Half-Integral Linkages in Highly Connected Directed Graphs*†

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Abstract

We study the half-integral k-Directed Disjoint Paths Problem ($\frac{1}{2}$kDDPP) in highly strongly connected digraphs. The integral kDDPP is NP-complete even when restricted to instances where $k = 2$, and the input graph is L-strongly connected, for any $L \geq 1$. We show that when the integrality condition is relaxed to allow each vertex to be used in two paths, the problem becomes efficiently solvable in highly connected digraphs (even with k as part of the input). Specifically, we show that there is an absolute constant c such that for each $k \geq 2$ there exists $L(k)$ such that $\frac{1}{2}$kDDPP is solvable in time $O(|V(G)|^c)$ for a $L(k)$-strongly connected directed graph G. As the function $L(k)$ grows rather quickly, we also show that $\frac{1}{2}$kDDPP is solvable in time $O(|V(G)|^{f(k)})$ in $(36k^3 + 2k)$-strongly connected directed graphs. We show that for each $\epsilon < 1$, deciding half-integral feasibility of kDDPP instances is NP-complete when k is given as part of the input, even when restricted to graphs with strong connectivity ϵk.

1998 ACM Subject Classification F.2.2 Nonnumerical Algorithms and Problems

Keywords and phrases linkage, directed graph, treewidth

Digital Object Identifier 10.4230/LIPIcs.ESA.2017.36

1 Introduction

Let $k \geq 1$ be a positive integer. An *instance of a directed k-linkage problem* is an ordered tuple (G, S, T) where G is a directed graph and $S = (s_1, \dots, s_k)$ and $T = (t_1, \dots, t_k)$ are each ordered sets of k distinct vertices in G. The instance is *integrally feasible* if there exist paths $P_1, \dots, P_k$ such that P_i is a directed path from s_i to t_i for $1 \leq i \leq k$ and the paths P_i are pairwise vertex disjoint. The paths $P_1, \dots, P_k$ will be referred to as an *integral solution* to the linkage problem.

The *k-Directed Disjoint Paths Problem* (**kDDPP**) takes as input an instance of a directed k-linkage problem. If the problem is integrally feasible, we output an integral solution and otherwise, return that the problem is not feasible. The kDDPP is notoriously difficult. The problem was shown to be NP-complete even under the restriction that $k = 2$ by Fortune, Hopcroft and Wyllie [4].

* A full version of this article is available at https://arxiv.org/abs/1611.01004.

† Supported by the European Research Council under the European Unions Seventh Framework Programme (FP7/2007-2013)/ERC Grant Agreement no. 279558.

25th Annual European Symposium on Algorithms (ESA 2017).
Editors: Kirk Pruhs and Christian Sohler; Article No. 36; pp. 36:1–36:12
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In an attempt to make the kDDPP more tractable, Thomassen [16] asked if the problem would be easier if we assume the graph is highly connected . Define a *separation* in a directed graph G as a pair (A, B) with $A, B \subseteq V(G)$ such that $A \cup B = V(G)$ and where there does not exist an edge (u, v) with $u \in A \setminus B$ and $v \in B \setminus A$. The *order* of the separation (A, B) is $|A \cap B|$. The separation is *trivial* if $A \subseteq B$ or $B \subseteq A$. The graph G is *strongly k-connected* if $|V(G)| \geq k + 1$ and there does not exist a nontrivial separation of order at most $k - 1$. Let $k \geq 1$ and define a directed graph G to be *integrally k-linked* if every linkage problem (G, S, T) is integrally feasible. Thomassen conjectured [16] that there exists a function f such that every $f(k)$-strongly connected digraph G is integrally k-linked. He later answered his own conjecture in the negative [17], showing that no such function $f(k)$ exists. Moreover, he also showed [17] for all $L \geq 1$, the 2DDPP is NP-complete even when restricted to problem instances where the graph is L-strongly connected.

In this article, we relax the kDDPP problem by requiring that a potential solution not use any vertex more than twice. Define a directed k-linkage problem (G, S, T) to be *half-integrally feasible* if $S = (s_1, \ldots, s_k)$, and $T = (t_1, \ldots, t_k)$ and there exist paths $P_1, \ldots, P_k$ such that:

- for all $1 \leq i \leq k$, P_i is a directed path from s_i to t_i, and
- for every vertex $v \in V(G)$, v is contained in at most two distinct paths P_i.

The paths $P_1, \ldots, P_k$ form a *half-integral solution.*

The main result of this article is that the $\frac{1}{2}$kDDPP is polynomial time solvable (even with k as part of the input) when the graph is sufficiently highly connected. Define a graph G to be *half-integrally k-linked* if every k disjoint paths problem (G, S, T) is half-integrally feasible.

▶ **Theorem 1.** *For all integers $k \geq 1$, there exists a value $L(k)$ such that every strongly $L(k)$-connected graph is half-integrally k-linked. Moreover, there exists an absolute constant c such that given an instance (G, S, T) of the $\frac{1}{2}$kDDPP where G is $L(k)$-connected, we can find a solution in time $O(|V(G)|^c)$.*

The assumption that G is highly connected in Theorem 1 cannot be omitted under the usual complexity assumptions.

▶ **Theorem 2.** *For all $\epsilon < 1$, it is NP-complete to determine whether a given kDDPP instance (G, S, T) is half-integrally feasible, even under the assumption that G is ϵk-strongly connected.*

The value for $L(k)$ in Theorem 1 grows extremely quickly. However, when we fix k, we can still efficiently solve the $\frac{1}{2}$kDDPP with a significantly weaker bound on the connectivity than that given in Theorem 1.

