Proper Functors and their Rational Fixed Point

Stefan Milius*

Lehrstuhl für Theoretische Informatik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Abstract

The rational fixed point of a set functor is well-known to capture the behaviour of finite coalgebras. In this paper we consider functors on algebraic categories. For them the rational fixed point may no longer be a subcoalgebra of the final coalgebra. Inspired by Ésik and Maletti's notion of proper semiring, we introduce the notion of a proper functor. We show that for proper functors the rational fixed point is determined as the colimit of all coalgebras with a free finitely generated algebra as carrier and it is a subcoalgebra of the final coalgebra. Moreover, we prove that a functor is proper if and only if that colimit is a subcoalgebra of the final coalgebra. These results serve as technical tools for soundness and completeness proofs for coalgebraic regular expression calculi, e.g. for weighted automata.

1998 ACM Subject Classification F.3.2 Semantics of Programming Languages

Keywords and phrases proper functor, proper semiring, coalgebra, rational fixed point

Digital Object Identifier 10.4230/LIPIcs.CALCO.2017.18

1 Introduction

Coalgebras allow to model many types of systems within a uniform and conceptually clear mathematical framework [25]. One of the key features of this framework is *final semantics*; the final coalgebra provides a fully abstract domain of system behaviour (i.e. it identifies precisely the behaviourally equivalent states). For example, the standard coalgebraic modelling of deterministic automata (without restricting to finite state sets) yields the set of formal languages as final coalgebra. Restricting to finite automata, one obtains precisely the regular languages [24]. It is well-known that this correspondence can be generalized to locally finitely presentable (lfp) categories [4], where *finitely presentable* objects play the role of finite sets. For a finitary functor F (modelling a coalgebraic system type) one then obtains the *rational fixed point* ϱF, which provides final semantics to all coalgebras with a finitely presentable carrier [17]. Moreover, the rational fixed point is fully abstract whenever the classes of finitely presentable and finitely generated objects agree in the base category and F preserves monomorphisms [7, Proposition 3.12]. While the latter assumption on F is very mild (and is not even needed in the case of a lifted set functor), the former one on the base category is more restrictive. However, it is still true for many categories used in the construction of coalgebraic system models (e.g. sets, posets, graphs, vector spaces, commutative monoids, nominal sets and convex sets).

In this paper we will consider rational fixed points in algebraic categories (a.k.a. finitary varieties), i.e. categories of algebras specified by a finitary signature of operation symbols and a set of equations (equivalently, these are precisely the Eilenberg-Moore categories for finitary monads on sets). Being the target of generalized determinization [28], these categories

* Supported by Deutsche Forschungsgemeinschaft (DFG) under project MI 717/5-1

7th Conference on Algebra and Coalgebra in Computer Science (CALCO 2017).
Editors: Filippo Bonchi and Barbara König; Article No. 18; pp. 18:1–18:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

provide a paradigmatic setting for coalgebraic modelling beyond sets. For example, non-deterministic automata, weighted or probabilistic ones [16], or context-free grammars [33] are coalgebraically modelled over the categories of join-semilattices, modules for a semiring, convex sets, and idempotent semirings, respectively. In algebraic categories one would like that the rational fixed point, in addition to being fully abstract, is determined already by those coalgebras carried by free finitely generated algebras, i.e. precisely those coalgebras arising by generalized determinization. In particular, this feature is used in completeness proofs for generalized regular expressions calculi [29, 28, 7]; there one proves that the quotient of syntactic expressions modulo axioms of the calculus is (isomorphic to) the rational fixed point by establishing its universal property as a final object for that quotient. A key feature of the settings in loc. cit. is that it suffices to verify the finality only w.r.t. coalgebras with a free finitely generated carrier.

The purpose of the present paper is to provide sufficient conditions on the algebraic base category and coalgebraic type functor that ensure such finality proofs are sound. More precisely, inspired by Ésik and Maletti's notion of a proper semiring (which is in fact a notion concerning weighted automata), we introduce *proper functors* (Definition 23), and we prove that for a proper functor on an algebraic category the rational fixed point is determined by the coalgebras with a free finitely presentable carrier. More precisely, let $T : \mathsf{Set} \to \mathsf{Set}$ be a finitary monad on sets and $F : \mathsf{Set}^T \to \mathsf{Set}^T$ be a finitary endofunctor preserving surjective T-algebra morphisms (note that the last assumption always holds if F is lifted from some endofunctor on Set). If F is proper, then the rational fixed point is the colimit φF of the inclusion functor of the full subcategory $\mathsf{Coalg}_{\mathsf{free}}\, F$ formed by all F-coalgebras of the form $TX \to FTX$, where X is a finite set (Theorem 27). Moreover, we show that a functor F is proper if and only if φF is a subcoalgebra of the final coalgebra νF (Theorem 26). As a consequence we also obtain that for a proper functor F finality of a given locally finitely presentable coalgebra can be established by only verifying that property for all coalgebras from $\mathsf{Coalg}_{\mathsf{free}}\, F$ (Corollary 29).

We also provide more easily established sufficient conditions on Set^T and F that ensure properness: F is proper if finitely generated algebras of Set^T are closed under kernel pairs and F maps kernel pairs to weak pullbacks in Set. For a lifting F this holds whenever the lifted functor on sets preserves weak pullbacks; in fact, in this case the above conditions were shown to entail Corollary 29 in previous work [7, Corollary 3.36]. However, the type functor (on the category of commutative monoids) of weighted automata with weights drawn from the semiring of natural numbers provides an example of a proper functor for which the above condition on Set^T fails.

Another recent related work concerns the so-called *locally finite fixed point* ϑF [19]; this provides a fully abstract behavioural domain whenever F is a finitary endofunctor on an lfp category preserving monomorphisms. In loc. cit. it was shown that ϑF captures a number of instances that cannot be captured by the rational fixed point, e.g. context free languages [33], constructively algebraic formal power-series [22, 34], Courcelle's algebraic trees [8, 2] and the behaviour of stack machines [15]. However, as far as we know, ϑF is not amenable to the simplified finality check mentioned above unless F is proper.

Putting everything together, in an algebraic category we obtain the following picture of fixed points of F (where $\twoheadrightarrow$ denotes quotient coalgebras and $\rightarrowtail$ a subcoalgebra):

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F. \tag{1}$$

We exhibit an example, where all four fixed points are different. However, if F is proper and preserves monomorphisms, then φF, ϱF and ϑF are isomorphic and fully abstract, i.e. they collapse to a subcoalgebra of the final one: $\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$.

The rest of the paper is structured as follows: in Section 2 we collect some technical preliminaries and recall the rational and locally finite fixed points more in detail. Section 3 introduces proper functors and presents all our results while in Section 4 we present the proof of our main result Theorem 26. Finally, Section 5 concludes the paper.

Due to space restrictions some proofs and details are omitted; these can be found in the full version of this paper [18].

2 Preliminaries

In this section we recall a few preliminaries needed for the subsequent development. We assume that readers are familiar with basic concept of category theory.

We denote the coproduct of two object X and Y of a category $\mathscr{A}$ by $X+Y$ with injections $\mathsf{inl} : X \to X+Y$ and $\mathsf{inr} : Y \to X+Y$.

▶ Remark 1. Recall that a *strong epimorphism* in a category $\mathscr{A}$ is an epimorphism $e : A \twoheadrightarrow B$ of $\mathscr{A}$ that has the unique diagonal property w.r.t. any monomorphism. More precisely, whenever we have a commutative square $m \cdot f = g \cdot e$, where $m : C \rightarrowtail D$ is a monomorphism, then there exists a unique diagonalization $d : B \to C$ with $d \cdot e = f$ and $m \cdot d = g$.

