Split Contraction: The Untold Story*

Akanksha Agrawal[1], Daniel Lokshtanov[2], Saket Saurabh[3], and Meirav Zehavi[4]

1 **Department of Informatics, University of Bergen, Bergen, Norway**
`akanksha.agrawal@uib.no`

1 **Department of Informatics, University of Bergen, Bergen, Norway**
`daniello@uib.no`

3 **Department of Informatics, University of Bergen, Bergen, Norway; and Institute of Mathematical Sciences, Chennai, India**
`saket@imsc.res.in`

4 **Department of Informatics, University of Bergen, Bergen, Norway**
`meirav.zehavi@uib.no`

Abstract

The edit operation that contracts edges, which is a fundamental operation in the theory of graph minors, has recently gained substantial scientific attention from the viewpoint of Parameterized Complexity. In this paper, we examine an important family of graphs, namely the family of split graphs, which in the context of edge contractions, is proven to be significantly less obedient than one might expect. Formally, given a graph G and an integer k, Split Contraction asks whether there exists $X \subseteq E(G)$ such that G/X is a split graph and $|X| \leq k$. Here, G/X is the graph obtained from G by contracting edges in X. It was previously claimed that Split Contraction is fixed-parameter tractable. However, we show that Split Contraction, despite its deceptive simplicity, is W[1]-hard. Our main result establishes the following conditional lower bound: under the Exponential Time Hypothesis, Split Contraction cannot be solved in time $2^{o(\ell^2)} \cdot n^{\mathcal{O}(1)}$ where ℓ is the vertex cover number of the input graph. We also verify that this lower bound is essentially tight. To the best of our knowledge, this is the *first* tight lower bound of the form $2^{o(\ell^2)} \cdot n^{\mathcal{O}(1)}$ for problems parameterized by the vertex cover number of the input graph. In particular, our approach to obtain this lower bound borrows the notion of harmonious coloring from Graph Theory, and might be of independent interest.

1998 ACM Subject Classification G.2.2 Graph Algorithms, I.1.2 Analysis of Algorithms

Keywords and phrases Split Graph, Parameterized Complexity, Edge Contraction

Digital Object Identifier 10.4230/LIPIcs.STACS.2017.5

1 Introduction

Graph modification problems have been extensively studied since the inception of Parameterized Complexity in the early 90's. The input of a typical graph modification problem consists of a graph G and a positive integer k, and the objective is to edit k vertices (or edges) so that the resulting graph belongs to some particular family, $\mathcal{F}$, of graphs. These problems are not only mathematically and structurally challenging, but have also led to the discovery of several important techniques in the field of Parameterized Complexity. It would be completely appropriate to say that solutions to these problems played a central role in the growth of the

* The research leading to these results has received funding from the European Research Council (ERC) via grant PARAPPROX, reference 306992.

34th Symposium on Theoretical Aspects of Computer Science (STACS 2017).
Editors: Heribert Vollmer and Brigitte Vallée; Article No. 5; pp. 5:1–5:14
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

field. In fact, just over the course of the last couple of years, parameterized algorithms have been developed for Chordal Editing [9], Unit Interval Editing [7], Interval Vertex (Edge) Deletion [10, 8], Proper Interval Completion [3], Interval Completion [4] Chordal Completion [20], Cluster Editing [19], Threshold Editing [16], Chain Editing [16], Trivially Perfect Editing [17, 15] and Split Editing [21]. This list is not comprehensive but rather illustrative.

The focus of all of these papers, and in fact, of the vast majority of papers on parameterized graph editing problems, has so far been limited to edit operations that delete vertices, delete edges or add edges. Using a different terminology, these problems can also be phrased as follows. For some particular family of graphs, $\mathcal{F}$, we say that a graph G belongs to $\mathcal{F} + kv$, $\mathcal{F} + ke$ or $\mathcal{F} - ke$ if some graph in $\mathcal{F}$ can be obtained by deleting at most k vertices from G, deleting at most k edges from G or adding at most k edges to G, respectively. Recently, a methodology for proving lower bounds on running times of algorithms for such parameterized graph editing problems was proposed by Bliznets et al. [2]. Furthermore, a well-known result by Cai [5] states that in case $\mathcal{F}$ is a hereditary family of graphs with a finite set of forbidden induced subgraphs, then the graph modification problem defined by $\mathcal{F}$ and the aforementioned edit operations admits a simple FPT algorithm.

In recent years, a different edit operation has begun to attract significant scientific attention. This operation, which is arguably the most natural edit operation apart from deletions/insertions of vertices/edges, is the one that contracts an edge. Here, given an edge (u, v) that exists in the input graph, we remove the edge from the graph and merge its two endpoints. Edge contraction is a fundamental operation in the theory of graph minors. Using our alternative terminology, we say that a graph G belongs to $\mathcal{F}/ke$ if some graph in $\mathcal{F}$ can be obtained by contracting at most k edges in G.[1] Then, given a graph G and a positive integer k, $\mathcal{F}$-Edge Contraction asks whether G belongs to $\mathcal{F}/ke$. For several families of graphs $\mathcal{F}$, early papers by Watanabe et al. [34, 35] and Asano and Hirata [1] showed that $\mathcal{F}$-Edge Contraction is NP-complete. In the framework of Parameterized Complexity, these problems exhibit properties that are quite different from those of problems where we only delete or add vertices and edges. Indeed, for these problems, the result by Cai [5] does not hold. In particular, Lokshtanov et al. [31] and Cai and Guo [6] independently showed that if $\mathcal{F}$ is either the family of P_ℓ-free graphs for some $\ell \geq 5$ or the family of C_ℓ-free graphs for some $\ell \geq 4$, then $\mathcal{F}$-Edge Contraction is W[2]-hard.

To the best of our knowledge, Heggernes et al. [26] were the first to explicitly study $\mathcal{F}$-Edge Contraction from the viewpoint of Parameterized Complexity. They showed that in case $\mathcal{F}$ is the family of trees, $\mathcal{F}$-Edge Contraction is FPT but does not admit a polynomial kernel, while in case $\mathcal{F}$ is the family of paths, the corresponding problem admits a faster algorithm and an $\mathcal{O}(k)$-vertex kernel. Golovach et al. [22] proved that if $\mathcal{F}$ is the family of planar graphs, then $\mathcal{F}$-Edge Contraction is again FPT. Moreover, Cai and Guo [6] showed that in case $\mathcal{F}$ is the family of cliques, $\mathcal{F}$-Edge Contraction is solvable in time $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$, while in case $\mathcal{F}$ is the family of chordal graphs, the problem is W[2]-hard. Heggernes et al. [25] developed an FPT algorithm for the case where $\mathcal{F}$ is the family of bipartite graphs. Later, a faster algorithm was proposed by Guillemot and Marx [23].

The recent paper [24] studied the case where $\mathcal{F}$ is the family of split graphs, which corresponds to the following problem.

[1] Here, it might be more appropriate to replace / (in $\mathcal{F}/ke$) by the operation opposite to edge contraction, but we believe that the current notation is clearer.

Split Contraction	**Parameter:** k
Input: A graph G and an integer k.	
Question: Does there exist $X \subseteq E(G)$ such that G/X is a split graph and $\|X\| \leq k$?	

The paper [24] claimed to design an algorithm that solves Split Contraction in time $2^{\mathcal{O}(k^2)} \cdot n^{\mathcal{O}(1)}$, which proves that the problem is FPT. Our initial objective was to either speed-up this algorithm or obtain a tight conditional lower bound. In fact, it seemed plausible that Split Contraction, like $\mathcal{F}$-Edge Contraction where $\mathcal{F}$ is the family of cliques, is solvable in time $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$. The algorithm in [24] first computes a set of vertices of small size whose removal renders the graph into a split graph. Then, it is based on case distinction. In case the remaining graph contains a large clique, the problem is solved in time $2^{\mathcal{O}(k \log k)} \cdot n^{\mathcal{O}(1)}$, and otherwise it is solved in time $2^{\mathcal{O}(k^2)} \cdot n^{\mathcal{O}(1)}$. In particular, in case the clique is small, the minimum size of a vertex cover of the input graph is small—it can be bounded by $\mathcal{O}(k)$. Thus, the bottleneck of the proposed algorithm is captured by graphs having small vertex covers. Interestingly, our first main result, given in Section 3, proves that it is unlikely to overcome the difficulty imposed by such graphs.

