_R$. Therefore, $R = ker(q_R) \subseteq ker(h_X)$. ◀

## 3.2 Precongruence relations across natural transformations

Suppose $\epsilon : H \to F$ is a natural transformation. Then each $H$-coalgebra $(X, d)$ carries an $F$-coalgebra structure $\epsilon_X \circ d$. The relationship between $H$ and $F$ leads to relationships between other coalgebraic notions, including precongruence relations, on $H$- and $F$-coalgebras.

▶ **Definition 12.** Let $\epsilon : H \to F$ be a natural transformation between Set endofunctors. We define a collection of equivalence relations $=_{\epsilon,X}$ as $ker(\epsilon_X)$. Typically the relevant set $X$ can be inferred from context, and we write $=_\epsilon$ instead.

If we assume further $\epsilon$ is epic, there is a very close relationship between the final $H$- and final $F$-coalgebras. This assumption holds in the context of finitary functors; recall Section 2.2.

▶ **Lemma 13** ([9], Lemma 2.3). *Suppose $\epsilon : H \to F$ is an epic natural transformation and that the final $H$-coalgebra, $(\nu H, t)$, exists. Then $\nu F$ exists and is the quotient of the $F$-coalgebra $(\nu H, \epsilon_{\nu H} \circ t)$ by the greatest $F$-precongruence relation $R$ on it.*

▶ **Theorem 14** (Coinduction Principle). *In the setting of Lemma 13, consider $\nu H$ as an $F$-coalgebra with structure $\epsilon_{\nu H} \circ t$. Let $\varphi : \nu H \to \nu F$ be the final $F$-coalgebra map. Let $R$ be any relation on $\nu H$ with the property that if $(u, v) \in R$, then $(Hq_R \circ t)(u) = (Hq_R \circ t)(v)$. Then for all pairs $(u, v) \in R$, $\varphi(u) = \varphi(v)$.*

**Proof.** We claim that $R$ is an $F$-precongruence on $(\nu H, \epsilon_{\nu H} \circ t)$. To see this, let $(u, v) \in R$. By naturality, $Fq_R \circ \epsilon_X \circ t = \epsilon_{X/R} \circ Hq_R \circ t$. So $u$ and $v$ have the same image under $(Fq_R \circ \epsilon_X) \circ t$. This proves our claim. By Lemma 13, $R \subseteq ker(\varphi)$. ◀

## 3.3 The final coalgebra for finitary functors: level cuts and precongruences

Recall that a finitary set functor $F$ may be described in terms of a signature functor $H$ and a natural transformation $\epsilon : H \to F$ with surjective components. Lemma 13 gives the relation between $\nu F$ and $\nu H$: $\nu F$ is the quotient of $\nu H$ by the greatest precongruence relation on $\nu H$. The idea in this section is that since we have a convenient representation of $\nu H$ as the set of all (finite and infinite trees) on $H$, we have a tool to study $\nu F$. The main point of this section is to mention two ways to carry out this study.

Adámek and Milius characterized $\nu F$ in [3] as the quotient of $\nu H$ by a relation $\sim^*$ based on level cuts. We briefly describe this relation and compare it to our characterization with an example.

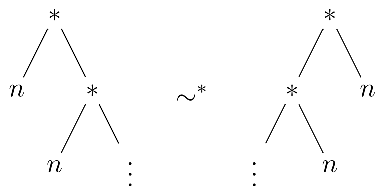
Let $\sim$ be the least congruence on finite trees which contains the relation $=_{\epsilon, X\cup\{\perp\}}$. The *level $k$ cutting* of a tree $\sigma \in \nu H$ is defined to be the biggest height $k$ subtree of $\sigma$ with the level $k$ leaves replaced by a fresh symbol $\perp$. We define the relation $\sim^*$ on $\nu H$ by $\sigma \sim^* \tau$ if and only if the level $k$ cuts of $\sigma$ and $\tau$ are related by $\sim$ for all $k \in \omega$.

