

k -Distinct In- and Out-Branchings in Digraphs^{*†}

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Abstract

An out-branching and an in-branching of a digraph D are called k -distinct if each of them has k arcs absent in the other. Bang-Jensen, Saurabh and Simonsen (2016) proved that the problem of deciding whether a strongly connected digraph D has k -distinct out-branching and in-branching is fixed-parameter tractable (FPT) when parameterized by k . They asked whether the problem remains FPT when extended to arbitrary digraphs. Bang-Jensen and Yeo (2008) asked whether the same problem is FPT when the out-branching and in-branching have the same root.

By linking the two problems with the problem of whether a digraph has an out-branching with at least k leaves (a leaf is a vertex of out-degree zero), we first solve the problem of Bang-Jensen and Yeo (2008). We then develop a new digraph decomposition called the rooted cut decomposition and using it we prove that the problem of Bang-Jensen et al. (2016) is FPT for all digraphs. We believe that the *rooted cut decomposition* will be useful for obtaining other results on digraphs.

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1 Introduction

While both undirected and directed graphs are important in many applications, there are significantly more algorithmic and structural results for undirected graphs than for directed ones. The main reason is likely to be the fact that most problems on digraphs are harder than those on undirected graphs. The situation has begun to change: recently there appeared a number of important structural results on digraphs, see e.g. [16, 17, 18]. However, the progress was less pronounced with algorithmic results on digraphs, in particular, in the area of parameterized algorithms.

In this paper, we introduce a new decomposition for digraphs and show its usefulness by solving an open parameterized problem on digraphs by Bang-Jensen, Saurabh and Simonsen [6]. We believe that our decomposition will prove to be helpful for obtaining further algorithmic and structural results on digraphs.

A digraph T is an *out-tree* (an *in-tree*) if T is an oriented tree with just one vertex s of in-degree zero (out-degree zero). The vertex s is the *root* of T . A vertex v of an out-tree

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(in-tree) is called a *leaf* if it has out-degree (in-degree) zero. If an out-tree (in-tree) T is a spanning subgraph of a digraph D , then T is an *out-branching* (an *in-branching*) of D . It is well-known that a digraph D contains an out-branching (in-branching) if and only if D has only one strongly connected component with no incoming (no outgoing) arc [3].

A well-known result in digraph algorithms, due to Edmonds, states that given a digraph D and a positive integer ℓ , we can decide whether D has ℓ arc-disjoint out-branchings in polynomial time [15]. The same result holds for ℓ arc-disjoint in-branchings. Inspired by this fact, it is natural to ask for a “mixture” of out- and in-branchings: given a digraph D and a pair u, v of (not necessarily distinct) vertices, decide whether D has an arc-disjoint out-branching T_u^+ rooted at u and an in-branching T_v^- rooted at v . We will call this problem ARC-DISJOINT BRANCHINGS.

Thomassen proved (see [2]) that the problem is NP-complete and remains NP-complete if we add the condition that $u = v$. The same result still holds for digraphs in which the out-degree and in-degree of every vertex equals two [7]. The problem is polynomial-time solvable for tournaments [2] and for acyclic digraphs [8, 10]. The single-root special case (i.e., when $u = v$) of the problem is polynomial time solvable for quasi-transitive digraphs¹ [4] and for locally semicomplete digraphs² [5].

An out-branching T^+ and an in-branching T^- are called *k-distinct* if $|A(T^+) \setminus A(T^-)| \geq k$. Bang-Jensen, Saurabh and Simonsen [6] considered the following parameterization of ARC-DISJOINT BRANCHINGS.

k-DISTINCT BRANCHINGS parametrised by k

Input: A digraph D , an integer k .

Problem: Are there k -distinct out-branching T^+ and in-branching T^- ?

They proved that *k*-DISTINCT BRANCHINGS is fixed-parameter tractable (FPT)³ when D is strongly connected and conjectured that the same holds when D is an arbitrary digraph. Earlier, Bang-Jensen and Yeo [9] considered the version of *k*-DISTINCT BRANCHINGS where T^+ and T^- must have the same root and asked whether this version of *k*-DISTINCT BRANCHINGS, which we call SINGLE-ROOT *k*-DISTINCT BRANCHINGS, is FPT.

The first key idea of this paper is to relate *k*-DISTINCT BRANCHINGS to the problem of deciding whether a digraph has an out-branching with at least k leaves via a simple lemma (see Lemma 4). The lemma and the following two results on out-branchings with at least k leaves allow us to solve the problem of Bang-Jensen and Yeo [9] and to provide a shorter proof for the above-mentioned result of Bang-Jensen, Saurabh and Simonsen [6] (see Theorem 6).

► **Theorem 1** ([1]). *Let D be a strongly connected digraph. If D has no out-branching with at least k leaves, then the (undirected) pathwidth of D is bounded by $O(k \log k)$.*

► **Theorem 2** ([12, 19]). *We can decide whether a digraph D has an out-branching with at least k leaves in time $O^*(4^k)$.*

¹ A digraph $D = (V, A)$ is quasi-transitive if for every $xy, yz \in A$ there is at least one arc between x and z , i.e. either $xz \in A$ or $zx \in A$ or both.

² A digraph $D = (V, A)$ is locally semicomplete if for every $xy, xz \in A$ there is at least one arc between y and z and for every $yx, zx \in A$ there is at least one arc between y and z . Tournaments and directed cycles are locally semicomplete digraphs.

³ Fixed-parameter tractability of *k*-DISTINCT BRANCHINGS means that the problem can be solved by an algorithm of runtime $O^*(f(k))$, where O^* omits not only constant factors, but also polynomial ones, and f is an arbitrary computable function. The books [11, 13] are excellent recent introductions to parameterized algorithms and complexity.

