

# Divergence and Unique Solution of Equations\*

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## Abstract

We study proof techniques for bisimilarity based on *unique solution of equations*. We draw inspiration from a result by Roscoe in the denotational setting of CSP and for failure semantics, essentially stating that an equation (or a system of equations) whose infinite unfolding never produces a divergence has the unique-solution property. We transport this result onto the operational setting of CCS and for bisimilarity. We then exploit the operational approach to: refine the theorem, distinguishing between different forms of divergence; derive an abstract formulation of the theorems, on generic LTSs; adapt the theorems to other equivalences such as trace equivalence, and to preorders such as trace inclusion. We compare the resulting techniques to enhancements of the bisimulation proof method (the ‘up-to techniques’). Finally, we study the theorems in name-passing calculi such as the asynchronous  $\pi$ -calculus, and revisit the completeness proof of Milner’s encoding of the  $\lambda$ -calculus into the  $\pi$ -calculus for Lévy-Longo Trees.

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## 1 Introduction

In this paper we study the technique of *unique solution of equations* for (weak) behavioural relations. We mainly focus on bisimilarity but we also consider other equivalences, such as trace equivalence, as well as preorders such as trace inclusion. Roughly, the technique consists in proving that two tuples of processes are componentwise in a given behavioural relation by establishing that they are solutions of the same system of equations.

In this work, behavioural relations, hence also bisimilarity, are meant to be *weak* because they abstract from internal moves of terms, as opposed to the *strong* relations, which make no distinctions between the internal moves and the external ones (i.e., the interactions with the environment). Weak equivalences are, practically, the most relevant ones: e.g., two equal programs may produce the same result with different numbers of evaluation steps. Further, the problems tackled in this paper only arise in the weak case.

The technique of unique solution has been proposed by Milner in the setting of CCS, and plays a prominent role in proofs of examples in his book [13]. The method is important in verification techniques and tools based on algebraic reasoning [22, 2, 10]. Not all equations have a unique solution: for instance any process trivially satisfies  $X = X$ . In Milner’s theorem [13], uniqueness of solutions is subject to some limitations: the equations must

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be ‘strongly guarded and sequential’, that is, the variables of the equations may only be used underneath a visible prefix and preceded, in the syntax tree, only by the sum and prefix operators. This limits the expressiveness of the technique (since occurrences of other operators above the variables, such as parallel composition and restriction, in general cannot be removed), and its transport onto other languages (e.g., languages for distributed systems or higher-order languages usually do not include the sum operator, which makes the theorem essentially useless). A comparable technique, involving similar limitations, has been proposed by Hoare in his CSP book [12].

Trying to overcome such limitations, a variant of the technique, called *unique solution of contractions* has been proposed [25]. The technique is for behavioural equivalences; however the meaning of ‘solution’ is defined in terms of the contraction of the chosen equivalence. Contraction is, intuitively, a preorder that conveys an idea of efficiency on processes, where efficiency is measured on the number of internal actions needed to perform a certain activity. The condition for applicability of the technique is, as for Milner’s, purely syntactic: each variable in the body of an equation should be underneath a prefix. The technique has two main disadvantages: for proving an equivalence one needs also the theory of the associated contraction preorder; there may be processes for which the technique is not applicable simply because the contraction is strictly finer than the equivalence, and therefore one of the processes fails to be a solution.

In this paper we explore a different approach, inspired by results by Roscoe in CSP [21, 20], essentially stating that a guarded equation (or system of equations) whose infinite unfolding never produces a divergence has the unique-solution property. The theorem is presented, as usual in CSP, with respect to denotational semantics and failure based equivalence [5, 6]. In such a setting, where divergence is catastrophic (e.g., it is the bottom element of the domain), the theorem has an elegant and natural formulation. (Indeed, Roscoe develops a denotational model [20] in which the proof of the theorem is just a few lines.)

We draw inspiration from Roscoe’s work to formulate the counterpart of these results in the operational setting of CCS and bisimilarity. In comparison with the denotational CSP proof, the operational CCS proof is more complex. The operational setting offers however a few advantages. First, we can formulate more refined versions of the theorem, in which we distinguish between different forms of divergence. (These refinements would look less natural in the denotational and trace-based setting of CSP, where any divergence causes a process to be considered undefined.) A second and more important advantage comes as a consequence of the flexibility of the operational approach: the unique-solution theorems can be tuned to other behavioural relations (both equivalences and preorders), and to other languages.

To highlight the latter aspect, we present abstract formulations of the theorems, on a generic LTS (i.e., without reference to CCS), where the body of an equation becomes a function on the states of the LTS. The CCS theorems are instances of the abstract formulations. Similarly we can derive analogous theorems for other languages. Indeed we can do so for all languages whose constructs have an operational semantics with rules in the GSOS format [3] (assuming appropriate hypotheses, among which congruence properties). In contrast, the analogous theorems fail for languages whose constructs follow the *tyft/tyxt* [11] format, due to the possibility of rules with a *lookahead*. We also consider extensions of the theorems to name-passing calculi such as the  $\pi$ -calculus. The abstract version of our main unique-solution theorem has been formalised using the Coq proof assistant [8].

Today, for concrete proofs of bisimilarity results, the bisimulation proof method is predominant, also thanks to enhancements of the method provided by the so called ‘up-to techniques’ [19]. Powerful enhancements are ‘up to context’, whereby in the derivatives of

two terms a common context can be erased, ‘up to expansion’, whereby two derivatives can be rewritten using the expansion preorder, and ‘up to transitivity’, whereby the matching between two derivatives is made with respect to the transitive closure of the candidate relation (rather than the relation alone). Different enhancements can sometimes be combined, though care is needed to preserve soundness. One of the most powerful combinations is Pous ‘*up to transitivity and context*’ technique, which relies on a termination hypothesis. This technique generalises ‘up to expansion’ and combines it with ‘up to context’ and ‘up to transitivity’. We show that, under an additional side condition, our techniques are at least as powerful as this up-to technique: any up-to relation can be turned into a system of equations of the same size as the up-to relation and that satisfies the hypothesis of our theorems.

An important difference between unique solution of equations and up-to techniques arises in the (asynchronous)  $\pi$ -calculus. In this setting, forms of bisimulation enhancements that involve ‘up to context’ require closure of the candidate relation under substitutions (or instantiation of the parameters of an abstraction with arbitrary values). It is an open problem whether this closure is necessary in the asynchronous  $\pi$ -calculus, where bisimilarity is closed under substitutions. Our unique-solution techniques are strongly reminiscent of up to context techniques (the body of an equation acts like a context that is erased in a proof using ‘up to context’); yet, surprisingly, no closure under substitutions is required.

As an example of application of our techniques in the  $\pi$ -calculus we revisit the completeness part of the proof of full abstraction for the encoding of the  $\lambda$ -calculus into the  $\pi$ -calculus [24, 27] with respect to Levy-Longo Trees (LTs). The proof in [24, 27] uses ‘up to expansion and context’. Such up-to techniques seem to be essential: without them, it would be hard even to define the bisimulation candidate. For our proof using unique-solution, there is one equation for each node of a given LT, describing the shape of such node.

