Exact Algorithms for List-Coloring of Intersecting Hypergraphs

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Abstract

We show that list-coloring for any intersecting hypergraph of m edges on n vertices, and lists drawn from a set of size at most k, can be checked in quasi-polynomial time $(mn)^{o(k^2 \log(mn))}$.

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1 Introduction

Hypergraph k-Coloring is the problem of checking whether the vertex-set of a given hypergraph (family of sets) can be colored with at most k colors such that every edge receives at least two *distinct* colors. It is a basic problem in theoretical computer science and discrete mathematics which has received considerable attention (see, e.g. [3, 4, 11, 12, 26, 29, 38]). The problem is NP-complete already for $k = 2$, and in fact, it is quasi-NP-hard[1] to decide if a 2-colorable hypergraph can be (properly) colored with $2^{(\log n)^{\Omega(1)}}$ colors [26]. On the positive side, there exist polynomial time algorithms that can color an $O(1)$-colorable hypergraph with $n^{O(1)}$ colors, where n is the number of vertices (see, e.g., [1, 9, 30]). Several generalizations of the problem have also been considered, for example, List-coloring where every vertex can take only colors from a given list of colors [20, 37].

Given the intrinsic difficulty of the problem, it is natural to consider special classes of hypergraphs for which the problem is easier. Some better results exist for special classes, e.g., better approximation algorithms for hypergraphs of low discrepancy and rainbow-colorable hypergraphs [5], polynomial time algorithms for bounded-degree linear hypergraphs [4, 8], for random 3-uniform 2-colorable hypergraphs [34], as well as for some special classes of graphs [14, 25, 27, 10].

In this paper, we consider the special class of *intersecting* hypergraphs, i.e., those in which every pair of edges have a non-empty intersection (also considered in [35]). While this may seem as a strong restriction at a first thought, the problem is still actually highly non-trivial. In fact, the case $k = 2$ is equivalent to the well-known Monotone Boolean Duality Testing, which is the problem of checking for a given pair of monotone CNF and DNF formulas if they represent the same monotone Boolean function [15, 35]. Determining the exact complexity of this duality testing problem is an outstanding open question, which has been referenced in a number of complexity theory retrospectives, e.g., [31, 32], and has been the subject of many papers, see, e.g., [6, 7, 13, 19, 15, 16, 17, 18, 21, 23, 22, 24, 28, 36].

[1] More precisely, there is no polynomial time algorithm unless NP$\subseteq$ DTIME($2^{\text{polylog}\, n}$).

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Fredman and Khachiyan [21] gave an algorithm for solving this problem with running time $n^{o(\log n)}$, where n is the size of the input, thus providing strong evidence that this decision problem is unlikely to be NP-hard.

The reduction from Boolean Duality Testing to checking 2-colorability is essentially obtained by a construction from [35] which reduces the problem to checking if a monotone Boolean function given by its CNF representation is *self-dual*. However, almost all the known algorithms for solving Boolean duality testing cannot work directly with the self-duality (and hence the 2-colorabilty) problem, due to their recursive nature which results in subproblems that do not involve checking self-duality. The only algorithm we are aware of that works directly on the 2-colorability version is the one given in [22], but it yields weaker bounds $n^{O(\log n)}$ than those given in [21]. In this paper, we provide bounds that (almost) match those given in [21] and show that those can be in fact extended to the list-colorabilty version[2].

It is also worth mentioning that intersecting hypergraphs have been considered in [33, Section 2.4.1] (with a slight generalization), where it was shown that if such a hypergraph is 2-colorable then it is also list-colorable for any lists of size 2. It is not clear whether such result extends to the case $k > 2$.

2 Basic Notation and Main Result

Let $\mathcal{H} \subseteq 2^V$ be a hypegraph on a finite set V, $k \geq 2$ be a positive integer, and $\mathcal{L} : V \to 2^{[k]}$ be a mapping that assigns to each vertex $v \in V$ a non-empty list of *admissible colors* $\mathcal{L}(v) \subseteq [k] := \{1, \ldots, k\}$. An $\mathcal{L}$-*(list) coloring* of $\mathcal{H}$ is an assignment $\chi : V \to [k]$ of colors to the vertices of $\mathcal{H}$ such that $\chi(v) \in \mathcal{L}(v)$ for all $v \in V$. An $\mathcal{L}$-coloring is said to be *proper* if it results in no *monochromatic* edges, that is, if $|\chi(H)| \geq 2$, for all $H \in \mathcal{H}$, where $\chi(H) := \{\chi(v) : v \in H\}$.

A hypergraph $\mathcal{H}$ is said to be *intersecting* if

$$H \cap H' \neq \emptyset \text{ for all } H,\ H' \in \mathcal{H}. \tag{1}$$

In this paper, we are interested in the following problem:

> Proper-$\mathcal{L}$-Coloring: Given a hypergraph $\mathcal{H} \subseteq 2^V$ satisfying (1) and a mapping $\mathcal{L} : V \to 2^{[k]}$, either find a proper $\mathcal{L}$-coloring of $\mathcal{H}$, or declare that no such coloring exists.

We denote by $n := |V|$, $m := |\mathcal{H}|$, $\nu := \min_{v \in V} |\mathcal{L}(v)|$, $\rho := \max_{v \in V} |\mathcal{L}(v)|$, and $\kappa := \max_{u,v \in V,\ u \neq v} |\mathcal{L}(u) \cap \mathcal{L}(v)|$. We assume without loss of generality that $\nu \geq 2$.

For a set $S \subseteq V$, let $H_S := \{H \in \mathcal{H} : H \subseteq S\}$ be the subhypergraph of $\mathcal{H}$ induced by set S, $\mathcal{H}^S = \{H \cap S : H \in \mathcal{H}\}$ be the projection (or trace) of $\mathcal{H}$ into S, and $\mathcal{H}(S) := \{H \in \mathcal{H} : H \cap S \neq \emptyset\}$. For simplicity, we allow $\mathcal{H}^S$ to be a multi-hypergraph (some edges may be repeated). For $v \in V$, we define $\deg_{\mathcal{H}}(v) := |\{H \in \mathcal{H} : v \in H\}|$.

The main result of the paper is the following.

▶ **Theorem 1.** *Problem* Proper-$\mathcal{L}$-Coloring *can be solved in time* $(mn)^{o(k^2 \log(mn))}$.

[2] Note that is an intersecting hypergragh (with more than one edge) is trivially 3-colorable; so the generalization to k colors would only be interesting if we consider list-coloring.

In the following, we will consider partial $\mathcal{L}$-colorings $\chi : V \to [0:k] := \{0, 1, \ldots, k\}$ of $\mathcal{H}$, where $\chi(v) = 0$ is used to mean that the vertex v is not assigned any color yet; we say that such coloring is proper if no edge is monochromatic with this coloring. Given a proper partial $\mathcal{L}$-coloring χ of a hypergraph $\mathcal{H} \subseteq 2^V$, we will use the following notation: $V_0(\chi) := \{v \in V : \ \chi(v) = 0\}$ and $\mathcal{H}_i(\chi) := \{H \in \mathcal{H} : \ \chi(H) = \{0, i\}\}$ for $i \in [0:k]$, and shall simply write V_0 and $\mathcal{H}_i$ when χ is clear from the context. For $i \in [0:k]$, we write $\bar{\mathcal{H}}_i := \bigcup_{j \neq i} \mathcal{H}_j$. For a set $S \subseteq V$, we write $\bar{S} := V \setminus S$ and denote by $\chi[S]$ the restriction of χ on S. For two $\mathcal{L}$-colorings $\chi : S \to [k]$ and $\chi' : S' \to [k]$, where $S \cap S' = \emptyset$, we denote by $\chi'' := \chi \cup \chi' : S \cup S' \to [k]$ the k-coloring that assigns $\chi''(v) := \chi(v)$ for $v \in S$ and $\chi''(v) := \chi'(v)$ for $v \in S'$. If there is an $H \in \mathcal{H}$ such that $|H| \leq 1$, we shall assume that $\mathcal{H}$ is *not* properly $\mathcal{L}$-colorable for any $\mathcal{L} : V \to 2^{[k]}$. Also, by assumption, an empty hypergraph (that is, $\mathcal{H} = \emptyset$) is properly $\mathcal{L}$-colorable.