The general case of k -DISTINCT BRANCHINGS seems to be much more complicated. We first introduce a version of k -DISTINCT BRANCHINGS called k -ROOTED DISTINCT BRANCHINGS, where the roots s and t of T^+ and T^- are fixed, and add the arc ts to D (provided the arc is not in D) to make D strongly connected. This introduces a complication: we may end up in a situation where D has an out-branching with many leaves, and thereby potentially unbounded pathwidth, but the root of the out-branching is not s . To deal with this situation, our goal will be to *reconfigure* the out-branching into an out-branching rooted at s . In order to reason about this process, we develop a new digraph decomposition we call the *rooted cut decomposition*. The cut decomposition of a digraph D rooted at a given vertex r consists of a tree \hat{T} rooted at r whose nodes are some vertices of D and subsets of vertices of D called *diblocks* associated with the nodes of \hat{T} .

Our strategy is now as follows. If \hat{T} is *shallow* (i.e., it has bounded height), then any out-branching with sufficiently many leaves can be turned into an out-branching rooted at s without losing too many of the leaves. On the other hand, if \hat{T} contains a path from the root of \hat{T} with sufficiently many non-degenerate diblocks (diblocks with at least three vertices), then we are able to show immediately that the instance is positive. The remaining and most difficult issue is to deal with digraphs with decomposition trees that contain long paths of diblocks with only two vertices, called *degenerate* diblocks. In this case, we employ two reduction rules which lead to decomposition trees of bounded height.

The paper is organized as follows. In the next section, we provide some terminology and notation on digraphs used in this paper. In Section 3, we prove Theorem 6. Section 4 is devoted to proving that ROOTED k -DISTINCT BRANCHINGS is FPT for all digraphs using cut decomposition and Theorems 1 and 2. We conclude the paper in Section 5, where some open parameterized problems on digraphs are mentioned. Due to space constraints, we only provide a short version of Section 4.3 in the main text. A complete version is available online⁴.

2 Terminology and Notation

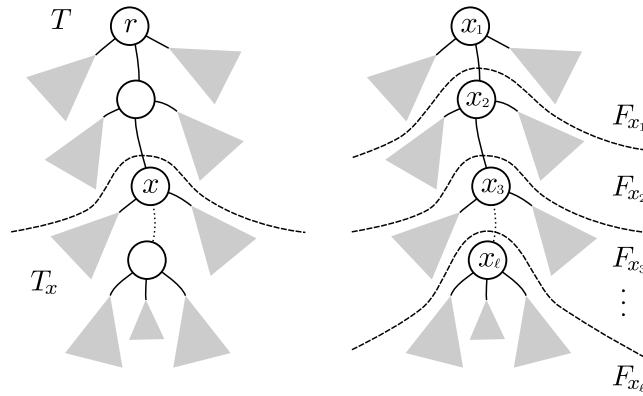
Let us recall some basic terminology of digraph theory, see [3]. A digraph D is *strongly connected* (*connected*) if there is a directed (oriented) path from x to y for every ordered pair x, y of vertices of D . Equivalently, D is connected if the underlying graph of D is connected. A vertex v is a *source* (*sink*) if its in-degree (out-degree) is equal to zero. It is well-known that every acyclic digraph has a source and a sink [3].

In this paper, we exclusively work with digraphs, therefore we assume all our graphs, paths, and trees to be directed unless otherwise noted. For a path $P = x_1x_2 \dots x_k$ of length $k - 1$ we will employ the following notation for subpaths of P : $P[x_i, x_j] := x_i \dots x_j$ for $1 \leq i \leq j \leq k$ is the *infix* of P from x_i to x_j . For paths $P_1 := x_1 \dots x_kv$ and $P_2 := vy_1 \dots y_\ell$ we denote by $P_1P_2 := x_1 \dots x_kv y_1 \dots y_\ell$ their *concatenation*. For rooted trees T and some vertex $x \in T$, T_x stands for the subtree of T rooted at x (see Figure 1).

We will frequently partition the nodes of a tree around a path in the following sense (cf. Figure 1): Let T be a tree rooted at r and $P = x_1 \dots x_\ell$ a path from $r = x_1$ to some node $x_\ell \in T$. The *fins* of P are the sets $\{F_{x_i}\}_{x_i \in P}$ defined as $F_{x_i} := V(T_{x_i}) \setminus V(T_{x_{i+1}})$ for $i < \ell$ and $F_{x_\ell} := V(T_{x_\ell})$.

► **Definition 3** (Bi-reachable Set). A set B in a digraph D is *bi-reachable* from a vertex r if for all $v \in B$ there exist two internally vertex-disjoint paths from r to v .

⁴ <https://arxiv.org/abs/1612.03607>



■ **Figure 1** Subtree notation T_x for $x \in T$ (left) and the fins $F_{x_1}, \dots, F_{x_\ell}$ for a path $x_1 \dots x_\ell$ in T (right).

Given a digraph D and a vertex r , we can compute the set of vertices that are bi-reachable from r in polynomial time using network flows.

3 Strongly Connected Digraphs

Let us prove a simple fact on a link between out/in-branchings with many leaves and k -DISTINCT BRANCHINGS, which together with a structural result of Alon *et al.* [1] and an algorithmic result for the maximum leaf out-branching problem [12, 19] gives a short proof that both versions of k -DISTINCT BRANCHINGS are FPT for strongly connected digraphs.