**Outline of the paper.** Section 2 provides background about CCS and behavioural relations. We formulate our main results for CCS in Section 3, and generalise them in an abstract setting in Section 4. Section 5 shows how our results can be applied to the  $\pi$ -calculus.

## 2 Background

**CCS.** We assume an infinite set of *names*  $a, b, \dots$  and a set of *constant identifiers* (or simply *constants*) to write recursively defined processes. The special symbol  $\tau$  does not occur in the names and in the constants. We recall the grammar of CCS:

$$P := P_1 \mid P_2 \mid \Sigma_{i \in I} \mu_i. P_i \mid \nu a P \mid K \qquad \mu := a \mid \bar{a} \mid \tau$$

where  $I$  is a countable indexing set. We write  $\mathbf{0}$  when  $I$  is empty, and  $P + Q$  for binary sums. Each constant  $K$  has a definition  $K \triangleq P$ . We sometimes omit trailing  $\mathbf{0}$ , e.g., writing  $a \mid b$  for  $a. \mathbf{0} \mid b. \mathbf{0}$ . The operational semantics is given by means of an LTS, and is given in Figure 1 (the symmetric versions of the rules `parL` and `comL` have been omitted).

Some standard notations for transitions:  $\Longrightarrow$  is the reflexive and transitive closure of  $\xrightarrow{\tau}$ , and  $\xRightarrow{\mu}$  is  $\Longrightarrow \xrightarrow{\mu} \Longrightarrow$  (the composition of the three relations). Moreover,  $P \xrightarrow{\hat{\mu}} P'$  holds if  $P \xrightarrow{\mu} P'$  or  $(\mu = \tau \text{ and } P = P')$ ; similarly  $P \xRightarrow{\hat{\mu}} P'$  holds if  $P \xRightarrow{\mu} P'$  or  $(\mu = \tau \text{ and } P = P')$ . Letters  $\mathcal{R}, \mathcal{S}$  range over relations. We use the infix notation for relations, e.g.,  $P \mathcal{R} Q$  means that  $(P, Q) \in \mathcal{R}$ , and denote as  $\mathcal{RS}$  the composition of  $\mathcal{R}$  and  $\mathcal{S}$ . A relation *terminates* if there is no infinite sequence  $P_1 \mathcal{R} P_2 \mathcal{R} \dots$ . We use a tilde to denote a tuple, with countably many elements; thus the tuple may also be infinite. All notations are extended to tuples

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$$\begin{array}{c}
 \text{sum} \frac{}{\Sigma_{i \in I} \mu_i. P_i \xrightarrow{\mu_i} P_i} \quad \text{parL} \frac{P \xrightarrow{\mu} P'}{P \mid Q \xrightarrow{\mu} P' \mid Q} \quad \text{comL} \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\bar{a}} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'} \\
 \text{res} \frac{P \xrightarrow{\mu} P'}{\nu a P \xrightarrow{\mu} \nu a P'} \quad \mu \neq a, \bar{a} \quad \text{const} \frac{P \xrightarrow{\mu} P'}{K \xrightarrow{\mu} P'} \quad \text{if } K \triangleq P
 \end{array}$$

■ **Figure 1** The LTS for CCS

componentwise; e.g.,  $\tilde{P} \mathcal{R} \tilde{Q}$  means that  $P_i \mathcal{R} Q_i$ , for each component  $i$  of the tuples  $\tilde{P}$  and  $\tilde{Q}$ . We use  $\stackrel{\text{def}}{=}$  for abbreviations; in contrast,  $\triangleq$  is used for the definition of constants, and  $=$  for syntactic equality and for equations. We focus on *weak* behavioural equivalences, which abstract from the number of internal steps performed.

► **Definition 1** (Bisimilarity). A relation  $\mathcal{R}$  is a *bisimulation* if, whenever  $P \mathcal{R} Q$ , we have:

1.  $P \xrightarrow{\mu} P'$  implies that there is  $Q'$  such that  $Q \xrightarrow{\mu} Q'$  and  $P' \mathcal{R} Q'$ ;
2. the converse, on the actions from  $Q$ .

$P$  and  $Q$  are *bisimilar*, written  $P \approx Q$ , if  $P \mathcal{R} Q$  for some bisimulation  $\mathcal{R}$ .

**Systems of equations.** Unique solution of equations [13] intuitively says that if a context  $C$  obeys certain conditions, then all processes  $P$  that satisfy the equation  $P \approx C[P]$  are bisimilar with each other.

We need variables to write equations. We use capital letters  $X, Y, Z$  for these variables and call them *equation variables*. The body of an equation is a CCS expression possibly containing equation variables. We use  $E, E', \dots$  to range over *equation expressions*; these are process expressions that may contain occurrences of variables; that is, the grammar for processes is extended with a production for variables.

► **Definition 2.** Assume that, for each  $i$  of a countable indexing set  $I$ , we have a variable  $X_i$ , and an expression  $E_i$ , possibly containing some variables. Then  $\{X_i = E_i\}_{i \in I}$  (sometimes written  $\tilde{X} = \tilde{E}$ ) is a *system of equations*. (There is one equation for each variable  $X_i$ .)

$E[\tilde{P}]$  is the process resulting from  $E$  by replacing each variable  $X_i$  with the process  $P_i$ , assuming  $\tilde{P}$  and  $\tilde{X}$  have the same length. (This is syntactic replacement.) The components of  $\tilde{P}$  need not be different from each other, while this must hold for the variables  $\tilde{X}$ .

► **Definition 3.** Suppose  $\{X_i = E_i\}_{i \in I}$  is a system of equations. We say that:

- $\tilde{P}$  is a *solution of the system of equations for  $\approx$*  if for each  $i$  it holds that  $P_i \approx E_i[\tilde{P}]$ .
- The system has a *unique solution for  $\approx$*  if whenever  $\tilde{P}$  and  $\tilde{Q}$  are both solutions for  $\approx$ , then  $\tilde{P} \approx \tilde{Q}$ .

For instance, the system  $X_1 = a.X_2$ ,  $X_2 = b.X_1$  has a unique solution, whereas the equations  $X = X$ , or  $X = \tau.X$ , or  $X = a \mid X$  do not.

A system of equations is *guarded* (resp. *strongly guarded*) if each occurrence of a variable in the body of an equation is underneath a prefix (resp. a visible prefix, i.e., different from  $\tau$ ).

**Divergences.** We consider the standard notion of divergence:

► **Definition 4** (Divergence). A process  $P$  *diverges* if it can perform an infinite sequence of internal moves, possibly after some visible ones; i.e., there are processes  $P_i$ ,  $i \geq 0$ , and some  $n$ , such that  $P = P_0 \xrightarrow{\mu_0} P_1 \xrightarrow{\mu_1} P_2 \xrightarrow{\mu_2} \dots$  and for all  $i > n$ ,  $\mu_i = \tau$ . We call a *divergence of  $P$*  the sequence of transitions  $(P_i \xrightarrow{\mu_i} P_{i+1})_i$ .