▶ **Theorem 15** (Adámek and Milius, [3]). *$(\nu H)/\sim^*$ is the final $F$-coalgebra.*

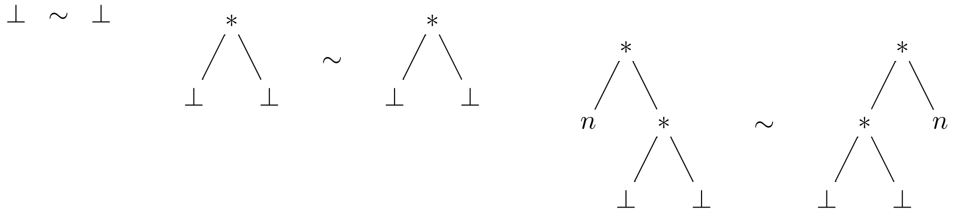
We compare these two characterizations with an example. It is based on the presentation in Example 1.

▶ **Example 16.** Let $X = \{x, y, z\}$, and let $d$ be the $H$-coalgebra $d(x) = z * x$, $d(y) = y * z$ and $d(z) = n$. Then $d$ extends to a $\mathcal{P}_3$-coalgebra via $\epsilon_X$ in the usual way. We claim that $x$ and $y$ have the same image in the final $\mathcal{P}_3$-coalgebra.

The tree unfolding of $x$ and $y$ in $\nu H$, the final $H$-coalgebra, look like the following:



Intuitively, these two trees are the same as $\mathcal{P}_3$-coalgebras since $\epsilon$ makes the $*$ operation commutative. To show this with level cuts, we have to show that the level $k$ cuts of the above trees are pairwise related by $\sim$ for all $k \in \omega$. This sequence starts:



This is easy enough to do by induction, but it does require an induction.

We mentioned that the purpose of this section is to present two ways of studying $\nu F$ using $\nu H$. The second way is to use Theorem 14.

▶ **Example 17.** We revisit Example 16 from just above. Again, we wish to see that $x$ and $y$ have the same images in the final $F$-coalgebra. With precongruence relations, we need only exhibit a relation such that all pairs $(u, v) \in R$ satisfy $Hq_R(d(u)) =_\epsilon Hq_R(d(v))$. We suggest $R = \{(x, y)\}$. Here are the calculations that verify $R$ is a precongruence:

$$Hq_R(d(x)) = Hq_R(z * x) = [z]_{e(R)} * [x]_{e(R)}$$
$$Hq_R(d(y)) = Hq_R(y * z) = [y]_{e(R)} * [z]_{e(R)} = [x]_{e(R)} * [z]_{e(R)}$$

and in fact $[z]_{e(R)} * [x]_{e(R)} =_\epsilon [x]_{e(R)} * [z]_{e(R)}$. (The reason comes from Example 1: the operation $*$ is interpreted by a commutative function on every quotient set $H(X)/\equiv_X$.) So $R$ is a precongruence. By Theorem 14, $x$ and $y$ have the same image in the final $\mathcal{P}_3$-coalgebra.

There is no induction in Example 17, and the argument has a flavor closer to a coinductive proof. That is, we start the proof assuming that $(x, y) \in R$, and we use this very fact when we claim that $[x]_{e(R)} = [y]_{e(R)}$. Of course, we hasten to add that by a "coinductive" (or "circular") proof we mean a proof that is justified by a sound principle. In our setting, this sound principle is Proposition 11.

The rest of this paper elaborates the observation in Example 17 into a proof system and studies it in connection with the coinduction principles which we have seen already.

## 3.4 Precongruence iterates and parametrized precongruence relations

For any set $X$, the set of all relations on $X$ is a complete lattice. Its top element is $X \times X$. Given an $F$-coalgebra $(X, f)$, define the following operation on this lattice:

$$\begin{aligned} R^- &= \{(x, y) \in X \times X : Fq_R(f(x)) = Fq_R(f(y))\} \\ &= ker(Fq_R \circ f) \end{aligned}$$

Note that the post-fixed points of $R \mapsto R^-$ are precisely the precongruence relations. This is our motivation in investigating this operation.