▶ **Theorem 3.** *There exists a function f satisfying the following. Let $k \geq 1$ be a positive integer. Given a k-linkage problem (G, S, T) such that G is $(36k^3 + 2k)$-strongly connected, we can determine if the problem is half-integrally feasible and if so, output a half-integral solution, in time $O(|V(G)|^{f(k)})$.*

Given that the kDDPP is NP-complete even in the case $k = 2$, previous work on the problem has focused on various relaxations of the problem. Schrijver [14] showed that for fixed k, the kDDPP is polynomial time solvable when the input graph is assumed to be planar. Later, Cygan et al. [1] improved this result, showing that the kDDPP is fixed parameter tractable with the assumption that the input graph is planar. In their recent series of articles [8, 7, 10] leading to the breakthrough showing the grid theorem holds for directed graphs, Kawarabayashi and Kreutzer and Kawarabayashi et al. showed the following relaxation of the kDDPP can be efficiently resolved for fixed k. They showed that there exists a polynomial time algorithm which, given an instance $(G, S = (s_1, \ldots, s_k), T = (t_1, \ldots, t_k))$ of the kDDPP, does one of the following:

- find directed paths P_i, $1 \le i \le k$, such that P_i links s_i to t_i and for every vertex v of G, v is in at most four distinct P_i, or
- determine that no integral solution to (G, S, T) exists.

In terms of hardness results, Slivkins [15] showed that the kDDPP is $W[1]$-complete even when restricted to acyclic graphs. Kawarabayashi et al. [7] announced that the proof of Slivkins result can be extended to show that the $\frac{1}{2}$kDDPP is also $W[1]$-complete.

There are two primary steps in the proof of Theorem 1. First, we show that any highly connected graph contains a large structure which we can use to connect up the appropriate pairs of vertices. The exact structure we use is a *bramble of depth two*. A bramble is a set of pairwise touching, connected (strongly connected) subgraphs; they are widely studied certificates of large tree-width both in directed and undirected graphs. See Sections 2 and 3 for the exact definitions and further details. The existence of such a bramble of depth two follows immediately from Kawarabayashi and Kreutzer's proof of the grid theorem [9]; however, the algorithm given in [9] only runs in polynomial time for fixed size of the bramble. We show in Section 4 that from appropriate assumptions which will hold both in the proof of Theorem 1 and Theorem 3, we are able to find a large bramble of depth two in time $O(n^c)$ for a graph on n vertices and some absolute constant c.

The second main step in the proof of Theorem 1 is to show how we can use such a bramble of depth two to find the desired solution to a given instance of the $\frac{1}{2}$kDDPP. Define a *linkage* to be a set of pairwise disjoint paths. We show in Section 5 that given an instance (G, S, T) and a large bramble $\mathcal{B}$ of depth two, we can find a smaller, sub-bramble $\mathcal{B}' \subseteq \mathcal{B}$ along with a linkage $\mathcal{P}$ of order k such that every element of $\mathcal{P}$ is a path from an element of S to a distinct subgraph in $\mathcal{B}'$. Moreover, the linkage $\mathcal{P}$ is internally disjoint from $\mathcal{B}'$. At the same time, we find a linkage $\mathcal{Q}$ from distinct subgraphs of $\mathcal{B}'$ to the vertices T. Thus, by linking the appropriate endpoints of $\mathcal{Q}$ and $\mathcal{P}$ in the bramble $\mathcal{B}'$, we are able to find the desired solution to (G, S, T). The fact that the bramble $\mathcal{B}'$ has depth two ensures that the solution we find uses each vertex at most twice. This result is given as Theorem 11; the statement and proof are presented in Section 5.

Linking to a well-behaved structure (the bramble of depth two in the instance above) is a common technique in disjoint path and cycle problems in undirected graphs. See [6, 13] for examples. The main contribution of Theorem 11 is to extend the technique to directed graphs, and in particular, simultaneously find the linkage from S to $\mathcal{B}'$ and the linkage $\mathcal{Q}$ from $\mathcal{B}'$ to T. This is made significantly more difficult in the directed case by the directional nature of separations in directed graphs and the fact that there is no easy way to control how the separations between S and $\mathcal{B}'$ and those between $\mathcal{B}'$ and T cross.

The proofs of Theorems 1 and 3 are given in Section 6. The construction showing NP-completeness in Theorem 2 is given in the full version of this article [3], Section 7. Due to space constraints, some of the more technical proofs are also found in that version; see Sections 4 and 5.2 in particular.

2 Directed tree-width

An *arborescence* is a directed graph R such that R has a vertex r_0, called the *root* of R, with the property that for every vertex $r \in V(R)$ there is a unique directed path from r_0 to r. Thus every arborescence arises from a tree by selecting a root and directing all edges away from the root. If $r, r' \in V(R)$ we write $r' > r$ if $r' \neq r$ and there exists a directed path in R from r to r'. If $(u,v) \in E(R)$ and $r \in V(R)$, we write $r > (u,v)$ if $r > v$ or $r = v$. Let G be a directed graph and $Z \subseteq V(G)$. A set $S \subseteq V(G) \setminus Z$ is *Z-normal* if there is no directed walk in $G - Z$ with the first and last vertex in S which also contains a vertex of $V(G) \setminus (S \cup Z)$. Note that every Z-normal set is a union of strongly connected components of $G - Z$.

Let G be a directed graph. A *tree decomposition* of G is a triple (R, β, γ), where R is an arborescence, $\beta : V(R) \to 2^{V(G)}$ and $\gamma : E(R) \to 2^{V(G)}$ are functions such that:

1. $\{\beta(r) : r \in V(R)\}$ is a partition of $V(G)$ into non-empty sets and
2. if $e \in E(R)$, then $\{\beta(r) : r \in V(R), r > e\}$ is $\gamma(e)$-normal.

The sets $\beta(r)$ are called the *bags* of the decomposition and the sets $\gamma(e)$ are called the *guards* of the decomposition. For any $r \in V(R)$, we define $\Gamma(r) := \beta(r) \cup \{\gamma(e) :\ e \text{ incident to } r\}$. The *width* of (R, β, γ) is the smallest integer w such that $|\Gamma(r)| \leq w + 1$ for all $r \in V(R)$. The *directed tree-width* of G is the minimum width of a tree decomposition of G.

Johnson, Robertson, Seymour, and Thomas showed that if we assume k and w are fixed positive integers, then we can efficiently resolve the kDDPP when restricted to directed graphs of tree-width at most w [5].