► **Lemma 4.** *Let D be a digraph containing an out-branching and an in-branching. If D contains an out-branching (in-branching) T with at least $k + 1$ leaves, then every in-branching (out-branching) T' of D is k -distinct from T .*

Proof. We will consider only the case when T is an out-branching since the other case can be treated similarly. Let T' be an in-branching of D and let L be the set of all leaves of T apart from the one which is the root of T' . Observe that all vertices of L have outgoing arcs in T' and since in T the incoming arcs of L are the only arcs incident to L in T , the sets of the outgoing arcs in T' and incoming arcs in T do not intersect. ◀

We will use the following standard dynamic programming result (see, e.g., [6]).

► **Lemma 5.** *Let H be a digraph of (undirected) treewidth τ . Then k -DISTINCT BRANCHINGS and SINGLE-ROOT k -DISTINCT BRANCHINGS on H can be solved in time $O^*(2^{O(\tau \log \tau)})$.*

Note that if a digraph D is a positive instance of SINGLE-ROOT k -DISTINCT BRANCHINGS then D must be strongly connected as an out-branching and an in-branching rooted at the same vertex form a strongly connected subgraph of D .

► **Theorem 6.** *k -DISTINCT BRANCHINGS and SINGLE-ROOT k -DISTINCT BRANCHINGS on strongly connected digraphs can be solved in time $O^*(2^{O(k \log^2 k)})$.*

Proof. The proof is essentially the same for both problems and we will give it for SINGLE-ROOT k -DISTINCT BRANCHINGS. Let D be an input strongly connected digraph. By Theorem 2 using an $O^*(4^k)$ -time algorithm we can find an out-branching T^+ with at least $k + 1$ leaves, or decide that D has no such out-branching. If T^+ is found, the instance of

SINGLE-ROOT k -DISTINCT BRANCHINGS is positive by Lemma 4 as any in-branching T^- of D is k -distinct from T^+ . In particular, we may assume that T^- has the same root as T^+ (a strongly connected digraph has an in-branching rooted at any vertex). Now suppose that T^+ does not exist. Then, by Theorem 1 the (undirected) pathwidth of D is bounded by $O(k \log k)$. Thus, by Lemma 5 the instance can be solved in time $O^*(2^{O(k \log^2 k)})$. ◀

4 The k -Distinct Branchings Problem

In this section, we fix a digraph D with terminals s, t and simply talk about *rooted out-branchings* (*in-branchings*) whose root we implicitly assume to be s (t). Similarly, unless otherwise noted, a *rooted out-tree* (*in-tree*) is understood to be rooted at s (t).

The problem k -DISTINCT BRANCHINGS in which T^+ and T^- must be rooted at s and t , respectively, will be called the ROOTED k -DISTINCT BRANCHINGS problem. Clearly, to show that both versions of k -DISTINCT BRANCHINGS are FPT it is sufficient to prove the following:

► **Theorem 7.** ROOTED k -DISTINCT BRANCHINGS is FPT for arbitrary digraphs.

In the rest of this section, (D, s, t) will stand for an instance of ROOTED k -DISTINCT BRANCHINGS (in particular, D is an input digraph of the problem) and H for an arbitrary digraph. As noted in the previous section, the case in which $s = t$ implies strong connectivity and is therefore already solved. Consequently, we will assume that $s \neq t$ in the following. Let us start by observing what further restrictions on D can be imposed by polynomial-time preprocessing.

4.1 Preprocessing

Let (D, s, t) be an instance of ROOTED k -DISTINCT BRANCHINGS with $s \neq t$. Recall that D contains an out-branching (in-branching) if and only if D has only one strongly connected component with no incoming (no outgoing) arc. As a first preprocessing step, we can decide in polynomial time whether D has a rooted out-branching and a rooted in-branching. If not, we reject the instance. Note that this in particular means that in a non-rejected instance, every vertex in D is reachable from s and t is reachable from every vertex.

Next, we test for every arc $a \in D$ whether there exists at least one rooted in- or out-branching that uses a as follows: since a maximal-weight out- or in-branching for an arc-weighted digraph can be computed in polynomial time [14], we can force the arc a to be contained in a solution by assigning it a weight of 2 and every other arc weight 1. If we verify that a indeed neither appears in any rooted out-branching or in-branching, we remove a from D and obtain an equivalent instance of ROOTED k -DISTINCT BRANCHINGS.

After this polynomial-time preprocessing, our instance has the following properties: there exists a rooted out-branching, there exists a rooted in-branching, and every arc of D appears in some rooted in- or out-branching. We call such a digraph with a pair s, t *reduced*.

Lastly, by the following lemma we may assume that our instance is strongly connected by incurring a factor of two in the application of Lemma 4.

► **Lemma 8.** Let (D, s, t) be reduced and let D' be the digraph obtained from D by adding the arc ts to it unless ts is already in D in which case $D' = D$. Then (D', s, t) is a positive instance of ROOTED k -DISTINCT BRANCHINGS if and only if so is (D, s, t) . Furthermore, if D' contains an out-tree (in-tree) with at least ℓ leaves, then D contains an out-tree (in-tree) with at least $\ell/2$ leaves.

Proof. We may assume that $D' \neq D$. For the first claim, simply note that any rooted out-branching of D' cannot use the arc ts and the same holds for rooted in-branchings. For the second claim, assume T is an out-tree in D with ℓ leaves. Assume $ts \in T$ (otherwise the claim follows trivially). Let T_1 and T_2 be the two out-trees obtained by deleting the arc ts from T . Both are out-trees in D and one of them contains at least $\ell/2$ leaves, as claimed. ◀

The first claim of Lemma 8 shows that we may assume that D is strongly connected in ROOTED k -DISTINCT BRANCHINGS. This implies the following simple claim required for further references.