In the following two sections we give two algorithms for solving the problem. They are inspired by the two corresponding algorithms in [21] and can be thought of as generalizations. The first algorithm is simpler and exploits the idea of the existence of a high-degree vertex in any non-colorable instance. By considering all possible admissible colorings of such a vertex we can remove a large fraction of the edges and recurse on substantially smaller subproblems. Unfortunately, the degree of the high-degree vertex is only large enough to guarantee a bound of $m^{O(\log^2 m)}$ (assuming $k = O(1)$). The second algorithm is more complicated and considers both scenarios when there is a high-degree vertex and there are none (where now the threshold for "high" is actually higher). If there is no high-degree vertex, then we can find a "balanced-set" which contains a constant fraction of edges. Then a decomposition can be obtained based on this set.

3 Solving Proper-$\mathcal{L}$-Coloring in time $n^{O(k^3)} m^{O(k^2 \log^2 m)}$

We give two lemmas which show the existence of a high-degree vertex, unless the hypergraph is easily colorable.

▶ **Lemma 2.** *Let $\mathcal{H} \subseteq 2^V$ be a given hypergraph satisfying (1) of minimum edge-size 2, $\mathcal{L} : V \to 2^{[k]}$, and $\chi : V \to [0:k]$ be a proper partial $\mathcal{L}$-coloring of $\mathcal{H}$. Then either* (i) *there is a vertex $v \in V_0$ with $\deg_{\mathcal{H}_0}(v) \geq \frac{|\mathcal{H}_0|}{\log_\nu (m\kappa)}$, or* (ii) *an $\mathcal{L}$-coloring $\chi_0 : V_0 \to [k]$, such that $\chi[V \setminus V_0] \cup \chi_0$ is a proper $\mathcal{L}$-coloring of $\mathcal{H}$, can be found in $O(\rho |V_0| m)$ time.*

Proof. We use the probabilistic method [2]. Let $H_{\min}$ be an edge in $\bigcup_{i=0}^k \mathcal{H}_i^{V_0}$ of minimum size. Pick a random $\mathcal{L}$-coloring $\chi_0 : V_0 \to [k]$ by assigning, independently for each $v \in V_0$, $\chi_0(v) = i \in \mathcal{L}(v)$ with probability $\frac{1}{|\mathcal{L}(v)|}$. Then, for an edge $H \in \mathcal{H}_0$,

$$\Pr[H \text{ is monochromatic}] = |\bigcap_{v \in H} L(v)| \cdot \prod_{v \in H} \frac{1}{|\mathcal{L}(v)|} \leq \kappa \cdot \left(\frac{1}{\nu}\right)^{|H|},$$

and for $H \in \mathcal{H}_i$, $i \in [k]$,

$$\Pr[H \text{ is monochromatic}] \leq \prod_{v \in H \cap V_0} \frac{1}{|\mathcal{L}(v)|} \leq \left(\frac{1}{\nu}\right)^{|H \cap V_0|}.$$

It follows that

$$\begin{aligned} \mathbb{E}[\# \text{ monochromatic } H \in \mathcal{H}] &= \sum_{H \in \mathcal{H}} \Pr[H \text{ is monochromatic}] \\ &\leq \kappa \sum_{H \in \mathcal{H}_0} \left(\frac{1}{\nu}\right)^{|H|} + \sum_{i=1}^{k} \sum_{H \in \mathcal{H}_i} \left(\frac{1}{\nu}\right)^{|H \cap V_0|} \leq m\kappa \left(\frac{1}{\nu}\right)^{|H_{\min}|}. \end{aligned}$$

Thus if $m\kappa\left(\frac{1}{\nu}\right)^{|H_{\min}|} < 1$, then there is a proper $\mathcal{L}$-coloring $\chi' := \chi[V \setminus V_0] \cup \chi_0$ of $\mathcal{H}$, which can be found by the method of conditional expectations in time $O(\rho|V_0|m)$. Let us therefore assume for the rest of this proof that $|H_{\min}| \le \log_\nu(m\kappa)$.

Let $v_{\max}$ be a vertex of maximizing $\deg_{\mathcal{H}_0}(v)$ over $v \in H_{\min}$. Then (1) implies that

$$\begin{aligned}|\mathcal{H}_0| &= \left|\bigcup_{v\in H_{\min}} \{H \in \mathcal{H}_0 : \ v \in H\}\right| \le \sum_{v\in H_{\min}} |\{H \in \mathcal{H}_0 : \ v \in H\}| = \sum_{v\in H_{\min}} \deg_{\mathcal{H}_0}(v)\\ &\le |H_{\min}| \deg_{\mathcal{H}_0}(v_{\max}).\end{aligned}$$

Consequently, $\deg_{\mathcal{H}_0}(v_{\max}) \ge \frac{|\mathcal{H}_0|}{|H_{\min}|} \ge \frac{|\mathcal{H}_0|}{\log_\nu(m\kappa)}$. ◀

▶ **Lemma 3.** *Let $\mathcal{H} \subseteq 2^V$ be a given hypergraph satisfying (1) of minimum edge-size 2, $\mathcal{L} : V \to 2^{[k]}$ be a mapping, and $\chi : V \to [0 : k]$ be a proper partial $\mathcal{L}$-coloring of $\mathcal{H}$. Then either* (i) *there is a vertex $v \in V_0$ and $i, j \in [k]$, $j \ne i$, such that $\deg_{\mathcal{H}_i}(v) \ge \frac{|\mathcal{H}_i|}{\log_\nu m}$ and $\deg_{\mathcal{H}_j}(v) \ge 1$, or* (ii) *an $\mathcal{L}$-coloring $\chi_0 : V_0 \to [k]$, such that $\chi[V \setminus V_0] \cup \chi_0$ is a proper $\mathcal{L}$-coloring of $\mathcal{H}$, can be found in $O(\rho|V_0|m)$ time.*

Proof. Let $H_{\min}$ be an edge in $\bigcup_{i=1}^k \mathcal{H}_i^{V_0}$ of minimum size. Note that (1) implies:

$$\forall H \in \mathcal{H}_i : \ H \cap H' \cap V_0 \ne \emptyset \ \text{ for all } H' \in \bar{\mathcal{H}}_i, \tag{2}$$

since $\{i\} = \chi(H \setminus V_0) \ne \chi(H' \setminus V_0) = \{j\}$ for all $H \in \mathcal{H}_i$ and $H' \in \mathcal{H}_j$, for $i \ne j$.

If there is an $i \in [k]$ such that $\mathcal{H}_j = \emptyset$ for all $j \in [k] \setminus \{i\}$ then an $\mathcal{L}$-coloring satisfying (ii) can be found by choosing arbitrarily $\chi(v) \in \mathcal{L}(v) \setminus \{i\}$ for $v \in V_0$. Assume therefore that $\mathcal{H}_i \ne \emptyset$ for at least two distinct indices $i \in [k]$. Pick a random $\mathcal{L}$-coloring $\chi_0 : V_0 \to [k]$ by assigning, independently for each $v \in V_0$, $\chi(v) = i \in \mathcal{L}(v)$ with probability $\frac{1}{|\mathcal{L}(v)|}$. Then

$$\begin{aligned}\Pr[\exists i \in [k],\ H \in \mathcal{H}_i : \ \chi(H_i) = \{i\}] &\le \sum_{i=1}^k \sum_{H\in\mathcal{H}_i} \Pr[\chi(H) = \{i\}]\\ &\le \sum_{i=1}^k \sum_{H\in\mathcal{H}_i} \prod_{v\in H\cap V_0} \frac{1}{|\mathcal{L}(v)|} \le m\left(\frac{1}{\nu}\right)^{|H_{\min}|}.\end{aligned}$$

Thus if $m\left(\frac{1}{\nu}\right)^{|H_{\min}|} < 1$, then there is an $\mathcal{L}$-coloring satisfying (ii), which can be found by the method of conditional expectations in time $O(\rho|V_0|m)$. Let us therefore assume for the rest of this proof that $|H_{\min}| \le \log_\nu m$.

