Finding a Maximum 2-Matching Excluding Prescribed Cycles in Bipartite Graphs

Kenjiro Takazawa*

Department of Industrial and Systems Engineering, Faculty of Science and Engineering, Hosei University, Tokyo 184-8584, Japan
takazawa@hosei.ac.jp

Abstract

We introduce a new framework of restricted 2-matchings close to Hamilton cycles. For an undirected graph (V, E) and a family $\mathcal{U}$ of vertex subsets, a 2-matching F is called $\mathcal{U}$-feasible if, for each $U \in \mathcal{U}$, F contains at most $|U| - 1$ edges in the subgraph induced by U. Our framework includes $C_{\leq k}$-free 2-matchings, i.e., 2-matchings without cycles of at most k edges, and 2-factors covering prescribed edge cuts, both of which are intensively studied as relaxations of Hamilton cycles. The problem of finding a maximum $\mathcal{U}$-feasible 2-matching is NP-hard. We prove that the problem is tractable when the graph is bipartite and each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. This case generalizes the $C_{\leq 4}$-free 2-matching problem in bipartite graphs. We establish a min-max theorem, a combinatorial polynomial-time algorithm, and decomposition theorems by extending the theory of $C_{\leq 4}$-free 2-matchings. Our result provides the first polynomially solvable case for the maximum $C_{\leq k}$-free 2-matching problem for $k \geq 5$. For instance, in bipartite graphs in which every cycle of length six has at least two chords, our algorithm solves the maximum $C_{\leq 6}$-free 2-matching problem in $\mathrm{O}(n^2 m)$ time, where n and m are the numbers of vertices and edges, respectively.

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1 Introduction

The Hamilton cycle problem is one of the most fundamental NP-hard problems in various research fields such as graph theory, computational complexity, and combinatorial optimization. One successful approach to the Hamilton cycle problem is to utilize matching theory. In a graph $G = (V, E)$, an edge set $F \subseteq E$ is a *2-matching* (resp., *2-factor*) if it has at most (resp., exactly) two edges incident to each vertex in V. Since a Hamilton cycle is a special kind of 2-matching (or 2-factor) and a 2-matching of maximum size can be found in polynomial time, it is reasonable to put restrictions on 2-matchings to provide a tight relaxation of Hamilton cycles to which matching theory can be applied. Examples include the following two kinds of restricted 2-matchings:

$C_{\leq k}$-free 2-matchings. For a positive integer k, a 2-matching is called *$C_{\leq k}$-free* if it contains no cycles of length at most k. The larger k becomes, the closer a $C_{\leq k}$-free 2-factor becomes to a Hamilton cycle. If $k \geq |V|/2$, a $C_{\leq k}$-free 2-factor is a Hamilton cycle, whereas a $C_{\leq 2}$-free 2-matching is nothing other than a 2-matching.

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2-factors covering prescribed edge cuts. An *edge cut* is a minimal set of edges whose removal makes the graph disconnected. Given a family $\mathcal{K}$ of edge cuts, an edge subset is called $\mathcal{K}$*-covering* if it intersects every edge cut in $\mathcal{K}$. A Hamilton cycle is exactly a $\mathcal{K}$-covering 2-factor, where $\mathcal{K}$ is the family of all edge cuts.

Recently both $C_{\leq k}$-free and $\mathcal{K}$-covering 2-factors have been intensively studied and applied to designing approximation algorithms for NP-hard problems related to the Hamilton cycle problem, such as the graph-TSP and the minimum 2-edge connected spanning subgraph problem [4, 5, 8, 12, 20, 30, 32, 33].