▶ **Theorem 4** ([5], Theorem 4.8). *For all $t \geq 1$, there exists a function f satisfying the following. Let $k \geq 1$, and let (G, S, T) be an k-linkage problem such that the directed tree-width of G is at most t. Then we can determine if (G, S, T) is integrally feasible and if so, output an integral solution, in time $O(|V(G)|^{f(k)})$.*

A simple construction shows that the same result holds to efficiently resolve k-linkage problems half-integrally when k and the tree-width of the graph are fixed. We first define the following operation. To *double* a vertex v in a directed graph G, we create a new vertex v' and add the edges (u, v') for all edges $(u,v) \in E(G)$, the edges (v', u) for all edges $(v,u) \in E(G)$ and the edges (v, v') and (v', v).

▶ **Corollary 5.** *For all $t \geq 1$, there exists a function f satisfying the following. Let $k \geq 1$, and let (G, S, T) be an instance of a k-linkage problem such that the directed tree-width of G is at most t. Given in input (G, S, T) and a directed tree-decomposition of G of width at most t, we can determine if the problem is half-integrally feasible and if so, output a half-integral solution, in time $O(|V(G)|^{f(k)})$.*

Proof. Fix $w \geq 1$ to be a positive integer. Let $(G, S = (s_1, \ldots, s_k), T = (t_1, \ldots, t_k))$ be an instance of a k-linkage problem where G has tree-width at most w. Let G' be the directed graph obtained by doubling every vertex $v \in V(G)$. Define the k-linkage problem $(G', S^* = (s_1^*, \ldots, s_k^*), T^* = (t_1^*, \ldots, t_k^*))$ by letting $s_i^* = s_i$ and $t_i^* = t_i'$ for $1 \leq i \leq k$. Thus, (G, S, T) is half-integrally feasible if and only if (G', S^*, T^*) is integrally feasible. Moreover, any integral solution to (G', S^*, T^*) can be easily converted to a half-integral solution for the original problem (G, S, T).

Let (R, β, γ) be a tree decomposition of G of width w. Observe that (R, β', γ') defined by $\beta'(r) = \{\{v, v'\} : v \in \beta(r)\}$ and $\gamma'(r) = \{\{v, v'\} : v \in \gamma(r)\}$ yields a tree decomposition of G' of width at most $2w$. Thus, by Theorem 4, we can determine if $(G', S^* = (s_1^*, \ldots, s_k^*), T^* = (t_1^*, \ldots, t_k^*))$ is integrally feasible and find an solution when it is, in polynomial time assuming k and w are fixed, proving the claim. ◀

3 Certificates for large directed tree-width

A *bramble* in a directed graph G is a set $\mathcal{B}$ of strongly connected subgraphs $B \subseteq G$ such that if $B, B' \in \mathcal{B}$, then $V(B) \cap V(B') \neq \emptyset$ or there exists edges $e, e' \in E(G)$ such that e links B to B' and e' links B' to B. A cover of $\mathcal{B}$ is a set $X \subseteq V(G)$ such that $V(B) \cap X \neq \emptyset$ for all

$B \in \mathcal{B}$. The *order* of a bramble is the minimum size of a cover of $\mathcal{B}$. The *bramble number*, denoted $bn(G)$, is the maximum order of a bramble in G. The elements of a bramble are called *bags*, and the *size* of a bramble, denoted $|\mathcal{B}|$, is the number of bags it contains.

The bramble number of a directed graph gives a good approximation of the tree-width, as seen by the following theorem of [12] as formulated by [10].

▶ **Theorem 6** ([12],[10]). *There exist constants c, c' such that for all directed graphs G, it holds that*

$$bn(G) \leq c \cdot tw(G) \leq c' \cdot bn(G).$$

Johnson, Robertson, Seymour, and Thomas showed one can efficiently (in fixed-parameter time) either find a large bramble in a directed graph or explicitly find a directed tree-decomposition. Note that the result is not stated algorithmically, but that the algorithm follows from the construction in the proof. Additionally, they looked at an alternate certificate of large tree-width, namely havens, but a haven of order $2t$ immediately gives a bramble of order t by the definitions.

▶ **Theorem 7** ([5], 3.3). *There exist constants c_1, c_2 such that for all t and directed graphs G, we can algorithmically find in time $O(|V(G)|^{c_1})$ either a bramble in G of order t or a tree-decomposition of G of order at most $c_2 t$. Moreover, if we find the bramble, it has at most $|V(G)|^{2t}$ elements.*

A long open question of Johnson, Robertson, Seymour, and Thomas [5] was whether sufficiently large tree-width in a directed graph would force the presence of a large directed grid minor. Let $r \geq 2$ be a positive integer. The *directed r-grid J_r* (or *cylindrical grid*) is the graph defined as follows. Let $C_1, \ldots, C_r$ be directed cycles of length $2r$. Let the vertices of C_i be labeled $v_1^i, \ldots, v_{2r}^i$ for $1 \leq i \leq r$. For $1 \leq i \leq 2r$, i odd, let P_i be the directed path $v_i^1, v_i^2, \ldots, v_i^r$. For $1 \leq i \leq 2r$, i even, let P_i be the directed path $v_i^r, v_i^{r-1}, \ldots, v_i^1$. The directed grid $J_r = \bigcup_1^r C_i \cup \bigcup_1^{2r} P_i$.

In a major recent breakthrough, Kreutzer and Kawarabayashi have confirmed the conjecture of Johnson et al.

▶ **Theorem 8** ([10]). *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that given any directed graph and any fixed constant k, in polynomial time, we can obtain either*
1. *a cylindrical grid of order k as a butterfly minor, or*
2. *a directed tree decomposition of width at most $f(k)$.*

For our purposes, we will use brambles when attempting to solve the $\frac{1}{2}$kDDPP. However, in order to ensure that the paths we find don't use any vertex more than twice, we require the bramble to have *depth* two. Define the *depth* of a bramble $\mathcal{B} = \{B_1, \ldots, B_t\}$ in a directed graph G to be the $\max_{v \in V(G)} |\{i : v \in V(B_i)\}|$; in other words, a bramble has depth at most k for some positive integer k if no vertex is contained in more than k distinct subgraphs in the bramble. Note that if $\mathcal{B}$ has depth k and size t, then it has order at least $\lceil t/k \rceil$.