► **Example 5.** The process  $L \triangleq a.\nu a(L \mid \bar{a})$  diverges, since  $L \xrightarrow{a} \nu a(L \mid \bar{a})$ , and (leaving aside  $\mathbf{0}$  and useless restrictions)  $\nu a(L \mid \bar{a})$  has a  $\tau$  transition onto itself.

### 3 Main Results

#### 3.1 Divergences and Unique Solution

This section is devoted to our main results for bisimilarity, in the case of CCS. We need to reason with the unfoldings of the given equation  $X = E$ : we define the  $n$ -th unfolding of  $E$  to be  $E^n$ ; thus  $E^1$  is defined as  $E$ ,  $E^2$  as  $E[E]$ , and  $E^{n+1}$  as  $E^n[E]$ . The *infinite* unfolding represents the simplest and most intuitive solution to the equation. In the CCS grammar, such a solution is obtained by turning the equation into a constant definition, namely the constant  $K_E$  with  $K_E \triangleq E[K_E]$ . We call  $K_E$  the *syntactic solution of the equation*.

For a system of equations  $\tilde{X} = \tilde{E}[\tilde{X}]$ , the unfoldings are defined accordingly (where  $E_i$  replaces  $X_i$  in the unfolding), and the syntactic solutions are defined to be the set of mutually recursive constants  $\{K_{\tilde{E},i} \triangleq E_i[\tilde{K}_{\tilde{E}}]\}_i$ .

► **Theorem 6** (Unique solution). *A guarded system of equations whose syntactic solutions do not diverge has a unique solution for  $\approx$ .*

We explain the schema of the proof, considering, for simplicity, a single equation  $X = E$ . We take a solution  $P$  of the equation and a transition  $P \xrightarrow{\mu} P'$ . The goal is to find an  $n$  such that  $E^n[P]$  can match this transition *without the need of  $P$* ; i.e., there is  $E'$  with  $E^n[P] \xrightarrow{\hat{\mu}} E'[P]$ , and for any process  $Q$  also  $E^n[Q] \xrightarrow{\hat{\mu}} E'[Q]$  holds.

We look for this  $n$  incrementally. If the matching transition  $E^m[P] \xrightarrow{\hat{\mu}} P_m$  (recall the  $P$  is solution) involves some transitions of  $P$ , then  $E^m[P]$  does not work. We then consider a matching transition emanating from  $E^{m+1}[P]$ , which starts with the transitions in  $E^m[P] \xrightarrow{\hat{\mu}} P_m$  that do not involve  $P$ . We observe that there are at least  $m$  of these, because  $P$  is underneath at least  $m$  prefixes in  $E^m[P]$ .

This procedure necessarily stops: otherwise, we could build an infinite sequence of transitions involving only the unfoldings of  $E$ , and with at most one visible transition: this would yield a divergence in the syntactic solution of  $E$ . With this construction at hand, given another solution  $Q$  of the equation, we construct a bisimulation containing the pair  $(P, Q)$ .

#### 3.2 Innocuous Divergences

In the remainder of the section we refine Theorem 6 by taking into account only certain forms of divergence. To introduce the idea, consider the equation  $X = a.X \mid K$ , for  $K \triangleq \tau.K$ : the divergences induced by  $K$  do not prevent uniqueness of the solution, as any solution  $P$  necessarily satisfies  $P \approx a.P$ . Indeed the variable of the equation is strongly guarded and a visible action has to be produced before accessing the variable. These divergences are not dangerous because they do not percolate through the infinite unfolding of the equation; in other words, a finite unfolding may produce the same divergence, therefore it is not necessary to go to the infinite unfolding to diverge. We call such divergences *innocuous*. Formally, these divergences are derived by applying only a finite number of times rule `const` of the LTS (see Figure 1) to the constant that represents the syntactic solution of the equation.

► **Definition 7** (Innocuous divergence). Consider a guarded system of equations  $\tilde{X} = \tilde{E}$  and its syntactic solutions  $\tilde{K}_{\tilde{E}}$ . A divergence of  $K_{\tilde{E},i}$  (for some  $i$ ) is called *innocuous* when

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summing up all usages of rule `const` with one of the  $K_{\tilde{E},j}$ s (including  $j = i$ ) in all derivation proofs of the transitions belonging to the divergence, we obtain a finite number.

► **Theorem 8** (Unique solution with innocuous divergences). *Let  $\tilde{X} = \tilde{E}$  be a system of guarded equations, and  $\tilde{K}_{\tilde{E}}$  be its syntactic solutions. If all divergences of any  $K_{\tilde{E},i}$  are innocuous, then  $\tilde{E}$  has a unique solution for  $\approx$ .*

► **Remark.** The conditions for unique solution in Theorems 6 and 8 are a mixture of syntactic (guardedness) and semantic (divergence-free) conditions. A purely semantic condition can be used if rule `const` of Figure 1 is modified so that the unfolding of a constant yields a  $\tau$ -transition:

$$\frac{}{K \xrightarrow{\tau} P} \text{ if } K \triangleq P$$

Thus in the theorems the condition that the equations are guarded could be dropped. The resulting theorems would actually be more powerful because they would accept equations not all of which are guarded: it is sufficient that each equation has a finite unfolding that is guarded. For instance the system of equations  $X = b \mid Y, Y = a.X$  would be accepted, although the first equation is not guarded.

The next lemma states a condition to ensure that all divergences produced by a system of equations are innocuous. This condition will be sufficient in all examples in the paper.

► **Lemma 9.** *In a system of equations  $\tilde{X} = \tilde{E}$ , suppose for each  $i$  there is  $n_i$  such that in  $E_i^{n_i}$ , each variable is underneath a visible prefix (say,  $a$  or  $\bar{a}$ ) whose complementary prefix ( $\bar{a}$  or  $a$ ) does not appear in any equation. Then the system has only innocuous divergences.*

### 3.3 An example: lazy and eager servers

We now show an example of application of our technique, taken from [25]. The example also illustrates the relative strengths of the two unique solution theorems (Theorems 6 and 8).

For the sake of readability, we use a version of CCS with value passing; this could be translated into pure CCS [13]. In a value-passing calculus,  $a(x).P$  is an input at  $a$  in which  $x$  is the placeholder for the value received, whereas  $\bar{a}\langle n \rangle.P$  is an output at  $a$  of the value  $n$ ; and  $A\langle n \rangle$  is a parametrised constant. This example consists of two implementations of a server; this server, when interrogated by clients at a channel  $c$ , should start a certain interaction protocol with the client, after consulting an auxiliary server  $A$  at  $a$ .