1.1 Previous Work

In general graphs, the $C_{\leq k}$-free 2-matching problem is much more difficult than the 2-matching problem. For the cases $k \geq 3$, no algorithm is known other than Hartvigsen's $C_{\leq 3}$-free 2-matching algorithm [14]. NP-hardness for the case $k \geq 5$ is proved by Papadimitriou (see [7]). More generally, Hell et al. [17] proved that the problem is NP-hard, unless the excluded length of a cycle is a subset of $\{3, 4\}$. The case $k = 4$ is still open, and conjectured to be solvable in polynomial time [9]. Discrete convexity shown in [22] supports this conjecture.

While only a few positive results are known for the $C_{\leq k}$-free 2-matching problem in general graphs, in bipartite graphs the $C_{\leq 4}$-free 2-matching problem is efficiently solvable, and fundamental theorems in matching theory are extended. Motivated by a stimulating paper of Hartvigsen [15], Király [21] gave a min-max theorem for the $C_{\leq 4}$-free 2-matching problem in bipartite graphs, followed by a different min-max theorem by Frank [11]. Comparison of these two theorems is discussed in [31], together with decomposition theorems corresponding to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions. Polynomial combinatorial algorithms are designed by Hartvigsen [16] and Pap [25], which are again slightly different and followed by an improvement in time complexity by Babenko [1]. For the weighted version, while the NP-hardness of the weighted $C_{\leq 4}$-free 2-matching problem in bipartite graphs is proved by Király (see [11]), positive results such as a linear programming formulation with dual integrality [23], a combinatorial algorithm [29], and discrete convexity [22] are established when the edge weights satisfy a certain property. Since the $C_{\leq 6}$-free 2-matching problem is NP-hard even in bipartite graphs [13], the $C_{\leq 4}$-free 2-matching problem in bipartite graphs is one of the few cases where the $C_{\leq k}$-free 2-matching problem is tractable.

For a set of positive integers $A \subseteq \mathbf{Z}$, denote the set of edge cuts whose sizes belong to A by $\mathcal{K}_A$. Kaiser and Škrekovski [18] proved that every bridgeless planar cubic graph has a $\mathcal{K}_{\{3,4\}}$-covering 2-factor, which is extended to a stronger result that every bridgeless cubic graph has a $\mathcal{K}_{\{3,4\}}$-covering 2-factor [19]. While the proof in [19] was not algorithmic, Boyd, Iwata, and Takazawa [4] designed a combinatorial algorithm for finding a $\mathcal{K}_{\{3,4\}}$-covering 2-factor in bridgeless cubic graphs, together with a combinatorial algorithm for finding a minimum-weight $\mathcal{K}_{\{3\}}$-covering 2-factor in bridgeless cubic graphs. Čada et al. [6] exhibited a family of graphs which has no $\mathcal{K}_{\{4,5\}}$-covering edge subset with even degree at every vertex, disproving a conjecture in [19].

1.2 Our Contribution

In the present paper, we introduce a new framework of restricted 2-matchings which commonly generalizes $C_{\leq k}$-free 2-matchings and $\mathcal{K}$-covering 2-factors. Let $G = (V, E)$ be a graph. For $U \subseteq V$, let $G[U] = (U, E[U])$ denote the subgraph induced by U, i.e., $E[U] = \{uv \in E \colon u, v \in U\}$. For $F \subseteq E$, let $F[U] = F \cap E[U] = \{uv \in F \colon u, v \in U\}$.

▶ **Definition 1** ($\mathcal{U}$-feasible 2-matching)**.** Let $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets. A 2-matching $F \subseteq E$ is called *$\mathcal{U}$-feasible* if $|F[U]| \leq |U| - 1$ for each $U \in \mathcal{U}$.

Equivalently, a 2-matching F is $\mathcal{U}$-feasible if and only if F does not contain a 2-factor in $G[U]$ for each $U \in \mathcal{U}$. We remark that F does not only excludes a Hamilton cycle in $G[U]$, but also any 2-factor in $G[U]$ consisting of possibly multiple cycles.