2.1 Algebras and Coalgebras

We assume that readers are familiar with algebras and coalgebras for an endofunctor. Given an endofunctor F on some category $\mathscr{A}$ we write $(\nu F, t)$ for the final F-coalgebra (if it exists). Recall, that the final F-coalgebra exists under mild assumptions on $\mathscr{A}$ and F, e.g. whenever $\mathscr{A}$ is locally presentable and F an accessible functor (see [4]). For any coalgebra $c : C \to FC$ we will write $\dagger c : C \to \nu F$ for the unique coalgebra morphism.

If $\mathscr{A}$ is a concrete category, i.e. equipped with a faithful functor $|\cdot| : \mathscr{A} \to \mathsf{Set}$, one defines *behavioural equivalence* as the following relation $\sim$: given two F-coalgebras (X, c) and (Y, d) then $x \sim y$ holds for $x \in |X|$ and $y \in |Y|$ if there is another F-coalgebra (Z, e) and F-coalgebra morphisms $f : X \to Z$ and $g : Y \to Z$ with $|f|(x) = |g|(y)$.

The base categories $\mathscr{A}$ of interest in this paper are the *algebraic categories*, i.e. categories of Eilenberg-Moore algebras (or T-algebras, for short) for a finitary monad T on Set. Equivalently, those categories are precisely the finitary varieties, i.e. category of Σ-algebras for a finitary signature Σ satisfying the a set of equations (e.g. the categories of monoids, groups, vector spaces, join-semilattices).

Given a monad T with unit $\eta : \mathsf{Id} \to T$ and multiplication $\mu : TT \to T$, we will sometimes make use of its *Kleisli extension*, i.e. the operation $(-)^\star$ that takes any morphism $f : X \to TY$ to $f^\star = \mu_Y \cdot Tf : TX \to TY$. Note that $f^\star$ is the unique T-algebra morphism from (TX, μ_X) to (TY, μ_Y) such that $f^\star \cdot \eta_X = f$.

▶ **Example 2.** The leading example in this paper are weighted automata considered as coalgebras. Let $(\mathbb{S}, +, \cdot, 0, 1)$ be a semiring, i.e. $(\mathbb{S}, +, 0)$ is a commutative monoid, $(\mathbb{S}, \cdot, 1)$ a monoid and the usual distributive laws hold: $r \cdot 0 = 0 = 0 \cdot r$, $r \cdot (s+t) = r \cdot s + r \cdot t$ and $(r+s) \cdot t = r \cdot t + s \cdot t$. We just write $\mathbb{S}$ to denote a semiring. As base category $\mathscr{A}$ we consider the category $\mathbb{S}$-Mod of $\mathbb{S}$-semimodules; recall that a (left) $\mathbb{S}$-*semimodule* is a commutative monoid $(M, +, 0)$ together with an action $\mathbb{S} \times M \to M$, written as juxtaposition sm for $r \in \mathbb{S}$ and $m \in M$, such that for every $r, s \in \mathbb{S}$ and every $m, n \in M$ the following laws hold:

$$
\begin{aligned}
(r+s)m &= rm + sm & \quad 0m &= 0 & \quad 1m &= m \\
r(m+n) &= rm + rn & \quad r0 &= 0 & \quad r(sm) &= (r \cdot s)m
\end{aligned}
$$

An $\mathbb{S}$-semimodule morphism is a monoid homomorphism $h\colon M_1 \to M_2$ such that $h(rm) = rh(m)$ for each $r \in \mathbb{S}$ and $m \in M_1$.

Now consider the functor $FX = \mathbb{S} \times X^A$ on $\mathbb{S}$-Mod, where A is an input alphabet. Then it is easy to see that an $\mathbb{S}$-weighted automaton with n states is precisely a coalgebra on the free $\mathbb{S}$-semimodule on n generators, i.e. $\mathbb{S}^n \to \mathbb{S} \times (\mathbb{S}^n)^A$. The final $\mathbb{S}$-coalgebra is carried by the set $\mathbb{S}^{A^*}$ of all *formal power series* (or *weighted languages*) over A with the obvious (coordinatewise) $\mathbb{S}$-semimodule structure and with the F-coalgebra structure given by $\langle o, t\rangle : \mathbb{S}^{A^*} \to \mathbb{S} \times (\mathbb{S}^{A^*})^A$ with $o(L) = L(\varepsilon)$ and $t(L)(a) = \lambda w.L(aw)$; it is straightforward to verify that o and t are $\mathbb{S}$-semimodule morphisms and form a final coalgebra.

An important special case of $\mathbb{S}$-weighted automata are ordinary nondeterministic automata. One takes $\mathbb{S} = \{0, 1\}$ the Boolean semiring for which the category of $\mathbb{S}$-semimodules is (isomorphic to) the category of join-semilattices. Then $FX = \{0, 1\} \times X^A$ is the coalgebraic type functor of deterministic automata with input alphabet A, and there is a bijective correspondence between an F-coalgebra on a free join-semilattice and non-deterministic automata. In fact in one direction one restricts $\mathcal{P}_{\mathsf{f}}X \to \{0, 1\} \times (\mathcal{P}_{\mathsf{f}}X)^A$ to the set X of generators, and in the other direction one performs the well-known subset construction. The final coalgebra is carried by the set of all formal languages on A in this case.

Another special case is where $\mathbb{S}$ is a field. In this case, $\mathbb{S}$-semimodules are precisely the vector spaces over the field $\mathbb{S}$. Moreover, since every field is freely generated by its basis, it follows that the $\mathbb{S}$-weighted automata are precisely those F-coalgebras whose carrier is a finite dimensional vector space over $\mathbb{S}$.

We will now recall a few properties of algebraic categories Set^T, where T is a finitary set monad, needed for our proofs.

▶ Remark 3.

1. Recall that every strong epimorphisms e in Set^T is regular, i.e. e is the coequalizer of some pair of T-algebra morphisms. It follows that the classes of strong and regular epimorphisms coincide, and these are precisely the surjective T-algebra morphisms.
2. We will later use that every free T-algebra TX is *(regular) projective*, i.e. given any surjective T-algebra morphism $q : A \twoheadrightarrow B$ then for every T-algebra morphism $h : TX \to B$ there exists a T-algebra morphism $g : TX \to A$ such that $q \cdot g = h$.
3. Furthermore, note that every finitely presentable T-algebra A is a regular quotient of a free T-algebra TX with a finite set X of generators. Indeed, A is presented by finitely many generators and relations. So by taking X as a finite set of generators of A, the unique extension of the embedding $X \hookrightarrow A$ yields a surjective T-algebra morphism $TX \twoheadrightarrow A$.

2.2 The Rational Fixed Point

As we mentioned in the introduction the canonical domain of behaviour of 'finite' coalgebras is the rational fixed point of an endofunctor on F. Its theory can be developed for every finitary endofunctor on a locally finitely presentable category. We will now recall the necessary background material.

A *filtered colimit* is the colimit of a diagram $\mathscr{D} \to \mathscr{C}$ where $\mathscr{D}$ is a filtered category (i.e. every finite subdiagram has a cocone in $\mathscr{D}$), and a *directed colimit* is a colimit whose diagram scheme $\mathscr{D}$ is a directed poset. A functor is called *finitary* if it preserves filtered (equivalently directed) colimits. An object C is called *finitely presentable* (fp) if the hom-functor $\mathscr{C}(C, -)$ preserves filtered (equivalently directed) colimits, and *finitely generated* (fg) if $\mathscr{C}(C, -)$ preserves directed colimits of monos (i.e. colimits of directed diagrams $D : \mathscr{D} \to \mathscr{C}$

where all connecting morphisms Df are monic in $\mathscr{C}$). Clearly any fp object is fg, but the converse fails in general. In addition, fg objects are closed under strong epis (quotients), which fails for fp objects in general.