▶ **Lemma 9.** *For all $t \geq 2$, the directed t-grid contains a model of a bramble $\mathcal{B}$ of size t and depth two.*

Proof. Let the cycles $C_1, \ldots, C_t$, paths $P_1, \ldots, P_{2t}$, and vertex labels v_i^j, $1 \leq i \leq 2t$, $1 \leq j \leq t$, be as in the definition of the directed t-grid. For every l, $1 \leq l \leq t$, and for every i, $1 \leq i \leq 2t$, let $P_i(l)$ be the subpath of P_i with endpoints v_i^1 and v_i^l. For $1 \leq i \leq t-1$, let C_i' be the (unique) cycle in $C_i \cup C_1 \cup P_{2i-1}(i) \cup P_{2i}(i)$ which contains all the vertices v_1^j, $1 \leq j \leq 2k$. Let $C_t' = C_t$. The cycles $C_1', \ldots, C_t'$ form a bramble of depth two and size t, as desired. ◀

4 Finding a bramble of depth two

As described in the introduction, we can actually find a bramble of depth two in time $O(n^c)$ for some absolute constant c without appealing to the full power of the directed grid theorem of [10]. Indeed, we can show that in a graph with large enough directed treewidth, we find what is called a sufficiently large *well-linked* set of vertices in a directed graph, and from that we are able to efficiently find a large bramble of depth two.[1]

▶ **Theorem 10.** *There exists a function f which satisfies the following. Let G be a directed graph on n vertices and $t \geq 1$ a positive integer. Let P be a directed path and $X \subseteq V(P)$ a well-linked set with $|X| \geq f(t)$. Then G contains a bramble $\mathcal{B} = B_1, \dots, B_t$ of depth two. Moreover, given G, P, and X in input, we can find $\mathcal{B}$ in time $O(n^c)$ for some absolute constant c.*

The proof of Theorem 10 is given in [3], Section 4. The argument in many ways follows Diestel et al.'s proof of Robertson and Seymour's grid theorem (see [2] for the proof) for undirected graphs.

5 Linking in a bramble of depth two

The main result of this section is the following which shows that if we have a sufficiently large bramble of depth two, we can use it to efficiently resolve a given instance of the $\frac{1}{2}$kDDPP under a modest assumption on the connectivity of the graph.

▶ **Theorem 11.** *For all $k \geq 1$, there exists a positive integer t such that if G is a $(36k^3 + 2k)$-strongly connected directed graph, and G contains a bramble $\mathcal{B}$ of depth two and size t, then for every k-linkage problem instance (G, S, T) is half-integrally feasible. Moreover, given (G, S, T) and the bags of $\mathcal{B}$, we can find a solution in time $O(k^4 n^2)$.*

We begin with some notation. Recall that the doubling of a vertex in a directed graph was defined in Section 2. To *contract* a set of vertices U inducing a strongly connected subgraph of G is to delete U and create a new vertex v, then add edges (w, v) for all edges $(w, u) \in E(G)$ with $u \in U, w \notin U$ and edges (u, w) for all edges $(v, u) \in E(G)$ with $u \in U, w \notin U$.

Let $\mathcal{B}$ be a depth two bramble in a directed graph G and $\mathcal{B}_1 \subseteq \mathcal{B}$. Define the graph $G(\mathcal{B}_1; \mathcal{B})$ as follows: First, let G' be the graph obtained from G by doubling every vertex belonging to two bags of $\mathcal{B}$ and to at least one bag of $\mathcal{B}_1$. For each such vertex v, denote its double by v'. Let $\mathcal{B}'$ be the collection of $|\mathcal{B}_1|$ subsets of $V(G')$ obtained from $\mathcal{B}_1$ by replacing each vertex v belonging to a bag of $\mathcal{B}$ with v' in exactly one of the bags it belongs to. Thus, the elements of $\mathcal{B}'$ are pairwise disjoint and each induces a strongly connected subgraph of G', so $\mathcal{B}'$ is a depth 1 bramble in G'. Let $G(\mathcal{B}_1; \mathcal{B})$ be the graph obtained from G' by contracting each element of $\mathcal{B}'$. Denote by $K_{\mathcal{B}_1}$ the set of contracted vertices in $G(\mathcal{B}_1; \mathcal{B})$; note that the vertices of $K_{\mathcal{B}_1}$ form a bidirected clique. Observe that every double of a vertex of G' gets contracted, so $V(G(\mathcal{B}_1; \mathcal{B})) \setminus K_{\mathcal{B}_1} \subseteq V(G)$. For a vertex $v \in K_{\mathcal{B}_1}$, we write $\mathrm{im}(v)$ for the bag of $\mathcal{B}_1$ corresponding to the vertices contracted to v. We stress that each $\mathrm{im}(v)$ is a bag of $\mathcal{B}_1$; in particular $\mathrm{im}(v) \subseteq V(G)$.

Let S, T be disjoint subsets of the vertices of a directed graph G. A separation (A, B) *separates* S from T if $S \subseteq A$ and $T \subseteq B$. The separation (A, B) *properly separates* S from T if $S \setminus B$ and $T \setminus A$ are both nonempty. For a positive integer α, we say S is *α-connected* to T if every separation separating S from T has order at least α.

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[1] A subset $X \subseteq V(G)$ of vertices of a directed graph G is *well-linked* if for any pair of subsets $U_1, U_2 \subseteq X$ with $|U_1| = |U_2|$, there exists a directed U_1 to U_2 linkage of order $|U_1|$.

Let G be a directed graph, and $\mathcal{B}' \subseteq \mathcal{B}$ be brambles of depth two. Let $X \subseteq V(G(\mathcal{B}';\mathcal{B})) \setminus K_{\mathcal{B}'}$. We say an $X - K_{\mathcal{B}'}$ or $K_{\mathcal{B}'} - X$ linkage $P_1, \ldots, P_{|X|}$ is $\mathcal{B}$-*minimal* if none of the paths contains internally a vertex in $K_{\mathcal{B}'}$ or in $\mathrm{im}(v)$ for some $v \in K_{\mathcal{B}} \setminus \cup_i P_i$.