We consider the two following implementations of this server: the first one,  $L$ , is ‘lazy’, and consults  $A$  only *after* a request from a client has been received. In contrast, the other one,  $E$ , is ‘eager’, and consults  $A$  *beforehand*, so then to be ready in answering a client:

$$\begin{aligned} L &\triangleq c(z).a(x).(L \mid R\langle x, z \rangle) & A\langle n \rangle &\triangleq \bar{a}\langle n \rangle.A\langle n + 1 \rangle \\ E &\triangleq a(x).c(z).(E \mid R\langle x, z \rangle) \end{aligned}$$

Here  $R\langle x, z \rangle$  represents the interaction protocol that is started with a client, and can be any process. It may use the values  $x$  and  $z$  (obtained from the client and the auxiliary server  $A$ ); the interactions produced may indeed depend on the values  $x$  and  $z$ . We assume for now that  $R\langle x, z \rangle$  may not use channel  $c$  and  $a$ ; that is, the interaction protocol that has been spawned need not come back to the main server or to the auxiliary server. Moreover we assume  $R$  may not diverge. We want to prove that the two servers, when composed with  $A$ , yield bisimilar processes. We thus define  $LS\langle n \rangle \triangleq \nu a(A\langle n \rangle \mid L)$  and  $ES\langle n \rangle \triangleq \nu a(A\langle n \rangle \mid E)$ . A proof that  $LS\langle n \rangle \approx ES\langle n \rangle$  using the plain bisimulation proof method would be long and

tedious, due to the differences between the lazy and the eager server, and to the fact that  $R$  is nearly an arbitrary process.

For a proof using our technique, the equations are:  $\{X_n = c(z).(X_{n+1} \mid R\langle n, z \rangle)\}_n$ . The proofs that the two servers are solutions can be carried out using a few algebraic laws: expansion law, structural laws for parallel composition and restriction, one  $\tau$ -law. To apply Theorem 6, we also have to check that the equations may not produce divergences. This check is straightforward, as no silent move may be produced by interactions along  $c$ , and any two internal communications at  $a$  are separated by a visible input at  $c$ . Moreover, by assumption, the protocol  $R$  does not produce internal divergences.

If however the hypothesis that  $R$  may not diverge is lifted, then Theorem 6 is not applicable anymore, and divergences are possible. However, such divergences are innocuous: the equation need not be unfolded an infinite number of times for the divergence to occur. We can therefore still prove the result, by appealing to the more powerful Theorem 8.

### 3.4 Comparison with other techniques

Milner's syntactic condition for unique solution of equations essentially allows only equations in which variables are underneath prefixes and sums. The technique is not complete [25]; for instance it cannot handle the server example of Section 3.3.

The technique of 'unique solution of contractions' [25] relies on the theory of an auxiliary preorder (contraction), needed to establish the meaning of 'solution'; and the soundness theorems in [25] use a purely syntactic condition (guarded variables). In contrast, our techniques with equations do not rely on auxiliary relations and their theory, but the soundness theorems use a semantic condition (divergence). The two techniques are incomparable. Considering the server example of Section 3.3, the contraction technique is capable of handling also the case in which the protocol  $R$  is freely allowed to make calls back to the main server, including the possibility that, in doing this, infinitely many copies of  $R$  are spawned. This possibility is disallowed for us, as it would correspond to a non-innocuous divergence. On the other hand, when using contraction, a solution is evaluated with respect to the contraction preorder, that conveys an idea of efficiency (measured against the number of silent transitions performed). Thus, while two bisimilar processes are solutions of exactly the same set of equations, they need not be solutions of the same contractions. For instance, we can use our techniques to prove that processes  $K \triangleq \tau.a.a.K$  and  $H \triangleq a.H$  are bisimilar because solutions of the equation  $X = a.X$ ; in contrast, only  $H$  is a solution of the corresponding contraction.

**Up-to techniques.** We compare our unique-solution techniques with one of the most powerful forms of enhancement of the bisimulation proof method, namely Pous 'up to transitivity and context' technique [17]. That technique allows us to use 'up to weak bisimilarity', 'up to transitivity', and 'up to context' techniques together. While 'up to weak bisimilarity' and 'up to transitivity' are known to be unsound techniques [19], here they are combined at the price of a termination hypothesis over a 'control relation', below written  $\succ$ .

We write  $\mathcal{C}(\mathcal{R})$  for the context closure of a relation  $\mathcal{R}$  (the set of all pairs  $(C[\tilde{P}], C[\tilde{Q}])$  with  $\tilde{P} \mathcal{R} \tilde{Q}$ ). Moreover,  $\bar{\mathcal{R}}$  stands for  $(\approx \cup \mathcal{C}(\mathcal{R}))$ , and  $\mathcal{R}^+$  for the transitive closure of  $\mathcal{R}$ .

► **Definition 10.** Let  $\succ$  be a relation that is transitive, closed under contexts, and such that  $\succ (\xrightarrow{\tau}^+)$  terminates. A relation  $\mathcal{R}$  is a *bisimulation up to  $\succ$  and context* if, whenever  $P \mathcal{R} Q$ :

1. if  $P \xrightarrow{\mu} P'$  then  $Q \xrightarrow{\mu} Q'$  for some  $Q'$  with  $P' (\succ \cap \bar{\mathcal{R}})^+ \mathcal{C}(\mathcal{R}) \approx Q'$ ;
2. the converse, on the transitions from  $Q$ .



If  $\mathcal{R}$  is a relation then we can also view  $\mathcal{R}$  as an ordered sequence of pairs (e.g., assuming some lexicographical ordering). Then  $\mathcal{R}_i$  indicates the tuple obtained by projecting the pairs in  $\mathcal{R}$  on the  $i$ -th component ( $i = 1, 2$ ). The *size* of a relation is the number of pairs it contains. The size of a system of equations is the number of equations it consists of.

► **Theorem 11** (Completeness with respect to up-to techniques). *Suppose  $\mathcal{R}$  is a bisimulation up to  $\succ$  and context. Then there is a guarded system of equations, of the same size as  $\mathcal{R}$ , with only innocuous divergences, admitting  $\mathcal{R}_1$  and  $\mathcal{R}_2$  as solutions.*

In the proof, the equations are defined by exploiting the expansion law in CCS [13]. The proof then involves a rather delicate analysis in which the termination hypothesis is used to prove that the syntactic solutions of this system have only innocuous divergences (they may indeed have divergences). We may then apply Theorem 8 (rather than Theorem 6).

► **Remark.** In [17], the transitive closure  $(\succ \cap \overline{\mathcal{R}})^+$  in Definition 10 is actually a *reflexive* and transitive closure. We do not know if relaxing this technical condition breaks Theorem 11.

## 4 Abstract Formulation

### 4.1 Abstract LTS and Operators

In this section we propose generalisations of the unique-solution theorems. For this we introduce abstract formulations of them, which are meant to highlight their essential ingredients. When instantiated to the specific case of CCS, such abstract formulations yield the theorems in Section 3. The proofs are adapted from those of the corresponding theorems in Section 3. The results of this section, up to Theorem 15, have been formalised in Coq [8].

The abstract formulation is stated on a generic LTS, that is, a triple  $\mathcal{T} = (S, \Lambda, \rightarrow)$  where:  $S$  is the set of states;  $\Lambda$  the set of action labels, containing the special label  $\tau$  accounting for silent actions;  $\rightarrow$  is the transition relation. As usual, we write  $s_1 \xrightarrow{\mu} s_2$  when  $(s_1, \mu, s_2) \in \rightarrow$ . The definition of weak bisimilarity  $\approx$  is as in Section 2. We omit explicit reference to  $\mathcal{T}$  when there is no ambiguity.