A cocomplete category $\mathscr{C}$ is called *locally finitely presentable* (lfp) if there is a set of finitely presentable objects in $\mathscr{C}$ such that every object of $\mathscr{C}$ is a filtered colimit of objects from that set. We refer to [4] for further details.

Examples of lfp categories are the categories of sets, posets and graphs, with finitely presentable objects precisely the finite sets, posets, and graphs, respectively. The category of vector spaces over the field k is lfp with finite-dimensional spaces being the fp-objects. Every algebraic category is lfp. The finitely generated objects are precisely the finitely generated algebras (in the sense of general algebra), and finitely presentable objects are precisely those algebras specified by finitely many generators and finitely many relations.

▶ **Assumptions 4.** For the rest of this section we assume that F denotes a finitary endofunctor on the lfp category $\mathscr{A}$.

The rational fixed point is a fully abstract model of behaviour for all F-coalgebras whose carrier is an fp-object. We now recall its construction [1].

▶ **Notation 5.** Denote by $\mathsf{Coalg}\,F$ the full subcategory of all F-coalgebras on fp carriers, and let $(\varrho F, r)$ be the colimit of the inclusion functor of $\mathsf{Coalg}_{\mathsf{fp}}\,F$ into $\mathsf{Coalg}\,F$: $(\varrho F, r) = \mathrm{colim}(\mathsf{Coalg}_{\mathsf{fp}}\,F \hookrightarrow \mathsf{Coalg}\,F)$ with the colimit injections $a^\sharp : A \to \varrho F$ for every coalgebra $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\,F$.

We call $(\varrho F, r)$ the *rational fixed point* of F; indeed, it is a fixed point:

▶ **Proposition 6** ([1]). *The coalgebra structure $r : \varrho F \to F(\varrho F)$ is an isomorphism.*

The rational fixed point can be characterized by a universal property both as a coalgebra and as an algebra for F: as a coalgebra ϱF is the *final locally finitely presentable coalgebra* [17], and as an algebra it is the *initial iterative algebra* [1]. We will not recall the latter notion as it is not needed for the technical development in this paper. Locally finitely presentable (lfp, for short) coalgebras for F can be characterized as precisely those F-coalgebra obtained as a filtered colimit of a diagram of coalgebras from $\mathsf{Coalg}_{\mathsf{fp}}\,F$:

▶ **Proposition 7** ([17], Corollary III.13). *An F-coalgebra is lfp if and only if it is a colimit of some filtered diagram $\mathscr{D} \to \mathsf{Coalg}_{\mathsf{fp}}\,F \hookrightarrow \mathsf{Coalg}\,F$.*

For $\mathscr{A} = \mathsf{Set}$ an F-coalgebra (X, c) is lfp iff it is *locally finite*, i.e. every element of X is contained in a finite subcoalgebra. Analogously, for $\mathscr{A}$ the category of vector spaces over the field k an F-coalgebra (X, c) is lfp iff it is *locally finite dimensional*, i.e. every element of X is contained in a finite dimensional subcoalgebra.

Of course, there is a unique coalgebra morphism $\varrho F \to \nu F$. Moreover, in many cases ϱF is *fully abstract* for lfp coalgebras, i.e. besides being the final lfp coalgebra the above coalgebra morphism is monic; more precisely, if the classes of fp- and fg-objects coincide and F preserves monos, then ϱF is fully abstract (see [7, Proposition 3.12]). The assumption that the two object classes coincide is often true:

▶ **Example 8.**

1. In the category of sets, posets, and graphs, fg-objects are fp and those are precisely the finite sets, posets, and graphs, respectively.
2. A *locally finite variety* is a variety of algebras, where every free algebra on a finite set of generators is finite. It follows that fp- and fg-objects coincide and are precisely the finite

algebras. Concrete examples are the categories of Boolean algebras, distributive lattices and join-semilattices.

3. In the category of $\mathbb{S}$-semimodules for a semiring $\mathbb{S}$ the fp- and fg-objects need not coincide in general. However, if the semiring $\mathbb{S}$ is *Noetherian* in the sense of Ésik and Maletti [11], i.e. every subsemimodule of a finitely generated $\mathbb{S}$-semimodule is itself finitely generated, then fg- and fp-semimodules coincide. Examples of Noetherian semirings are: every finite semiring, every field, every principal ideal domain such as the ring of integers and therefore every finitely generated commutative ring by Hilbert's Basis Theorem. The tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is not Noetherian [10]. The usual semiring of natural numbers is also not Noetherian: the $\mathbb{N}$-semimodule $\mathbb{N} \times \mathbb{N}$ is finitely generated but its subsemimodule generated by the infinite set $\{(n, n+1) \mid n \geq 1\}$ is not. However, $\mathbb{N}$-semimodules are precisely the commutative monoids, and for them fg- and fp-objects coincide (this is known as Redei's theorem [23]; see Freyd [13] for a very short proof).
4. Recently, it was established by Sokolova and Woracek [30] that in the category of convex sets, i.e. the Eilenberg-Moore category for the (sub)distribution monad on sets, the classes of fp- and fg-objects coincide.

▶ **Example 9.** We list a number of examples of rational fixed points for cases where they do form subcoalgebras of the final coalgebra.

1. For the functor $FX = \{0,1\} \times X^A$ on Set the finite coalgebras are deterministic automata, and the rational fixed point is carried by the set of regular languages on the alphabet A.
2. For any signature $\Sigma = (\Sigma_n)_{n<\omega}$ of operation symbols with prescribed arity we have the associated polynomial endofunctor on sets given by $F_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n$. Its final coalgebra is carried by the set of all (finite and infinite) Σ-trees, i.e. rooted and ordered trees where each node with n-children is labelled by an n-ary operation symbol. The rational fixed point is the subcoalgebra given by rational (or regular [8]) Σ-trees, i.e. those Σ-trees that have only finitely many different subtrees (up to isomorphism) – this characterization is due to Ginali [14]. For example, for the signature Σ with a binary operation symbol $*$ and a constant c the following infinite Σ-tree (here written as an infinite term) is rational:

$$c * (c * (c * \cdots)));$$

in fact, its only subtrees are the whole tree and the single node tree labelled by c).
3. For the functor $FX = \mathbb{R} \times X$ on Set the final coalgebra is carried by the set $\mathbb{R}^\omega$ of real streams, and the rational fixed point is carried by its subset of eventually periodic streams (or lassos). Considered as a functor on the category of vector spaces over $\mathbb{R}$, the final coalgebra νF remains the same, but the rational fixed point ϱF consists of all rational streams [26].
4. For the functor $FX = \mathbb{S} \times X^A$ on the category $\mathbb{S}$-Mod of $\mathbb{S}$-semimodules for the semiring $\mathbb{S}$ we already mentioned that $\nu F = \mathbb{S}^{A^*}$ consists of all formal power-series. Whenever the classes of fg- and fp-semimodules coincide, e.g. for every Noetherian semiring $\mathbb{S}$ or the semiring of natural numbers, then ϱF is formed by the *recognizable* formal power-series; from the Kleene-Schützenberger theorem [27] (see also [6]) it follows that these are, equivalently, the *rational* formal power-series.
5. On the category of presheaves $\mathsf{Set}^{\mathcal{F}}$, where $\mathcal{F}$ is the category of all finite sets and maps between them, consider the functor $FX = V + X \times X + \delta(X)$, where $V : \mathcal{F} \hookrightarrow \mathsf{Set}$ is the embedding and $\delta(X)(n) = X(n+1)$. This is a paradigmatic example of a functor arising from a *binding signature* for which initial semantics was studied by Fiore et al. [12].