If F is a 2-factor, then F is $\mathcal{U}$-feasible if and only if $F \cap \delta(U) \neq \emptyset$ for every $U \in \mathcal{U}$, where $\delta(U)$ denotes the set of edges having exactly one endpoint in U. From these viewpoints, it is not difficult to see that Hamilton cycles, $C_{\leq k}$-free 2-matchings, and $\mathcal{K}$-covering 2-factors are special cases of $\mathcal{U}$-feasible 2-factors or 2-matchings. That is, if we put $\mathcal{U} = \{U \subseteq V : |U| \leq |V|/2\}$ and $\mathcal{U} = \{U \subseteq V : \delta(U) \in \mathcal{K}\}$, then the set of $\mathcal{U}$-feasible 2-factors are exactly that of Hamilton cycles and $\mathcal{K}$-covering 2-factors, respectively. If putting $\mathcal{U} = \{U \subseteq V : |U| \leq k\}$, then the set of $\mathcal{U}$-feasible 2-factors is exactly that of $C_{\leq k}$-free 2-matchings.

The *$\mathcal{U}$-feasible 2-matching problem* is defined as a problem of finding a $\mathcal{U}$-feasible 2-matching of maximum size for given G and $\mathcal{U}$. In order to discuss the time complexity of the $\mathcal{U}$-feasible 2-matching problem, we should notice how $\mathcal{U}$ is given. In some cases, the size of $\mathcal{U}$ might be exponential in $|V|$, e.g., $\mathcal{U} = \{U \subseteq V : |U| \leq |V|/2\}$. Nevertheless, in many cases it is efficiently determined whether a given edge set is $\mathcal{U}$-feasible, such as the $C_{\leq k}$-free 2-matching case and the $\mathcal{K}$-covering 2-factor case. Therefore, we denote by γ the time for determining whether an edge set is $\mathcal{U}$-feasible, and we seek an algorithm with running time polynomial in $|V|$ and γ.

Since the Hamilton cycle problem is a special case of the $\mathcal{U}$-feasible 2-matching problem, the $\mathcal{U}$-feasible 2-matching problem is NP-hard in general. Thus, we need some assumption in order to obtain a tractable class of the $\mathcal{U}$-feasible 2-matching problem, such as the cases where G is bipartite and $\mathcal{U} = \{U \subseteq V : |U| \leq 4\}$, and G is bridgeless cubic and $\mathcal{U} = \{U \subseteq V : \delta(U) \in \mathcal{K}_{\{3,4\}}\}$.

A main objective of this paper is to provide a broader tractable class of the $\mathcal{U}$-feasible 2-matching problem by extending the theory of $C_{\leq 4}$-free 2-matchings in bipartite graphs. For this purpose, we exploit a graph-theoretic concept of *Hamilton-laceable graphs*. For a bipartite graph (V, E), we denote the two color classes by V^+ and V^-. For $X \subseteq V$, let $X^+ = X \cap V^+$ and $X^- = X \cap V^-$.

▶ **Definition 2** (Hamilton-laceable graph [26])**.** A bipartite graph $G = (V, E)$ is *Hamilton-laceable* if (i) $|V^+| = |V^-|$ and G has a Hamilton path between an arbitrary pair of $u \in V^+$ and $v \in V^-$, or (ii) $|V^+| = |V^-| - 1$ and G has a Hamilton path between an arbitrary pair of distinct vertices $u, v \in V^-$.

In what follows, we work on the $\mathcal{U}$-feasible 2-matching problem under the assumption that G is bipartite and $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. We note that, for a 2-factor F, $|F[U]| = |U|$ implies that $|U^+| = |U^-|$. Thus, we assume $|U^+| = |U^-|$ for each $U \in \mathcal{U}$, and hence only the case (i) in Definition 2 occurs in our arguments.

If we take into account the original motivation, i.e., finding a 2-factor close to a Hamilton cycle, then $\mathcal{U}$ should be a vertex set family as large as possible. Thus a natural setting would be to define $\mathcal{U}$ as the family of all vertex sets $U \subseteq V$ such that $G[U]$ is Hamilton-laceable. This indeed provides a new framework of 2-matchings closer to Hamilton cycles.