▶ **Proposition 18.** *If $R \subseteq S$, then $R^- \subseteq S^-$.*

**Proof.** Suppose that $R \subseteq S$, and let $(x, y) \in R^-$. This means $Fq_R(f(x)) = Fq_R(f(y))$. From $e(R) \subseteq e(S)$ and we infer a morphism $q$ such that $q_S = q \circ q_R$. Applying $Fq$ to both sides yields $(x, y) \in S^-$. ◀

Having a monotone operator on a complete lattice, we can instantiate the results of Section 2.3 in this context. Following that approach, we define additional monotone operators on $\mathcal{P}(X \times X)$: for each relation $S$,

$$R^-_S = R^- \cup S = \{(x, y) : Fq_R(f(x)) = Fq_R(f(y))\} \cup S.$$

Conflating $S$ and $(\cdot)^-_S$ slightly, we have *iterates* $S^\alpha$ for ordinals $\alpha$ as defined in Section 2.3.

$$S^0 = X \times X \quad \text{and} \quad S^{\alpha+1} = (S^\alpha)^-_S \quad \text{and} \quad S^\lambda = \bigcap_{\alpha<\lambda} S^\alpha$$

We define $S^*$ to be the greatest fixed point of $(\cdot)^-_S$.

When $F$ is a finitary set functor presented by a signature functor $H$ and an epic natural transformation $\epsilon$, the $F$-coalgebra $(X, f)$ has a related $H$-coalgebra, which we denote $(X, d)$, satisfying $\epsilon_X \circ d = f$.

Then we can alternatively define $R^- \triangleq ker(Fq_R \circ f)$ using $Fq_R \circ \epsilon_X = \epsilon_{X/R} \circ Hq_R$ by the naturality of $\epsilon$:

$$R^- = \{(x, y) : Hq_R(d(x)) =_\epsilon Hq_R(d(y))\}. \tag{1}$$

This is a more useful formulation in what follows.

▶ **Proposition 19.** *Let $R$ and $S$ be any relations on $X$. If $S \subseteq ((R \cup S)^*)^-$, then $(R \cup S)^* \subseteq R^*$.*

**Proof.** This is just a restatement of Lemma 3 in our current context. ◀

To illustrate the value of the iterative approach, we present a different proof of Aczel and Mendler's Theorem 8. The operator $(\cdot)^-$ is a monotone operator on $\mathcal{P}(X \times X)$, and hence some iterate $\emptyset^\alpha$ is a greatest post-fixed point. This relation $\emptyset^\alpha$ is a precongruence, and indeed it includes all precongruences. An easy induction shows that each iterate $\emptyset^\beta$ is an equivalence relation, using the facts that each $R^-$ is the kernel relation of some morphism, and also that the intersection of any family of equivalence relations is an equivalence relation.

▶ Remark. We think of the iteration of $(\cdot)^-_S$ as a partition refinement process. We start with the complete relation on $X$, and at each step we retain only the pairs which satisfy $Hq_R(d(x)) =_\epsilon Hq_R(d(y))$. The iteration converges in $\omega$ steps for a finitary functor.

The relation $\emptyset^*$ is the greatest precongruence relation. The variant operations $(\cdot)^-_S$ also have greatest fixed points; these are not necessarily precongruences. They include this greatest precongruence relation, but in a somewhat controlled way. We call these *parametrized precongruences*. As we shall see in Example 20 below, parametrized precongruences need not be equivalence relations.

The relation $S$ in the $(\cdot)^-_S$ operation provides a "forgiveness parameter": all $(x, y) \in S$ remain in all stages of the $S^i$ iteration even if they do not satisfy the refinement predicate $Hq_R(d(x)) =_\epsilon Hq_R(d(y))$. Since pairs in $S$ stubbornly remain in each iterate $S^i$, other pairs which separate in $\emptyset^j$ to fail to separate in $S^j$, even if those pairs themselves are not in $S$! We illustrate such a scenario in the next example.