Let j be such that $H_{\min} \in \mathcal{H}_j^{V_0}$, and $v_{\max}$ be a vertex maximizing $\deg_{\bar{\mathcal{H}}_j}(v)$ over $v \in H_{\min}$. Then (2) implies that

$$\begin{aligned}|\bar{\mathcal{H}}_j| &= \left|\bigcup_{v\in H_{\min}} \{H \in \bar{\mathcal{H}}_j : \ v \in H\}\right| \le \sum_{v\in H_{\min}} |\{H \in \bar{\mathcal{H}}_j : \ v \in H\}| = \sum_{v\in H_{\min}} \deg_{\bar{\mathcal{H}}_j}(v)\\ &\le |H_{\min}| \deg_{\bar{\mathcal{H}}_j}(v_{\max}).\end{aligned}$$

Consequently, $\sum_{i\ne j} \deg_{\mathcal{H}_i}(v_{\max}) = \deg_{\bar{\mathcal{H}}_j}(v_{\max}) \ge \frac{|\bar{\mathcal{H}}_j|}{|H_{\min}|} \ge \frac{|\bar{\mathcal{H}}_j|}{\log_\nu m} = \frac{\sum_{i\ne j}|\mathcal{H}_i|}{\log_\nu m}$, from which it follows that $\max_{i\ne j} \frac{\deg_{\mathcal{H}_i}(v_{\max})}{|\mathcal{H}_i|} \ge \frac{\sum_{i\ne j}\deg_{\mathcal{H}_i}(v_{\max})}{\sum_{i\ne j}|\mathcal{H}_i|} \ge \frac{1}{\log_\nu m}$. ◀

If the number of edges in each $\mathcal{H}_i$ is small, the problem is easily solvable in polynomial time.

▶ **Lemma 4.** *Given a hypergraph $\mathcal{H} \subseteq 2^V$ such that $\max_{i=0}^k |\mathcal{H}_i| \le \delta$ or $|\{i: \ \mathcal{H}_i \neq \emptyset\}| = 1$, a mapping $\mathcal{L}: V \to 2^{[k]}$, and a proper partial $\mathcal{L}$-coloring $\chi: V \to [0:k]$ of $\mathcal{H}$ such that $\mathcal{H}_0 = \emptyset$, there is a procedure* Proper-$\mathcal{L}$-Coloring-*simple*$(\mathcal{H}, L, \chi)$ *that checks if there is a proper $\mathcal{L}$-coloring of $\mathcal{H}$ extending χ, in time $O((|V_0|\rho)^{(k+1)\delta})$.*

Proof. If $\mathcal{H}_i \neq \emptyset$ for exactly one i, then assigning any color $j \neq i$ to the uncolored vertices yields a proper $\mathcal{L}$-coloring for $\mathcal{H}$. On the other hand, if $|\mathcal{H}_i| \le \delta$ for all i, we can simply try all possibilities: for each edge $H \in \mathcal{H}_i$, for $i = 1, \ldots, k$ (resp., $H \in \mathcal{H}_0$), we choose a vertex $v \in H \cap V_0$ and a color for v among the colors in $\mathcal{L}(v) \setminus \{i\}$ (resp., two distinct vertices $v, v' \in H \cap V_0$ and two distinct colors $i \in \mathcal{L}(v)$ and $j \in \mathcal{L}(v')$). For each such choice, if the resulting coloring, combined with χ, is a proper partial $\mathcal{L}$-coloring for $\mathcal{H}$, then it can be extended to a proper $\mathcal{L}$-coloring by coloring any remaining uncolored vertices arbitrarily; otherwise, we conclude that no such coloring exists if we run out of choices. Since we have at most $(k+1)\delta$ edges in $\bigcup_{i=0}^k \mathcal{H}_i$, the total number of choices is at most $(|V_0|\rho)^{(k+1)\delta}$. ◀

The algorithm for solving Proper-$\mathcal{L}$-Coloring is given as Algorithm 1, which is called initially with $\chi \equiv 0$. The algorithm terminates either with a proper $\mathcal{L}$-coloring of $\mathcal{H}$, or with a partial $\mathcal{L}$-coloring with some unassigned vertices, in which case we conclude that no proper $\mathcal{L}$-coloring of $\mathcal{H}$ exists.

The algorithm proceeds in two phases. As long as there is an edge with no assigned colors, that is $|\mathcal{H}_0| \ge 1$, the algorithm is still in phase I; otherwise it proceeds to phase II. In a general step of phase I (resp. phase II), the algorithm picks a vertex v satisfying condition (i) of Lemma 2 (resp., Lemma 3) and iterates over all feasible assignments of colors to v, that result in no monochromatic edges; if no such v can be found, the algorithm concludes with a proper $\mathcal{L}$-coloring. In each iteration, any edge that becomes non-monochromatic is removed and the algorithm recurses on the updated sets of hypergraphs. If non of the recursive calls yields a feasible extension of the current proper partial $\mathcal{L}$-coloring χ, we unassign vertex v and return that there are no proper $\mathcal{L}$-colorings (line 11).

To analyze the running time of the algorithm, let us measure the "volume" of a subproblem with input $(\mathcal{H}, \mathcal{L}, \chi)$, in phase I by $\mu_1 = \mu_1(\mathcal{H}, \chi) := |\mathcal{H}_0(\chi)|$, and in phase II by

$$\mu_2 = \mu_2(\mathcal{H}, \chi) := |\{i \in [k]: \ |\mathcal{H}_i(\chi)| \ge 1\}| \cdot \prod_{i=1}^k \max\{|\mathcal{H}_i(\chi)|, 1\}. \tag{3}$$

The recursion stops when the volume $\mu_2(\mathcal{H})$ becomes sufficiently small, or an $\mathcal{L}$-coloring satisfying condition (ii) of Lemmas 2 or 3 is found.

Lemma 4 implies that problem Proper-$\mathcal{L}$-Coloring can be solved in time $O((\rho n)^{(k+1)\rho^2})$ if $m \le \delta := \rho^2$. Algorithm Proper-$\mathcal{L}$-Coloring-A can be used to solve the problem in case $m > \delta$.

▶ **Lemma 5.** *Algorithm 1 solves problem* Proper-$\mathcal{L}$-Coloring *in time*

$$(\rho n)^{O(k\rho^2)} \rho^3 m^{O(k^2 \frac{\log^2 m}{\log \nu})}.$$

Proof. Let $\epsilon := \min\{\frac{1}{\log_\nu m}, \frac{1}{k}\}$, $\alpha = \frac{1}{1-\epsilon}$ and $\delta = \rho^2$. Note that $\delta \ge \alpha^2$ since $\epsilon \le \frac{1}{k} \le \frac{1}{2}$ and thus $\rho \ge 2 \ge \frac{1}{1-\epsilon} = \alpha$.

Consider the recursion tree $\mathbf{T}$ of the algorithm. Let $\mathbf{T}_1$ (reps., $\mathbf{T}_2$) be the subtree (resp., sub-forest) of $\mathbf{T}$ belonging to phase I (resp., phase II) of the algorithm. Note that $\mathbf{T}_2$ consists of maximal sub-trees of $\mathbf{T}$, each of which is rooted at a leaf in $\mathbf{T}_1$. Let $A_1(\mu)$ (resp., $A_2(\mu)$) be the total number of nodes in $\mathbf{T}_1$ (resp., $\mathbf{T}_2$) that result from a subproblem of volume μ.