We now give a quick outline of how the proof will proceed. Let us denote $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$. Our approach to proving half-integral feasibility is in two steps. We find three sets of paths, one set of k paths linking S to the bramble $\mathcal{B}$, another set linking $\mathcal{B}$ to T, and a third linking the appropriate ends of paths in the first two sets to each other inside of $\mathcal{B}$. To get the first two sets of paths, we take advantage of the high connectivity of the graph. Linking half-integrally inside of the bramble is easy, and its structure allows us to link any pairs of vertices we like half-integrally. We need the union of the three sets of paths to form a half-integral solution, so we will choose the first and second sets each to be (almost) vertex-disjoint, and to intersect the bramble $\mathcal{B}$ in a very limited way. The third set of paths will be half-integral and completely contained in $\mathcal{B}$.

The underlying idea behind our approach to finding the first two sets of paths is to contract each bag of the bramble (after doubling vertices in two bags) and try to apply Menger's theorem. In trying to do this, some issues arise. First, we want the ends of all $2k$ paths to belong to distinct bags of $\mathcal{B}$. More concerningly, contracting the bags of the bramble may destroy the connectivity between the bramble and the terminals S and T. We solve this by throwing away a bounded number of bags from the bramble until we are left with a sub-bramble that is highly connected to S and from T. In Subsection 5.1, we will show how to find the first two sets of paths (Lemma 12), modulo finding the sub-bramble (Lemma 13), and the third set of paths (Lemma 14). Then we show how to put these pieces together to prove Theorem 11. The proof of Lemma 13 can be found in the full version of this paper [3], Section 5.2.

5.1 Linking into and inside of a depth two bramble

▶ **Lemma 12.** *Let G be a $(36k^3 + 2k)$-strongly connected directed graph and $\mathcal{B}$ be a bramble of depth two and size $> 188k^3$ in G. Let $(G, S = (s_1, \ldots, s_k), T = (t_1, \ldots, t_k))$ be a k-linkage problem instance. Then we can find paths $P_1^s, \ldots, P_k^s, P_1^t, \ldots, P_k^t$ and $\mathcal{B}' \subseteq \mathcal{B}$ satisfying the following:*

A1: *For each i, P_i^s is a directed path from s_i to some vertex s_i', and P_i^t is a directed path from some vertex t_i' to t_i.*

A2: *The vertices $s_1', \ldots, s_k', t_1', \ldots, t_k'$ belong to distinct bags of $\mathcal{B}'$, say $B_1^s, \ldots, B_k^s, B_1^t, \ldots, B_k^t$, respectively.*

A3: *Every vertex belongs to at most two of $P_1^s, \ldots, P_k^s$, and if a vertex v does belong to two paths, say P_i^s and $P_j^s (i \neq j)$, then $v = s_i'$ or $v = s_j'$.*

A4: *Similarly, every vertex belongs to at most two of $P_1^t, \ldots, P_k^t$, and if a vertex v does belong to two paths, say P_i^t and $P_j^t (i \neq j)$, then $v = t_i'$ or $v = t_j'$.*

A5: *For each i, the internal vertices of P_i^s and of P_i^t belong to at most one bag of $\mathcal{B}'$.*

A6: *For each i, j, ℓ all distinct , $P_i^s \cap P_j^t \cap (B_\ell^s \cup B_\ell^t) = \emptyset$.*

A7: *Every vertex belongs to at most two of $P_1^s, \ldots, P_k^s, P_1^t, \ldots, P_k^t$.*

Moreover, given the bags of $\mathcal{B}$, we can find the paths $P_1^s, \ldots, P_k^s, P_1^t, \ldots, P_k^t$ in time $O(k^4 n^2)$.

We will prove the following lemma as an intermediate step to Lemma 12.

▶ **Lemma 13.** *Let G be a $(36k^3 + 2k)$-strongly connected directed graph and $\mathcal{B}$ be a bramble of depth two and size $> 188k^3$ in G. Let (G, S, T) be a k-linkage problem instance. Assume $\mathcal{B}$ is disjoint from $\{s_i, t_i; 1 \leq i \leq k\}$. Then there exist brambles $\mathcal{B}_S$ and $\mathcal{B}_T$ with $\mathcal{B}_T \subseteq \mathcal{B}_S \subseteq \mathcal{B}$ such that S is $(36k^3 + 2k)$-connected to $K_{\mathcal{B}_S}$ in $G(\mathcal{B}_S;\mathcal{B})$ and T is $3k$-connected to $K_{\mathcal{B}_T}$ in $G(\mathcal{B}_T;\mathcal{B}_S)$. Also $|\mathcal{B}_S| - |\mathcal{B}_T| < 36k^3$. Moreover we can find $\mathcal{B}_S$ and $\mathcal{B}_T$ in time $O(k^4 n^2)$.*

The proof of Lemma 13 is found in [3], Section 5.2. But first, let's see how Lemma 13 implies Lemma 12.

Proof of Lemma 12. Consider the brambles $\mathcal{B}_S$ and $\mathcal{B}_T$ given by Lemma 13. Denote by W the vertices in $G(\mathcal{B}_T;\mathcal{B})$ that belong to exactly one bag in $\mathcal{B}_T$ and to two bags in $\mathcal{B}_S$.

▶ **Claim.** *There exist k vertex-disjoint paths $P_1,\dots,P_k$ in $G(\mathcal{B}_T;\mathcal{B})\setminus W$ where P_i links s_i to v_i, for some $v_i \in K_{\mathcal{B}_T}$.*

Suppose not; then by Menger's theorem there exists a separation (A,B) of order $< k$ in $G(\mathcal{B}_T;\mathcal{B})\setminus W$ separating S from $K_{\mathcal{B}_T}$. But then consider the following separation in $G(\mathcal{B}_S)$. Let

$$A' = (A\cap V(G(\mathcal{B}_S;\mathcal{B}))) \cup \{v \in K_{\mathcal{B}_S} : \mathrm{im}(v)\cap A \neq \emptyset\} \cup (K_{\mathcal{B}_S}\setminus K_{\mathcal{B}_T})$$

and

$$B' = (B\cap V(G(\mathcal{B}_S;\mathcal{B}))) \cup \{v \in K_{\mathcal{B}_S} : \mathrm{im}(v)\cap B \neq \emptyset\} \cup (K_{\mathcal{B}_S}\setminus K_{\mathcal{B}_T}).$$