$$\frac{}{R \vdash \sigma = \sigma}\ r \qquad \frac{R \vdash \tau = \sigma}{R \vdash \sigma = \tau}\ s \qquad \frac{R \vdash \sigma = \tau \quad R \vdash \tau = \rho}{R \vdash \sigma = \rho}\ t$$

$$\frac{}{\{(x,y)\} \cup R \vdash x = y}\ a \qquad \frac{R \vdash x_1 = y_1 \quad \dots \quad R \vdash x_{ar(f)} = y_{ar(f)}}{R \vdash f(x_1, \dots, x_{ar(f)}) = f(y_1, \dots, y_{ar(f)})}\ c$$

$$\frac{\alpha =_\epsilon \beta}{R \vdash \alpha = \beta}\ \epsilon \qquad \frac{R \cup S \vdash \sigma = \tau \quad \forall (x,y) \in S. R \cup S \vdash d(x) = d(y)}{R \vdash \sigma = \tau}\ i$$

**Figure 1** Our proof system.

▶ **Example 20.** Recall the signature for the functor $\mathcal{P}_3$ from Example 1. Define a $\mathcal{P}_3$-coalgebra structure on $X = \{x, y, z\}$ by extending the following signature coalgebra with $\epsilon_X$.

$$d(x) = z * y \qquad\qquad d(y) = z * z \qquad\qquad d(z) = n$$

We consider the $\emptyset^*$ iteration for this coalgebra. We know $\emptyset^1 = \Delta_X \cup \{(x,y),(y,x)\}$ since $[-]_{\emptyset^0} * [-]_{\emptyset^0} \neq_\epsilon [n]$ for any variables in the left hand side, so $x$ and $y$ are separated from $z$. Then $\emptyset^2 = \Delta_X$ since $[z]_{\emptyset^1} * [y]_{\emptyset^1} \neq_\epsilon [z]_{\emptyset^1} * [z]_{\emptyset^1}$. After this the sequence stabilizes at $\Delta_X$. We shall see that this implies that all three of these variables have different images in the final coalgebra.

Next we take $S = \{(y,z)\}$ and consider the $S^*$ iterates for the same coalgebra. We know $S^1 = \emptyset^1 \cup S = \Delta_X \cup \{(x,y),(y,x)\} \cup \{(y,z)\}$. Notice the equivalence closure of $S^1$ is still the total relation on $X$. As a consequence, $S^2 = S^1$, so we have stabilized after only one step.

We certainly would have expected $(y,z) \in S$ since that pair is returned to the relation during each application of $(\cdot)^-_S$. However, $(x,y)$ is also present in $S^*$ despite not being in $S$. Intuitively, this is because their separation in the $\emptyset^*$ iteration required $y$ and $z$ to separate in the $\emptyset^1$ iterate. This never happens in the $S^i$ process.

## 4 Logic

Let $F$ be finitary, and assume that it is presented via $\epsilon : H = H_\Sigma \to F$ which has a set $E$ of $\Sigma$-equations, as in Section 2.2.

Let $X$ be any set. We again think of the elements of $X$ as variables, but they could be any objects. For each $H$-coalgebra $(X, d)$, we have a proof system that is intended to talk about the final $F$-coalgebra map of $\epsilon_X \circ d$.

In the syntax of our logic, $x, y \in X$ are variables, $\alpha, \beta \in HX$ are flat terms, and $\sigma, \tau$ are either both variables or both flat terms. We do not have terms that are "deeper" than flat terms. The judgments are of the form $R \vdash \sigma = \tau$ where $R \subseteq X \times X$ is a relation on variables. The role of $R$ on the left side of the turnstile is to track the precongruence relation we are verifying through the course of the proof.

### 4.1 Proof system

For each $H$-coalgebra $(X, d)$, Figure 1 contains a proof system for determining behavioral equivalence in $\nu F$. The function $d$ enters into the $i$ rule. (That is, the proof system depends on the underlying coalgebra.)

The first three rules, $r$, $s$, and $t$, are the obvious equivalence properties, and $c$ just says that each function symbol is a congruence. In $\epsilon$, we mean to take as axioms all of the instances of the equations of $E$ in the carrier set $X$ of the coalgebra at hand.