The final coalgebra νF is carried by the presheaf of all λ-trees modulo α-equivalence: $\nu F(n)$ is the set of (finite and infinite) λ-trees in n free variables (note that such a tree may have infinitely many bound variables). And ϱF is carried by the rational λ-trees, where an α-equivalence class is called *rational* if it contains at least one λ-tree which has (up to isomorphism) only finitely many different subtrees (see [3]). Rational λ-trees also appear as the rational fixed point of a very similar functor on the category of nominal sets [21]. Similarly, for any functor on nominal sets arising from a binding signature [20].

As we mentioned previously, whether fg- and fp-objects coincide is currently unknown in some base categories used in the coalgebraic modelling of systems, for example, in idempotent semirings (used in the treatment of context-free grammars [33]), in algebras for the stack monad (used for modelling configurations of stack machines [15]); or it even fails, for example in the category of finitary monads on sets (used in the categorical study of algebraic trees [2]) or in Eilenberg-Moore categories for a monad in general (the target categories of generalized determinization [28]).

As a remedy, in recent joint work with Pattinson and Wissmann [19], we have introduced the *locally finite fixed point* which provides a fully abstract model of finitely generated behaviour. Its construction is very similar to that of the rational fixed point but based on fg- in lieu of fp-objects. In more detail, one considers the full subcategory $\mathsf{Coalg}_{\mathsf{fg}}\, F$ of all F-coalgebras carried by an fg-object and takes the colimit of its inclusion functor:

$$(\vartheta F, \ell) = \mathrm{colim}(\mathsf{Coalg}_{\mathsf{fg}}\, F \hookrightarrow \mathsf{Coalg}\, F).$$

▶ **Theorem 10** ([19], Theorems 3.10 and 3.12)**.** *Suppose that the finitary functor $F : \mathscr{A} \to \mathscr{A}$ preserves monos. Then $(\vartheta F, \ell)$ is a fixed point for F, and it is a subcoalgebra of νF.*

Furthermore, like its brother, the rational fixed point, ϑF is characterized by a universal property both as a coalgebra and as an algebra: it is the final locally finitely generated coalgebra and the initial fg-iterative algebra [19, Theorems 3.8 and Corollary 3.18].

Under additional assumptions, which all hold in any algebraic category, we have a close relation between ϱF and ϑF; in fact, the following is a consequence of [19, Theorem 3.22]:

▶ **Theorem 11.** *Suppose that $\mathscr{A}$ is an algebraic category and that the finitary functor $F : \mathscr{A} \to \mathscr{A}$ preserves monos. Then ϑF is the image of ϱF in the final coalgebra.*

More precisely, taking the (strong-epi, mono)-factorization of the unique F-coalgebra morphism $\varrho F \to \nu F$ yields ϑF, i.e. for F preserving monos on an algebraic category we have the following picture:

$$\varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F.$$

If furthermore, fg- and fp-objects coincide, then $\vartheta F \cong \varrho F$, i.e. the left-hand morphism is an isomorphism.

In the introduction we briefly mentioned a number of interesting instances of ϑF that are not (known to be) instances of the rational fixed point; see [19] for details.

A concrete example, where ϱF is not a subcoalgebra of νF (and hence not isomorphic to ϑF) was given in [7, Example 3.15]. We present a new, simpler example based on similar ideas:

▶ **Example 12.**

1. Let $\mathscr{A}$ be the category of algebras for the signature Σ with two unary operation symbols u und v. The natural numbers $\mathbb{N}$ with the successor function as both operations $u^{\mathbb{N}}$ and $v^{\mathbb{N}}$ form an object of $\mathscr{A}$. We consider the functor $FX = \mathbb{N} \times X$ on $\mathscr{A}$. Coalgebras for F are automata carried by an algebra A in $\mathscr{A}$ equipped with two Σ-algebra morphisms: an output morphism $A \to \mathbb{N}$ and a next state morphism $A \to A$. The final coalgebra is carried by the set $\mathbb{N}^\omega$ of streams of naturals with the coordinatewise algebra operations and with the coalgebra structure given by the usual head and tail functions.
   Note that the free Σ-algebra on a set X of generators is $TX \cong \{u, v\}^* \times X$; we denote its elements by $w(x)$ for $w \in \{u, v\}^*$ and $x \in X$. The operations are given by prefixing words by the letters u and v, respectively: $s^{TX} : w(x) \mapsto sw(x)$ for $s = u$ or v.
   Now one considers the F-coalgebra $a : A \to FA$, where $A = T\{x\}$ is free Σ-algebra on one generator x and a is determined by $a(x) = (0, u(x))$. Clearly, $\dagger a(x)$ is the stream $(0, 1, 2, 3, \cdots)$ of all natural numbers, and since $\dagger a$ is a Σ-algebra morphism we have

$$\dagger a(u(x)) = \dagger a(v(x)) = (1, 2, 3, 4, \cdots).$$

   Since A is (free) finitely generated, it is of course, finitely presentable as well. Thus, (A, a) is a coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$. However, one can prove that the (unique) F-coalgebra morphism $a^\sharp : A \to \varrho F$ satisfies $a^\sharp(u(x)) \neq a^\sharp(v(x))$, see the full paper for details [18].
2. In this example we also have that ϑF and νF do not coincide. To see this we use that ϑF is the union of images of all $\dagger a : TX \to \nu F$ where (TX, a) ranges over those F-coalgebras whose carrier TX is free finitely generated (i.e. TX is a term algebra over some finite set X) [19, Theorem 4.4].
   Note that being a Σ-algebra morphism any coalgebra structure $a : TX \to FTX$ is determined by its action on the generators. And from the form of any TX we know that for any $x \in X$ there exist $k, n_i \in \mathbb{N}$, $w_i \in \{u, v\}^*$ and $x_i \in X$, $i = 1, \dots k$, such that $x = x_0$ and

$$\begin{aligned} a(x_i) &= (n_i, w_i(x_{i+1})) && \text{for } i = 0, \dots, k-1 \text{ and} \\ a(x_k) &= (n_k, w_k(x_j)) && \text{for some } j \in \{0, \dots, k\}. \end{aligned}$$

   Now let $m_i = |w_i|$, $i = 1, \dots, k$, be the lengths of words. Then it follows that

$$\dagger a(x_0) = (n_0, m_0 + n_1, m_0 + m_1 + n_2, \cdots, m_0 + \cdots + m_{k-1} + n_k, m_0 + \cdots m_k + n_j, \cdots).$$

   Let m be the maximum of all n_i and m_i. Then it is clear that the n-th entry of $\dagger a(x_0)$ can be at most $(n+1) \cdot m$. It follows that for any $w \in \{u, v\}^*$ the n-th entry of $\dagger a(w(x))$ is bounded above by $(n + 1) \cdot m + |w|$. Thus, the entries of every stream in ϑF grow at most linearly. However, there are streams in νF for which this is not the case, e.g. the stream $(1, 2, 4, 8, \cdots)$ of powers of 2. Hence ϑF does not coincide with νF.

3 Proper Functors and Coalgebras Carried by Free Algebras

The purpose of this section is to study the situation where the rational fixed point for a functor F on an algebraic category Set^T coincides with the locally finite one, and moreover, both can be constructed just from those coalgebras whose carrier is a free finitely generated coalgebra. The latter coalgebras are precisely those coalgebras arising as the results of the generalized determinization [28].