We reason about *state functions*, i.e., functions from  $S$  to  $S$ , and use  $f, f', g$  to range over them. We recall that  $(f \circ g)(x) = f(g(x))$  for all  $x$ . The CCS processes of Section 2 correspond here to the states of an LTS; and CCS contexts correspond to state functions.

► **Definition 12** (Autonomy). For state functions  $f, f'$  we say that there is an *autonomous  $\mu$ -transition from  $f$  to  $f'$* , written  $f \xrightarrow{\mu} f'$ , if for all states  $x$  it holds that  $f(x) \xrightarrow{\mu} f'(x)$ .

Likewise, given a set  $\mathcal{F}$  of state functions and  $f \in \mathcal{F}$ , we say that a transition  $f(x) \xrightarrow{\mu} y$  is *autonomous on  $\mathcal{F}$*  if, for some  $f' \in \mathcal{F}$  we have  $f \xrightarrow{\mu} f'$  and  $y = f'(x)$ . Moreover, we say that *function  $f$  is autonomous on  $\mathcal{F}$*  if all the transitions emanating from  $f$  (that is, all transitions of the form  $f(x) \xrightarrow{\mu} y$ , for some  $x, \mu, y$ ) are autonomous on  $\mathcal{F}$ .

When  $\mathcal{F}$  is clear, we omit it, and we simply say that a function *is autonomous*.

Thus,  $f$  is autonomous on  $\mathcal{F}$  if, for some indexing set  $I$ , there are  $\mu_i$  and  $f_i \in \mathcal{F}$  such that for all  $x$  we have  $f(x) \xrightarrow{\mu_i} f_i(x)$ , for each  $i$ ; the set of all transitions emanating from  $f(x)$  is precisely  $\cup_i \{f(x) \xrightarrow{\mu_i} f_i(x)\}$ . Autonomous functions correspond to CCS guarded contexts, which do not need their process argument to perform the first transition. We now formulate conditions under which, intuitively, a state function behaves like a CCS context. Functions satisfying these conditions are called *operators*.

► **Definition 13** (Set of operators). Consider an LTS  $\mathcal{T}$ , and a set  $\mathcal{O}$  of functions from  $S$  to  $S$ . We say that  $\mathcal{O}$  is a *set of operators on  $\mathcal{T}$*  if the following conditions hold:



1.  $\mathcal{O}$  contains the identity function;
2.  $\mathcal{O}$  is closed by composition (that is,  $f \circ g \in \mathcal{O}$  whenever  $f, g \in \mathcal{O}$ );
3. composition preserves autonomy (i.e., if  $g$  is autonomous on  $\mathcal{O}$ , then so is  $f \circ g$ );
4. all functions in  $\mathcal{O}$  respect  $\approx$ , i.e.,  $x \approx y$  and  $f \in \mathcal{O}$  imply  $f(x) \approx f(y)$ .

A ‘symmetric variant’ of clause 3 always holds: if  $f$  is autonomous, then so is  $f \circ g$ . The autonomous transitions of a set of operators yield an LTS whose states are the operators themselves. Such transitions are of the form  $f \xrightarrow{\mu} g$ . Where the underlying set  $\mathcal{O}$  of operators is clear, we simply call a function belonging to  $\mathcal{O}$  an *operator*.

We use state functions to express equations, such as  $X = f(X)$ . We thus look at conditions under which such an equation has a unique solution (again, the generalisation to a system of equations is easy, using  $n$ -ary functions).

Thinking of functions as equation expressions, to formulate our abstract theory about unique solution of equations, we have to define the divergences of finite and infinite unfoldings of state functions. The  $n$ th unfolding of  $f$  (for  $n \geq 1$ ),  $f^n$ , is the function obtained by  $n$  applications of  $f$ . An operator is *well-behaved* if there is  $n$  with  $f^n$  autonomous (the well-behaved operators correspond, in CCS, to equations some finite unfolding of which yields a guarded expression). We also have to reason about the infinite unfolding of an equation  $X = f(X)$ . For this, given a set  $\mathcal{O}$  of operators, we consider the infinite terms obtained by infinite compositions of operators in  $\mathcal{O}$ , that is, the set coinductively defined by the grammar:

$$F := f \circ F \quad \text{where } f \in \mathcal{O} \text{ (i.e., } f \text{ is a metavariable for the elements in } \mathcal{O}\text{)}.$$

(We do not need finite compositions, as  $\mathcal{O}$  itself is closed under finite compositions.) In particular, we write  $f^\infty$  for the infinite term  $f \circ f \circ f \circ \dots$ .

We define the autonomous transitions for such infinite terms using the following rules:

$$\frac{g \xrightarrow{\mu} g'}{g \circ F \xrightarrow{\mu} g' \circ F} \quad g \text{ autonomous} \qquad \frac{(g \circ f) \circ F \xrightarrow{\mu} F'}{g \circ (f \circ F) \xrightarrow{\mu} F'} \quad g \text{ not autonomous}$$

Intuitively a term is ‘unfolded’, with the second rule, until an autonomous function is uncovered, and then transitions are computed using the first rule (we disallow unnecessary unfoldings; these would complicate our abstract theorems, by adding duplicate transitions, since the transitions of  $g \circ f$  duplicate those of  $g$  when  $g$  is autonomous). An infinite term has no transitions if none of its finite unfoldings ever yields an autonomous function. This situation does not arise for terms of the form  $f^\infty$  or  $g \circ f^\infty$ , where  $f$  is well-behaved, which are the terms we are interested in. Note that no infinite term belongs to a set of operators.

► **Definition 14** (Operators and divergences). Let  $f, f', f_i$  be operators in a set  $\mathcal{O}$  of operators, and consider the LTS induced by the autonomous transitions of operators in  $\mathcal{O}$ . A sequence of transitions  $f_1 \xrightarrow{\mu_1} f_2 \xrightarrow{\mu_2} f_3 \dots$  is a *divergence* if for some  $n \geq 1$  we have  $\mu_i = \tau$  whenever  $i \geq n$ . We also say that  $f_1$  *diverges*. We apply these notations and terminology also to infinite terms (such as  $f_1 \circ f_2 \circ \dots$ ), as expected.

In the remainder of the section we fix a set  $\mathcal{O}$  of operators and we only consider autonomous transitions on  $\mathcal{O}$ . We now state the ‘abstract version’ of Theorem 6 (the proof being similar).