▶ **Theorem 1.** *Unless the* ETH *fails,* Split Contraction *parameterized by ℓ, the size of a minimum vertex cover of the input graph, does not have an algorithm running in time $2^{o(\ell^2)} \cdot n^{\mathcal{O}(1)}$. Here, n denotes the number of vertices in the input graph.*

To the best of our knowledge, under the Exponential Time Hypothesis (ETH) [12, 27], this is the *first* tight lower bound of this form for problems parameterized by the vertex cover number of the input graph. Lately, there has been increasing scientific interest in the examination of lower bounds of forms other than $2^{o(s)} \cdot n^{\mathcal{O}(1)}$ for some parameters s. For example, lower bounds that are "slightly super-exponential", i.e. of the form $2^{o(s \log s)} \cdot n^{\mathcal{O}(1)}$ for various parameters s, have been studied in [30]. Cygan et al. [13] obtained a lower bound of the form $2^{2^{o(k)}} \cdot n^{\mathcal{O}(1)}$, where k is the solution size, for the Edge Clique Cover problem. Very recently, Marx and Mitsoue [32] have further obtained lower bounds of the forms $2^{2^{o(w)}} \cdot n^{\mathcal{O}(1)}$ and $2^{2^{2^{o(w)}}} \cdot n^{\mathcal{O}(1)}$, where w is the treewidth of the input graph, for choosability problems.

In order to derive our main result, we make use of a partitioning of the vertex set $V(G)$ into independent sets $C_1, \ldots, C_t$ such that for each $i, j \in [t]$, $i \neq j$, $|E(G[C_i \cup C_j]) \cap E(G)| \leq 1$. Essentially, this coloring can be viewed as a proper coloring $f : V(G) \to [t]$ with the additional property that between any two color classes we have at most one edge. (Here, we use the standard notation $[t] = \{1, 2, \ldots, t\}$.) This kind of coloring, called *harmonious coloring* [29, 33, 18], has been studied extensively in the literature. We are not aware of uses of harmonious coloring in deriving lower bound results and believe that this approach could be of independent interest.

After we had established Theorem 1, we took a closer look at the paper [24], and were not able to verify some of their arguments. We next prove that unless FPT=W[1]-hard, the algorithm in [24] is incorrect, as the problem is W[1]-hard (Section 4).

▶ **Theorem 2.** Split Contraction *parmeterized by the size of a solution is* W[1]-hard.

We find this result surprising: one might a priori expect that "contraction to split graphs" should be easy as split graphs have structures that seem relatively simple. Indeed, many NP-hard problems admit simple polynomial-time algorithms if restricted to split graphs. Consequently, our result can also be viewed as a strong evidence of the inherent complexity of the edge operation which contracts edges. Furthermore, some of the ideas underlying the constructions of this reduction, such as the exploitation of properties of a special case of the Perfect Code problem to analyze budget constraints involving edge contractions, might be used to establish other W[1]-hard results for problems of similar flavors. We remark that despite errors in the paper [24], it can be verified that the lower bound given by Theorem 1 is tight. For the sake of completeness, we design a standalone FPT algorithm for Split Contraction that runs in time $2^{\mathcal{O}(\ell^2)} \cdot n^{\mathcal{O}(1)}$, the details of which are omitted due to space constraints.

2 Preliminaries

We consider only finite simple graphs. A *split graph* is a graph G whose vertex set $V(G)$ can be partitioned into two sets, A and B, such that $G[A]$ is a clique while B is an independent set, i.e. $G[B]$ is an edgeless graph. We say that two disjoint vertex subsets, say $S, S' \subseteq V(G)$, are *adjacent* if there exist $v \in S$ and $v' \in S'$ such that $(v, v') \in E(G)$. Further, an edge $(u, v) \in E(G)$ is *between* S, S' if $u \in S$ and $v \in S'$ (or $v \in S$ and $u \in S'$). For $(v, u) \in E(G)$, the result of *contracting* the edge (v, u) in G is the graph obtained by the following operation. We add a vertex $vu^\star$ and make it adjacent to the vertices in $(N(v) \cup N(u)) \setminus \{v, u\}$, and delete v, u from the graph. We often call such an operation a *contraction* of the edge (v, u). For $E' \subseteq E(G)$, we denote by G/E' the graph obtained by contracting the edges of E' in G.

A graph G is *isomorphic* to a graph H if there exists a *bijective* function $\phi : V(G) \rightarrow V(H)$ such that for $v, u \in V(G)$, $(v, u) \in E(G)$ if and only if $(\phi(v), \phi(u)) \in E(H)$. A graph G is *contractible* to a graph H if there exists $E' \subseteq E(G)$ such that G/E' is isomorphic to H. In other words, G is contractible to H if there exists a *surjective* function $\varphi : V(G) \rightarrow V(H)$ with the following properties.

- For all $h, h' \in V(H)$, $(h, h') \in E(H)$ if and only if $W(h), W(h')$ are *adjacent* in G. Here, $W(h) = \{v \in V(G) \mid \varphi(v) = h\}$.
- For all $h \in V(H)$, $G[W(h)]$ is *connected*.

Let $\mathcal{W} = \{W(h) \mid h \in V(H)\}$. Observe that $\mathcal{W}$ defines a partition of the vertex set of G. We call $\mathcal{W}$ a *H-witness structure* of G. The sets in $\mathcal{W}$ are called *witness-sets*.

3 Lower Bound for Split-Contraction Parameterized by Vertex Cover

In this section we show that unless the ETH fails, Split Contraction does not admit an algorithm running in time $2^{o(\ell^2)} n^{\mathcal{O}(1)}$, where ℓ is the size of a minimum vertex cover of the input graph G on n vertices. We complement it by designing an algorithm (whose details are omitted) for Split Contraction parameterized by ℓ, running in time $2^{\mathcal{O}(\ell^2)} n^{\mathcal{O}(1)}$.

To obtain our lower bound, we give an appropriate reduction from Vertex Cover on sub-cubic graphs. For this we utilize the fact that Vertex Cover on sub-cubic graphs does not have an algorithm running in time $2^{o(n)} n^{\mathcal{O}(1)}$ unless the ETH fails [27, 28]. For the ease of presentation we split the reduction into two steps. The first step comprises of reducing a special case of Vertex Cover on sub-cubic graphs, which we call Sub-Cubic Partitioned Vertex Cover (Sub-Cubic PVC) to Split Contraction. In the second step we show that there does not exist an algorithm for Sub-Cubic PVC running in time $2^{o(n)} n^{\mathcal{O}(1)}$ for Sub-Cubic PVC. We remark that the reduction from Vertex Cover on sub-cubic graphs to Sub-Cubic PVC is a Turing reduction.

3.1 Reduction from Sub-Cubic Partitioned Vertex Cover to Split Contraction

In this section we give a reduction from SUB-CUBIC PARTITIONED VERTEX COVER to SPLIT CONTRACTION. Next we formally define SUB-CUBIC PARTITIONED VERTEX COVER.

> SUB-CUBIC PARTITIONED VERTEX COVER (SUB-CUBIC PVC)
> **Input:** A sub-cubic graph G; an integer t; for $i \in [t]$, an integer $k_i \geq 0$; a partition $\mathcal{P} = \{C_1, \ldots, C_t\}$ of $V(G)$ such that $t \in \mathcal{O}(\sqrt{|V(G)|})$ and for all $i \in [t]$, C_i is an independent set and $|C_i| \in \mathcal{O}(\sqrt{|V(G)|})$. Furthermore, for $i, j \in [t], i \neq j$, $|E(G[C_i \cup C_j]) \cap E(G)| = 1$.
> **Question:** Does G have a vertex cover X such that for all $i \in [t]$, $|X \cap C_i| \leq k_i$?