▶ **Assumptions 13.** Throughout the rest of the paper we assume that $\mathscr{A}$ is an *algebraic category*, i.e. $\mathscr{A}$ is (equivalent to) the Eilenberg-Moore category Set^T for a finitary monad T on Set. Furthermore, we assume that $F : \mathscr{A} \to \mathscr{A}$ is a finitary endofunctor preserving surjective T-algebra morphisms.

▶ Remark 14.

1. The most common instance is when F is a lifting of an endofunctor $F_0 : \mathsf{Set} \to \mathsf{Set}$, i.e. we have a commutative square $F_0 \cdot U = U \cdot F$, where $U : \mathscr{A} \to \mathsf{Set}$ is the forgetful functor. Then F preserves surjective T-algebra morphisms since every set functor F_0 preserves surjections (which are split epis in Set). In addition, F is finitary whenever F_0 is so because filtered colimits in Set^T are created by U. Furthermore, observe that the assumption that F preserves monomorphisms in Theorems 10 and 11 as well as in Corollary 28 is not needed. Indeed, inspection of the proofs in [19] reveals that it suffices to assume that non-empty monomorphisms are preserved, and this holds for every lifted F since it does for every F_0 on Set.
2. Let $F : \mathsf{Set} \to \mathsf{Set}$ have a lifting to Set^T (also denoted by F for simplicity). *Generalized determinization* [28] is the process of turning a given coalgebra $c : X \to FTX$ in Set into the coalgebra $c^\star : TX \to FTX$ for the lifting of F on Set^T. For example, for the functor $FX = \{0,1\} \times X^\Sigma$ on Set and the finite power-set monad $T = \mathcal{P}_\mathsf{f}$, FT-coalgebras are precisely non-deterministic automata and generalized determinization is the construction of a deterministic automaton by the well-known subset construction. The unique F-coalgebra morphism $\dagger(c^\star)$ assigns to each state $x \in X$ the language accepted by x in the given nondeterministic automaton (whereas the final semantics for FT on Set provides a kind of process semantics taking the nondeterministic branching into account).
   Thus studying the behaviour of F-coalgebras whose carrier is a free finitely generated T-algebra TX is precisely the study of a *coalgebraic language semantics* of finite FT-coalgebras.

▶ **Notation 15.** We denote by $\mathsf{Coalg}_\mathsf{free}\, F$ the full subcategory of $\mathsf{Coalg}\, F$ given by all coalgebras $c : TX \to FTX$ whose carrier is a free finitely generated T-algebra, i.e. where X is a finite set X.

The colimit of the inclusion functor of $\mathsf{Coalg}_\mathsf{free}\, F$ into the category of all F-coalgebras is denoted by $(\varphi F, \zeta) = \mathrm{colim}(\mathsf{Coalg}_\mathsf{free}\, F \hookrightarrow \mathsf{Coalg}\, F)$ with the colimit injections $\mathsf{in}_c : TX \to \varphi F$ for every $c : TX \to FTX$.

▶ **Notation 16.** Since $\mathsf{Coalg}_\mathsf{free}\, F$ is a full subcategory of $\mathsf{Coalg}_\mathsf{fp}\, F$, the universal property of the colimit φF induces a coalgebra morphism denoted by $h : \varphi F \to \varrho F$. Furthermore we write $m : \varphi F \to \nu F$ for the unique F-coalgebra morphisms into the final lfp coalgebra and the final coalgebra, respectively.

▶ Remark 17. We shall show in Propositon 21 that h is a strong epimorphism. Thus, whenever F preserves monos, we have the picture (1) from the introduction.

Urbat [31] shows that φF is always a fixed point of F. However, φF does not have a universal property similar to the coalgebras ϱF and ϑF. In fact, Urbat gives the following example of a coalgebra $c : TX \to FTX$ where $\mathsf{in}_c : TX \to \varphi F$ is not the only F-coalgebra morphism:

▶ **Example 18.**

1. Let $\mathscr{A}$ be the category of algebras for the signature with one unary operation symbol u (and no equations), and let $F = \mathsf{Id}$ be the identity functor on $\mathscr{A}$. Let A be the free (term) algebra on one generator x, and let B be the free algebra on one generator y

(i.e. both A and B are isomorphic to $\mathbb{N}$). We equip A and B with the F-coalgebra structures $a = \mathsf{id} : A \to A$ and $b : B \to B$ given by $b(y) = u(y)$. Define $g : A \to \varphi F$ by $g(x) = \mathsf{in}_b(y)$. Then one can show that g is an F-coalgebra morphism different from the F-coalgebra morphism $\mathsf{in}_a : A \to \varphi F$.

2. Using similar ideas as in the previous point one can show that, for the category $\mathscr{A}$ and $FX = \mathbb{N} \times X$ from Example 12, φF and ϱF do not coincide, see the full paper [18]. Consequently, in this example, none of the arrows in (1) is an isomorphism.

In this section we are going to investigate when the first three fixed points in (1) collaps to one, i.e. $\varphi F \cong \varrho F \cong \vartheta F$. As a consequence, it follows that finality of a given lfp coalgebra for F can be established by checking the universal property only for the coalgebras in $\mathsf{Coalg}_{\mathsf{free}}\, F$ (Corollary 29).

▶ **Lemma 19.** *The category* $\mathsf{Coalg}_{\mathsf{free}}\, F$ *is closed under finite coproducts.*

Proof. The empty map $0 \to FT0$ extends uniquely to a T-algebra morphism $T0 \to FT0$, i.e. an F-coalgebra, and this coalgebra is the initial object of $\mathsf{Coalg}_{\mathsf{free}}\, F$.

Given coalgebras $c : TX \to FTX$ and $d : TY \to FTY$ one uses that $T(X + Y)$ together with the injections $T\mathsf{inl} : TX \to T(X + Y)$ and $T\mathsf{inr} : TY \to T(X + Y)$ form a coproduct in Set^T. This implies that forming the coproduct of (TX, c) and (TY, d) in $\mathsf{Coalg}\, F$ we obtain an F-coalgebra on $T(X + Y)$, and this is an object of $\mathsf{Coalg}_{\mathsf{free}}\, F$ since $X + Y$ is finite. ◀

▶ Remark 20. We will use later that the colimit φF is a sifted colimit.

1. Recall that a small category $\mathscr{D}$ is called *sifted* [5] if finite products commute with colimits over $\mathscr{D}$ in Set. More precisely, $\mathscr{D}$ is sifted iff given any diagram $D : \mathscr{D} \times \mathscr{J} \to \mathsf{Set}$, where $\mathscr{J}$ is a finite discrete category, the canonical map

$$\operatorname*{colim}_{d \in \mathscr{D}} \Big(\prod_{j \in \mathscr{J}} D(d, j) \Big) \to \prod_{j \in \mathscr{J}} \big(\operatorname*{colim}_{d \in \mathscr{D}} D(d, j)\big)$$

is an isomorphism. A *sifted colimit* is a colimit of a diagram with a sifted diagram scheme.

2. It is well-known that the forgetful functor $\mathsf{Set}^T \to \mathsf{Set}$ preserves sifted colimits; this follows from [5, Proposition 2.5].
3. Further recall [5, Example 2.16] that every small category $\mathscr{D}$ with finite coproducts is sifted. Thus, following Lemma 19, $\mathscr{D} = \mathsf{Coalg}_{\mathsf{free}}\, F$ is sifted, and φF is a sifted colimit.