► **Theorem 15** (Unique solution, abstract formulation). *Let  $f$  be a well-behaved operator on  $\mathcal{O}$ . If  $f^\infty$  does not diverge, then the equation  $X = f(X)$  either has no solution or has a unique solution for  $\approx$ .*

The equation in the statement of the theorem might have no solution at all. For example, consider the LTS  $(\mathbb{N}, \{a\}, \rightarrow)$  where for each  $n$  we have  $n + 1 \xrightarrow{a} n$ . The function  $f$  with

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$f(n) = n + 1$  is an operator of the set  $\mathcal{O} = \{f^n\}_{n \in \mathbb{N}}$  (with  $f^0 = \text{Id}$ , the identity function). The function  $f$  is autonomous because, for all  $n$ , the only transition of  $f(n)$  is  $f(n) \xrightarrow{a} n$  (this transition is autonomous because  $f \xrightarrow{a} \text{Id}$ ). A fixpoint of  $f$  would be an element  $x$  with  $x \xrightarrow{a} x$ , and there is no such  $x$  in the LTS.

Theorem 15 can be refined along the lines of Theorem 8. For this, we have to relate the divergences of any  $f^n$  (for  $n \geq 1$ ) to divergences of  $f^\infty$ , in order to distinguish between innocuous and non-innocuous divergences. To build a divergence of  $f^\infty$  from a divergence of  $f^n$ , for  $n \geq 1$ , we need to reason up to (finite) unfoldings of  $f$ : thus we set  $=_f$  to be the symmetric reflexive transitive closure of the relation that relates  $g$  and  $g'$  whenever  $g = g' \circ f$ .

► **Lemma 16.** *Consider an autonomous operator  $f$  on  $\mathcal{O}$  and a divergence of  $f^n$*

$$f^n \xrightarrow{\mu_1} f_1 \xrightarrow{\mu_2} \dots \xrightarrow{\mu_i} f_i \xrightarrow{\tau} f_{i+1} \xrightarrow{\tau} \dots$$

*This yields a divergence of  $f^\infty$ :  $f^\infty \xrightarrow{\mu_1} g_1 \circ f^\infty \xrightarrow{\mu_2} \dots \xrightarrow{\mu_i} g_i \circ f^\infty \xrightarrow{\tau} g_{i+1} \circ f^\infty \xrightarrow{\tau} \dots$  such that for all  $i \geq 1$ ,  $g_i$  is an operator and  $g_i =_f f_i$ .*

Given a divergence  $\Delta$  of  $f^n$ , we write  $\Delta^\infty$  to indicate the divergence of  $f^\infty$  obtained from  $\Delta$  as in Lemma 16. We call a divergence of  $f^\infty$  *innocuous* when it can be described in this way, that is, as a divergence  $\Delta^\infty$  obtained from a divergence  $\Delta$  of  $f^n$ , for some  $n$ .

► **Theorem 17** (Unique solution with innocuous divergences, abstract formulation). *Let  $f \in \mathcal{O}$  be a well-behaved operator. If all divergences of  $f^\infty$  are innocuous, then the equation  $X = f(X)$  either has no solution or has a unique solution for  $\approx$ .*

## 4.2 Reasoning with other Behavioural Relations

**Trace-based Equivalences.** We can adapt the results of the previous section about bisimilarity to other settings, including both preorders and non-coinductive relations. As an example, we consider trace-based relations. We write  $\approx_{tr}$  for trace equivalence, that equates two processes having the same set of (finite) traces. The definitions from previous sections are the same as for  $\approx$ , replacing  $\approx$  with  $\approx_{tr}$ . All theorems presented for  $\approx$  can be adapted to  $\approx_{tr}$ , and the proofs are similar. As an example, Theorem 17 becomes:

► **Theorem 18.** *Let  $f \in \mathcal{O}$  be a well-behaved operator. If all divergences of  $f^\infty$  are innocuous, then the equation  $X = f(X)$  either has no solution or has a unique solution for  $\approx_{tr}$ .*

In contrast, the theorems fail for *infinitary trace equivalence*,  $\approx_{tr}^\infty$  (whereby two processes are equated if they have the same traces, including the infinite ones), for the same reason why the ‘unique solution of contraction’ technique fails in this case [25]. As a counterexample, we consider equation  $X = a + a.X$ , whose syntactic solution has no divergences. The process  $P \triangleq \sum_{n>0} a^n$  is a solution, yet it is not  $\approx_{tr}^\infty$ -equivalent to the syntactic solution of the equation, because the syntactic solution has an infinite trace involving  $a$  transitions.

**Preorders.** We show how the theory for equivalences can be transported onto *preorders*. This means moving to *systems of pre-equations*,  $\{X_i \leq E_i\}_{i \in I}$ . With preorders, our theorems have a different shape: we do not use pre-equations to reason about unique solution – we expect interesting pre-equations to have many solutions, some of which may be incomparable with each other. We rather derive theorems to prove that, in a given preorder, any solution of a pre-equation is below its syntactic solution.

► **Remark.** The opposite direction for pre-equations, namely  $\{X_i \geq E_i\}_{i \in I}$  is less interesting. It would mean aiming to prove that the syntactic solution is below other solutions. This is usually a trivial property for a behavioural preorder, without any hypothesis such as autonomy or non divergence (a possible exception is a preorder with infinitary observables, such as infinitary trace inclusion, where the property may fail).

We write  $\subseteq_{tr}$  for trace inclusion,  $\subseteq_{tr^\infty}$  for infinitary trace inclusion, and  $\leq_s$  for weak simulation. These preorders are standard from the literature [28].

In the abstract setting, the body of the pre-equations are functions. Then the theorems give us conditions under which, given a pre-equation  $X \leq f(X)$  and a behavioural preorder  $\preceq$ , a solution  $r$ , i.e., a state for which  $r \preceq f(r)$  holds, is below the syntactic solution  $f^\infty$ . We present the counterpart of Theorem 17; other theorems are transported in a similar manner.

► **Theorem 19.** *Let  $f \in \mathcal{O}$  be a well-behaved operator. If all divergences of  $f^\infty$  are innocuous, then, given  $\leq \in \{\subseteq_{tr}, \subseteq_{tr^\infty}, \leq_s\}$ , whenever  $x \leq f(x)$  we also have  $x \leq f^\infty$ , for any state  $x$ .*

Theorem 19 intuitively says that the syntactic solution of a pre-equation is *maximal* among all solutions. Note that, in contrast with equations, Theorem 19 and the theory of pre-equations also work for *infinitary trace inclusion*.

### 4.3 Rule formats

A way to instantiate the results in Sections 4.1 and 4.2 is to consider *rule formats* [1, 16]. These provide a specification for the form of the SOS rules used to describe the constructs of a language. To fit a rule format into the abstract formulation of the theory from Section 4.1, we view the constructs of a language as functions on the states of the LTS (the processes of the language). One of the most common formats is GSOS [3, 29, 9]. For lack of space we only show the instantiation to GSOS of Theorem 8. In the statement, we consider an extension of a GSOS language with constants, in the same way as they appear in CCS, so that the definitions of ‘syntactic solution of equations’ and of ‘innocuous divergence’ can be taken to be the same as for CCS (these are easier to grasp than their formulation in Section 4.1).