Algorithm 1 PROPER-$\mathcal{L}$-COLORING-A$(\mathcal{H}, \mathcal{L}, \chi)$

Input: hypergraph $\mathcal{H} \subseteq 2^V$, a mapping $\mathcal{L} : V \to 2^{[k]}$, and a proper partial $\mathcal{L}$-coloring $\chi : V \to [0 : k]$

Output: TRUE (resp,. FALSE) if a proper $\mathcal{L}$-coloring $\chi : V \to [k]$ of $\mathcal{H}$ is (resp., cannot be) found

1: $V_0 := V_0(\chi)$; $\mathcal{H}_i := \mathcal{H}_i(\chi)$ for $i \in [0 : k]$
2: **if** $\mathcal{H}_0 \neq \emptyset$ **then** /* Phase I */
3:     **if** there is $v \in V_0$ satisfying condition (i) of Lemma 2 **then**
4:         **for** each $j \in \mathcal{L}(v)$ **do**
5:             $\chi(v) := j$
6:             **if** there is no $H \in \mathcal{H}$ such that $\chi(H) = \{j\}$ **then** /* if no edge becomes monochromatic */
7:                 $\mathcal{H}' := \mathcal{H} \setminus \bigcup_{i \in [k],\ i \neq j} \{H \in \mathcal{H}_i :\ j \in \chi(H)\}$ /* delete non-monochromatic edges */
8:                 **return** PROPER-$\mathcal{L}$-COLORING-A$(\mathcal{H}', \mathcal{L}, \chi)$
9:             **end if**
10:         **end for**
11:         $\chi(v) := 0$; **return** FALSE
12:     **else**
13:         Let $\chi_0 : V_0 \to [k]$ be a coloring computed as in (ii) of Lemma 2
14:         Set $\chi := \chi[V \setminus V_0] \cup \chi_0$ and stop /* A proper $\mathcal{L}$-coloring has been found */
15:     **end if**
16: **else**/* Phase II */
17:     **if** $\mu_2(\mathcal{H}, \chi) \leq \delta := \rho^2$ or $|\{i :\ \mathcal{H}_i \neq \emptyset\}| = 1$ **then**
18:         **if** PROPER-$\mathcal{L}$-COLORING-simple$(\mathcal{H}, L, \chi)$ **then**
19:             Stop /* A proper $\mathcal{L}$-coloring has been found */
20:         **else**
21:             **return** FALSE
22:         **end if**
23:     **end if**
24:     **if** there is $v \in V$ satisfying condition (i) of Lemma 3 **then**
25:         Same as in steps 4-11 of Phase I
26:     **else**
27:         Let $\chi_0 : V \to [k]$ be a coloring computed as in (ii) of Lemma 3
28:         Set $\chi := \chi[V \setminus V_0] \cup \chi_0$ and stop /* A proper $\mathcal{L}$-coloring has been found */
29:     **end if**
30: **end if**

▶ **Claim 6.** *The number of nodes in* $\mathbf{T}_1$ *is at most* $m^{\log \rho \cdot \log_\nu(m\kappa) + O(1)}$.

Proof. For a non-leaf node of $\mathbf{T}_1$, we have the recurrence:

$$A_1(\mu_1) \leq \rho \cdot A_1((1-\epsilon)\mu_1) + 1. \tag{4}$$

At leaves we have $\mu_1 = 0$. It follows that the depth $d(\mu_1)$ of the recursion subtree of a node (in $\mathbf{T}_1$) of volume μ_1 is at most $\log_{\frac{1}{1-\epsilon}} \mu_1 + 1$, and hence the total number of tree nodes N_1 is bounded by $\frac{\rho^{d(\mu_1)+1} - 1}{\rho - 1} \leq \mu_1^{\log_{\frac{1}{1-\epsilon}} \rho + 2}$. Using $\mu_1 \leq m$, we get $N_1 = O(m^{\frac{\log \rho}{\log(1+1/\log_\nu(m\kappa))}}) = m^{\log \rho \cdot \log_\nu(m\kappa) + O(1)}$. ◀

▶ **Claim 7.** *The number of nodes in any sub-tree of* $\mathbf{T}_2$ *is at most* $m^{\log\rho\cdot\log_\nu(m\kappa)+O(1)}$.

Proof. Suppose that the algorithm proceeds to line 25 during the current recursive call corresponding to a subproblem of volume μ_2, and let $v \in V$ be the vertex chosen at step 24, and $i, j \in [k]$ be such that $i \neq j$, $\deg_{\mathcal{H}_i}(v) \geq \epsilon|\mathcal{H}_i|$ and $\deg_{\mathcal{H}_j}(v) \geq 1$. There are $|\mathcal{L}(v)|$ recursive calls that will be initiated from this point, corresponding to $\ell \in \mathcal{L}(v)$; consider the ℓth recursive call. If $\ell \neq i$ then setting $\chi(v) = \ell$ will result in deleting all the edges containing v from $\mathcal{H}_i$. Thus the new volume μ_2' will satisfy $\mu_2' \leq (1 - \frac{1}{\log_\nu m})\mu_2$ if $|\mathcal{H}_i| > 1$ and $\mu_2' \leq (1 - \frac{1}{k})\mu_2$ if $|\mathcal{H}_i| = 1$; in both cases, $\mu_2' \leq (1-\epsilon)\mu_2$. On the other hand, if $\ell = i$, then at least one edge in $\mathcal{H}_j$ will be deleted, yielding $\mu_2' \leq \mu_2 - 1$. Consequently we get the recurrence:

$$A_2(\mu_2) \leq (\rho - 1)\cdot A(\lfloor(1-\epsilon)\mu_2\rfloor) + A(\mu_2 - 1) + 1. \tag{5}$$

By the stopping criterion in line 17, we have $A_2(\mu_2) = 1$ for $\mu_2 \leq \delta$. We will prove by induction on $\mu_2 > \delta$ that $A_2(\mu_2) \leq C \cdot \mu_2^{\log_\alpha \mu_2}$, where $C := (2\delta + 1)$. We consider 3 cases:

Case 1. $\mu_2 - 1 \leq \delta$: Then $\lfloor(1-\epsilon)\mu_2\rfloor \leq \delta$ and (5) reduces to $A_2(\mu_2) \leq \rho + 1 < C$.

Case 2. $(1-\epsilon)\mu_2 \leq \delta$: Then (5) reduces to

$$A_2(\mu_2) \leq \rho + A_2(\mu_2 - 1).$$

Iterating we get $A_2(\mu_2) \leq r\rho + A_2(\mu_2 - r) \leq r\rho + 1$, for $r = \mu_2 - \delta \leq \frac{\epsilon}{1-\epsilon}\delta$. Thus, $A_2(\mu_2) \leq \frac{\epsilon}{1-\epsilon}\delta\rho + 1 \leq \frac{1}{k-1}\rho\delta + 1 \leq C$.

Case 3. $\mu_2 - 1 > \delta$ and $(1-\epsilon)\mu_2 > \delta$: We apply induction:

$$\begin{aligned}
A_2(\mu_2) &\leq C(\rho-1)\cdot((1-\epsilon)\mu_2)^{\log_\alpha((1-\epsilon)\mu_2)} + C(\mu_2-1)^{\log_\alpha(\mu_2-1)} + 1 \\
&\leq C\frac{\rho-1}{(1-\epsilon)\mu_2}\cdot\frac{1}{\mu_2}\cdot\mu_2^{\log_\alpha \mu_2} + C(\mu_2-1)^{\log_\alpha \mu_2} + 1 \\
&= C\cdot\mu_2^{\log_\alpha \mu_2}\left(\frac{\rho-1}{\delta\mu_2} + \left(1 - \frac{1}{\mu_2}\right)^{\log_\alpha \mu_2} + \frac{1}{C\cdot\mu_2^{\log_\alpha \mu_2}}\right) \\
&\leq C\cdot\mu_2^{\log_\alpha \mu_2}\left(\frac{1}{\mu_2} + \left(1 - \frac{1}{\mu_2}\right)^2 + \frac{1}{\mu_2^2}\right) \quad \text{(since } \mu_2 \geq \delta \geq \alpha^2 \text{ and hence } \log_\alpha \mu_2 \geq 2\text{)} \\
&\leq C\cdot\mu_2^{\log_\alpha \mu_2} \quad \text{(since } \mu_2 \geq \delta > 2\text{).}
\end{aligned}$$

Using the bound

$$\mu_2(\mathcal{H}, \chi) \leq k\cdot\prod_{i=1}^{k}|\mathcal{H}_i| \leq k\cdot\left(\frac{\sum_{i=1}^{k}|\mathcal{H}_i|}{k}\right)^k \leq k\cdot\left(\frac{m}{k}\right)^k,$$

we get the claim. ◀

Putting these Claims 6 and 7 together, and noting that at internal nodes the running time is $O(nm\rho)$, and that the roots of the maximal sub-trees in $\mathbf{T}_2$ are the leaves of $\mathbf{T}_1$, the lemma follows. ◀

4 Solving Proper-$\mathcal{L}$-Coloring in time $(nm)^{o(k^2 \log(nm))}$

For a hypergraph $\mathcal{H} \subseteq 2^V$ and a positive number $\epsilon \in (0,1)$, denote by $T(\mathcal{H}, \epsilon)$ the subset $\{v \in V : \deg_{\mathcal{H}}(v) > \epsilon|\mathcal{H}|\}$ of "high" degree vertices in $\mathcal{H}$. Given $\epsilon', \epsilon'' \in (0,1)$, let us call an (ϵ', ϵ'')-*balanced set* with respect to $\mathcal{H}$, any set $S \subseteq V$ such that $\epsilon'|\mathcal{H}| \leq |\mathcal{H}_S| \leq \epsilon''|\mathcal{H}|$.