► **Lemma 9.** *Let D be an input digraph of ROOTED k -DISTINCT BRANCHINGS. Then every rooted out-tree with q leaves can be extended into a rooted out-branching with at least q leaves.*

In summary, we enforce the following properties for (D, s, t) by polynomial-time preprocessing:

1. Every arc of D is contained in at least one rooted in-branching or rooted out-branching,
2. D is strongly connected.

4.2 Decomposition and Reconfiguration

We work towards the following win-win scenario: either we find an out-tree with $\Theta(k)$ leaves that can be turned into a rooted out-tree with at least $k + 1$ leaves, or we conclude that every out-tree in D has less than $\Theta(k)$ leaves. We refer to the process of turning an out-tree into a rooted out-tree as a *reconfiguration*. In the process we will develop a new digraph decomposition, the *rooted cut-decomposition*, which will aid us in reasoning about reconfiguration steps and ultimately lead us to a solution for the problem.

To make the notion of a bi-reachable set easier to use, the decomposition will employ a slightly broader notion as follows.

► **Definition 10.** Let H be a digraph with at least two vertices, and let $r \in V(H)$ such that every vertex of H is reachable from r . Let $B \subseteq V(H)$ be the set of all vertices that are bi-reachable from r . The *directed block (diblock) B_r of r in H* is the set $B \cup N^+[r]$, i.e., the bi-reachable vertices together with all out-neighbors of r and r itself.

Note that according to the above definition a diblock must have at least two vertices.

The following statement provides us with an easy case in which a reconfiguration is successful, that is, we can turn an arbitrary out-tree into a rooted out-tree without losing too many leaves. Later, the obstructions to this case will be turned into building blocks of the decomposition.

► **Lemma 11.** *Let $B_s \subseteq V(D)$ be the diblock of s and let T be an out-tree of D whose root r lies in B_s with ℓ leaves. Then there exists a rooted out-tree with at least $(\ell - 1)/2$ leaves.*

Proof. We may assume that $r \neq s$. In case T contains s as a leaf, we remove s from T for the remaining argument and hence will argue about the $\ell - 1$ remaining leaves.

If r is bi-reachable from s , consider two internally vertex-disjoint paths P, Q from s to r . One of the two paths necessarily avoids half of the $\ell - 1$ leaves of T ; let without loss of generality this path be P . Let further L be the set of those leaves of T that do *not* lie on P . If $r \in N^+(s)$, let $P = sr$.

We construct the required out-tree T' as follows: first, add all arcs and vertices of P to T' . Now for every leaf $v \in L$, let P_v be the unique path from r to v in T and let P'_v be the segment of P_v from the last vertex x of P_v contained in T . Add all arcs and vertices

of P'_v to T' . Observe that $x \neq v$ as v cannot be in T' already. Since P_v and thus P'_v contains no leaf of L other than v , in the end of the process, all vertices of L are leaves of T' . Since $|L| \geq (\ell - 1)/2$, the claim follows. \blacktriangleleft

Note that the definition of diblocks can be understood in terms of network flows. Let $v \neq r$. Consider the vertex-capacitated version of H where r and v both have capacity 2, and every other vertex has capacity 1, for some $v \in V(H) \setminus \{r\}$. Then v is contained in the diblock of r in H if and only if the max-flow from r to v equals 2 (note that if $v \in N^+(r)$, then such a flow exists trivially since arcs have infinite capacity). Dually, by Menger's theorem, v is *not* contained in the diblock if and only if there is a vertex $u \notin \{r, v\}$ such that all r - v paths P intersect u . This has the following simple consequence regarding connectivity inside a diblock.

► **Lemma 12.** *Fix $r \in V(H)$ and let $B_r \subseteq V(H)$ be the diblock of r in H . Then for every pair of distinct vertices $x, y \in B_r$, there exist an r - x -path P_x and an r - y -path P_y that intersect only in r .*

Proof. If $r \in \{x, y\}$, then clearly the claim holds since every vertex in B_r is reachable from r . Otherwise, add a new vertex z with arcs xz and yz , and note that the lemma holds if and only if z is bi-reachable from r . If this is not true, then by Menger's theorem there is a vertex $v \in B_r$, $v \neq r$, such that all paths from r to z , and hence to x and y , go through v . But as noted above, there is no cut-vertex $v \notin \{x, r\}$ for r - x paths, and no cut-vertex $v \notin \{y, r\}$ for r - y paths. We conclude that z is bi-reachable from r , hence the lemma holds. \blacktriangleleft

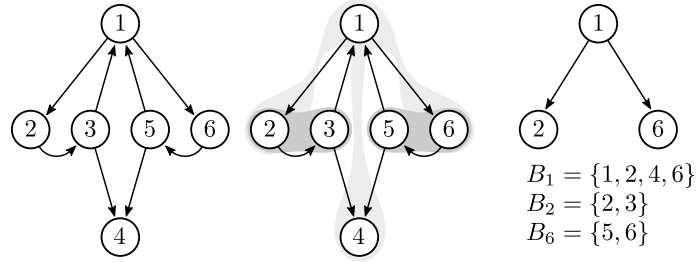
Next, we will use Lemma 12 to show that given a vertex r , the set of vertices not in the diblock B_r of r in H partitions cleanly around B_r .