We first explain (informally) the ideas behind our reduction. Let X be a *hypothetical* vertex cover we are looking for. Recall that we assume the ETH holds and thus we are allowed to use $2^{o(n)} n^{\mathcal{O}(1)}$ time to obtain our reduction. We will use this freedom to design our reduction and to construct an instance (G', k') of SPLIT CONTRACTION. For $i \in [t]$, in $V(G')$, we have a *vertex* corresponding to each possible intersection of X with C_i on at most k_i vertices. Further, we have a vertex $c_i \in V(G')$ corresponding to each C_i, for $i \in [t]$. We want to make sure that for each $(u, v) \in E(G)$, we choose an edge of $E(G')$ (in the solution to SPLIT CONTRACTION) that is incident to a vertex which corresponds to a subset containing one of u or v and one of c_i or c_j. Furthermore, we want to force these selected vertices to be contracted to the clique side in the resulting split graph. We crucially exploit the fact that there is exactly one edge between every C_i, C_j pair, where $i, j \in [t], i \neq j$. Finally, we will add a clique, say Γ, of size $3t$ and make each of its vertices adjacent to many pendant vertices, which ensures that after contracting the solution edges, the vertices of Γ remain in the clique side. We will assign appropriate adjacencies between the vertices of Γ and c_i, for $i \in [t]$. This will guide us in selecting edges for the solution of the contraction problem. We now move to the formal description of the construction used in the reduction.

Construction. Let $(G, \mathcal{P} = \{C_1, C_2, \ldots C_t\}, k_1, \ldots k_t)$ be an instance of SUB-CUBIC PVC and $n = |V(G)|$. We create an instance of SPLIT CONTRACTION (G', k') as follows. For $i \in [t]$, let $S_i = \{v_Y \mid Y \subseteq C_i \text{ and } |Y| \leq k_i\}$. That is, S_i comprises of vertices corresponding to subsets of C_i of size at most k_i. For each $i \in [t]$, we add five vertices b_i, c_i, x_i, y_i, z_i to $V(G')$. The vertices $\{x_i, y_i, z_i \mid i \in [t]\}$ induce a clique (on $3t$ vertices) in G'. We add the edges $(b_i, s_Y), (c_i, s_Y), (x_i, s_Y), (y_i, s_Y), (z_i, s_Y)$ for all $s_Y \in S_i$ to $E(G')$. For $i, j \in [t]$, $i \neq j$, we add the edges $(c_i, x_j), (c_i, y_j), (c_i, z_j)$ to $E(G')$. For $i, j \in [t]$, $i \neq j$ and $s_Y \in S_j$, we add the edge (c_i, s_Y) in $E(G')$ if and only if Y covers the unique edge between C_i and C_j. For all $i \in [t]$, we add $4t + 2$ pendant vertices, b'^i_j, $j \in [4t + 2]$, to b_i. Similarly, for all $i \in [t]$, we add $4t + 2$ pendant vertices c'^i_j, x'^i_j, y'^i_j, and z'^i_j, $j \in [4t + 2]$, to c_i, x_i, y_i and z_i, respectively. The pendant vertices are added in order to make sure that the vertices resulting after the contraction of their witness sets belong to the clique side. This completes the construction of the graph G'. Observe that $\{b_i, c_i, x_i, y_i, z_i \mid i \in [t]\}$ forms a minimum vertex cover of G' of size $5t$. Finally, we set $k' = 2t$. The resulting instance of SPLIT CONTRACTION is (G', k'). We refer the reader to Figure 1 for an illustration of the construction.

In the next few lemmata (Lemmata 3 to 8) we prove certain properties of the instance (G', k') of SPLIT CONTRACTION. This will be helpful later for establishing the equivalence between the original instance $(G, \mathcal{P} = \{C_1, C_2, \ldots C_t\}, k_1, \ldots k_t)$ of SUB-CUBIC PVC and the instance (G', k') of SPLIT CONTRACTION. In Lemmas 3 to 8 we will use the following notations. We use T to denote a solution to SPLIT CONTRACTION in (G', k') and $H = G'/T$

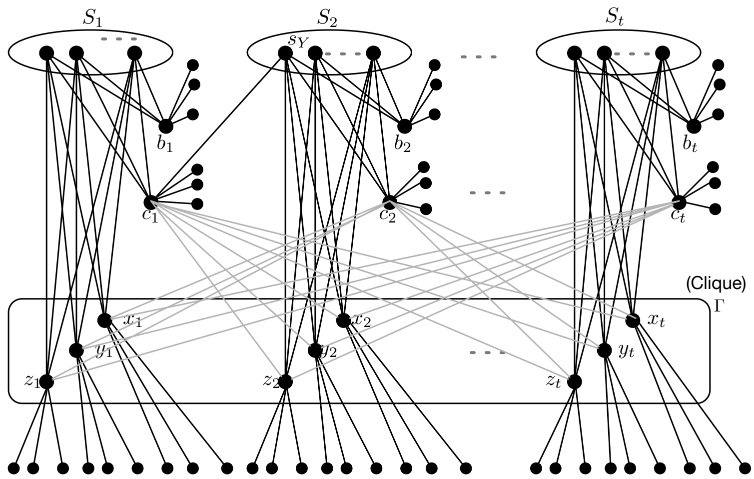

Figure 1 Reduction from Sub-Cubic PVC to Split Contraction.

with $\hat{C}$, $\hat{I}$ being a partition of $V(H)$ inducing a clique and an independent set, respectively, in H. We let $\varphi : V(G') \to V(H)$ be the surjective function defining the contractibility of G' to H, and $\mathcal{W}$ be the H-*witness structure* of G'.

▶ **Lemma 3.** *Let* (G', k') *be a* YES *instance of* Split Contraction*. Then, for all* $v \in \{b_i, c_i, x_i, y_i, z_i \mid i \in [t]\}$*, we have* $\varphi(v) \in \hat{C}$*.*

Proof. Consider $v \in \{b_i, c_i, x_i, y_i, z_i \mid i \in [t]\}$. Recall that there are $4t + 2 = 2k' + 2$ pendant vertices $v_j'^i$, for $j \in [2k' + 2]$ adjacent to v. At most k' edges in $\{(v_j'^i, v) \mid j \in [2k' + 2]\}$ can belong to T. Therefore, there exist $j_1, j_2 \in [2k' + 2]$, $j_1 \neq j_2$ such that no edge incident to $v_{j_1}'^i$ or $v_{j_2}'^i$ is in T. In other words, for $h_1 = \varphi(v_{j_1}'^i)$ and $h_2 = \varphi(v_{j_2}'^i)$, $W(h_1)$ and $W(h_2)$ are singleton sets. Since $\mathcal{W}$ is a H-witness structure of G', $(h_1, h_2) \notin E(H)$. Therefore, at least one of h_1, h_2 belongs to $\hat{I}$, say $h_1 \in \hat{I}$. This implies that $\varphi(v) \in \hat{C}$. ◀

▶ **Lemma 4.** *Let* (G', k') *be a* YES *instance of* Split Contraction*. Then, for all* $i \in [t]$*, there exists* $s_{Y_i} \in S_i$ *such that* $(b_i, s_{Y_i}) \in T$*.*

Proof. Towards a contradiction assume that there is $i \in [t]$ such that for all $s_Y \in S_i$, $(b_i, s_Y) \notin T$. Recall that $N_{G'}(b_i) = S_i \cup \{b_j'^i \mid j \in [4t + 2]\}$. Let $h = \varphi(b_i)$ and $A = \{b_j, c_j, x_j, y_j, z_j \mid j \in [t], j \neq i\}$. There exists $v \in A$ such that $|W(h')| = 1$, where $h' = \varphi(v)$. This follows from the fact that at most $2k' = 4t$ vertices in A can be incident to an edge in T, although $|A| = 5(t-1) > 4t$, as t can be assumed to be larger than 6, else the graph has constantly many edges and we can solve the problem in polynomial time. From Lemma 3 it follows that $(h, h') \in E(H)$, but $W(h), W(h')$ are not adjacent in G', contradicting that $\mathcal{W}$ is an H-*witness structure* of G'. Hence the claim follows. ◀

For each $i \in [t]$, we arbitrarily choose a vertex $s^\star_{Y_i} \in S_i$ such that $(b_i, s^\star_{Y_i}) \in T$. The existence of such a vertex is guaranteed by Lemma 4.