Intuitively, (A',B') is the separation (A,B) viewed in the graph $G(\mathcal{B}_S;\mathcal{B})$, plus we add the vertices of $K_{\mathcal{B}_S}\setminus K_{\mathcal{B}_T}$ to each side. It's easy to check that (A',B') is a separation in $G(\mathcal{B}_S;\mathcal{B})$, since every vertex in $V(G(\mathcal{B}_T;\mathcal{B}))\setminus V(G(\mathcal{B}_S;\mathcal{B}))$ belongs to $\mathrm{im}(v)$ for some $v\in K_{\mathcal{B}_S}$. Also, we have $|A'\cap B'| \le 2|A\cap B| + 36k^3$ because every vertex belongs to at most two bags of $\mathcal{B}_S$ and every vertex in W belongs to one bag of $\mathcal{B}_S\setminus\mathcal{B}_T$. But this contradicts Lemma 13 and proves the claim.

Choose the paths $P_1,\dots,P_k$ so that they are $\mathcal{B}_T$-minimal in $G(\mathcal{B}_T;\mathcal{B})$. Let us now view these as paths in the original graph G: Since $V(G(\mathcal{B}_T;\mathcal{B}))\setminus K_{\mathcal{B}_T}\subseteq V(G)$, each vertex in P_i except v_i is a vertex of G, for each $1\le i\le k$. So choose $s_i'\in\mathrm{im}(v_i)$ such that there exists an edge from the second to last vertex of P_i to s_i'. Then let P_i^s be the path obtained from P_i by replacing v_i with s_i'. Notice that P_i^s is a path in G. The paths $P_1^s,\dots,P_k^s$ are internally disjoint, so they satisfy A3 .

▶ **Claim.** *There exist vertex-disjoint paths $Q_1,\dots,Q_k$ in $G(\mathcal{B}_T;\mathcal{B}_S)\setminus\{v_1,\dots,v_k,s_1',\dots,s_k'\}$ where Q_i links w_i to t_i for some $w_i\in K_{\mathcal{B}_T}$. Moreover, the vertices $v_1,\dots,v_k,w_1,\dots,w_k$ are distinct.*

Suppose not; then by Menger's theorem, in the graph $G(\mathcal{B}_T;\mathcal{B}_S)\setminus\{v_1,\dots,v_k,s_1',\dots,s_k'\}$ there is a separation (A,B) of order $< k$ properly separating $K_{\mathcal{B}_T}$ from T. But then $(A\cup\{v_1,\dots,v_k,s_1',\dots,s_k'\}, B\cup\{v_1,\dots,v_k,s_1',\dots,s_k'\})$ has order $< 3k$ and properly separates $K_{\mathcal{B}_T}$ from T in $G(\mathcal{B}_T;\mathcal{B}_S)$, contradicting Lemma 13. This proves the claim.

We may also choose the paths $Q_1,\dots,Q_k$ to be $\mathcal{B}_T$-minimal in $G(\mathcal{B}_T;\mathcal{B}_S)$. Viewing these paths as paths in G as above (symmetrically), we obtain paths $P_1^t,\dots,P_k^t$, with P_i^t joining t_i' to t_i. These paths satisfy A4 .

Let $\mathcal{B}' = \{\mathrm{im}(v) : v\in\{v_1,\dots,v_k,w_1,\dots,w_k\}\}$. For each i, set $B_i^s = \mathrm{im}(v_i)$ and $B_i^t = \mathrm{im}(w_i)$. We now check that the paths $P_1^s,\dots,P_k^s,P_1^t,\dots,P_k^t$ satisfy the seven assertions in the lemma statement. A1 , A2 , A3 and A4 have already been established.

To see that A5 holds, note that each of $P_1,\dots,P_k$ is internally disjoint from $K_{\mathcal{B}_T}$ in $G(\mathcal{B}_T;\mathcal{B})$. Similarly, the $Q_1,\dots,Q_k$ paths are internally disjoint from $K_{\mathcal{B}_T}$ in $G(\mathcal{B}_T;\mathcal{B}_S)$. Moreover, by the definition of $G(\mathcal{B}_T;\mathcal{B})$ and $G(\mathcal{B}_T;\mathcal{B}_S)$, every vertex not in $K_{\mathcal{B}_T}$ in either of those graphs belongs to at most one bag of $\mathcal{B}_T$ and therefore to at most one bag of $\mathcal{B}'$. It follows that for each i, each internal vertex of P_i^s and P_i^t belongs to at most one bag of $\mathcal{B}'$, proving A5 .

To see A6 , let $1 \leq i, j, \ell \leq k$ be distinct. Suppose for contradiction that some vertex v belongs to $P_i^s \cap P_j^t \cap (B_\ell^s \cup B_\ell^t)$. If v is an internal vertex of either P_i^s or P_j^t then v belongs to only one bag of $\mathcal{B}'$ by A5 . Also, if $v = s'_i$ or t'_j then v belongs to two bags of $\mathcal{B}'$. We deduce that v is an internal vertex of both P_i^s and P_j^t. Since we found P_i in the graph $G(\mathcal{B}_T; \mathcal{B}) \setminus W$, we know $v \notin W$ so v belongs to one bag in $\mathcal{B}_T$ and one bag of $\mathcal{B}_S$. But we found Q_j in the graph $G(\mathcal{B}_T; \mathcal{B}_S)$, so v belongs to one bag of $\mathcal{B}_T$ and two bags of $\mathcal{B}_S$. This is a contradiction, proving A6 .