► **Lemma 13.** *Let $r \in V(H)$ be given, such that every vertex of H is reachable from r . Let $B_r \subseteq V(H)$ be the diblock of r in H . Then $V(H) \setminus B_r$ partitions according to cut vertices in B_r , in the following sense: For every $v \in V(H) \setminus B_r$, there is a unique vertex $x \in B_r \setminus \{r\}$ such that every path from r to v intersects B_r for the last time in x . Furthermore, this partition can be computed in polynomial time.*

Proof. Assume towards a contradiction that for $v \in V(H) \setminus B_r$ there exist two r - v -paths P_1, P_2 that intersect B_r for the last time in distinct vertices x_1, x_2 , respectively. We first observe that $r \notin \{x_1, x_2\}$, since the second vertices of P_1 and P_2 are contained in B_r by definition. By Lemma 12, we may assume that $P_1[r, x_1] \cap P_2[r, x_2] = \{r\}$. But then P_1 and P_2 intersect for the first time outside of B_r in some vertex v' (potentially in $v' = v$). This vertex is, however, bi-reachable from r , contradicting our construction of B_r . Hence there is a vertex $x \in B_r$ such that every path from r to v intersects B_r for the last time in x , with $x \neq r$, and clearly this vertex is unique. Finally, the set B_r can be computed in polynomial time, and given B_r it is easy to compute for each $x \in B_r$ the set of all vertices $v \in V(H)$ (if any) for which x is a cut vertex. \blacktriangleleft

We refer to the vertices $x \in B_r$ that are cut vertices in the above partition as the *bottlenecks* of B_r . Note that r itself is not considered a bottleneck in B_r . Using these notions, we can now define a *cut decomposition* of a digraph H .

► **Definition 14** (Rooted cut decomposition and its tree). Let H be a digraph and r a vertex such that every vertex in H is reachable from r . The (r -rooted) *cut decomposition* of H is a pair (\hat{T}, \mathcal{B}) where \hat{T} is a rooted tree with $V(\hat{T}) \subseteq V(H)$ and $\mathcal{B} = \{B_x\}_{x \in \hat{T}}$, $B_x \subseteq V(H)$ for each $x \in \hat{T}$, is a collection of diblocks associated with the nodes of \hat{T} , defined and computed recursively as follows.



■ **Figure 2** An example of a rooted cut decomposition.

1. Let B_r be the diblock of r in H , and let $L \subseteq B_r \setminus \{r\}$ be the set of bottlenecks in B_r . Let $\{X_x\}_{x \in L}$ be the corresponding partition of $V(H) \setminus B_r$.
2. For every $x \in L$, let $(\hat{T}_x, \mathcal{B}_x)$ be the x -rooted cut decomposition of $H[X_x \cup \{x\}]$.
3. \hat{T} is the tree with root node r , where L is the set of children of r , and for every $x \in L$ the subtree of \hat{T} rooted at x is \hat{T}_x .
4. $\mathcal{B} = \{B_r\} \cup \bigcup_{x \in L} \mathcal{B}_x$.

Furthermore, for every node $x \in \hat{T}$, we define $B_x^* = \bigcup_{y \in \hat{T}_x} B_y$ as the set of all vertices contained in diblocks associated with nodes of the subtree \hat{T}_x .

Figure 2 provides an illustration to Definition 14.

► **Lemma 15.** *Let a digraph H and a root $r \in V(H)$ be given, such that every vertex of H is reachable from r . Then the r -rooted cut decomposition $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ of H is well-defined and can be computed in polynomial time. Furthermore, the diblocks cover $V(H)$, i.e., $\bigcup_{x \in \hat{T}} B_x = V(H)$, and for every node $x \in \hat{T}$, every vertex of B_x^* is reachable from x in $H[B_x^*]$.*

Proof. By Lemma 13, the root diblock B_r as well as the set $L \subseteq B_r$ of bottlenecks and the partition $\{X_x\}_{x \in L}$ are well-defined and can be computed in polynomial time. Also note that for each $x \in L$, $r \notin X_x \cup \{x\}$, and every vertex of $H_x := H[X_x \cup \{x\}]$ is reachable from x in H_x by the definition of the partition. Hence the collection of recursive calls made in the construction is well-defined, and every digraph H_x used in a recursive call is smaller than H , hence the process terminates. Finally, for any two distinct bottlenecks $x, y \in L$ we have $V(H_x) \cap V(H_y) = \emptyset$. Thereby, distinct nodes of \hat{T} are associated with distinct vertices of H , $|\hat{T}| \leq |V(H)|$, and the map $x \mapsto B_x$ is well-defined. It is also clear that the whole process takes polynomial time. ◀

We collect some basic facts about cut decompositions.

► **Lemma 16.** *Let a digraph H and a vertex $r \in V(H)$ be given, and let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be the r -rooted cut decomposition of H . Then the following hold.*

1. The sets $\{B_x \setminus \{x\}\}_{x \in \hat{T}}$ are all non-empty and partition $V(H) \setminus \{r\}$.
2. For distinct nodes $x, y \in \hat{T}$, if x is the parent of y in \hat{T} then $B_x \cap B_y = \{y\}$; in every other situation, $B_x \cap B_y = \emptyset$.
3. For every node $x \in \hat{T}$, the following hold:
 - (a) If y is a child of x in \hat{T} , then any arc leading into the set B_y^* from $V(H) \setminus B_y^*$ will have the form uy where $u \in B_x$.
 - (b) If y, y' are distinct children of x in \hat{T} , then there is no arc between B_y^* and $B_{y'}^*$.

In particular, every arc of H is either contained in a subgraph of H induced by a diblock B_x , or it is a back arc going from a diblock B_y to a diblock B_x , where x is an ancestor of y in \hat{T} .

Proof. For the first claim, the sets $B_x \setminus \{x\}$ are non-empty by definition; we show the partitioning claim. By Lemma 13, for every $v \in V(H) \setminus \{r\}$ either $v \in B_r \setminus \{r\}$ or there is exactly one bottleneck $x \in B_r$ such that $v \in X_x$ in the construction of the decomposition. Also note that in the latter case, $v \neq x$ since $x \in B_r$. Applying the argument recursively and using that the diblocks cover $V(H)$, by Lemma 15, we complete the proof of the partitioning claim.

For the second claim, the partitioning claim implies that if $v \in B_x \cap B_y$ for distinct nodes $x, y \in \hat{T}$, then either $v = x$ or $v = y$, i.e., v must be a bottleneck. This is only possible in the situation described.