▶ **Lemma 5.** *Let* (G', k') *be a* YES *instance of* Split Contraction *and* $(b_i, s^\star_{Y_i}) \in T$ *for* $i \in [t]$*. Then, for* $h_i = \varphi(s^\star_{Y_i})$*, we have* $|W(h_i)| \geq 3$*. Furthermore, there is an edge in* T *incident to* b_i *or* $s^\star_{Y_i}$ *other than* $(b_i, s^\star_{Y_i})$*.*

Proof. Suppose there exists $i \in [t]$, $h_i = \varphi(s^\star_{Y_i})$ such that $|W(h_i)| < 3$. Recall that $|W(h_i)| \geq 2$, since $b_i \in W(h_i)$. Let $A = \{x_j, y_j, z_j \mid j \in [t], j \neq i\}$. From Lemma 4, it follows that for each $j \in [t]$, there is an edge $(b_j, s^{\star}_{Y_j}) \in T$, therefore the number of edges in T incident to a vertex in A is bounded by $k' - t = t$. But $|A| = 3t - 3 > 2t$, therefore, there exists $a \in A$ such that for $h_a = \varphi(a)$, $|W(h_a)| = 1$. From Lemma 3, $(h_i, h_a) \in E(H)$, therefore $W(h_i)$ and $W(h_a)$ must be adjacent in G'. But $a \notin N(\{b_i, s^{\star}_{Y_i}\})$, hence $W(h_i)$ and $W(h_a)$ are not adjacent in G', contradicting that $\mathcal{W}$ is an *H-witness structure* of G'.

Since $|W(h_i)| \geq 3$ and $G[W(h_i)]$ is connected, at least one of $s^{\star}_{Y_i}, b_i$ must be adjacent to an edge in T which is not $(s^{\star}_{Y_i}, b_i)$. ◀

▶ **Lemma 6.** *Let (G', k') be a* YES *instance of* Split Contraction*. Then, for all $i \in [t]$, we have $|W(h_i)| \geq 2$ where $h_i = \varphi(c_i)$.*

Proof. Towards a contradiction assume that there exists $i \in [t]$, $h_i = \varphi(c_i)$, such that $|W(h_i)| < 2$. Let $A = \{c_j \mid j \in [t], j \neq i\} \cup \{x_i, y_i, z_i\}$. From Lemma 4 it follows that the edge $(b_j, s^{\star}_{Y_j}) \in T$, for each $j \in [t]$. By Lemma 5 it follows that there is an edge in T that is adjacent to exactly one of $\{b_j, s^{\star}_{Y_j}\}$ in T, for all $j \in [t]$. Therefore, at most t vertices in A can be adjacent to an edge in T, while $|A| = t + 2$. This implies that there exists $a \in A$, $h_a = \varphi(a)$ such that $|W(h_a)| = 1$. Observe that none of the vertices in A are adjacent to c_i in G'. Therefore, it follows that $W(h_i), W(h_a)$ are not adjacent in G'. But Lemma 3 implies that $(h_i, h_a) \in E(H)$, a contradiction to $\mathcal{W}$ being an *H-witness structure* of G'. ◀

▶ **Lemma 7.** *Let (G', k') be a* YES *instance of* Split Contraction *and $(b_i, s^{\star}_{Y_i}) \in T$ for $i \in [t]$. Then, for each $i \in [t]$, we have $|W(h_i)| = 3$ where $h_i = \varphi(s^{\star}_{Y_i})$.*

Proof. For $i \in [t]$, let $h_i = \varphi(s^{\star}_{Y_i})$. From Lemma 5 we know that $|W(h_i)| \geq 3$. Let $C = \{c_i \mid i \in [t]\}$ and $\mathcal{S} = \{\{b_i, s^{\star}_{Y_i}\} \mid i \in [t]\}$. From Lemmata 5 and 6 it follows that each $c \in C$ must be incident to an edge in T and each $S \in \mathcal{S}$ must have a vertex which is incident to an edge in T with the other endpoint not in S. Since $|C| = |\mathcal{S}| = t$ and $(b_i, s^{\star}_{Y_i}) \in T$, for all $i \in [t]$, there are at most t edges in T that are incident to a vertex in C and a vertex in $S \in \mathcal{S}$. Therefore, each $c \in C$ is incident to exactly one edge in T. Similarly, each $S \in \mathcal{S}$ is incident to exactly one edge with one endpoint in S and the other not in S. This implies that exactly one vertex $c \in C$ belongs to $W(h_i)$ for $i \in [t]$, and c does not belong to $W(h_j)$, where $i \neq j$, $i, j \in [t]$. Also note that none of the vertices in $\{x_i, y_i, z_i \mid i \in [t]\}$ can be incident to an edge in T. Similarly, none of the vertices in $\{b'^i_j, c'^i_j, x'^i_j, y'^i_j, z'^i_j \mid i \in [t], j \in [4t+2]\}$ can be incident to an edge in T. Hence, we get that $|W(h_i)| = 3$, concluding the proof. ◀

▶ **Lemma 8.** *Let (G', k') be a* YES *instance of* Split Contraction *and $(b_i, s^{\star}_{Y_i}) \in T$ for $i \in [t]$. Then, for all $i \in [t]$, we have $c_i \in W(h_i)$ where $h_i = \varphi(s^{\star}_{Y_i})$.*

Proof. Suppose for some $i \in [t]$, $c_i \notin W(h_i)$ where $h_i = \varphi(s^{\star}_{Y_i})$. From Lemmata 5, 6 and $k' = 2t$, it follows that there exists some $j \in [t]$ such that $c_i \in W(h_j)$, where $h_j = \varphi(s^{\star}_{Y_j})$. By our assumption, $j \neq i$. From Lemma 7 we know that $|W(h_j)| = 3$, therefore $W(h_j) = \{b_j, s^{\star}_{Y_j}, c_i\}$. Moreover, by Lemmata 6 and 7 and since $k' = 2t$, $|W(x_i)| = 1$. However, we then get that $W(h_j), W(x_i)$ are not adjacent in G'. By Lemma 3, we obtain a contradiction to the assumption that $\mathcal{W}$ is an *H-witness structure* of G'. This completes the proof. ◀

We are now ready to prove the main equivalence lemma of this section.

▶ **Lemma 9.** *$(G, \mathcal{P} = \{C_1, C_2, \ldots C_t\}, k_1, \ldots k_t)$ is a* YES *instance of* Sub-Cubic PVC *if and only if (G', k') is a* YES *instance of* Split Contraction*.*

Proof. In the forward direction, let Y be a vertex cover in G such that for each $i \in [t]$, $|Y \cap C_i| \leq k_i$. For $i \in [t]$, we let $Y_i = Y \cap C_i$. Let $T = \{(b_i, s_{Y_i}), (c_i, s_{Y_i}) \mid i \in [t]\}$. Let