▶ **Lemma 8** ([19]). *Let $\epsilon_1, \epsilon_2 \in (0,1)$ be two given numbers such that, $\epsilon_1 < \epsilon_2$ and $T = T(\mathcal{H}, \epsilon_1)$ satisfies $|\mathcal{H}_T| \leq (1-\epsilon_2)|\mathcal{H}|$. Then there exists a $(1-\epsilon_2, 1-(\epsilon_2-\epsilon_1))$-balanced set $S \supseteq T$ with respect to $\mathcal{H}$. Such a set S can be found in $O(nm)$ time.*

▶ **Lemma 9.** *Let $\mathcal{H} \subseteq 2^V$ be a hypergraph satisfying (1), $\mathcal{L} : V \to 2^{[k]}$ be a mapping, $\chi : V \to [0:k]$ be a proper partial $\mathcal{L}$-coloring of $\mathcal{H}$, and $S \subseteq V_0$ be a given set of vertices such that $(\mathcal{H}_0)_S \neq \emptyset$. Then, χ is extendable to a proper $\mathcal{L}$-coloring of $\mathcal{H}$ if and only if either*

$$\chi \text{ is extendable to a proper } \mathcal{L}\text{-coloring for } \bar{\mathcal{H}}_0 \cup \mathcal{H}_0^S, \text{ or} \tag{6}$$

$$\exists Y \in \mathcal{H}_0^S \setminus (\mathcal{H}_0)_S,\ j \in \bigcap_{v \in Y} \mathcal{L}(v) :\ \chi \text{ is extendable to a proper } \mathcal{L}\text{-coloring } \chi' \text{ for } \mathcal{H} \text{ such that } \chi'(Y) = \{j\}. \tag{7}$$

Proof. Suppose that χ is extendable to a proper $\mathcal{L}$-coloring χ' for $\mathcal{H}$. If (6) is not satisfied then (since $\emptyset \notin \mathcal{H}_0^S$ by (1)) there is a $Y \in \mathcal{H}_0^S \setminus (\mathcal{H}_0)_S$, such that (in any proper extension χ' of χ) $\chi'(Y) = \{j\}$, for some $j \in \bigcap_{v \in Y} \mathcal{L}(v)$, and hence (7) is satisfied.

Conversely, if either (6) or (7) holds then there is an $\mathcal{L}$-coloring extension χ' of χ that properly colors $\mathcal{H}$. ◀

▶ **Lemma 10.** *Let $\mathcal{H} \subseteq 2^V$ be a hypergraph satisfying (1), $\mathcal{L} : V \to 2^{[k]}$ be a mapping, $\chi : V \to [0:k]$ be a proper partial $\mathcal{L}$-coloring of $\mathcal{H}$, and $S \subset V_0$ be a given set of vertices such that, for some $i \in [k]$, $(\mathcal{H}_i)_{S \cup (V \setminus V_0)} \neq \emptyset$. Then χ is extendable to a proper $\mathcal{L}$-coloring of $\mathcal{H}$ if and only if either*

$$\chi \text{ is extendable to a proper } \mathcal{L}\text{-coloring for } \bar{\mathcal{H}}_i^{S \cup (V \setminus V_0)} \cup (\mathcal{H}_i)_{S \cup (V \setminus V_0)}, \text{ or} \tag{8}$$

$$\exists j \neq i,\ Y \in \mathcal{H}_j^{S \cup (V \setminus V_0)} \setminus (\mathcal{H}_j)_{S \cup (V \setminus V_0)} :\ j \in \bigcap_{v \in Y} \mathcal{L}(v),\ \chi \text{ is extendable to a proper } \mathcal{L}\text{-coloring } \chi' \text{ for } \mathcal{H} \text{ such that } \chi'(Y) = \{j\}. \tag{9}$$

Proof. Suppose that χ is extendable to a proper $\mathcal{L}$-coloring χ' for $\mathcal{H}$. If (8) is not satisfied then (since $\emptyset \notin \bar{\mathcal{H}}_i^{S \cup (V \setminus V_0)}$ by(1)) there is a $Y \in \mathcal{H}_j^{S \cup (V \setminus V_0)}$ for some $j \neq i$, such that $\chi'(Y) = \{j\}$ and $j \in \bigcap_{v \in Y} \mathcal{L}(v)$, and hence (9) is satisfied.

Conversely, suppose that either (8) or (9) holds. If (9) is satisfied then χ' is a proper $\mathcal{L}$-coloring of $\mathcal{H}$ which extends χ. On the other hand, if (8) holds then there is an $\mathcal{L}$-coloring $\chi' : S \cup (V \setminus V_0) \to [k]$ such that $|\chi(H)| \geq 2$ for all $H \in \bar{\mathcal{H}}_i^{S \cup (V \setminus V_0)} \cup (\mathcal{H}_i)_{S \cup (V \setminus V_0)}$. Then χ' can be extended to a proper $\mathcal{L}$-coloring for $\mathcal{H}$ by setting $\chi'(v) \in \mathcal{L}(v) \setminus \{i\}$ arbitrarily for $v \in V_0 \setminus S$ (as $H \cap (V_0 \setminus S) \neq \emptyset$ for all $H \in \mathcal{H}_i \setminus (\mathcal{H}_i)_{S \cup (V \setminus V_0)}$). ◀

▶ **Lemma 11.** *Let $\mathcal{H} \subseteq 2^V$ be a hypergraph satisfying (1), $\mathcal{L} : V \to 2^{[k]}$ be a mapping, $\chi : V \to [0:k]$ be a proper partial $\mathcal{L}$-coloring of $\mathcal{H}$ such that $\mathcal{H}_i \neq \emptyset$ for at least two i's, and $\epsilon_1, \epsilon_2 \in (0,1)$ be two given numbers such that, $\epsilon_1 < \epsilon_2$. Then either* (i) *there is $v \in V_0$ and $i \neq j$ such that $\deg_{\mathcal{H}_i}(v) \geq \epsilon_1|\mathcal{H}_i|$ and $\deg_{\mathcal{H}_j}(v) \geq \epsilon_1|\mathcal{H}_j|$, or* (ii) *there is a $(1-\epsilon_2, 1-(\epsilon_2-\epsilon_1))$-balanced set $S \subseteq V_0$ with respect to $\mathcal{H}_j^{V_0}$ for some $j \in [k]$.*

Proof. For any $i \neq j$ such that $\mathcal{H}_i \neq \emptyset$ and $\mathcal{H}_i \neq \emptyset$, let $T_i := T(\mathcal{H}_i^{V_0}, \epsilon_1)$ and $T_j := T(\mathcal{H}_j^{V_0}, \epsilon_1)$. If $T_i \cap T_j \neq \emptyset$ then any v in this intersection will satisfy (i). Otherwise, (1) implies that either $(\mathcal{H}_i^{V_0})_{T_i} = \emptyset$ or $(\mathcal{H}_j^{V_0})_{T_j} = \emptyset$, in which case a $(1-\epsilon_2, 1-(\epsilon_2-\epsilon_1))$-balanced set with respect to $\mathcal{H}_i^{V_0}$ or $\mathcal{H}_j^{V_0}$, respectively, can be obtained by Lemma 8. ◀

Algorithm 2 is more sophisticated than Algorithm 1 as it does not require the existence of a large-degree vertex, but uses more complicated decomposition rules, given by lemmas 9 and 10. As before, the algorithm proceeds in two phases. As long as there is a large "volume" of edges with no assigned colors, that is $|\mathcal{H}_0|\sum_{H\in\mathcal{H}_0}|H| \geq \delta$, the algorithm is still in phase I; otherwise it proceeds to phase II. In a general step of phase I (resp., phase II), the algorithm tries, in step 3 (resp., step 35), to find a vertex v of large degree in $\mathcal{H}_0$ (resp., in $\mathcal{H}_i$ and $\mathcal{H}_j$ for some $i \neq j$) and iterates over all feasible assignments of colors to v, that result in no monochromatic edges; if no such v can be found then Lemma 8 guarantees the existence of a $(1-\epsilon_2, 1-(\epsilon_2-\epsilon_1))$-balanced set with respect to $\mathcal{H}_0$ (resp., with respect to either $\mathcal{H}_i^{V_0}$ or $\mathcal{H}_j^{V_0}$ as in Lemma 11), which is found in step 13 (resp, 38). Lemma 9 (resp., Lemma 10) then reduces the problem in the latter case to checking (6) and (7) (resp., (8) and (9)), which is done in steps 14, and 17-23 (resp., in steps 39, and 43-45), respectively. If none of the recursive calls yields a feasible extension of the current proper partial $\mathcal{L}$-coloring χ, we return that there are no proper $\mathcal{L}$-colorings (lines 11, 24, 32 and 46).