For Claim 3(b), first consider the diblock B_r and the partition $\{X_x\}$ given by Lemma 13. We show that for any two distinct sets X_x, X_y of the partition, there is no arc between X_x and X_y . Suppose for a contradiction that there is such an arc uv , $u \in X_x$, $v \in X_y$. By Lemma 12, there are paths P_x and P_y in B_r that intersect only in r , and by Lemma 15, there are paths P_u from x to u in X_x and P_v from y to v in X_y . But then the paths $P_x P_u uv$ and $P_y P_v$ form two r - v paths that are internally vertex-disjoint, showing that $v \in B_r$, contrary to our assumptions. Since the decomposition is computed recursively, this also holds in every internal node of \hat{T} .

For Claim 3(a), let uv be an arc such that $u \notin B_y^*$ and $v \in B_y^*$. Moreover, let $u \in B_{x'}$ and $v \in B_{y'}$. By construction of cut decomposition, there is a path \hat{P} from x' to y' in \hat{T} containing nodes x and y . Let x'' be the second node in \hat{P} (just after x'). Thus, there is a path P from x'' to v in H containing the vertices of \hat{P} apart from x' .

Assume that $u \neq x''$. Then by Lemma 12, there is an x' - u -path P' and an x' - x'' -path P'' of H which intersect only at x' . Then $x'P'uv$ and $P''P$ are internally vertex-disjoint paths from x' to v . This means that v must be in $B_{x'}$, a contradiction unless $x' = x$, $u \in B_x$ and $v = y$. If $u = x''$, then P and uv are internally vertex-disjoint paths from u to v . This means that v must be in $B_{x''}$, a contradiction unless $x' = x$ and $v = y$. ◀

As we saw, for every diblock B_y , $y \in \hat{T}$, any path “into” the diblock must go via the bottleneck vertex y . By induction, for any $v \in B_y$, every node of \hat{T} from r to y represents a bottleneck vertex that is unavoidable for paths from r to v . More formally, the following holds in cut decompositions:

► **Lemma 17.** *Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be the cut decomposition of H rooted at r . Fix a diblock B_x for $x \in \hat{T}$. Consider a path P in H from r to $v \in B_x$ and let $x_1 \dots x_\ell$ be the sequence of bottleneck vertices that P encounters. Then $\hat{P} = x_0 x_1 \dots x_\ell$ with $x_0 = r$ is the path from r to x in \hat{T} .*

Proof. We prove the claim by induction over the depth d of x in \hat{T} . If $r = x$ then any path from r to $v \in B_r$ contains r itself and hence the base case for $d = 0$ holds.

Consider a diblock B_x , $x \in \hat{T}$ where x has distance d to r in \hat{T} and let y be the parent of x in \hat{T} . We assume the induction hypothesis holds for diblocks at depth $d - 1$, hence it holds for B_y in particular. Because $x \in B_y$, this implies that every path from r to x will contain all ancestors of x in \hat{T} . Since by construction every path from r to a vertex $v \in B_x$ needs to pass through x , the inductive step holds. This proves the claim. ◀

As an immediate consequence, we can identify arcs in cut decompositions that cannot participate in any rooted out-branching.

► **Corollary 18.** *Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be the cut decomposition of rooted at r and let $R := \{uv \in A(H) \mid u \in B_x \text{ and } x \in \hat{T}_v\}$ be all the arcs that originate in a diblock B_x and end in an ancestor v of x on \hat{T} . Then for every out-tree T rooted at r we have $A(T) \cap R = \emptyset$.*

Proof. Fix a bottleneck vertex $v \in \hat{T}$ of the decomposition and let the arc uv be in an out-tree T rooted at r . There must exist a path P_{su} from s to u that is part of T . By Lemma 17, this path will contain the vertex v . But then v is an ancestor of u in T and therefore the arc uv cannot be part of T , which is a contradiction. \blacktriangleleft

The decomposition actually restricts paths even further: a path that starts at the root and visits two bottleneck vertices x, y (in this order) cannot intersect any vertex of B_y^* before visiting y and cannot return to any set B_z^* , $z \in \hat{T}$, after having left it.

► **Lemma 19.** *Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be the cut decomposition of H rooted at r . Fix a diblock B_x for $x \in \hat{T}$. Consider a path P from r to $v \in B_x$ and let $\hat{P} = x_0 \dots x_\ell$ be the path from $r = x_0$ to $x = x_\ell$ in \hat{T} . Let further F_0, \dots, F_ℓ be the fins of \hat{P} in \hat{T} . Then the subpath $P[x_i, x_{i+1}] \setminus \{x_{i+1}\}$ is contained in the union of diblocks of F_i for $0 \leq i < \ell$.*

Proof. By Lemma 17 we know that the nodes of \hat{P} appear in P in the correct order, hence the subpath $P[x_i, x_{i+1}]$ is well-defined. Let us first show that the subpath $P[x_i, x_{i+1}] \setminus \{x_{i+1}\}$ cannot intersect any diblock associated with $\hat{T}_{x_{i+1}}$. By Lemma 16, the only arcs from B_{x_i} into diblocks below x_{i+1} connect to the bottleneck x_{i+1} itself. Since x_{i+1} is already the endpoint of $P[x_i, x_{i+1}]$, this subpath cannot intersect the diblocks of $\hat{T}_{x_{i+1}}$. This already proves the claim for x_0 ; it remains to show that it does not intersect diblocks of $V(\hat{T}) \setminus V(\hat{T}_{x_i})$ for $i \geq 1$. The reason is similar: since the bottleneck x_i is already part of $P[x_i, x_{i+1}]$, this subpath could not revisit B_{x_i} if it enters any diblock B_y for a proper ancestor y of x_i in \hat{T} . We conclude that therefore it must be, with the exception of the vertex x_{i+1} , inside the diblocks of the fin F_i . \blacktriangleleft