Finally, let us check A7 . Suppose for contradiction's sake that some vertex $v \in V(G)$ belongs to three paths. By A3 and A4 , we must have $v \in P_i^s \cap P_j^s \cap P_\ell^t$ or $v \in P_i^t \cap P_j^t \cap P_\ell^s$ for some $1 \leq i, j, \ell \leq k$. If $v \in P_i^s \cap P_j^s \cap P_\ell^t$, then by A3 we may assume without loss of generality that $v = s'_i$. But the path Q_ℓ was found in a graph not containing v_i or s'_i, so we must have $s'_i \in B_\ell^t \cap B_i^s$. Since $\mathcal{B}'$ is depth two, $v \notin B_j^s$ so v is an internal vertex of P_j^s, contradicting A5 . If $v \in P_i^t \cap P_j^t \cap P_\ell^s$, then without loss of generality $v = t'_i \in B_i^t = \text{im}(w_i)$. By the $\mathcal{B}_T$-minimality of $P_1, \ldots, P_k$, v cannot be an internal vertex of P_ℓ so we have $v = s'_\ell \in B_\ell^s$. Since v belongs to two bags, A5 implies that $v = t'_j$, a contradiction.

It remains to check that we can indeed find these paths in time $O(k^4 n^2)$. Indeed finding the brambles $\mathcal{B}_S$ and $\mathcal{B}_T$ takes time $O(k^4 n^2)$ using Lemma 13. Then, the sets of paths $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ can be found in time $O(n^2)$ according to Menger's Theorem (see [11]), and from these we can easily get $P_1^s, \ldots, P_k^s, P_1^t, \ldots, P_k^t$ in linear time. ◀

The following lemma shows how to solve any linkage problem half-integrally in a depth two bramble, provided the terminals belong to distinct bags.

▶ **Lemma 14.** *For all $k \geq 2$, let G be a directed graph and let $S' = (s'_1, \ldots, s'_k)$ and $T' = (t'_1, \ldots, t'_k))$ be two ordered k-tuples of vertices in G. Suppose $\mathcal{B}$ is a bramble of depth two in G, and $s'_1, \ldots, s'_k, t'_1, \ldots, t'_k$ belong to distinct bags $B_1^s, \ldots, B_k^s, B_1^t, \ldots, B_k^t$, respectively of $\mathcal{B}$. Then there exist paths $P_1, \ldots, P_k$ such that P_i links s'_i to t'_i and, additionally, every vertex of G is in at most two distinct paths P_i. Finally, it also holds that $P_i \subseteq B_i^s \cup B_i^t$ for each i, and we can find the paths $P_1, \ldots, P_k$ in time $O(kn^2)$.*

Proof. For each i, we obtain P_i as follows. By the definition of a bramble, there exist vertices $v_i \in B_i^s$ and $w_i \in B_i^t$ with either $v_i = w_i$ or $(v_i, w_i) \in E(G)$. Since B_i^s and B_i^t are both strongly connected, there exist a directed path from s'_i to v_i contained in B_i^s and a directed path from w_i to t'_i contained in B_i^t. Take P_i to be the concatenation of these two paths. By construction, each P_i belongs to $B_i^s \cup B_i^t$. Further, since the bags $B_1^s, \ldots, B_k^s, B_1^t, \ldots, B_k^t$ are distinct, and every vertex in G belongs to at most two distinct bags, it follows that $P_1, \ldots, P_k$ is the desired collection of paths. Each P_i can be found in time $O(n^2)$, and so the overall running time of $O(kn^2)$ follows. ◀

We can deduce Theorem 11 from Lemmas 12 and 14 as follows.

Proof of Theorem 11. Let $P_1^s, \ldots, P_k^s, P_1^t, \ldots, P_k^t$ and $s'_1, \ldots, s'_k, t'_1, \ldots, t'_k$ and $\mathcal{B}' = B_1^s, \ldots, B_k^s, B_1^t, \ldots, B_k^t$ satisfy A1 - A7 , as given by Lemma 12.

By A2 , G, $S' = (s'_1, \ldots, s'_k)$ and $T' = (t'_1, \ldots, t'_k))$ satisfying the hypothesis of Lemma 14. Let $P_1, \ldots, P_k$ be the paths guaranteed by that lemma.

For each $1 \leq i \leq k$, let $Q_i = P_i^s P_i P_i^t$ be the concatenation of these three paths. Clearly, each Q_i is a directed walk linking s_i to t_i and therefore contains a directed path from s_i to t_i. We just need to check that the k paths are half-integral. Suppose for contradiction's sake that some vertex $v \in Q_i \cap Q_j \cap Q_\ell$ for some $1 \leq i, j, \ell \leq k$ all distinct. By symmetry, we can consider four cases.

Case 1: $v \in P_i \cap P_j$.
Then, by Lemma 14, $v \in (B_i^s \cup B_i^t) \cap (B_j^s \cup B_j^t)$, so v belongs to two bags of $\mathcal{B}'$. Then by A5 v is not an internal vertex of P_ℓ^s or P_ℓ^t, a contradiction.

Case 2: $v \in P_i^s \cap P_j^s \cap P_\ell$.
By A3 in Lemma 12, we may assume $v = s_i'$, so $v \in P_i$. Since $v \in P_\ell$, it follows $v \in B_i^s \cap (B_\ell^s \cup B_\ell^t)$. By A5 , v is not an internal vertex of P_j^s, so $v \in B_j^s$ as well, a contradiction.

Case 3: $v \in P_i^t \cap P_j^t \cap P_\ell$.
By A4 in Lemma 12, we may assume $v = t_i'$, so $v \in P_i$. Again, since $v \in P_\ell$, it follows $v \in B_i^t \cap (B_\ell^s \cup B_\ell^t)$. By A5 , v is not an internal vertex of P_j^t so $v \in B_j^s$ as well, a contradiction.

Case 4: $v \in P_i^s \cap P_j^t \cap P_\ell$.
By Lemma 14 $P_\ell \subseteq (B_\ell^s \cap B_\ell^t)$, but this contradicts A6 . By A6 $v \notin B_\ell^s \cap B_\ell^t$ so $v \notin P_\ell$, a contradiction.

The running time bound of $O(k^4 n^2)$ follows from the bounds given by Lemmas 12, 13 and 14. ◀

6 Proofs of Theorems 1 and 3

Given Theorems 10 and 11, it is now easy to complete the proofs of Theorems 1 and 3. We begin with Theorem 1.

Proof of Theorem 1. Let f be the function from Theorem 10. Let $t = t(k)$ be the value necessary for the size of the bramble in order to apply Theorem 11 and resolve an instance of $\frac{1}{2}$kDDPP.