► **Theorem 20.** *Consider a language whose constructs have SOS rules in the GSOS format and preserve  $\approx$ , and an equation  $X = E$  for the language. If  $E$  is autonomous (over the set of functions corresponding to the contexts of the language), and if, in the language extended with constants, the syntactic solution of the equation only has innocuous divergences, then either the equation has no solution or it has a unique solution for  $\approx$ .*

We briefly discuss the hypotheses in the theorem. Some GSOS rule formats guarantee congruence for weak bisimilarity [29, 9], which allows one to remove the corresponding condition (the condition could actually be weakened, by considering the syntactic positions in which the variables can occur in the equations). Checking the autonomy property is often straightforward; for instance, it holds if, in the body of an equation, all variables are underneath an axiom construct, that is, a construct that (like prefix in CCS) is defined by means of SOS rules in which the set of premises is empty. In the tyft/tyxt formats [11], lookahead is possible. Lookahead allows one to write rules that ‘look into the future’ (a transition is allowed if certain sequences of actions are possible); this breaks autonomy (condition 3 of Definition 13), hence Theorem 17 is not applicable.

## 5 Name Passing: the $\pi$ -calculus

### 5.1 Unique solution in the asynchronous $\pi$ -calculus

In this section, we port our results onto the asynchronous  $\pi$ -calculus,  $A\pi$  (addressing the full  $\pi$ -calculus would also be possible, but somewhat more involved). To allow constants (recursive definitions) and equations, we enrich the syntax of  $A\pi$  [26] with parametrised processes  $(\tilde{a})P$ . These and the constants form the set of *abstractions*, ranged over by  $F, G$ . We omit the definitions of free and bound names. An expression is *closed* if it does not have free names. Constant definitions are of the form  $K \triangleq F$ , where  $F$  is a closed abstraction. The grammar for processes includes also the application construct  $F(\tilde{a})$ , used to instantiate the formal parameters of the abstraction  $F$  with the actual parameters  $\tilde{a}$ .

The body of an equation is also a closed abstraction, possibly containing equation variables. Since the calculus is polyadic, we rely on a sorting system [15] to avoid disagreements in the arities of the tuples of names carried by a given name, and in the parameters of abstractions and equations. For lack of space, we omit the full grammar and the operational semantics (see [24]). When writing examples, for readability, we assume that the syntax contains the guarded replication operator  $!a(\tilde{b}).P$  (it could be encoded, using constants).

In bisimulations or similar coinductive relations for the asynchronous  $\pi$ -calculus, no name instantiation is required in the input clause or elsewhere (provided  $\alpha$ -convertible processes are identified); i.e., the *ground* versions of the relations are congruence relations [26]. Similarly, the extension of bisimilarity to abstractions only considers fresh names:  $F \approx F'$  if  $F(\tilde{a}) \approx F'(\tilde{a})$  where  $\tilde{a}$  is a tuple of fresh names (as usual, of the appropriate sort).

Theorems 6 and 8 for CCS can be adapted to the asynchronous  $\pi$ -calculus. The definitions concerning transitions and divergences are transported to  $A\pi$  as expected. In the case of an abstraction, one first has to instantiate the parameters with fresh names; thus  $F$  has a divergence if the process  $F(\tilde{a})$  has a divergence, where  $\tilde{a}$  are fresh names.

► **Theorem 21** (Unique solution in  $A\pi$ ). *A guarded system of equations whose syntactic solutions do not contain divergences has a unique solution for  $\approx$ .*

► **Theorem 22** (Unique solution with innocuous divergences in  $A\pi$ ). *A guarded system of equations whose syntactic solutions only have innocuous divergences has a unique solution for  $\approx$ .*

We pointed out in earlier sections on CCS the connection between techniques based on unique solution of equations and ‘up to context’ enhancements of the bisimulation proof method. The same connection is less immediate in name-passing calculi, where indeed there are noticeable differences. In particular, ‘up to context’ enhancements for the ground bisimilarity of the  $\pi$ -calculus require closure under name instantiation, even when ground bisimilarity is known to be preserved by substitutions (it is an open problem whether the closure can be lifted). Thus, when comparing two derivatives  $C[P]$  and  $C[Q]$ , in general it is not sufficient that  $P$  and  $Q$  alone are in the candidate relation: one is required to include also all their closures under name substitutions (or, if the terms in the holes are abstractions, instantiation of their parameters with arbitrary tuples of names). In contrast, the two unique solution theorems above are ‘purely ground’:  $F = (\tilde{x})P$  is solution of an equation  $X = (\tilde{x})E$  if  $P$  and  $E\{F/X\}$  are ground bisimilar – a single ground instance of the equation is evaluated.

## 5.2 An application: encoding of the call-by-name $\lambda$ -calculus

To show an extended application of our techniques for the  $\pi$ -calculus, we revisit the proof of full abstraction for Milner's encoding of the call-by-name (or lazy)  $\lambda$ -calculus into  $\mathcal{A}\pi$  [14] with respect to Lévy Longo Trees (LTs), precisely the completeness part. We use  $M, N$  to range over the set  $\Lambda$  of  $\lambda$ -terms, and  $x, y, z$  to range over  $\lambda$  variables. If  $M$  is an open  $\lambda$ -term, then either  $M$  diverges, or  $M \Rightarrow \lambda x. M'$ , or  $M \Rightarrow x M_1 \dots M_n$ .

► **Definition 23** (Lévy-Longo Tree). The *Lévy-Longo Tree* (LT) of an open  $\lambda$ -term  $M$ , written  $LT(M)$ , is the (possibly infinite) tree defined coinductively as follows.

1. If  $M$  diverges, then  $LT(M)$  is the tree with a single node labelled  $\perp$ .
2. If  $M \rightarrow \lambda x. M'$ , then  $LT(M)$  is the tree with a root labelled with " $\lambda x.$ ", and  $LT(M')$  as its unique descendant.
3. If  $M \rightarrow x M_1 \dots M_n$ , then  $LT(M)$  is the tree with a root labelled with " $x$ ", and  $LT(M_1), \dots, LT(M_n)$  (in this order) as its  $n$  descendants.

LT equality (two  $\lambda$ -terms are identified if their LTs are equal) can also be presented as a bisimilarity (*open bisimilarity*,  $\simeq^o$ ), defined as the largest *open bisimulation*.

► **Definition 24.** A relation  $\mathcal{R}$  on  $\Lambda$  is an *open bisimulation* if, whenever  $M \mathcal{R} N$ :

1.  $M \Rightarrow \lambda x. M'$  implies  $N \Rightarrow \lambda x. N'$  with  $M' \mathcal{R} N'$ ;
2.  $M \Rightarrow x M_1 \dots M_n$  with  $n \geq 0$  implies  $N \Rightarrow x N_1 \dots N_n$  and  $M_i \mathcal{R} N_i$  for all  $1 \leq i \leq n$ .
3. The converse of clauses 1 and 2 on the challenges from  $N$ .