To analyze the running time of the algorithm, let us measure the volume of a subproblem in phase I by $\mu_2 = \mu_2(\mathcal{H},\chi) = |\mathcal{H}_0|\sum_{H\in\mathcal{H}_0}|H|$, and in phase II by $\mu_2 = \mu_2(\mathcal{H},\chi)$ given by (3). Phase II (and hence the recursion) stops when $\mu_2(\mathcal{H},\chi) \leq \delta$, or an $\mathcal{L}$-coloring has been found.

Given a subproblem of volume μ, let $\epsilon(\mu) := \frac{\ln(e\rho)}{\xi(\mu)}$, where $\xi(\mu)$ is the unique positive root of the equation:

$$\left(\frac{\xi(\mu)}{2\ln(e\rho)}\right)^{\xi(\mu)} = \mu^2. \tag{10}$$

Note that (for constant ρ) $\chi(\mu) \approx O\left(\frac{\log\mu}{\log\log\mu}\right)$. We set $\delta \geq 2\rho$ such that $\xi(\delta) \geq 2k\ln(e\rho)$. Note that $\xi(\mu) \geq 2$ and $\epsilon(\mu) \leq \frac{1}{2k}$, for $\mu \geq \delta$. We use in the algorithm: $\epsilon_1(\mu) := \epsilon(\mu)$ and $\epsilon_2(\mu) := 2\epsilon(\mu)$.

▶ **Lemma 12.** *Algorithm 2 solves problem* Proper-$\mathcal{L}$-Coloring *in time* $(mn)^{o(k^2\log(mn))}$.

Consider the recursion tree $\mathbf{T}$ of the algorithm. Let $\mathbf{T}_1$ (reps., $\mathbf{T}_2$) be the subtree (resp., sub-forest) of $\mathbf{T}$ belonging to phase I (resp., phase II) of the algorithm. Let $B_1(\mu)$ (resp., $B_2(\mu)$) be the total number of nodes in $\mathbf{T}_1$ (resp., $\mathbf{T}_2$) that result from a subproblem of volume μ. The lemma follows from the following two claims whose proofs are given in the appendix.

▶ **Claim 13.** *The number of nodes in* $\mathbf{T}_1$ *is at most* $(\delta(\rho\delta+1)+1)(m^2n)^{\xi(m^2n)}$.

Proof. If there is a vertex $v \in V_0$ such that $\deg_{\mathcal{H}_0}(v) \geq \epsilon_1|\mathcal{H}_0|$ then the algorithm proceeds with steps 4-11 and we get the recurrence:

$$B_1(\mu_1) \leq \rho \cdot B_1(\lfloor(1-\epsilon_1)\mu_1\rfloor) + 1, \tag{11}$$

since we recurse in step 8 on a hypergraph $\mathcal{H}'$ that excludes all the edges containing v from $\mathcal{H}'_0$. On the other hand, if no such v can be found then Lemma 8 implies that there is a $(1-\epsilon_2, 1-(\epsilon_2-\epsilon_1))$-balanced set S, with respect to some $\mathcal{H}_0$, which is found in

Algorithm 2 PROPER-$\mathcal{L}$-COLORING-B$(\mathcal{H}, \mathcal{L}, \chi)$

Input: hypergraph $\mathcal{H} \subseteq 2^V$, a mapping $\mathcal{L} : V \to 2^{[k]}$, and a proper partial $\mathcal{L}$-coloring $\chi : V \to [0:k]$
Output: TRUE (resp,. FALSE) if a proper $\mathcal{L}$-coloring $\chi : V \to [k]$ of $\mathcal{H}$ is (resp., cannot be) found

1: $V_0 := V_0(\chi)$; $\mathcal{H}_i := \mathcal{H}_i(\chi)$ for $i \in [0:k]$
2: **if** $\mu_1 := \mu_1(\mathcal{H}, \chi)| > \delta$ **then** /* Phase I */
3: **if** there is $v \in V_0$ such that $\deg_{\mathcal{H}_0}(v) \geq \epsilon_1(\mu_1)|\mathcal{H}_0|$ **then**
4: **for** each $j \in \mathcal{L}(v)$ **do**
5: $\chi(v) := j$
6: **if** there is no $H \in \mathcal{H}$ such that $\chi(H) = \{j\}$ **then** /* if no edge becomes monochromatic */
7: $\mathcal{H}' := \mathcal{H} \setminus \bigcup_{i \in [k],\ i \neq j} \{H \in \mathcal{H}_i :\ j \in \chi(H)\}$ /* delete non-monochromatic edges */
8: **return** PROPER-$\mathcal{L}$-COLORING-B$(\mathcal{H}', \mathcal{L}, \chi)$
9: **end if**
10: **end for**
11: $\chi(v) := 0$; **return** FALSE
12: **else**
13: Let S be a $(1 - \epsilon_2, 1 - (\epsilon_2(\mu_1) - \epsilon_1(\mu_1)))$-balanced set computed as in Lemma 8 w.r.t $\mathcal{H}_0$
14: **if** PROPER-$\mathcal{L}$-COLORING-B$(\bar{\mathcal{H}}_0 \cup \mathcal{H}_0^S, \mathcal{L}, \chi)$ **then**
15: stop /* A proper $\mathcal{L}$-coloring has been found */
16: **else**
17: **for** each $Y \in \mathcal{H}_0^S \setminus (\mathcal{H}_0)_S$ and $j \in \bigcap_{v \in Y} \mathcal{L}(v)$ **do**
18: $\chi(Y) := \{j\}$
19: **if** there is no $H \in \mathcal{H}$ such that $\chi(H) = \{j\}$ **then** /* if no edge becomes monochromatic */
20: $\mathcal{H}' := \mathcal{H} \setminus \bigcup_{i \in [k],\ i \neq j} \{H \in \mathcal{H}_i :\ j \in \chi(H)\}$ /* delete non-monochromatic edges */
21: **return** PROPER-$\mathcal{L}$-COLORING-B$(\mathcal{H}', \mathcal{L}, \chi)$
22: **end if**
23: **end for**
24: $\chi(Y) := \{0\}$; **return** FALSE
25: **end if**
26: **end if**
27: **else**/* Phase II */
28: **if** $\mu_2 := \mu_2(\mathcal{H}, \chi) \leq \delta$ or $|\{i :\ \mathcal{H}_i \neq \emptyset\}| = 1$ **then**
29: **if** PROPER-$\mathcal{L}$-COLORING-simple$(\mathcal{H}, L, \chi)$ **then**
30: Stop /* A proper $\mathcal{L}$-coloring has been found */
31: **else**
32: **return** FALSE
33: **end if**
34: **else**
35: **if** there is $v \in V_0$ and $i \neq j$ such that $\deg_{\mathcal{H}_i}(v) \geq \epsilon_1(\mu_2)|\mathcal{H}_i|$ and $\deg_{\mathcal{H}_j}(v) \geq \epsilon_1|\mathcal{H}_j|$ **then**
36: Same as in steps 4-11 of Phase I
37: **else**
38: Let S be a $(1 - \epsilon_2, 1 - (\epsilon_2(\mu_2) - \epsilon_1(\mu_2)))$-balanced set computed as in Lemma 8 w.r.t $\mathcal{H}_i^{V_0}$ for some $i \in [k]$
39: **if** PROPER-$\mathcal{L}$-COLORING-B$(\bar{\mathcal{H}}_i^{S \cup (V \setminus V_0)} \cup (\mathcal{H}_i)_{S \cup (V \setminus V_0)}, \mathcal{L}, \chi)$ **then**
40: Set $\chi(v) \in \mathcal{L}(v) \setminus \{i\}$ arbitrarily for $v \in V_0 \setminus S$
41: stop /* A proper $\mathcal{L}$-coloring has been found */
42: **else**
43: **for** each $j \neq i$ and $Y \in \mathcal{H}_j^{S \cup (V \setminus V_0)} \setminus (\mathcal{H}_j)_{S \cup (V \setminus V_0)}$ s.t. $j \in \bigcap_{v \in Y} \mathcal{L}(v)$ **do**
44: Same as in steps 18-22 of Phase I
45: **end for**
46: $\chi(Y) := \{0\}$; **return** FALSE
47: **end if**
48: **end if**
49: **end if**
50: **end if**