$H = G'/T$, $\varphi : V(G') \to V(H)$ be the underlying surjective map and $\mathcal{W}$ be the H-*witness structure* of G'. To show that T is a solution to Split Contraction in (G', k'), it is enough to show that H is a split graph. Let $I = \cup_{i\in[t]}(S_i \setminus \{s_{Y_i}\}) \cup \{b'^i_j, c'^i_j, x'^i_j, y'^i_j, z'^i_j \mid i \in [t], j \in [4t+2]\}$. Recall that for each $v \in I$, $|W(\varphi(v))| = 1$. Furthermore, for $v, v' \in I$, $(v, v') \notin E(G')$. Hence, it follows that $\hat{I} = \{\varphi(v) \mid v \in I\}$ induces an independent set in H. Let $\mathcal{C}_1 = \{x_i, y_i, z_i \mid i \in [t]\}$. Recall that $G'[\mathcal{C}_1]$ is a clique and from the construction of T, $|W(\varphi(c))| = 1$ for all $c \in \mathcal{C}_1$. Therefore, $\hat{\mathcal{C}}_1 = \{\varphi(c) \mid c \in \mathcal{C}_1\}$ induces a clique in H. Let $\mathcal{C}_2 = \{s_{Y_i} \mid i \in [t]\}$, $h_i = \varphi(s_{Y_i})$ for $i \in [t]$, and $\hat{\mathcal{C}}_2 = \{h_i \mid i \in [t]\}$. From the construction of T, we have $W(h_i) = \{b_i, c_i, s_{Y_i}\}$ for all $i \in [t]$. Observe that for $c_1 \in \hat{\mathcal{C}}_1$ and $c_2 \in \hat{\mathcal{C}}_2$, $W(c_1), W(c_2)$ are adjacent in G', therefore, $(c_1, c_2) \in E(H)$. Consider $h_i, h_j \in \hat{\mathcal{C}}_2$, where $i, j \in [t], i \neq j$. Recall $W(h_i) = \{b_i, s_{Y_i}, c_i\}$ and $W(h_j) = \{b_j, s_{Y_j}, c_j\}$. Since Y is a vertex cover, at least one of Y_i or Y_j covers the unique edge between C_i and C_j in G, say Y_i covers the edge between C_i and C_j. But then $(s_{Y_i}, c_j) \in E(G')$, therefore $(h_i, h_j) \in E(H)$. The above argument implies that $\hat{\mathcal{C}} = \hat{\mathcal{C}}_1 \cup \hat{\mathcal{C}}_2$ induces a clique in H. Furthermore, $V(H) = \hat{I} \cup \hat{\mathcal{C}}$. This implies that H is a split graph.

In the reverse direction, let T be a solution to Split Contraction in (G', k'). Let $H = G'/T$, $\varphi : V(G') \to V(H)$ be the underlying surjective map and $\mathcal{W}$ be the H-*witness structure* of G'. From Lemma 4, it follows that for all $i \in [t]$, there exists $s_{Y_i} \in S_i$ such that $(b_i, s_{Y_i}) \in T$. For $i \in [t]$, let Y_i be the set such that $(b_i, s_{Y_i}) \in T$. We let $Y = \cup_{i\in[t]} Y_i$. For $i \in [t]$, from the definition of the vertices in S_i, it follows that $|Y \cap C_i| \leq k_i$. We will show that Y is a vertex cover in G. Towards a contradiction assume that there exists $i, j \in [t], i \neq j$, such that Y does not cover the unique edge between C_i and C_j. From Lemmas 4 and 8 it follows that $W(h_i) = \{b_i, s_{Y_i}, c_i\}$ and $W(h_j) = \{b_j, s_{Y_j}, c_j\}$, where $h_i = \varphi(s_{Y_i})$ and $h_j = \varphi(s_{Y_j})$. From Lemma 3 it follows that $(h_i, h_j) \in E(H)$. Therefore, $W(h_i)$ and $W(h_j)$ are adjacent in G'. Recall that $N_{G'}(b_i) \cap W(h_j) = \emptyset$, $N_{G'}(b_j) \cap W(h_i) = \emptyset$, $(c_i, c_j), (s_{Y_i}, s_{Y_j}) \notin E(G')$. Therefore, at least one of $(c_i, s_{Y_j}), (c_j, s_{Y_i})$ must belong to $E(G')$, say $(c_i, s_{Y_j}) \in E(G')$. But then by construction it follows that $Y_j \subseteq Y$ covers the unique edge between C_i and C_j in G, a contradiction. This completes the proof. ◀

Finally, we restate Theorem 1 and prove its correctness.

▶ **Theorem 10.** *Unless the* ETH *fails,* Split Contraction *parameterized by ℓ, the size of a minimum vertex cover of the input graph, does not have an algorithm running in time $2^{o(\ell^2)} \cdot n^{\mathcal{O}(1)}$. Here, n denotes the number of vertices in the input graph.*

Proof. Towards a contradiction assume that there is an algorithm $\mathcal{A}$ for Split Contraction, parameterized by ℓ, the size of a minimum vertex cover, running in time $2^{o(\ell^2)} n^{\mathcal{O}(1)}$. Let $(G, \mathcal{P} = \{C_1, C_2, \ldots C_t\}, k_1, \ldots k_t)$ be an instance of Sub-Cubic PVC. We create an instance (G', k') of Split Contraction as described in the **Construction**, running in time $2^{o(n)} \cdot n^{\mathcal{O}(1)}$, where $n = |V(G)|$. Recall that in the instance created, the size of a minimum vertex cover is $\ell = 5t = \mathcal{O}(\sqrt{n})$. Then we use algorithm $\mathcal{A}$ for deciding if (G', k') is a YES instance of Split Contraction and return the same answer for Sub-Cubic PVC on $(G, \mathcal{P}, k_1, \ldots k_t)$. The correctness of the answer returned follows from Lemma 9. But then we can decide whether $(G, \mathcal{P}, k_1, \ldots k_t)$ is a YES instance of Sub-Cubic PVC in time $2^{o(n)} \cdot n^{\mathcal{O}(1)}$, which contradicts ETH assuming Theorem 11. This concludes the proof. ◀

3.2 Reduction from Sub-Cubic VC to Sub-Cubic PVC

Finally, to complete our proof we show that Sub-Cubic PVC on graphs with n vertices can not be solved in time $2^{o(n)} n^{\mathcal{O}(1)}$ unless the ETH fails. In this section we give a Turing reduction from Sub-Cubic VC to Sub-Cubic PVC that will imply our desired assertion.

Let (G, k) be an instance of SUB-CUBIC VC and $n = |V(G)|$. We first create a new instance (G', k') of SUB-CUBIC VC satisfying certain properties. We start by computing a harmonious coloring of G using $t \in \mathcal{O}(\sqrt{n})$ color classes such that each color class contains at most $\mathcal{O}(\sqrt{n})$ vertices. A harmonious coloring on bounded degree graphs can be computed in polynomial time using at most $\mathcal{O}(\sqrt{n})$ colors with each color class having at most $\mathcal{O}(\sqrt{n})$ vertices [29, 33, 18]. Let $C_1, \ldots, C_t$ be the color classes. Recall that between each pair of the color classes, C_i, C_j for $i, j \in [t]$, $i \neq j$, we have at most one edge. If for some $i, j \in [t], i \neq j$, there is no edge between a vertex in C_i and a vertex in C_j, then we add a new vertex x_{ij} in C_i and a new vertex x_{ji} in C_j and add the edge (x_{ij}, x_{ji}). Observe that we add a matching corresponding to a missing edge between a pair of color classes. In this process we can add at most $t-1$ new vertices to a color class C_i, for $i \in [t]$. Therefore, the number of vertices in C_i for $i \in [t]$ after addition of new vertices is also bounded by $\mathcal{O}(\sqrt{n})$. We denote the resulting graph by G' with partition of vertices $C_1, \cdots, C_t$ (including the newly added vertices, if any). Observe that the number of vertices n' in G' is at most $\mathcal{O}(n)$. Let m be the number of matching edges added in G to obtain G' and let $k' = k + m$. It is easy to see that (G, k) is a YES instance of SUB-CUBIC VC if and only if (G', k') is a yes instance of SUB-CUBIC VC.

We will now be working with the instance (G', k') of SUB-CUBIC VC with the partition of vertices $C_1, \ldots, C_t$ obtained by extending the color classes of the harmonious coloring of G we started with. We guess the size of the intersection of the vertex cover in G' with each C_i, for $i \in [t]$. That is, for $i \in [t]$, we guess an integer $0 \leq k'_i \leq \min(|C_i|, k')$, such that $\sum_{i \in [t]} k'_i = k'$. Finally, we let $(G', \mathcal{P} = \{C_1, \cdots, C_t\}, k'_1, \cdots k'_t)$ be an instance of SUB-CUBIC PVC. Notice that G' and $\mathcal{P}$ satisfies all the requirements for it to be an instance of SUB-CUBIC PVC. It is easy to see that (G', k') is a YES instance of SUB-CUBIC VC if and only if for some guess of k_i, for $i \in [t]$, $(G', \mathcal{P} = \{C_1, \cdots, C_t\}, k'_1, \cdots, k'_t)$ is a YES instance of SUB-CUBIC PVC. This finishes the reduction from SUB-CUBIC VC to SUB-CUBIC PVC.