► **Corollary 20.** *For every vertex $u \in V(H)$ and every set $X \subseteq V(H) \setminus (V(\hat{T}) \cup \{u\})$ of non-bottleneck vertices there exists a path P from r to u such that $|P \cap X| \leq |X|/2$.*

Proof. Assume that $u \in B_x$ and let $\hat{P} = x_0 \dots x_\ell$ be a path from $x_0 = r$ to $x_\ell = x$ in \hat{T} . Let further F_0, \dots, F_ℓ be the fins of \hat{P} in \hat{T} . We partition the set X into X_1, \dots, X_ℓ where $X_i = X \cap F_i$ for $0 \leq i \leq \ell$. Lemma 19 allows us to construct the path P iteratively: any path that leads to u will pass through bottlenecks x_i, x_{i+1} in succession and visit only diblocks associated with F_i in the process. Since there are two internally vertex-disjoint paths between x_i, x_{i+1} for $1 \leq i \leq \ell$, we can always choose the path that has the smaller intersection with X_i . Stringing these paths together, we obtain the claimed path P . \blacktriangleleft

We want to argue that one of the following cases must hold: either the cut decomposition has bounded height and we can ‘re-root’ any out-tree with many leaves into a rooted out-tree with a comparable number of leaves, or we can directly construct a rooted out-tree with many leaves. In both cases we apply Lemmas 4 and 9 to conclude that the instance has a solution. This approach has one obstacle: internal diblocks of the decomposition that contain only two vertices.

► **Definition 21** (Degenerate diblocks). Let $\{B_x\}_{x \in \hat{T}}$ be the cut decomposition rooted at s . We call a diblock B_x *degenerate* if x is an internal node of \hat{T} and $|B_x| = 2$.

Let us first convince ourselves that a long enough sequence of non-degenerate diblocks provides us with a rooted out-branching with many leaves.

► **Lemma 22.** *Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be the cut decomposition rooted at s of H and let y be a node in \hat{T} such that the path \hat{P}_{sy} from s to y in \hat{T} contains at least ℓ nodes whose diblocks are non-degenerate. Then H contains an out-tree rooted at s with at least ℓ leaves.*

Proof. We construct an s -rooted out-tree T by repeated application of Lemma 12. Let x_1, \dots, x_ℓ be a sequence of nodes in \hat{P}_{sy} whose diblocks are non-degenerate, and for each $1 \leq i < \ell$ let x_i^+ be the node after x_i in \hat{P}_{sy} . We construct a sequence of s -rooted out-trees T_1, \dots, T_ℓ such that for $1 \leq i \leq \ell$, the vertex x_i is a leaf of T_i , and T_i contains i leaves. First construct T_1 as a path from s to x_1 , then for every $1 \leq i < \ell$ we construct an out-tree T_{i+1} from T_i as follows. Let $v_i \in B_{x_i} \setminus \{x_i, x_i^+\}$, which exists since B_{x_i} is non-degenerate, and let $P_{x_i x_i^+}, P_{x_i v_i}$ be a pair of paths in $H[B_{x_i}^*]$ from x_i to x_i^+ and to v_i respectively, which intersect only in x_i . Such paths exist by Lemma 12, and since x_i is a leaf of T_i , Lemma 17 implies that T_i is disjoint from $B_{x_i}^* \setminus \{x_i\}$. Hence the paths can be appended to T_i to form a new r -rooted out-tree T_{i+1} in H which contains a leaf in every diblock B_{x_j} , $1 \leq i$. Finally, note that the final tree T_ℓ contains two leaves in $B_{x_{\ell-1}}$, hence T_ℓ is an r -rooted out-tree with ℓ leaves. \blacktriangleleft

The next lemma is the last assertion that we will use to prove shortly that ROOTED k -DISTINCT BRANCHINGS is FPT for digraphs D whose cut decomposition rooted at s contains no degenerate diblocks.

► **Lemma 23.** *Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be the cut decomposition of D rooted at s such that \hat{T} is of height d and let T be an out-tree rooted at some vertex r with ℓ leaves. Then we can construct an out-tree T_s rooted at s with at least $(\ell - d)/2$ leaves.*

Proof. Assume that r is contained in the diblock B_x of the decomposition and let $x_p \dots x_1 = \hat{P}_{sx}$ be a path from $s = x_p$ to $x = x_1$ in \hat{T} . Let L be the leaves of T and let $L' := L \setminus \hat{P}_{sx}$. Clearly, $|L'| \geq \ell - d$. Applying Corollary 20 with $X = L'$ and $u = r$, we obtain a path P_{sr} in D from s to r that avoids half of L' . We construct T_s in a similar fashion to the proof of Lemma 11. We begin with $T_s = P_{sr}$, then for every leaf $v \in L' \setminus P_{sr}$, proceed as follows: let P_v be the unique path from r to v in T and let P'_v be the segment of P_v from the last vertex x of P_v contained in T_s . Add all arcs and vertices of P'_v to T_s . Since P_v and thus P'_v contains no leaf of L' other than v , in the end of the process, all vertices of $L' \setminus P_{sr}$ are leaves of T_s . Since $|L' \setminus P_{sr}| \geq |L'|/2$, we conclude that T_s contains at least $(\ell - d)/2$ leaves, as claimed. \blacktriangleleft

The next lemma demonstrates that using Lemma 23 and a number of other results we can prove that if the height d of the cut decomposition of D is upper-bounded by a function in k , then ROOTED k -DISTINCT BRANCHINGS on D is FPT. This shows that to prove that ROOTED k -DISTINCT BRANCHINGS in general it suffices to consider separately the cases of bounded d and unbounded d . To provide an appropriate bound on d we will use further results on degenerate diblocks proved in Section 4.3.