The way to think about the $i$ rule as a form of *strong coinduction*. Just as proof by induction frequently appears in strong forms, the same is true of coinduction. The reader will want to look at examples, since this rule is the main innovation in our system. It is a proof-theoretic version of what we saw in Lemma 3.

We write $R \vdash \sigma = \tau$ to mean that there is a tree of finite height whose leaves are blank, whose root is $R \vdash \sigma = \tau$, and all of whose internal nodes are justified by one of the rules in the system. We point out that instances of the $i$ rule with infinite $S$ will create "infinitely wide" proofs. These are allowable in our system; we only require finite height.

▶ **Example 21.** Here is the formal version of the argument presented in Example 17. Consider the coalgebra $d : X \to H_\Sigma X$, where $X = \{x, y, z\}$ and where

$$d(x) = x * z \qquad\qquad d(y) = z * y \qquad\qquad d(z) = n$$

Let $S = \{(x, y)\}$. The proof tree below witnesses $\vdash x = y$.

$$\dfrac{\dfrac{}{S \vdash x = y}\,a \qquad \dfrac{\dfrac{}{S \vdash x * z = z * x}\,\epsilon \qquad \dfrac{\dfrac{}{S \vdash z = z}\,r \quad \dfrac{}{S \vdash x = y}\,a}{S \vdash z * x = z * y}\,c}{S \vdash x * z = z * y}\,t}{\vdash x = y}\,i$$

The $i$ rule here is applied with $R = \emptyset$ and $S$ as above.

▶ Remark. This proof system is very similar to the one developed in [17], but the $i$ rule is slightly different. The system in that paper instead had the rule $b$

$$\dfrac{S \vdash \sigma = \tau \qquad \forall (x, y) \in S. S \vdash d(x) = d(y)}{\vdash \sigma = \tau}\,b$$

$b$ is derivable from $i$: take $R = \emptyset$ in $i$. Using the weaker rule $i$, we could obtain the completeness of the system with respect to semantics which we discuss below. However, the $i$ rule is not derivable from $b$, and is much easier to use. Rather than having to verify and discharge a precongruence relation all as one step, the $i$ rule allows the precongruence relation to be built up in multiple steps.

## 4.2 Soundness and completeness

Recall that we start with an $H$-coalgebra $(X, d)$, and we are interested in the images of points $x, y \in X$ in the final coalgebra of $F$ (not $H$). As Example 21 suggests, an assertion $\vdash x = y$ should mean that those images coincide. We formally define our semantics as behavioral equivalence: $\vDash x = y$ for two variables in $X$ means $x$ and $y$ have the same image in $\nu F$. But to prove soundness by induction on proof trees means proving something about inferences with hypotheses (the $R$ on the left), and this is our next result.

▶ **Proposition 22** (Strong soundness). *Suppose $(X, d)$ is an $H$-coalgebra, $x, y \in X$ and $\alpha, \beta \in HX$. Then $R \vdash x = y$ implies $(x, y) \in e(R^*) = ker(q_{R^*})$, and $R \vdash \alpha = \beta$ implies $Hq_{R^*}(\alpha) =_\epsilon Hq_{R^*}(\beta)$. (Equivalently, $(\alpha, \beta) \in ker(\epsilon_{X/e(R^*)} \circ Hq_{R^*})$.)*

**Proof.** By induction on the height of proof trees in our proof system. Both relations mentioned in the consequents above are equivalence relations; the $r$, $s$ and $t$ rules follow. The $a$ rule follows from $R \subseteq e(R^*)$.

To verify the soundness of the $c$ rule, fix an instance of the rule, say with a function symbol $f$ in $\Sigma$. Let $n = ar(f)$. Let us write $q$ for $q_{R^*}$ to save on notation in this paragraph. By induction hypothesis, for $i \leq n$, $q(x_i) = q(y_i)$. And so

$$Hq(f(x_1, \ldots, x_n)) = f(q(x_1), \ldots, q(x_n)) = f(q(y_1), \ldots, q(y_n)) = Hq(f(y_1, \ldots, y_n)).$$

For the $\epsilon$ rule, asume that $\alpha =_\epsilon \beta$. Then $Hq_\emptyset(\alpha) =_\epsilon Hq_\emptyset(\beta)$. So $Hq_{R^*}(\alpha) =_\epsilon Hq_{R^*}(\beta)$.