▶ **Theorem 11.** *Unless the* ETH *fails,* SUB-CUBIC PVC *does not admit an algorithm running in time* $2^{o(n)} \cdot n^{\mathcal{O}(1)}$*. Here, n is the number of vertices in the input graph.*

Proof. Towards a contradiction assume that there is an algorithm $\mathcal{A}$ for SUB-CUBIC PVC running in time $2^{o(n)} \cdot n^{\mathcal{O}(1)}$. Let (G, k) be an instance of SUB-CUBIC VC. We apply the above mentioned reduction to create an instance (G', k') of SUB-CUBIC VC with vertex partitions $C_1, \cdots, C_t$ such that $t \in \mathcal{O}(\sqrt{n})$ and $|C_i| \in \mathcal{O}(\sqrt{n})$, for all $i \in [t]$. Furthermore, there is exactly one edge between C_i, C_j, for $i, j \in [t]$, $i \neq j$, and C_i induces an independent set in G'. For each guess $0 \leq k'_i \leq \min(|C_i|, k')$ of the size of intersection of vertex cover with C_i, for $i \in [t]$, we solve the instance $(G', \mathcal{P}, k'_1, \cdots, k'_t)$. By the exhaustiveness of the guesses of the size of intersection for each partition, (G', k') is a YES instance of SUB-CUBIC VC if and only if for some guess $k'_1, \cdots k'_t$, $(G', \mathcal{P}, k'_1, \cdots, k'_t)$ is a YES instance of SUB-CUBIC PVC. We emphasize the fact that the number of guesses we make is bounded by $\sqrt{n}^{\mathcal{O}(\sqrt{n})} = 2^{o(n)}$, since $|C_i| \in \mathcal{O}(\sqrt{n})$ and $t \in \mathcal{O}(\sqrt{n})$. But then we have an algorithm for SUB-CUBIC VC running in time $2^{o(n)} \cdot n^{\mathcal{O}(1)}$, contradicting the ETH. This concludes the proof. ◀

4 W[1]-Hardness of Split Contraction

In this section we show that SPLIT CONTRACTION parameterized by the solution size is W[1]-hard. Towards this we first define an intermediate problem from which we give the desired reduction.

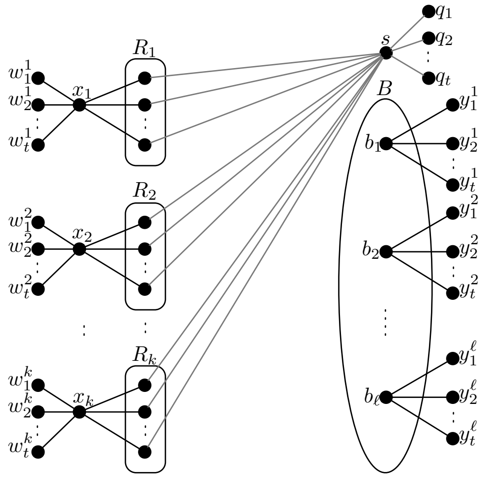

Figure 2 W[1]-Hardness of Split Contraction.

Special Red-Blue Perfect Code (SRBPC) **Parameter:** k
Input: A bipartite graph G with vertex set $V(G)$ partitioned into $\mathcal{R}$ (red set) and $\mathcal{B}$ (blue set). Furthermore, $\mathcal{R}$ is partitioned (disjoint) into $R_1 \uplus R_2 \uplus \cdots \uplus R_k$ and for all $r, r' \in \mathcal{R}$, $d_{\mathcal{B}}(r) = d_{\mathcal{B}}(r')$. That is, every vertex in $\mathcal{R}$ has same degree, say d.
Question: Does there exist $X \subseteq \mathcal{R}$, such that for all $b \in \mathcal{B}$, $|N(b) \cap X| = 1$ and for all $i \in [k]$, $|R_i \cap X| = 1$?

SRBPC is a variant of Perfect Code which is known to be W[1]-hard [14]. The W[1]-hardness proof of SRBPC is given by the following theorem. **Proofs of the results marked by an asterik are omitted due to space constraints.**

▶ **Theorem 12 (*).** SRBPC *parameterized by the number of parts in* $\mathcal{R}$ *is* W[1]*-hard.*

Let $(G, \mathcal{R} = R_1 \uplus, R_2 \uplus \cdots \uplus R_k, \mathcal{B})$ be an instance of SRBPC. We will assume that $|\mathcal{B}| = dk$, otherwise, the instance is a trivial NO instance of SRBPC. For technical reasons we assume that $|\mathcal{B}| = \ell > 4k$ (and hence $d > 4$). Such an assumption is valid because otherwise, the problem is FPT. Indeed, if $|\mathcal{B}| = \ell \leq 4k$ then for every partition $P_1, \ldots, P_k$ of $\mathcal{B}$ into k parts such that each part is non-empty, we first guess a permutation π on k elements and then for every $i \in [k]$, we check whether there exists a vertex $r_{\pi(i)} \in R_{\pi(i)}$ that dominates exactly all the vertices in P_i (and none in other parts P_j, $j \neq i$). Clearly, all this can be done in time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$. Furthermore, we also assume that $\boxed{k \geq 2}$, else the problem is solvable in polynomial time. Now we give the desired reduction. We construct an instance (G', k') of Split Contraction as follows. Initially, $V(G') = \mathcal{R} \cup \mathcal{B}$ and $E(G') = E(G)$. For all $b, b' \in \mathcal{B}$, $b \neq b'$, we add the edge (b, b') to $E(G')$. That is, we transform $\mathcal{B}$ into a clique. Let $\boxed{t = 2k + 2}$. For each $b_i \in \mathcal{B}$, we add a set of t vertices $y_1^i, \cdots, y_t^i$ each adjacent to b_i in G'. We add a vertex s adjacent to every vertex $r \in \mathcal{R}$ in G'. Also, we add a set of t vertices $q_1, \cdots, q_t$ each adjacent to s in G'. For each $i \in [k]$, we add a vertex x_i adjacent to each vertex $r \in R_i$. Finally, for all $i \in [k]$, we add a set of t vertices $w_1^i, \cdots w_t^i$ adjacent to x_i in G'. We set the new parameter k' to be $2k$. This completes the description of the reduction. We refer the reader to Figure 2 for an illustration of the reduction.

In the next four lemmata (Lemmata 13 to 16) we prove certain structural properties of the instance (G', k') of Split Contraction. These will later be used in showing that

$(G, \mathcal{R} = R_1 \uplus, R_2 \uplus \cdots \uplus R_k, \mathcal{B})$ is a YES instance of SRBPC if and only if (G', k') is a YES instance of Split Contraction. For the next four lemmata, we let S be a solution to Split Contraction in (G', k') and $H = G'/S$ with $\hat{C}$, $\hat{I}$ being a partition of $V(H)$ inducing a clique and an independent set, respectively, in H. Let $\varphi : V(G) \to V(H)$ denote the function defining the contractibility of G to H, and $\mathcal{W}$ be the *H-witness structure* of G.

▶ **Lemma 13 (*).** *Let (G', k') be a* YES *instance of* Split Contraction. *Then, for all $v \in (\{s\} \cup \mathcal{B} \cup \{x_i \mid i \in [k]\})$, we have $\varphi(v) \in \hat{C}$.*

▶ **Lemma 14 (*).** *Let (G', k') be a* YES *instance of* Split Contraction. *Then, for all $i \in [k]$, there exists $r_i \in R_i$ such that $(x_i, r_i) \in S$.*

For each $i \in [k]$ we arbitrarily choose a vertex $r_i^\star \in R_i$ such that $e_i^\star = (x_i, r_i^\star) \in S$. The existence of such a vertex is guaranteed by Lemma 14.