► **Lemma 24.** *Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ the cut decomposition rooted at s of height d . If $d \leq d(k)$ for some function $d(k) = \Omega(k)$ of k only, then we can solve ROOTED k -DISTINCT BRANCHINGS on D in time $O^*(2^{O(d(k) \log^2 d(k))})$.*

Proof. By Theorem 2, in time $O^*(2^{O(d(k))})$ we can decide whether D has an out-branching with at least $2k + 2 + d(k)$ leaves. If D has such an out-branching, then by Lemma 23 D has a rooted out-tree with at least $k + 1$ leaves. This out-tree can be extended to a rooted out-branching with at least $k + 1$ leaves by Lemma 9. So by Lemma 4, (D, s, t) is a positive instance if and only if D has a rooted in-branching, which can be decided in polynomial time.

If D has no out-branching with at least $2k + 2 + d(k)$ leaves, by Theorem 1 the pathwidth of D is $O(d(k) \log d(k))$ and thus by Lemma 5 we can solve ROOTED k -DISTINCT BRANCHINGS on D in time $O^*(2^{O(d(k) \log^2 d(k))})$. (Note that for the dynamic programming algorithm of

Lemma 5 we may fix roots of all out-branchings and all in-branchings of D by adding arcs $s's$ and tt' to D , where s' and t' are new vertices.) ◀

4.3 Handling degenerate diblocks

A full version of this section can be found in the complete version of the paper. Here is a key notion for our study of degenerate diblocks.

► **Definition 25** (Degenerate paths). Let $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ be a cut decomposition of D . We call a path \hat{P} in \hat{T} *monotone* if it is a subpath of a path from the root of \hat{T} to some leaf of \hat{T} . We call a path \hat{P} in \hat{T} *degenerate* if it is monotone and every diblock B_x , $x \in \hat{P}$ is degenerate.

Let (D, s, t) be a strongly connected reduced instance of ROOTED k -DISTINCT BRANCHINGS. As observed in Section 4.1, we can verify in polynomial time whether an arc participates in *some* rooted in- or out-branching. Let $R_s \subseteq A(D)$ be those arcs that do not participate in any rooted out-branching and $R_t \subseteq A(D)$ those that do not participate in any rooted in-branching. Since (D, s, t) is a reduced instance, we necessarily have that $R_s \cap R_t = \emptyset$.

► **Lemma 26.** *Let $\hat{P} = x_1 \dots x_\ell$ be a degenerate path of $(\hat{T}, \{B_x\}_{x \in \hat{T}})$ of D rooted at s . Then the following properties hold: every rooted out-branching contains $A(\hat{P})$, every arc $x_j x_i$ with $j > i$ is contained in R_s , and there is no arc from x_i ($i < \ell$) to B_y in D , where y is a descendant of x_i on \hat{T} , except for the arc $x_i x_{i+1}$.*

Let us fix a single degenerate path $\hat{P} = x_1 \dots x_\ell$. We categorize the arcs incident to \hat{P} as follows: let A^+ contain all ‘upward arcs’ that originate in \hat{P} and end in some diblock B_y where y is an ancestor of x_1 , let A^0 contain all ‘on-path arcs’ $x_j x_i$, $j > i$, and let A^- contain all ‘arcs from below’ that originate from some diblock B_y where y is a (not necessarily proper) descendant of x_ℓ . By the lemma above this categorization is complete.

We will need the following reduction rules for (D, s, t) :

Reduction Rule 1: If there are two arcs $x_i u, x_j u \in A^+ \cap R_t$ with $i < j$, remove $x_j u$.

Reduction Rule 2: If $\hat{P}[x, y] \subseteq \hat{P}$ is such that no vertex in $\hat{P}[x, y]$ is a tail of arcs in $A^+ \cup A^0$, contract $\hat{P}[x, y]$ into a single vertex.

Now we can state the main lemma of the section which finally enables us to proof the main result of this paper.

► **Lemma 27.** *Let \hat{P} be a degenerate path in an instance reduced with respect to Rules 1 and 2. If $t \notin \hat{P}$, then $|\hat{P}| \leq 14k + 3$. Otherwise, $|\hat{P}| \leq 28k + 7$.*

Proof of Theorem 7. By Lemma 8, we may assume that D is strongly connected. Consider the longest monotone path \hat{P} of \hat{T} . By Lemma 22, if \hat{P} has at least $k + 1$ non-degenerate diblocks, then D has a rooted out-tree with at least $k + 1$ leaves. This out-tree can be extended to a rooted out-branching with at least $k + 1$ leaves by Lemma 9. Thus, by Lemma 4, (D, s, t) is a positive instance if and only if D has a rooted in-branching, which can be decided in polynomial time.

Now assume that \hat{P} has at most k non-degenerate diblocks. By Lemma 27 we may assume that before, between and after the non-degenerate diblocks there are $O(k)$ degenerate diblocks. Thus, the height of \hat{T} is $O(k^2)$. Therefore, by Lemma 24, the time complexity for Theorem 7 is $O^*(2^{O(k^2 \log^2 k)})$. ◀

5 Conclusion

We showed that the ROOTED k -DISTINCT BRANCHINGS problem is FPT for general digraphs parameterized by k , thereby settling open question of Bang-Jensen *et al.* [6]. The solution in particular uses a new digraph decomposition, the *rooted cut decomposition*, that we believe might be useful for settling other problems as well. We did not try to optimize the running time of the algorithm of Theorem 7. Perhaps, a more careful handling of degenerate diblocks may lead to an algorithm of running time $O^*(2^{O(k \log^2 k)})$.

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