The most interesting point is the soundness of the $i$ rule. Fix an instance of this rule and assume our soundness result for all premises of our instance. In particular, from the rightmost premises in the $i$ rule, we know $Hq_{(R\cup S)^*}(d(x)) =_\epsilon Hq_{(R\cup S)^*}(d(y))$ for all $(x, y) \in S$. Thus $S \subseteq ((R \cup S)^*)^-$. By Proposition 19, $(R \cup S)^* \subseteq R^*$.

If $\sigma = \tau$ is an equality of variables, say $u = v$, then the induction hypothesis for the left premise of the $i$ rule is $q_{(R\cup S)^*}(u) = q_{(R\cup S)^*}(v)$. We just showed $(R \cup S)^* \subseteq R^*$, and thus we have $q_{R^*}(u) = q_{R^*}(v)$, as desired.

On the other hand, if $\sigma = \tau$ is an equality of flat terms, then the left induction hypothesis is $Hq_{(R\cup S)^*}(\alpha) =_\epsilon Hq_{(R\cup S)^*}(\beta)$. Again since $(R \cup S)^* \subseteq R^*$ we get our desired result of $Hq_{R^*}(\alpha) =_\epsilon Hq_{R^*}(\beta)$. ◀

▶ **Corollary 23** (Soundness). *If $\vdash x = y$, then $\vDash x = y$.*

**Proof.** Assume that $\vdash x = y$. By Proposition 22, $(x, y) \in ker(q_{\emptyset^*})$. But $\emptyset^*$ is a precongruence relation (indeed the largest such), and so we are done by Theorem 14. ◀

As usual for logics pertaining to (co-)recursively-defined terms, completeness is easier than soundness.

▶ **Proposition 24.** *Suppose that $R$ is a relation on $X$. Then the following hold:*

1. *If $(x, y) \in e(R)$, then $R \vdash x = y$.*
2. *If $Hq_R(\alpha) =_\epsilon Hq_R(\beta)$, then $R \vdash \alpha = \beta$.*
3. *If $R$ is a precongruence on $X$, then $\vdash x = y$ for all $(x, y) \in R$.*

**Proof.** 1. Repeated use of $r$, $s$ and $t$.

2. Let $r : X \to X$ send each variable $x$ to a canonical representative for the $[x]_{e(R)}$ equivalence class. Then $R \vdash \alpha = Hr(\alpha)$ and $R \vdash \beta = Hr(\beta)$ using rule $c$ and then item 1 repeatedly. We are also using the fact that all arities of symbols in the signature corresponding to $H$ are finite. The fact that $Hq_R(\alpha) =_\epsilon Hq_R(\beta)$ implies $Hr(\alpha) =_\epsilon Hr(\beta)$. So $R \vdash Hr(\alpha) = Hr(\beta)$ by the $\epsilon$ rule. Therefore, $R \vdash \alpha = \beta$ with $s$ and two applications of $t$.

3. $R \vdash x = y$ by the $a$ rule. By $R$ precongruence and item 2, we know $R \vdash d(z) = d(w)$ for all $(z, w) \in R$. Therefore by $i$ (or even by $b$), we have $\vdash x = y$. ◀

▶ **Corollary 25** (Completeness). *If $\vDash x = y$, then $\vdash x = y$.*

**Proof.** By Proposition 24(3), and the fact that $\{(x, y) | \vDash x = y\}$ is a precongruence on $(X, d)$. ◀

▶ Remark. We are mostly interested in this proof system when $X$ is a finite set. In that case, whenever $\vDash x = y$, there is a finite proof tree showing that $\vdash x = y$. But when $X$ is infinite, we might need a proof tree of "infinite width" (see Example 26 below).