▶ **Proposition 21.** *The above morphism* $h : \varphi F \twoheadrightarrow \varrho F$ *is a strong epimorphism in* $\mathscr{A}$.

▶ Remark 22.

1. Recall that a *zig-zag* in a category $\mathscr{A}$ is a diagram of the form

$$Z_0 \xrightarrow{f_0} Z_1 \xleftarrow{f_1} Z_2 \xrightarrow{f_2} Z_3 \xleftarrow{f_3} \cdots \xrightarrow{f_{n-2}} Z_{n-1} \xleftarrow{f_{n-1}} Z_n.$$

For $\mathscr{A} = \mathsf{Set}^T$, we say that the zig-zag *relates* $z_0 \in Z_0$ and $z_n \in Z_n$ if there exist $z_i \in Z_i$, $i = 1, \ldots, n-1$ such that $f_i(z_i) = z_{i+1}$ for i even and $f_i(z_{i+1}) = z_i$ for i odd.

2. Ésik and Maletti [10] introduced the notion of a *proper* semiring in order to obtain the decidability of the (language) equivalence of weighted automata. A semiring $\mathbb{S}$ is called *proper* if, whenever we have two $\mathbb{S}$-weighted automata A and B and two states x in A and y in B that accept the same weighted language, then there exists a zig-zag

$$A = M_0 \to M_1 \leftarrow M_2 \to M_3 \leftarrow \cdots \to M_{n-1} \leftarrow M_n = B$$

of simulations that *relates* x and y. They show that every Noetherian semiring is proper as well as the semiring $\mathbb{N}$ of natural numbers, which is not Noetherian. However, the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is not proper.

Recall from Example 2 that $\mathbb{S}$-weighted automata with input alphabet Σ are equivalently coalgebras with carrier $\mathbb{S}^n$, for some $n \geq 1$, for the functor $FX = \mathbb{S} \times X^\Sigma$ on the category $\mathbb{S}$-Mod. Thus, since simulations are precisely the F-coalgebra morphisms, one easily generalizes the notion of a proper semiring as follows. Recall that $\eta_X : X \to TX$ denotes the unit of the monad T.

▶ **Definition 23.** We call the functor $F : \mathscr{A} \to \mathscr{A}$ *proper* whenever for every pair of coalgebras $c : TX \to FTX$ and $d : TY \to FTY$ in $\mathsf{Coalg}_{\mathsf{free}}\, F$ and every $x \in X$ and $y \in Y$ such that $\eta_X(x) \sim \eta_Y(y)$ are behaviourally equivalent there exists a zig-zag in $\mathsf{Coalg}_{\mathsf{free}}\, F$ relating $\eta_X(x)$ and $\eta_Y(y)$.

▶ **Example 24.** A semiring $\mathbb{S}$ is proper iff the functor $FX = \mathbb{S} \times X^\Sigma$ on $\mathbb{S}$-Mod is proper.

▶ **Example 25.** Constant functors are always proper. Indeed, suppose that F is the constant functor on some algebra A. Then we have $\nu F = A$, and for any F-coalgebra B its coalgebra structure $c : B \to FB = A$ is also the unique F-coalgebra morphism from B to $\nu F = A$.

Now given any $c : TX \to FTX = A$ and $d : TY \to FTY = A$ and $x \in TX$, $y \in TY$ as in Definiton 23. Then $\eta_X(x) \sim \eta_Y(y)$ is equivalent to $c(\eta_X(x)) = d(\eta_Y(y))$. Let a be this element of A, and extend $x : 1 \to X$, $y : 1 \to Y$ and $a : 1 \to A$ to T-algebra morphisms $x^\star : T1 \to TX$, $y^\star : T1 \to TY$ and $a^\star : T1 \to A = FT1$ (the latter yielding an F-coalgebra). Then $TX \xleftarrow{x^\star} T1 \xrightarrow{y^\star} TY$ is the required zig-zag in $\mathsf{Coalg}_{\mathsf{free}}\, F$ relating $\eta_X(x)$ and $\eta_Y(y)$.

In general, it seems to be non-trivial to establish that a given functor is proper (even for the identity functor this may fail as we have seen in Example 18.1). However, we will provide in Proposition 30 sufficient conditions on $\mathscr{A}$ and F the entail properness using our main result:

▶ **Theorem 26.** *The functor F is proper iff the coalgebra φF is a subcoalgebra of νF.*

The latter condition states that the unique coalgebra morphism $m : \varphi F \to \nu F$ is a monomorphism in $\mathscr{A}$.

We present the proof of this theorem Section 4. Here we continue with a discussion of the consequences of this result.

▶ **Corollary 27.** *If F is proper, then φF is the rational fixed point of F.*

Proof. Let $u : \varrho F \to \nu F$ be the unique F-coalgebra morphism. Then we have a commutative triangle of F-coalgebra morphisms due to finality of νF: $m = (\varphi F \overset{h}{\twoheadrightarrow} \varrho F \overset{u}{\to} \nu F)$. Since F is proper m is a monomorphism in $\mathscr{A}$, hence so is h. Since h is also a strong epimorphism by Proposition 21, it is an isomorphism. Thus, $\varphi F \cong \varrho F$ is the rational fixed point of F. ◀

▶ **Corollary 28.** *If finitely generated and finitely presentable algebras coincide in $\mathscr{A}$ and F preserves monos, then F is proper iff $\varphi F \cong \varrho F \cong \vartheta F \rightarrowtail \nu F$.*

Indeed, this follows from Corollary 27 and Theorem 11. Note that this also entails full abstractness of $\varphi F \cong \varrho F$.

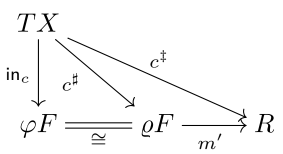
A key result for establishing soundness and completeness of coalgebraic regular expression calculi is the following corollary (cf. [7, Corollary 3.36] and its applications in Sections 4 and 5 of loc. cit.).

▶ **Corollary 29.** *Suppose that F is proper. Then an F-coalgebra (R, r) is a final lfp coalgebra if and only if (R, r) is lfp and for every coalgebra (TX, c) in $\mathsf{Coalg}_{\mathsf{free}}\, F$ there exists a unique F-coalgebra morphism from TX to R.*

Proof. The implication "⇒" clearly holds

For "⇐" it suffices to prove that for every $a : A \to FA$ in $\mathsf{Coalg}_{\mathsf{fp}}\, F$ there exists a unique F-coalgebra morphism from A to R. In fact, it then follows that R is the final lfp coalgebra. To see this write an arbitrary lfp coalgebra A as a filtered colimit of a diagram $D : \mathscr{D} \to \mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}\, F$ with colimit injections $h_d : Dd \to A$ (d an object in $\mathscr{D}$). Then the unique F-coalgebra morphisms $u_d : Dd \to R$ form a compatible cocone, and so one obtains a unique $u : A \to R$ such that $u \cdot h_d = u_d$ holds for every object d of $\mathscr{D}$. It is now straightforward to prove that u is a unique F-coalgebra morphism from A to R.