▶ **Lemma 15 (*).** *Let (G', k') be a* YES *instance of* Split Contraction. *Then, for all $i \in [k]$ and $h_i = \varphi(r_i^\star)$, we have $|W(h_i)| \geq 3$. Furthermore, there is an edge $e_i \neq e_i^\star$ in S incident to exactly one of $x_i, r_i^\star$ and not incident to the vertices in $\{w_1^i, \cdots, w_t^i\}$.*

From Lemma 14 we know that for each $i \in [k]$, we have $r_i^\star \in R_i$ such that $(x_i, r_i^\star) \in S$. Similarly, from Lemma 15 we know that, for each $i \in [k]$, there is an edge incident to one of x_i, r_i other than $e_i^\star = (x_i, r_i^\star)$ in every solution. Recall that for $i, j \in [k]$, $i \neq j$ none of x_i, r_i is adjacent to x_j, r_j. Hence, it follows that we have already used up our budget of $k' = 2k$ by forcing certain types of edges to be in S. Finally, we prove Lemma 16 that forces even more structure on the witness sets.

▶ **Lemma 16.** *Let (G', k') be a* YES *instance of* Split Contraction. *Then, for all $i \in [k]$, $r_i^\star \in W(\varphi(s))$.*

Proof. Let $h_s = \varphi(s)$ and $\hat{R} = \{r_i^\star \mid i \in [k], r_i^\star \in W(h_s)\}$. For a contradiction assume that $|\hat{R}| < k$, otherwise the claim trivially holds. By Lemma 14, for each $i \in [k]$, $e_i^\star = (x_i, r_i^\star) \in S$. This implies that for all $r_i^\star \in \hat{R}$, $x_i \in W(h_s)$ and hence $|W(h_s)| \geq 2|\hat{\mathcal{R}}| + 1$. From Lemma 15 we know that there exists an edge $e_i \neq e_i^\star \in S$ incident to either x_i or $r_i^\star$ and not incident to any vertex in $\{w_1^i, \cdots, w_t^i\}$. Thus, every edge in S is incident to either x_i or $r_i^\star$. This implies that for every vertex $z \in \{q_1, \ldots, q_t\} \cup \{y_1^j, \ldots, y_t^j \mid j \in [\ell]\}$, $|W(\varphi(z))| = 1$. Now we show that there exists a vertex in $\mathcal{B}$ that is not adjacent to any vertex in $W(h_s)$. Observe that the only vertices in $W(h_s)$ that can be adjacent to a vertex in $\mathcal{B}$ are in $\hat{R}$. However, every vertex in $\hat{R}$ has exactly d neighbours in $\mathcal{B}$. This together with the fact that $|\mathcal{B}| = \ell = dk > d|\hat{R}|$ implies that there exists a subset $\mathcal{B}'$ of size $d(k - |\hat{R}|)$ such that none of these vertices are adjacent to any vertex in $\hat{R}$. However, at most $(k - |\hat{R}|)$ vertices in $\mathcal{B}'$ can be incident to an edge in S. This implies that there exists a vertex $b \in \mathcal{B}'$ with $h = \varphi(b)$ such that it is not incident to any edge in S and thus $|W(h)| = 1$. But then we can conclude that $W(h)$ and $W(h_s)$ are not adjacent in G'. However, by Lemma 13 we know that $h_s, h \in \hat{C}$ and thus there is an edge $(h = \varphi(b), h_s) \in E(H')$, a contradiction. This contradicts our assumption that $|\hat{R}| < k$ and proves the claim. ◀

We are now ready to prove the equivalence between the instance $(G, \mathcal{R}, \mathcal{B})$ of SRBPC and the instance (G', k') of Split Contraction.

▶ **Lemma 17.** $(G, \mathcal{R} = R_1 \uplus \cdots R_k, \mathcal{B})$ *is a* YES *instance of* SRBPC *if and only if* (G', k') *is a* YES *instance of* Split Contraction.

Proof. In the forward direction, let $Z = \{r_i \mid r_i \in R_i, i \in [k]\} \subseteq \mathcal{R}$ be a solution to $(G, \mathcal{R}, \mathcal{B})$ of SRBPC. Let $Z' = \{(r_i, x_i), (r_i, s) \mid i \in [k]\}$. Observe that $|Z'| = 2k$. Let $T = \{r_i, x_i \mid i \in [k], r_i \in Z\}$. We define the following surjective function $\varphi : V(G') \to V(G') \setminus T$. If $v \in T \cup \{s\}$ then $\varphi(v) = s$, else $\varphi(v) = v$. Observe that $G'[W(s)]$ is connected and for all $v \in V(G') \setminus (T \cup \{s\})$, $W(v)$ is a singleton set. Consider the graph H with $V(H) = V(G') \setminus T$ and $(v, u) \in E(H)$ if and only if $\varphi^{-1}(v), \varphi^{-1}(u)$ are adjacent in G'. Note that the graphs G'/Z' and H are isomorphic, therefore we prove that H is a split graph. Let $\hat{C} = \{\varphi(v) \mid v \in \mathcal{B} \cup \{s\}\}$ and $\hat{I} = V(H) \setminus \hat{C}$. For $v, u \in \hat{I}$, $\varphi^{-1}(v) = \{v\}$ and $\varphi^{-1}(u) = \{u\}$ and $\{v\}, \{u\}$ are non-adjacent in G'. Therefore, $(v, u) \notin E(H)$. This proves that $\hat{I}$ is an independent set in H. For $b, b' \in \mathcal{B} \subset \hat{C}$, $(b, b') \in E(G')$, therefore $(\varphi(v), \varphi(u)) \in E(H)$. Since Z is a solution to SRBPC in $(G, \mathcal{R}, \mathcal{B})$, for each $b \in \mathcal{B}$, there exists $r_i \in Z$ such that $(b, r_i) \in E(G')$, therefore, $W(s)$ and b are adjacent in G'. Hence, $(\varphi(s), \varphi(b)) \in E(H')$. This finishes the proof that $\hat{C}$ induces a clique in H and that H is a split graph.

In the reverse direction, let S be a solution to (G', k') of Split Contraction, and denote $H = G'/S$. Let $\mathcal{W}$ be the *H-witness structure* of G', φ be the associated surjective function and $h_s = \varphi(s)$. From Lemmas 14 and 16 it follows that for all $i \in [k]$, there exists $r_i^\star \in R_i$ such that $(x_i, r_i^\star) \in S$ and $r_i^\star, x_i \in W(h_s)$. Let $Z = \{r_i^\star \mid i \in [k]\}$. We will show that Z is a solution to SRBPC in $(G, \mathcal{R}, \mathcal{B})$. Since $|W(h_s)| = k' + 1 = 2k + 1$, it holds that for all $v \in V(H) \setminus \{h_s\}$, $|W(v)| = 1$. This implies that for all $b \in \mathcal{B}$, $b \notin W(h_s)$. Also observe that since $x_i \in W(h_s)$ for all $i \in [k]$ and $|W(h_s)| = k' + 1 = 2k + 1$, we have that $|W(h_s) \cap R_i| = 1$. This implies that $|Z| = k$ and $|Z \cap R_i| = 1$, for all $i \in [k]$. To show that Z is indeed a solution, it is enough to show that for all $b_j \in \mathcal{B}$, $|Z \cap N(b_j)| = 1$. Towards a contradiction, assume there exists $b_j \in \mathcal{B}$ such that $|Z \cap N(b_j)| \neq 1$. Let $h_{b_j} = \varphi(b_j)$. We consider the following two cases.

- If $|Z \cap N(b_j)| < 1$. Recall that $W(h_{b_j}) = \{b_j\}$. Further, $N_{G'}(b_j) \subseteq \mathcal{R} \cup \{y_1^j, \ldots, y_t^j\}$, $Z = W(h_s) \cap \mathcal{R}$ and by our assumption $Z \cap N_{G'}(b_j) = \emptyset$. But then $W(h_s)$ and $W(h_{b_j})$ are not adjacent in G'. However, Lemma 13 implies that $(h_s, h_{b_j}) \in E(H)$, contradicting our assumption that $|Z \cap N(b_j)| < 1$.
- If $|Z \cap N(b_j)| > 1$, then there exists $j, j' \in [k]$, $j \neq j'$ such that $r_j^\star, r_{j'}^\star \in N_{G'}(b)$. Then it follows that $|\cup_{i \in [k]} N(r_i^\star)| < \ell = dk$. But then there exists $b' \in \mathcal{B}$ such that $W(\varphi(b'))$ and $W(h_s)$ are non-adjacent, contradicting that $(\varphi(b'), h_s) \in E(H)$ from Lemma 13.