step 13. Then we apply Lemma 9 which reduces the problem to one recursive call on the hypergraph $\bar{\mathcal{H}}_0 \cup \mathcal{H}_0^S$ in step 14, and at most $|\mathcal{H}_0^S \setminus (\mathcal{H}_0)_S|$ recursive calls (in step 21) on the hypergraphs obtained by fixing the color of one set $Y \in \mathcal{H}_0^S \setminus (\mathcal{H}_0)_S$. Note that S satisfies: $(1 - \epsilon_2)|\mathcal{H}_0| \leq |(\mathcal{H}_0)_S \leq |(1 - (\epsilon_2 - \epsilon_1))|\mathcal{H}_0|$. In particular, there is an $H \in \mathcal{H}_0$

such that $H \setminus S \neq \emptyset$. Hence, in step 14, we recurse on the hypergraph $\bar{\mathcal{H}}_0 \cup \mathcal{H}_0^S$ which excludes at least one vertex from H. Moreover, in step 21, we recurse on a hypergraph $\mathcal{H}'$ that has $\mathcal{H}_0' \subseteq \mathcal{H}_0 \setminus (\mathcal{H}_0)_S$, as all edges in $(\mathcal{H}_0)_S$ have non-empty intersections with the set $Y \in \mathcal{H}_0^S \setminus (\mathcal{H}_0)_S$ in the current iteration of the loop in step 17, all vertices of which are assigned color j. Since $|\mathcal{H}_0 \setminus (\mathcal{H}_0)_S| \leq \epsilon_2 |\mathcal{H}_0|$, we get the recurrence:

$$B_1(\mu_1) \leq B_1(\mu_1 - 1) + \rho\epsilon_2\mu_1 \cdot B_1(\lfloor \epsilon_2\mu_1 \rfloor) + 1. \tag{12}$$

By the termination condition of phase I (in line 2), we have $B_1(\mu_1) = 1$ for $\mu_1 \leq \delta$. We will prove by induction on $\mu_1 \geq \delta$ that $B_1(\mu_1) \leq C \cdot \mu_1^{\xi(\mu_1)}$, for $C := \delta(\rho\delta + 1) + 1$.

Let us consider first recurrence (11). If $(1 - \epsilon_1)\mu_1 \leq \delta$ then we get $B_1(\mu_1) \leq \rho + 1 < C$. Otherwise, we apply induction to get

$$\begin{aligned} B_1(\mu_1) &\leq C \cdot \rho((1-\epsilon_1)\mu_1)^{\xi(\mu_1)} + 1 \leq C \cdot \mu_1^{\xi(\mu_1)} \left(\rho(1-\epsilon_1)^{\chi(\mu_1)} + \frac{1}{\mu_1^{\xi(\mu_1)}} \right) \\ &\leq C \cdot \mu_1^{\xi(\mu_1)} \left(\rho e^{-\epsilon_1 \xi(\mu_1)} + \frac{1}{\mu_1^{\xi(\mu_1)}} \right) \\ &\leq C \cdot \mu_1^{\xi(\mu_1)} \left(\frac{1}{e} + \frac{1}{4} \right) < C \cdot \mu_1^{\xi(\mu_1)}, \quad (\text{as } \epsilon_1\xi(\mu_1) = \ln(e\rho) \text{ and } \xi(\mu_1) \geq 2 \text{ for } \mu_1 \geq \delta). \end{aligned}$$

Let us consider next recurrence (12). We consider 3 cases:

Case 1. $\mu_1 - 1 \leq \delta$: Then $\lfloor \epsilon_2\mu_1 \rfloor \leq \delta$ and (12) reduces to $B_1(\mu_1) \leq \rho\epsilon_2\mu_1 + 2 \leq \rho(\delta+1) + 2 < C$.

Case 2. $\epsilon_2\mu_1 \leq \delta$: Then (12) reduces to

$$B_1(\mu_1) \leq B_1(\mu_1 - 1) + \rho\epsilon_2\mu_1 + 1 \leq B_1(\mu_1 - 1) + \rho\delta + 1.$$

Iterating we get $B_1(\mu_1) \leq B_1(\mu_1 - r) + r(\rho\delta + 1) \leq r(\rho\delta + 1) + 1$, for $r = \mu_1 - \delta \leq \left(\frac{1}{\epsilon_2} - 1\right)\delta$. Thus, $B_1(\mu_1) \leq \left(\frac{1}{\epsilon_2} - 1\right)\delta(\rho\delta + 1) + 1$. As $\epsilon_2 = 2\epsilon(\mu_1) = \frac{2\ln(e\rho)}{\xi(\mu_1)}$, we get $B_1(\mu_1) \leq \left(\frac{\xi(\mu_1)}{2\ln(e\rho)} - 1\right)\delta(\rho\delta + 1) + 1 \leq (\mu_1 - 1)\,\delta(\rho\delta + 1) + 1 \leq C\mu_1^{\xi(\mu_1)}$, for $\mu_1 \geq \delta$.

Case 3. $\mu_1 - 1 > \delta$ and $\epsilon_2\mu_1 > \delta$: We apply induction:

$$\begin{aligned} B_1(\mu_1) &\leq C(\mu_1 - 1)^{\xi(\mu_1)} + C \cdot \rho\epsilon_2\mu_1(\epsilon_2\mu_1)^{\xi(\mu_2)} + 1 \\ &\leq C \cdot \mu_1^{\xi(\mu_1)} \left(\left(1 - \frac{1}{\mu_1}\right)^{\xi(\mu_1)} + \rho\epsilon_2\mu_1\epsilon_2^{\xi(\mu_1)} + \frac{1}{\mu_1^{\xi(\mu_1)}} \right) \\ &\leq C \cdot \mu_1^{\xi(\mu_1)} \left(\left(1 - \frac{1}{\mu_1}\right)^{\xi(\mu_1)} + \frac{\rho\epsilon_2}{\mu_1} + \frac{1}{\mu_1^{\xi(\mu_1)}} \right) \quad \left(\text{since } \epsilon_2^{\xi(\mu_1)} = \frac{1}{\mu_1^2} \text{ by (10)}\right) \\ &\leq C \cdot \mu_1^{\xi(\mu_1)} \left(\left(1 - \frac{1}{\mu_1}\right)^2 + \frac{1}{\mu_1} + \frac{1}{\mu_1^2} \right) \quad \left(\text{since } \epsilon_2 \leq \frac{1}{k} \text{ and } \xi(\mu_1) \geq 2 \text{ for } \mu_1 \geq \delta\right) \\ &\leq C \cdot \mu_1^{\xi(\mu_1)} \quad (\text{since } \mu_2 \geq \delta > 2). \end{aligned}$$

Using $\mu_1(\mathcal{H}, \chi) \leq m^2 n$, we get the claim. ◀

▶ **Claim 14.** *The number of nodes in any sub-tree of $\mathbf{T}_2$ is at most*

$$(\delta(\rho\delta + 1) + 1)(m^k/k^{k-1})^{\xi(m^k/k^{k-1})}.$$

Proof. If $\mu_2(\mathcal{H},\chi) \leq \delta$ (and already $\mu_1(\mathcal{H},\chi) \leq \delta$), then Lemma 4 implies that problem Proper-$\mathcal{L}$-Coloring can be solved in time $O((\rho n)^{(k+1)\delta})$, as $m \leq \delta$. If there is $v \in V_0$ and $i \neq j$ such that $\deg_{\mathcal{H}_i}(v) \geq \epsilon_1|\mathcal{H}_i|$ and $\deg_{\mathcal{H}_j}(v) \geq \epsilon_1|\mathcal{H}_j|$ then the algorithm proceeds similar to steps 4-11 and we get the recurrence:

$$B_2(\mu_2) \leq \rho \cdot B_2(\lfloor (1-\epsilon_1)\mu_2 \rfloor) + 1, \tag{13}$$

since we recurse (in the step similar to step 8) on a hypergraph $\mathcal{H}'$ that excludes either all the edges containing v from $\mathcal{H}'_i$, if we set the color of v to j, or all those containing v from $\mathcal{H}'_j$ if we set the color of v to i (or both, if we set the color of v to $\ell \notin \{i,j\}$).