Now let $a : A \to FA$ be a coalgebra in $\mathsf{Coalg}_{\mathsf{fp}}\, F$. For every (TX, c) in $\mathsf{Coalg}_{\mathsf{free}}\, F$ denote by $c^\ddagger : TX \to R$ the unique F-coalgebra morphism that exists by assumption. These morphisms $c^\ddagger$ form a compatible cocone of the diagram $\mathsf{Coalg}_{\mathsf{free}}\, F \hookrightarrow \mathsf{Coalg}\, F$. Thus, we obtain a unique F-coalgebra morphism $m' : \varphi F \cong \varrho F \to R$ such that the following diagram commutes for every $c : TX \to FTX$ in $\mathsf{Coalg}_{\mathsf{free}}\, F$:



Therefore we have an F-coalgebra morphism

$$h = (A \xrightarrow{a^\sharp} \varrho F \xrightarrow{m'} R).$$

To prove it is unique, assume that $g : A \to R$ is any F-coalgebra morphism. As in the proof of Proposition 21, we know that A is the quotient of some TX in $\mathsf{Coalg}_{\mathsf{free}}\, F$ via $q : TX \twoheadrightarrow A$, say. Then we have $m' \cdot a^\sharp \cdot q = g \cdot q$ because there is only one F-coalgebra morphism from TX to R by hypothesis. It follows that $h = m' \cdot a^\sharp = g$ since q is epimorphic. ◀

The next result provides sufficient conditions for properness of F. It can be seen as a category-theoretic generalization of Ésik's and Maletti's result [10, Theorem 4.2] that Noetherian semirings are proper.

For the special case of a lifting F of a set functor F_0 this is a corollary of a result of Winter [32, Proposition 7].

▶ **Proposition 30.** *Suppose that finitely generated algebras in $\mathscr{A}$ are closed under kernel pairs and that F maps kernel pairs to weak pullbacks in Set. Then F is proper.*

Note that closure of finitely generated algebras under kernel pairs can equivalently be stated in general algebra terms as follows: every congruence R of a finitely generated algebra A is finitely generated as a subalgebra $R \hookrightarrow A \times A$ (observe that this is *not* equivalent to stating that R is a finitely generated congruence).

Furthermore, for a lifting F of a set functor F_0, the above condition on F holds whenever F preserves weak pullbacks. Hence, all the functors on algebraic categories mentioned in Example 9 satisfy this assumption.

▶ **Examples 31.**
1. The first condition in Proposition 30 is not necessary for properness of F. In fact, it fails in the category of semimodules for $\mathbb{N}$, viz. the category of commutative monoids: the submonoid of $\mathbb{N}\times\mathbb{N}$ infinitely generated by $\{(n, n+1) \mid n \in \mathbb{N}\}$ is not a finitely generated submonoid. However, as we mentioned in Example 24, $FX = \mathbb{N} \times X^\Sigma$ is proper on the category of commutative monoids.
2. In Example 8.4 we mentioned that, in the category of convex sets (i.e. Eilenberg-Moore algebras for the distribution monad), fg- and fp-objects coincide. However, fg-objects are not closed under kernel pairs. In fact, the interval $[0, 1]$ is the free convex set on two generators, but $\{(0, 0), (1, 1)\} \cup (0, 1) \times (0, 1)$ is a congruence on $[0, 1]$ that is not an fg-object (i.e. a polytope) [30, Example 4.13]. It is an open problem whether coalgebraic type functors of interest on convex sets are proper, e.g. the functor $FX = [0, 1] \times X^\Sigma$.

4 Proof of Theorem 26

In this section we will present the proof of our main technical result Theorem 26. We start with two technical lemmas.

▶ Remark 32. Recall [5, Proposition 11.28.2] that every free T-algebra TX is *perfectly presentable*, i.e. the hom-functor $\mathsf{Set}^T(TX, -)$ preserves sifted colimits. It follows that for every sifted diagram $D : \mathscr{D} \to \mathsf{Set}^T$ and every T-algebra morphism $h : TX \to \operatorname{colim} D$ there exists some $d \in \mathscr{D}$ and $h' : TX \to Dd$ such that $h = \mathsf{in}_d \cdot h'$.

▶ **Lemma 33.** *For every finite set X and map $f : X \to \varphi F$ there exists an object (TY, d) in $\mathsf{Coalg}_{\mathsf{free}}\, F$ and a map $g : X \to Y$ such that $f = (X \xrightarrow{g} Y \xrightarrow{\eta_Y} TY \xrightarrow{\mathsf{in}_d} \varphi F)$.*

▶ Remark 34. Recall that a colimit of a diagram $D : \mathscr{D} \to \mathsf{Set}$ is computed as follows:

$$\operatorname{colim} D = \big(\coprod_{d\in\mathscr{D}} Dd\big)/{\sim},$$

where $\sim$ is the least equivalence on the coproduct (i.e. the disjoint union) of all Dd with $x \sim Df(x)$ for every $f : d \to d'$ in $\mathscr{D}$ and every $x \in Dd$. In other words, for every pair of objects c, d of $\mathscr{D}$ and $x \in Dc$, $y \in Dd$ we have $x \sim y$ iff there is a zig-zag in $\mathscr{D}$ whose D-image relates x and y (cf. Remark 22).

▶ **Lemma 35.** *Let (TX, c) and (TY, d) be coalgebras in $\mathsf{Coalg}_{\mathsf{free}}\, F$, $x \in TX$, and $y \in TY$. Then the following are equivalent:*
1. *$\mathsf{in}_c(x) = \mathsf{in}_d(y) \in \varphi F$, and*
2. *there is a zig-zag in $\mathsf{Coalg}_{\mathsf{free}}\, F$ relating x and y.*

Proof. By Remark 20, φF is a sifted colimit. Hence, the forgetful functor $\mathsf{Coalg}\, F \to \mathsf{Set}^T \to \mathsf{Set}$ preserves this colimit. Thus the colimit φF is formed as recalled in Remark 34:

$$\varphi F \cong \big(\coprod_c TX_c\big)/{\sim},$$

where $c : TX_c \to FTX_c$ ranges over the objects of $\mathsf{Coalg}_{\mathsf{free}}\, F$. Therefore, we have the desired equivalence. ◀

Proof of Theorem 26. "⇒" Suppose that for $m : \varphi F \to \nu F$ we have $x, y \in \varphi F$ with $m(x) = m(y)$. We apply Lemma 33 to $1 \xrightarrow{x} \varphi F$ and $1 \xrightarrow{y} \varphi F$, respectively, to obtain two objects $c : TX \to FTX$ and $d : TY \to FTY$ in $\mathsf{Coalg}_{\mathsf{free}}\, F$ with $x' \in X$ and $y' \in Y$ such that $\mathsf{in}_c(\eta_X(x')) = x$ and $\mathsf{in}_d(\eta_Y(y')) = y$. By the uniqueness of coalgebra morphisms into νF we have

$$\dagger c = m \cdot \mathsf{in}_c \qquad \text{and} \qquad \dagger d = m \cdot \mathsf{in}_d. \tag{2}$$

Thus we compute:

$$\dagger c(\eta_X(x')) = m \cdot \mathsf{in}_c \cdot \eta_X(x') = m(x) = m(y) = m \cdot \mathsf{in}_d \cdot \eta_Y(y') = \dagger d(\eta_Y(y')).$$

Since F is proper by assumption, we obtain a zig-zag in $\mathsf{Coalg}_{\mathsf{free}}\, F$ relating $\eta_X(x')$ and $\eta_Y(y')$. Thus, these two elements are merged by the colimit injections, and we have $x = \mathsf{in}_c(\eta_X(x')) = \mathsf{in}_d(\eta_Y(y') = y$. We conclude that m is monomorphic.