Let G be an $f(t)$-strongly connected graph on n vertices, and let $(G, S = (s_1, \dots, s_k), T = (t_1, \dots, t_k))$ be an instance of the $\frac{1}{2}$kDDPP. We can greedily find a path P with $|V(P)| \geq f(k)$. Note that any subset of at most $f(k)$ vertices is well-linked, and thus, $V(P)$ is a well-linked set. By Theorem 10, we can find in time $O(n^{c_1})$ a bramble $\mathcal{B}$ of size at least t. As $f(t) \geq 36k^3 + 2k$, by Theorem 11, we can find a solution to (G, S, T) in time $O(k^4 n^2)$, completing the proof of the theorem. ◀

For the proof of Theorem 3, we will need two additional results from [9]. Note that in [9], neither statement is algorithmic, but the existence of the algorithm follows immediately from the constructive proof.

▶ **Lemma 15** ([9], 4.3). *Let G be a directed graph on n vertices and $\mathcal{B}$ a bramble in G. Then there is a path P intersecting every element of $\mathcal{B}$ and given G and $\mathcal{B}$ in input, we can find the path P in time $O(|\mathcal{B}|n^2)$.*

▶ **Lemma 16** ([9], 4.4). *Let G be a directed graph graph on n vertices, $\mathcal{B}$ a bramble of order $k(k+2)$ and P a path intersecting every element of $\mathcal{B}$. Then there exists a set $X \subseteq V(P)$ of order $4k$ which is well-linked. Given P, $\mathcal{B}$, and G in input, we can algorithmically find X in time $|\mathcal{B}|n^{O(k)}$.*

Proof of Theorem 3. Let $(G, S = (s_1, \dots, s_k), T = (t_1, \dots, t_k))$ be an instance of the $\frac{1}{2}$kDDPP. Let $n = |V(G)|$. Let t be the necessary size of a bramble in order to apply Theorem 11 to resolve an instance of the $\frac{1}{2}$kDDPP. Let f be the function in Theorem 10.

By Theorem 7, we can either find a tree decomposition of G of width at most $c_2((f(t)+2)^2)$ or a bramble $\mathcal{B}$ of order $(f(t)+2)^2$. Given the tree decomposition, by Corollary 5, we can solve (G, S, T) in time $O(n^{f_1(c_2(f(t)+2)^2)})$ for some function f_1.

If instead we find the bramble $\mathcal{B}$, in order to apply Theorem 11, we will have to convert it to a bramble of depth two. By Theorem 7, we may assume that $|\mathcal{B}| \leq n^{2(f(t)+2)^2}$. Thus, in time $n^{O(f(t)^2)}$, we can find a path P intersecting every element of $\mathcal{B}$ by Lemma 15. By Lemma 16, again in time $n^{O(f(t)^2)}$, we can find a well-linked subset $X \subseteq V(P)$ with $|X| \geq f(t)$. Finally, applying Theorem 10, we find can find a bramble $\mathcal{B}'$ of size t and depth two. Finally, by Theorem 11, we can resolve (G, S, T) in time $O(k^4 n^2)$. In total, the algorithm takes time $O(n^{f_2(k)})$ for some function f_2, as desired. ◀

References

1. Marek Cygan, Daniel Marx, Marcin Pilipczuk, and Michal Pilipczuk. The planar directed k-vertex-disjoint paths problem is fixed-parameter tractable. In *Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on Foundations of Computer Science*, pages 197–206, Oct 2013. `doi:10.1109/FOCS.2013.29`.
2. Reinhard Diestel, Tommy R Jensen, Konstantin Yu Gorbunov, and Carsten Thomassen. Highly connected sets and the excluded grid theorem. *Journal of Combinatorial Theory, Series B*, 75(1):61–73, 1999.
3. Katherine Edwards, Irene Muzi, and Paul Wollan. Half-integral linkages in highly connected directed graphs, 2016. URL: `https://arxiv.org/abs/1611.01004`.
4. Steven Fortune, John Hopcroft, and James Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10(2):111–121, 1980.
5. Thor Johnson, Neil Robertson, Paul D Seymour, and Robin Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138–154, 2001.
6. Ken-ichi Kawarabayashi. An improved algorithm for finding cycles through elements. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 374–384. Springer, 2008.
7. Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Stephan Kreutzer. An excluded half-integral grid theorem for digraphs and the directed disjoint paths problem. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, STOC'14, pages 70–78, New York, NY, USA, 2014. ACM.
8. Ken-ichi Kawarabayashi and Stephan Kreutzer. An excluded grid theorem for digraphs with forbidden minors. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 72–81. `doi:10.1137/1.9781611973402.6`.
9. Stephan Kreutzer Ken-Ichi Kawarabayashi. The directed grid theorem, 2014. URL: `https://arxiv.org/abs/1411.5681`.
10. Stephan Kreutzer Ken-ichi Kawarabayashi. The directed grid theorem. In *Proceedings of the Forty-Seventh Annual ACM on Symposium on Theory of Computing*, pages 655–664. ACM, 2015.
11. Karl Menger. Zur allgemeinen Kurventheorie. *Fundamenta Mathematicae*, 1(10):96–115, 1927.
12. Bruce Reed. Introducing directed tree width. *Electronic Notes in Discrete Mathematics*, (3):1–8, 2000.
13. Neil Robertson and Paul D Seymour. Graph minors XIII. The disjoint paths problem. *Journal of Combinatorial Theory, Series B*, 63(1):65–110, 1995.
14. Alexander Schrijver. Finding k disjoint paths in a directed planar graph. *SIAM Journal on Computing*, 23(4):780–788, 1994.

15 Aleksandrs Slivkins. Parameterized tractability of edge-disjoint paths on directed acyclic graphs. *SIAM Journal on Discrete Mathematics*, 24(1):146–157, 2010.

16 Carsten Thomassen. 2-linked graphs. *European Journal of Combinatorics*, 1(4):371–378, 1980.

17 Carsten Thomassen. Highly connected non-2-linked digraphs. *Combinatorica*, 11(4):393–395, 1991.