We could generalize our proof system and results to consider *accesssible* set functors. These have a presentation in terms of signatures and sets of equations, but the arities in the signature may be infinite. The results of this paper generalize. If $\kappa$ is an infinite regular cardinal and $F$ a $\kappa$-accessible functor, we would need proof trees of height $\leq \kappa$.

▶ **Example 26.** Write $R \vdash_* \phi$ to mean that there is a *finite* proof tree showing $R \vdash \phi$. Since we are already assuming proofs are of finite height, this finiteness condition requires that additionally proofs are finitely wide. The purpose of this example is to show that our completeness result above may sometimes require these infinitely wide proofs.

Let $\Sigma$ consist of a unary function symbol, $f$. Let $H$ be the associated signature functor, and let $F = H$. In particular, this implies there are no $\epsilon$-axioms in our logic for $H$-coalgebras. Give $X = \{x_k : k \in \omega\}$ an $H$-coalgebra structure by $d(x_k) = f(x_{k+1})$.

We first note $\vdash x_0 = x_1$: set $R = \{(x_k, x_{k+1}) : k \in \omega\}$ and observe that $R \vdash_* f(x_{k+1}) = f(x_{k+2})$ for all $k$. Then one application of rule $i$ with infinitely many premises shows that $\vdash x_0 = x_1$.

In contrast, we also claim that $\nvdash_* x_0 = x_1$. The exact induction argument is a bit long for this section, but the idea is that $R \vdash_* x_j = x_k$ with $j \neq k$ requires that $j$ and $k$ are both less than the greatest index mentioned in $R$. This induction makes critical use of the fact that the $i$ rule can only be invoked with a finite sized $S$. Then taking $R = \emptyset$ gives our desired claim.

## 4.3 Further examples

As noted above, this proof system is an improvement in two ways over the system presented in [17]. First, it is sound and complete for **all** finitary functors, not just the ones which preserve weak pullbacks. Second, the $i$ rule is more flexible than the $b$ rule as it allows for more gradual assumption introduction. The next two example proofs showcase these capabilities.

▶ **Example 27.** The Aczel-Mendler functor $(-)^3_2$ is defined as $(X)^3_2 = \{(x, y, z) \in X^3 : |\{x, y, z\}| \leq 2\}$. This functor does not preserve weak pullbacks, but is finitary with presentation given by $\Sigma = \Sigma_2 = \{\sigma, \tau, \rho\}$. The associated natural transformation is given by

$$\epsilon_X : \sigma(x, y) \mapsto (x, x, y) \qquad \epsilon_X : \tau(x, y) \mapsto (y, x, x) \qquad \epsilon_X : \rho(x, y) \mapsto (x, y, x)$$

Therefore, the only equations for this presentation take the form $\sigma(x, x) =_\epsilon \tau(x, x) =_\epsilon \rho(x, x)$. The final coalgebra for this functor is a singleton set, so all states in all $(-)^3_2$-coalgebras are behaviorally equivalent. Therefore, we consider the functor $1 + (-)^3_2$ instead. The presentation for this functor adds a fresh constant $*$ for the left part of the coproduct. There are no additional equations.

Consider the following coalgebra on $X = \{a, b, c\}$:

$$d(a) = \sigma(a, b) \qquad d(b) = \rho(a, a) \qquad d(c) = \tau(b, a)$$

The greatest bisimulation on the related $(-)^3_2$-coalgebra is not transitive. In particular, it does not relate $a$ and $c$. However, $a$ and $c$ are behaviorally equivalent, as can be seen below. Take $R = \{(a, c), (a, b)\}$, then:

$$\dfrac{\dfrac{}{R \vdash a = c}\,a \qquad \dfrac{\dfrac{\dfrac{}{R \vdash a = a}\,r \quad \dfrac{\dfrac{}{R \vdash a = b}\,a}{R \vdash b = a}\,s}{R \vdash \sigma(a,b) = \sigma(a,a)}\,c \qquad \dfrac{}{R \vdash \sigma(a,a) = \rho(a,a)}\,\epsilon}{R \vdash \sigma(a,b) = \rho(a,a)}\,t \qquad \ldots \text{(proof continues below)}}{\vdash a = c}$$