On the other hand, if no such v can be found then Lemma 11 implies that there is a $(1-\epsilon_2, 1-(\epsilon_2-\epsilon_1))$-balanced set S, with respect to some $\mathcal{H}_i$, which is found in step 38. Then we apply Lemma 10 which reduces the problem to one recursive call on the hypergraph $\bar{\mathcal{H}}_i^{S\cup(V\setminus V_0)} \cup (\mathcal{H}_i)_{S\cup(V\setminus V_0)}$ in step 39, and at most $\sum_{j\neq i} |\mathcal{H}_j^{S\cup(V\setminus V_0)} \setminus (\mathcal{H}_j)_{S\cup(V\setminus V_0)}|$ recursive calls on the hypergraphs obtained by fixing the color of one set $Y \in \mathcal{H}_j^{S\cup(V\setminus V_0)} \setminus (\mathcal{H}_j)_{S\cup(V\setminus V_0)}$. As S satisfies: $(1-\epsilon_2)|\mathcal{H}_i| \leq |(\mathcal{H}_i)_{S\cup(V\setminus V_0)}| \leq (1-(\epsilon_2-\epsilon_1))|\mathcal{H}_i|$, in step 39 we recurse on the hypergraph $\mathcal{H}' := \bar{\mathcal{H}}_i^{S\cup(V\setminus V_0)} \cup (\mathcal{H}_i)_{S\cup(V\setminus V_0)}$ with $\mu_2(\mathcal{H}',\chi) \leq (1-(\epsilon_2-\epsilon_1))\mu_2(\mathcal{H},\chi)$. Moreover, for each Y satisfying the condition in step 43, we recurse on a hypergraph $\mathcal{H}'$ that has $\mathcal{H}'_i \subseteq \mathcal{H}_i \setminus (\mathcal{H}_i)_{S\cup(V\setminus V_0)}$, as all edges in $(\mathcal{H}_i)_{S\cup(V\setminus V_0)}$ have non-empty intersections with the set Y, all vertices of which are assigned color $j \neq i$. Since $|\mathcal{H}_i \setminus (\mathcal{H}_i)_{S\cup(V\setminus V_0)}| \leq \epsilon_2|\mathcal{H}_i|$, we get the recurrence:

$$B_2(\mu_2) \leq B_2(\lfloor (1-(\epsilon_2-\epsilon_1))\mu_2 \rfloor) + \rho\mu_2 \cdot B_2(\lfloor \epsilon_2\mu_2 \rfloor) + 1. \tag{14}$$

By the stopping criterion in line 28, we have $B_2(\mu_2) = 1$ for $\mu_2 \leq \delta$. We will prove by induction on $\mu_2 \geq \delta$ that $B_2(\mu_2) \leq C \cdot \mu_2^{\xi(\mu_2)}$, for $C := \delta(\rho\delta+1)+1$.

As recurrence (13) is the same as (11), we need only to consider recurrence (14). We consider 3 cases:

Case 1. $(1-(\epsilon_2-\epsilon_1))\mu_2 \leq \delta$: Then $\epsilon_2\mu_2 \leq \frac{2\epsilon(\mu_2)}{1-\epsilon(\mu_2)}\delta \leq \frac{2}{2k-1}\delta < \delta$ for $\mu_2 \geq \delta$ (recall that $\epsilon(\mu_2) \leq \frac{1}{2k}$ for $\mu_2 \geq \delta$), and hence (13) reduces to $B_2(\mu_2) \leq \rho\mu_2 + 2 \leq \frac{\rho\delta}{1-\epsilon(\mu_2)} + 2 \leq \frac{2k\rho\delta}{2k-1} + 2 < 2(\rho\delta+1) < C$.

Case 2. $\epsilon_2\mu_2 \leq \delta$: Then (12) reduces to

$$B_2(\mu_2) \leq B_2((1-\epsilon(\mu_2))\mu_2) + \rho\epsilon_2\mu_2 + 1 \leq B_2((1-\epsilon(\mu_2))\mu_2) + \rho\delta + 1.$$

Iterating we get $B_2(\mu_2) \leq B_2(\mu_2(1-\epsilon(\mu_2))^r) + r(\rho\delta+1) \leq r(\rho\delta+1)+1$, for $r = \frac{\ln(\mu_2/\delta)}{\epsilon(\mu_2)} \leq \frac{\ln(1/\epsilon(\mu_2))}{\epsilon(\mu_2)}$. Thus, $B_2(\mu_2) \leq \left(\frac{\ln(1/\epsilon(\mu_2))}{\epsilon(\mu_2)}\right)(\rho\delta+1)+1$. As $\frac{\ln(1/\epsilon(\mu_2))}{\epsilon(\mu_2)} \leq \delta\mu_2^{\xi(\mu_2)}$ for $\mu_2 \geq \delta \geq 2$ (since $\left(\frac{1}{2\epsilon(\mu_2)}\right)^{\xi(\mu_2)} = \mu_2^2$ by (10)), we get $B_2(\mu_2) \leq \delta(\rho\delta+1)+1 \leq C\mu_1^{\xi(\mu_2)}$.

Case 3. $(1-(\epsilon_2-\epsilon_1))\mu_2 > \delta$ and $\epsilon_2\mu_2 > \delta$: We apply induction:

$$\begin{aligned}
B_2(\mu_2) &\le C((1-\epsilon(\mu_2))\mu_2)^{\xi(\mu_2)} + C\cdot\rho\mu_2(\epsilon_2(\mu_2)\mu_2)^{\xi(\mu_2)} + 1\\
&\le C\cdot\mu_2^{\xi(\mu_2)}\left((1-\epsilon(\mu_2))^{\xi(\mu_2)} + \rho\mu_2(\epsilon_2(\mu_2))^{\xi(\mu_2)} + \frac{1}{\mu_2^{\xi(\mu_2)}}\right)\\
&\le C\cdot\mu_2^{\xi(\mu_2)}\left((1-\epsilon(\mu_2))^{\xi(\mu_2)} + \frac{\rho}{\mu_2} + \frac{1}{\mu_2^{\xi(\mu_2)}}\right) \quad (\text{since } (\epsilon_2(\mu_2))^{\xi(\mu_2)} = \frac{1}{\mu_2^2} \text{ by (10)})\\
&\le C\cdot\mu_2^{\xi(\mu_2)}\left(e^{-\epsilon(\mu_2)\xi(\mu_2)} + \frac{\rho}{\mu_2} + \frac{1}{\mu_2^{\xi(\mu_2)}}\right)\\
&\le C\cdot\mu_2^{\xi(\mu_2)}\left(\frac{1}{e\rho} + \frac{\rho}{\mu_2} + \frac{1}{\mu_2^{\xi(\mu_2)}}\right) \quad (\text{since } \epsilon(\mu_2) = \frac{\ln(e\rho)}{\xi(\mu_2)}\ )\\
&\le C\cdot\mu_2^{\xi(\mu_2)}\left(\frac{1}{4} + \frac{1}{2} + \frac{1}{16}\right) < C\cdot\mu_2^{\xi(\mu_2)}. \quad (\text{since } \xi(\mu_2)\ge 2 \text{ for } \mu_2\ge\delta\ge 2\rho)
\end{aligned}$$

Using the bound $\mu_1(\mathcal{H},\chi) \le k\cdot\left(\frac{m}{k}\right)^k$, we get the claim.

Putting these Claims 6 and 7 together, the lemma follows. ◀

▶ Remark. Theorem 1 can be extended to the case when the input hypergraph $\mathcal{H}$ is *almost intersecting*, that is, if for all $H \in \mathcal{H}$,

$H \cap H' = \emptyset$ for at most $O(1)$ edges $H' \in \mathcal{H}$.

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