This completes the proof. ◀

We now restate Theorem 2.

▶ **Theorem 2** (restated). Split Contraction *parmeterized by the size of a solution is* W[1]-hard.

Proof. Proof follows from construction, Lemma 17 and Theorem 12. ◀

5 Conclusion

In this paper, we have established two important results regarding the complexity of Split Contraction. First, we have shown that under the ETH, this problem cannot be solved in time $2^{o(\ell^2)} \cdot n^{\mathcal{O}(1)}$ where ℓ is the vertex cover number of the input graph, and this lower bound is tight. To the best of our knowledge, this is the first tight lower bound of the form $2^{o(\ell^2)} \cdot n^{\mathcal{O}(1)}$ for problems parameterized by the vertex cover number of the input graph. Second, we have proved that Split Contraction, despite its deceptive simplicity, is actually W[1]-hard with respect to the solution size. We believe that techniques integrated in our constructions can be used to derive conditional lower bounds and W[1]-hardness results in the context of other graph editing problems where the edit operation is edge contraction.

We would like to conclude our paper with the following intriguing question. In the exact setting, it is easy to see that Split Contraction can be solved in time $2^{\mathcal{O}(n \log n)}$. Can it be solved in time $2^{o(n \log n)}$? A negative answer would imply, for instance, that it is neither possible to find a topological clique minor in a given graph in time $2^{o(n \log n)}$, which is an interesting open problem [11]. It might be possible that tools developed in our paper, such as the usage of harmonious coloring, can be utilized to shed light on such problems.

References

1 Takao Asano and Tomio Hirata. Edge-contraction problems. *Journal of Computer and System Sciences*, 26(2):197–208, 1983.

2 Ivan Bliznets, Marek Cygan, Pawel Komosa, Lukas Mach, and Michal Pilipczuk. Lower bounds for the parameterized complexity of minimum fill-in and other completion problems. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1132–1151, 2016.

3 Ivan Bliznets, Fedor V. Fomin, Marcin Pilipczuk, and Michal Pilipczuk. A subexponential parameterized algorithm for proper interval completion. In *European Symposium on Algorithms (ESA)*, pages 173–184, 2014.

4 Ivan Bliznets, Fedor V. Fomin, Marcin Pilipczuk, and Michal Pilipczuk. A subexponential parameterized algorithm for interval completion. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1116–1131, 2016.

5 Leizhen Cai. Fixed-parameter tractability of graph modification problems for hereditary properties. *Information Processing Letters*, 58(4):171–176, 1996.

6 Leizhen Cai and Chengwei Guo. Contracting few edges to remove forbidden induced subgraphs. In *International Workshop on Parameterized and Exact Computation (IPEC)*, pages 97–109, 2013.

7 Yixin Cao. Unit interval editing is fixed-parameter tractable. In *International Colloquium on Automata, Languages and Programming (ICALP)*, pages 306–317, 2015.

8 Yixin Cao. Linear recognition of almost interval graphs. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1096–1115, 2016.

9 Yixin Cao and Dániel Marx. Chordal editing is fixed-parameter tractable. In *Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 214–225, 2014.

10 Yixin Cao and Dániel Marx. Interval deletion is fixed-parameter tractable. *ACM Transactions on Algorithms (TALG)*, 11(3):21, 2015.

11 Marek Cygan, Fedor V. Fomin, Alexander Golovnev, Alexander S. Kulikov, Ivan Mihajlin, Jakub Pachocki, and Arkadiusz Socala. Tight bounds for graph homomorphism and subgraph isomorphism. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1643–1649, 2016.

12 Marek Cygan, Fedor V. Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michal Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.

13 Marek Cygan, Marcin Pilipczuk, and Michal Pilipczuk. Known algorithms for edge clique cover are probably optimal. *SIAM Journal on Computing*, 45(1):67–83, 2016.

14 Rodney G. Downey and Michael R. Fellows. Fixed-parameter tractability and completeness II: On completeness for W[1]. *Theoretical Computer Science*, 141(1&2):109–131, 1995.

15 Pål Grønås Drange and Michal Pilipczuk. A polynomial kernel for trivially perfect editing. In *European Symposium on Algorithms (ESA)*, pages 424–436, 2015.

16 Pål Grønås Drange, Markus S. Dregi, Daniel Lokshtanov, and Blair D. Sullivan. On the threshold of intractability. In *European Symposium on Algorithms (ESA)*, pages 411–423, 2015.

17 Pål Grønås Drange, Fedor V. Fomin, Michal Pilipczuk, and Yngve Villanger. Exploring subexponential parameterized complexity of completion problems. In *Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 288–299, 2014.

18 Keith Edwards. The harmonious chromatic number and the achromatic number. *Surveys in Combinatorics*, pages 13–48, 1997.

19 Fedor V. Fomin, Stefan Kratsch, Marcin Pilipczuk, Michał Pilipczuk, and Yngve Villanger. Tight bounds for parameterized complexity of cluster editing with a small number of clusters. *Journal of Computer and System Sciences*, 80(7):1430–1447, 2014.

20 Fedor V. Fomin and Yngve Villanger. Subexponential parameterized algorithm for minimum fill-in. *SIAM Journal on Computing*, 42(6):2197–2216, 2013.

21 Esha Ghosh, Sudeshna Kolay, Mrinal Kumar, Pranabendu Misra, Fahad Panolan, Ashutosh Rai, and M. S. Ramanujan. Faster parameterized algorithms for deletion to split graphs. *Algorithmica*, 71(4):989–1006, 2015.

22 Petr A. Golovach, Pim van 't Hof, and Daniel Paulusma. Obtaining planarity by contracting few edges. *Theoretical Computer Science*, 476:38–46, 2013.

23 Sylvain Guillemot and Dániel Marx. A faster FPT algorithm for bipartite contraction. *Information Processing Letters*, 113(22–24):906–912, 2013.

24 Chengwei Guo and Leizhen Cai. Obtaining split graphs by edge contraction. *Theoretical Computer Science*, 607:60–67, 2015.

25 Pinar Heggernes, Pim van 't Hof, Daniel Lokshtanov, and Christophe Paul. Obtaining a bipartite graph by contracting few edges. *SIAM Journal on Discrete Mathematics*, 27(4):2143–2156, 2013.

26 Pinar Heggernes, Pim van 't Hof, Benjamin Lévêque, Daniel Lokshtanov, and Christophe Paul. Contracting graphs to paths and trees. *Algorithmica*, 68(1):109–132, 2014.

27 Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001.

28 Christian Komusiewicz. Tight running time lower bounds for vertex deletion problems. *arXiv preprint arXiv:1511.05449*, 2015.

29 Sin-Min Lee and John Mitchem. An upper bound for the harmonious chromatic number. *Journal of Graph Theory*, 11(4):565–567, 1987.

30 Daniel Lokshtanov, Dániel Marx, and Saket Saurabh. Slightly superexponential parameterized problems. In *ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 760–776, 2011.

31 Daniel Lokshtanov, Neeldhara Misra, and Saket Saurabh. On the hardness of eliminating small induced subgraphs by contracting edges. In *International Workshop on Parameterized and Exact Computation (IPEC)*, pages 243–254, 2013.

32 Dániel Marx and Valia Mitsou. Double-exponential and triple-exponential bounds for choosability problems parameterized by treewidth. In *International Colloquium on Automata, Languages and Programming (ICALP)*, pages 28:1–28:15, 2016.

33 Colin McDiarimid and Luo Xinhua. Upper bounds for harmonious coloring. *Journal of Graph Theory*, 15(6):629–636, 1991.

34 Toshimasa Watanabe, Tadashi Ae, and Akira Nakamura. On the removal of forbidden graphs by edge-deletion or by edge-contraction. *Discrete Applied Mathematics*, 3(2):151–153, 1981.

35 Toshimasa Watanabe, Tadashi Ae, and Akira Nakamura. On the NP-hardness of edge-deletion and-contraction problems. *Discrete Applied Mathematics*, 6(1):63–78, 1983.