$$\dfrac{\ldots \qquad \dfrac{\dfrac{\dfrac{}{R \vdash a = a}\,r \quad \dfrac{\dfrac{}{R \vdash a = b}\,a}{R \vdash b = a}\,s}{R \vdash \sigma(a,b) = \sigma(a,a)}\,c \qquad \dfrac{}{R \vdash \sigma(a,a) = \tau(a,a)}\,\epsilon \qquad \dfrac{\dfrac{}{R \vdash a = b}\,a \quad \dfrac{}{R \vdash a = a}\,r}{R \vdash \tau(a,a) = \tau(b,a)}\,c}{R \vdash \sigma(a,b) = \tau(b,a)}\,t \times 2}{}\,i$$

Finally, we introduce a toy signature to show the flexibility of the $i$ rule.

▶ **Example 28.** Let $\Sigma_1 = \{f, g\}$ and define an $H$-coalgebra on $\{x, y, w, z, u\}$ by

$$d(x) = f(w) \qquad d(y) = f(z) \qquad d(w) = g(y) \qquad d(z) = g(u) \qquad d(u) = f(w)$$

Our goal here is to show $x$ and $y$ have the same image in the final coalgebra, and we look forward enough to know that for this to be true we would need $f(w) = f(z)$. Being a bit impatient, we start the proof right away with a coinductive hypothesis $R = \{(x, y), (w, z)\}$.

$$\dfrac{\dfrac{}{R \vdash x = y}\,a \qquad \dfrac{\dfrac{}{R \vdash y = z}\,a}{R \vdash f(y) = f(z)}\,c \qquad \dfrac{\dfrac{}{R \vdash y = u}\,??}{R \vdash g(y) = g(u)}\,c}{\vdash x = y}\,i$$

(In the application of $i$ in this tree, our $R$ takes the role of $S$ in the rule statement and the $R$ in the rule statement is empty.) At this point we get stuck—there is nothing to help prove $y = u$. Due to $i$ though, we can add this pair to our coinduction hypothesis on the fly. Let $S = \{(y, u)\}$. We can then finish the proof as below:

$$\dfrac{\dfrac{}{R \vdash x = y}\,a \qquad \dfrac{\dfrac{}{R \vdash y = z}\,a}{R \vdash f(y) = f(z)}\,c \qquad \dfrac{\dfrac{\dfrac{}{R \cup S \vdash y = u}\,a \qquad \dfrac{\dfrac{\dfrac{}{R \cup S \vdash w = z}\,a}{R \cup S \vdash z = w}\,s}{R \cup S \vdash f(z) = f(w)}\,c}{R \vdash y = u}\,i}{R \vdash g(y) = g(u)}\,c}{\vdash x = y}\,i$$

## 5 Conclusion and future directions

We presented a sound and complete logic for behavioral equivalence. The core principles used in our soundness and completeness proofs were basic results on precongruence relations. We used precongruences in the context of finitary Set functors to do two things. First, we described the final coalgebra for a finitary functor as the quotient of a related final coalgebra by its greatest precongruence relation. Then we realized the greatest precongruence relation as the greatest fixed point of an operator on relations, and studied the a variant of the operator, leading to parametrized precongruences and then the the soundness of the main rule in our logical system.

Though we used several features of Set throughout this work, these features do not appear to be **unique** features of Set. We believe it would be possible to obtain similar results in a wider class of categories. In particular, we are interested in applications to order categories such as preorders and posets. Complete metric spaces would be another appropriate setting. One way to start would be to use results in [4] on generalizations of presentations of finitary functors to all locally presentable categories, and [6] for the specific case of order categories.

Another direction of generalization would be to enhance the expressivity of the logic by allowing more than just flat terms, removing the restrictions on the sets of equations, or otherwise expanding the set of available formulas. In particular, allowing partial specifications where some variables in the set of equations are undefined is upcoming work in the first author's thesis.

**Acknowledgements.** The authors are grateful to the anonymous CALCO referees for pointing out shortcomings of a previous version of this paper.

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