"$\Leftarrow$" Suppose that $m : \varphi F \rightarrowtail \nu F$ is a monomorphism. Let $c : TX \to FTX$ and $d : TY \to FTY$ be objects of $\mathsf{Coalg}_{\mathsf{free}}\, F$, and let $x \in X$ and $y \in Y$ be such that $\dagger c(\eta_X(x)) = \dagger d(\eta_Y(y))$. Using (2) and the fact that m is monomorphic we get $\mathsf{in}_c(\eta_X(x)) = \mathsf{in}_d(\eta_Y(y))$. By Lemma 35, we thus obtain a zig-zag in $\mathsf{Coalg}_{\mathsf{free}}\, F$ relating $\eta_X(x)$ and $\eta_Y(y)$. This proves that F is proper. ◀

5 Conclusions and Further Work

Inspired by Ésik and Maletti's notion of a proper semiring, we have introduced the notion of a proper functor. We have shown that, for a proper endofunctor F on an algebraic category preserving regular epis and monos, the rational fixed point ϱF is fully abstract and moreover determined by those coalgebras with a free finitely generated carrier (i.e. the target coalgebras of generalized determinization).

Our main result also shows that properness is necessary for this kind of full abstractness. For categories in which fg-objects are closed under kernel pairs we saw that when F maps kernel pairs to weak pullbacks in Set, then it is proper. This provides a number of examples of proper functors. However, in several categories of interest the condition on kernel pairs fails, e.g. in $\mathbb{N}$-semimodules (commutative monoids) and convex sets. There can still be proper functors, e.g. $FX = \mathbb{N} \times X^\Sigma$ on the former. But establishing properness of a functor without using Proposition 30 seems non-trivial, and we leave this task as an open problem for further work.

One immediate consequence of our results is that the soundness and completeness of the expression calculi for weighted automata [7] extend from Noetherian to proper semirings, see Ésik and Kuich [9] for a related result.

In the future, when additional proper functors are known, it will be interesting to study regular expression calculi for their coalgebras and use the technical machinery developed in the present paper for soundness and completeness proofs.

References

1 Jiří Adámek, Stefan Milius, and Jiří Velebil. Iterative algebras at work. *Math. Structures Comput. Sci.*, 16(6):1085–1131, 2006.

2 Jiří Adámek, Stefan Milius, and Jiří Velebil. On second-order iterative monads. *Theoret. Comput. Sci.*, 412:4969–4988, 2011.

3 Jiří Adámek, Stefan Milius, and Jiří Velebil. Semantics of higher-order recursion schemes. *Log. Methods Comput. Sci.*, 7(1:15):43 pp., 2011.

4 Jiří Adámek and Jiří Rosický. *Locally presentable and accessible categories*. Cambridge University Press, 1994.

5 Jiří Adámek, Jiří Rosický, and Enrico Vitale. *Algebraic Theories*. Cambridge University Press, 2011.

6 Jean Berstel and Christophe Reutenauer. *Rational Series and Their Languages*. Springer-Verlag, 1988.

7 Marcello Bonsangue, Stefan Milius, and Alexandra Silva. Sound and complete axiomatizations of coalgebraic language equivalence. *ACM Trans. Comput. Log.*, 14(1:7), 2013.

8 Bruno Courcelle. Fundamental properties of infinite trees. *Theoret. Comput. Sci.*, 25:95–169, 1983.

9 Zoltán Ésik and Werner Kuich. Free iterative and iteration k-semialgebras. *Algebra Univ.*, 67(2):141–162, 2012.

10 Zoltán Ésik and Andreas Maletti. Simulation vs. equivalence. In *Proc. 6th Int. Conf. Foundations of Computer Science*, pages 119–122. CSREA Press, 2010.

11 Zoltán Ésik and Andreas Maletti. Simulations of weighted tree automata. In *Proc. CIAA'11*, volume 6482 of *Lecture Notes Comput. Sci.*, pages 321–330. Springer, 2011.

12 Marcelo Fiore, Gordon D. Plotkin, and Daniele Turi. Abstract syntax and variable binding. In *Proc. LICS'99*, pages 193–202. IEEE Press, 1999.

13 Peter Freyd. Rédei's finiteness theorem for commutative semigroups. *Proc. Amer. Math. Soc.*, 19(4), 1968.

14 Susanna Ginali. Regular trees and the free iterative theory. *J. Comput. System Sci.*, 18:228–242, 1979.

15 Sergey Goncharov, Stefan Milius, and Alexandra Silva. Towards a coalgebraic Chomsky hierarchy (extended abstract). In *Proc. TCS'14*, volume 8705 of *Lecture Notes Comput. Sci.*, pages 265–280. Springer, 2014.

16 Bart Jacobs, Alexandra Silva, and Ana Sokolova. Trace semantics via determinization. *J. Comput. System Sci.*, 2015.

17 Stefan Milius. A sound and complete calculus for finite stream circuits. In *Proc. LICS'10*, pages 449–458. IEEE Computer Society, 2010.

18 Stefan Milius. Proper functors and their rational fixed point. Full version; available at `https://arxiv.org/abs/1705.09198`, 2017.

19 Stefan Milius, Dirk Pattinson, and Thorsten Wißmann. A new foundation for finitary corecursion: The locally finite fixpoint and its properties. In *Proc. FoSSaCS'16*, volume 9634 of *Lecture Notes Comput. Sci. (ARCoSS)*, pages 107–125. Springer, 2016.

20 Stefan Milius, Lutz Schröder, and Thorsten Wißmann. Regular behaviours with names: On rational fixpoints of endofunctors on nominal sets. *Appl. Categ. Structures*, 24(5):663–701, 2016.

21 Stefan Milius and Thorsten Wißmann. Finitary corecursion for the infinitary lambda calculus. In *Proc. CALCO'15*, volume 35 of *LIPIcs*, pages 336–351, 2015.

22 Ion Petre and Arto Salomaa. Algebraic systems and pushdown automata. In *Handbook of Weighted Automata*, pages 257–289. Springer, 2009.

23 Lásló Rédei. *The Theory of Finitely Generated Commutative Semigroups*. Pergamon, Oxford-Edinburgh-New York, 1965.

24 Jan Rutten. Automata and coinduction (an exercise in coalgebra). In *Proc. CONCUR 1998*, volume 1466 of *Lecture Notes Comput. Sci.*, pages 194–218. Springer, 1998.

25 Jan Rutten. Universal coalgebra: a theory of systems. *Theoret. Comput. Sci.*, 249(1):3–80, 2000.

26 Jan Rutten. Rational streams coalgebraically. *Log. Methods Comput. Sci.*, 4(3:9):22 pp., 2008.

27 Marcel Paul Schützenberger. On the definition of a family of automata. *Inform. and Control*, 4(2–3):275–270, 1961.

28 Alexandra Silva, Filippo Bonchi, Marcello Bonsangue, and Jan Rutten. Generalizing determinization from automata to coalgebras. *Log. Methods Comput. Sci*, 9(1:9), 2013.
29 Alexandra Silva, Marcello Bonsangue, and Jan Rutten. Non-deterministic Kleene coalgebras. *Log. Methods Comput. Sci.*, 6(3:23):39 pp., 2010.
30 Ana Sokolova and Harald Woracek. Congruences of convex algebras. *J. Pure Appl. Algebra*, 219(8):3110–3148, 2015.
31 Henning Urbat. Finite behaviours and finitary corecursion. In *Proc. CALCO'17*, volume 72 of *LIPIcs*, pages 24:1–24:15, 2017.
32 Joost Winter. A completeness result for finite λ-bisimulations. In *Proc. FoSSaCS'15*, volume 9034 of *Lecture Notes Comput. Sci.*, pages 117–132, 2015.
33 Joost Winter, Marcello Bonsangue, and Jan Rutten. Coalgebraic characterizations of context-free languages. *Log. Methods Comput. Sci.*, 9(3), September 2013.
34 Joost Winter, Marcello Bonsangue, and Jan Rutten. Context-free coalgebras. *J. Comput. System Sci.*, 81(5):911–939, 2015.