Milner's encoding is defined thus:  $\llbracket \lambda x. M \rrbracket \triangleq (p) p(x, q). \llbracket M \rrbracket \langle q \rangle$      $\llbracket x \rrbracket \triangleq (p) \bar{x}(p)$   
 $\llbracket M N \rrbracket \triangleq (p) \nu r, x (\llbracket M \rrbracket \langle r \rangle \mid \bar{r}(x, p) \mid !x(q). \llbracket N \rrbracket \langle q \rangle)$

The full abstraction theorem for the encoding [24, 27] states that two  $\lambda$ -terms have the same LT iff their encodings into  $\mathcal{A}\pi$  are weakly bisimilar terms. Full abstraction has two components: soundness, which says that if the encodings are weakly bisimilar then the original terms have the same LT; and completeness, which is the converse direction. The proof [24] first establishes some operational correspondence between the behaviour (visible and silent actions) of  $\lambda$ -terms and of their encodings. Then, exploiting this correspondence, soundness and completeness are proved using the bisimulation proof method. For soundness, this is just open bisimulation (Definition 24). In contrast, completeness involves enhancements of the proof method, notably 'bisimulation up to context and expansion'. As a consequence, the technique requires having developed the basic theory for the expansion preorder (e.g., precongruence properties and basic algebraic laws), and requires an operational correspondence fine enough in order to be able to reason about expansion.

Below we show that, by appealing to unique solution of equations, completeness can be proved by defining an appropriate system of equations, each of which having a simple shape, and without the need for auxiliary preorders. For this, the only results needed are: (i) validity of  $\beta$ -reduction for the encoding (Lemma 25), whose proof is simple and consists in the application of a few algebraic laws (including laws for replication); (ii) the property that if  $M$  diverges then  $\llbracket M \rrbracket \langle p \rangle$  may never produce a visible action [24].

► **Lemma 25** (Validity of  $\beta$ -reduction, [24]). *For  $M \in \Lambda$ , if  $M \rightarrow M'$  then  $\llbracket M \rrbracket \approx \llbracket M' \rrbracket$ .*

► **Theorem 26** (Completeness, [24]). *For  $M, N \in \Lambda$ ,  $LT(M) = LT(N)$  implies  $\llbracket M \rrbracket \approx \llbracket N \rrbracket$ .*

**Proof.** Suppose  $V$  and  $W$  are two  $\lambda$ -terms with the same LT. We define a system of equations, solutions of which are obtained from the encodings of  $V$  and  $W$ . We will then use Theorem 22 to deduce  $\llbracket V \rrbracket \approx \llbracket W \rrbracket$ . If  $V$  and  $W$  have the same LT, then there is an open bisimulation  $\mathcal{R}$

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containing the pair  $(V, W)$ . The variables of the equations are of the form  $X_{M,N}$  for  $M \mathcal{R} N$ , and there is one equation for each pair in  $\mathcal{R}$ . We show how the equation for a pair  $M \mathcal{R} N$  is built. We assume an ordering of the  $\lambda$ -calculus variables so to be able to view a finite set of variables as a tuple. Thus we write  $\tilde{x}$  for the variables appearing free in  $M$  or  $N$ .

Essentially, the equations are the translation of the clauses of Definition 24, assuming a generalisation of the encoding to equation variables:

If  $M, N$  are both divergent, then the equation is  $X_{M,N} = (\tilde{x}, p)! \tau$ .

If  $M, N$  satisfy clause 1 of Definition 24, the equation is  $X_{M,N} = (\tilde{x}, p)p(x, q) \cdot X_{M',N'}(\tilde{y}, q)$ , where  $\tilde{y}$  are the free variables in  $M', N'$ .

Finally, if  $M, N$  satisfy clause 2 of Definition 24, the equation is given by the translation of  $x X_{M_1, N_1} \dots X_{M_n, N_n}$ , which, rearranging restrictions and parallel compositions, is

$$X_{M,N} = (\tilde{x}, p)(\nu r_0, \dots, r_n) ( \bar{x}\langle r_0 \rangle \mid \bar{r}_0\langle r_1, x_1 \rangle \mid \dots \mid \bar{r}_{n-1}\langle r_n, x_n \rangle \mid \\ !x_1(q_1) \cdot X_{M_1, N_1}(\tilde{x}_1, q_1) \mid \dots \mid !x_n(q_n) \cdot X_{M_n, N_n}(\tilde{x}_n, q_n) ) .$$

where  $\tilde{x}_i$  are the free variables in  $M_i, N_i$ . Applying Lemma 9 (more precisely, its  $A\pi$  analogue, with additional reference to the sorting system to handle name instantiation in abstractions and inputs), we show that the equations may only produce innocuous divergences.

Now, for  $(M, N) \in \mathcal{R}$ , set  $F_{M,N}$  to be the abstraction  $(\tilde{x}, p)\llbracket M \rrbracket\langle p \rangle$ , and similarly  $G_{M,N} \triangleq (\tilde{x}, p)\llbracket N \rrbracket\langle p \rangle$ . The set of all such abstractions  $F_{M,N}$  yields a solution for the system of equations, and the same for the  $G_{M,N}$ 's. The proof that they are solutions (hence bisimilar) is a consequence of Lemma 25. For instance, for clause 1 of Definition 24, we have:

$$F_{M,N} = (\tilde{x}, p)\llbracket M \rrbracket\langle p \rangle \\ \approx (\tilde{x}, p)\llbracket \lambda x. M' \rrbracket\langle p \rangle \approx (\tilde{x}, p)\llbracket \lambda x. X_{M',N'} \rrbracket\langle p \rangle \{F_{M',N'}/X_{M',N'}\}$$

◀

## 6 Future Work

We have compared our techniques to one of the most powerful forms of enhancements of the bisimulation proof method, namely Pous ‘up to transitivity and context’, showing that, up to a technical condition, our techniques are at least as powerful. We believe that also the converse holds, though possibly under different side conditions. We leave a detailed analysis of this comparison, which seems non-trivial, for future work. In this respect, the goal of the work on unique solution of equations is to provide a way of better understanding up-to techniques and to shed light into the conditions for their soundness. Pous technique, for instance, is arguably more complex, both in its definition and its application.

Another aspect in which a deep comparison with up-to techniques might be useful is understanding the need for the closure under substitutions in up to context techniques for name-passing calculi such as the asynchronous  $\pi$ -calculus. Such a closure is rather heavy and is a long-standing open problem. However, currently it is unclear how to formally relate bisimulation enhancements and ‘unique solution of equations’ in name-passing calculi.

Up-to techniques have been analysed in an abstract setting using lattice theory [18] and category theory [4, 23]. It could be interesting to do the same for the unique-solution techniques, to study their connections with up-to techniques, and which equivalences can be handled (possibly using, or refining, the abstract formulation of Section 4).

In comparison with the enhancements of the bisimulation proof method, the main drawback of the techniques exposed in this paper is the presence of a semantic condition, involving divergence: the unfoldings of the equations should not produce divergences, or



only produce innocuous divergences. A syntactic condition for this has been proposed (Lemma 9). Various techniques for checking divergence exist in the literature, including type-based techniques [30, 7]. However, in general divergence is undecidable, and therefore, the check may sometimes be unfeasible. Nevertheless, the equations that one writes for proofs usually involve forms of ‘normalised’ processes, and as such they are divergence free (or at most, contain only innocuous divergences). More experiments are needed to validate this claim or to understand how limiting this problem is.

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