Furthermore, several types of the $C_{\leq k}$-free 2-matching problem are described as the $\mathcal{U}$-feasible 2-matching problem under our assumption. The smallest nontrivial example of a Hamilton-laceable graph would be a cycle of length four, and hence the $C_{\leq 4}$-free 2-matching problem in bipartite graphs is a special case of the $\mathcal{U}$-feasible 2-matching problem under our assumption. As for $C_{\leq 6}$-free-free 2-matchings, a cycle of length six is Hamilton-laceable if it has at least two chords. Thus, the $C_{\leq 6}$-free 2-matching problem in bipartite graphs in which every cycle of length six has at least two chords is described as the $\mathcal{U}$-feasible 2-matching problem under our assumption. In other words, in our setting a solution (a $\mathcal{U}$-feasible 2-matching) might contain a cycle of length six with at most one chord, but it can exclude all the cycles of length six with at least two chords. Further examples and previous work of Hamilton-laceable graphs are exhibited in § 2.

In the present paper, we exhibit that the theory of $C_{\leq 4}$-free 2-matching problem in bipartite graphs satisfactorily extends when $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. We first present a min-max theorem extending Király's min-max theorem [21]. We then design a combinatorial algorithm for finding a maximum $\mathcal{U}$-feasible 2-matching, which provides a constructive proof for our min-max theorem. In the design of our algorithm, we make use of both of Hartvigsen's and Pap's algorithms [16, 25]: the shrinking technique comes from Pap's algorithm; and the construction of a minimizer of the min-max theorem derives from Hartvigsen's method. Finally, we describe decomposition theorems extending those in [31] and corresponding to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions.

Here we summarize our algorithmic results. We denote the number of vertices and edges in the input graph by n and m, respectively. Recall that γ is the time for determining whether an edge set is $\mathcal{U}$-feasible.

► **Theorem 3.** *Let $G = (V, E)$ be a bipartite graph and $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets such that $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. Then a $\mathcal{U}$-feasible 2-matching of maximum size in G can be found in $\mathrm{O}(n^3\gamma + n^2m)$ time.*

We remark that, when our algorithm is applied to the $C_{\leq k}$-free 2-matching case, i.e., the case $\mathcal{U} = \{U \subseteq V : |U| \leq k,\ G[U]$ is Hamilton-laceable$\}$, γ becomes the time for determining if a specified edge is contained in a cycle of length at most k in a given 2-matching, and thus $\gamma = \mathrm{O}(k)$. (The detail is described in § 4.3.) Therefore, the following theorem is established.

► **Theorem 4.** *The $C_{\leq k}$-free 2-matching problem in bipartite graphs is solvable in $\mathrm{O}(kn^3 + n^2m)$ time if every cycle of length at most k induces a Hamilton-laceable graph.*

In particular, it holds that $\gamma = \mathrm{O}(1)$ if k is a constant. By setting $\mathcal{U} = \{U \subseteq V : |U| \leq 4,\ G[U]$ is Hamilton-laceable$\}$, we can see that Theorem 3 extends the solvability of the $C_{\leq 4}$-free 2-matching problem in bipartite graphs. Moreover, setting $\mathcal{U} = \{U \subseteq V : |U| \leq 6,\ G[U]$ is Hamilton-laceable$\}$ in Theorem 3 leads to the following corollaries on the $C_{\leq 6}$-free 2-matching problem in bipartite graphs.

► **Corollary 5.** *In a bipartite graph, a maximum 2-matching excluding any cycle of length six with at least two chords and any cycle of length four can be found in $\mathrm{O}(n^2m)$ time.*

► **Corollary 6.** *In a bipartite graph in which every cycle of length six has at least two chords, the $C_{\leq 6}$-free-free 2-matching problem can be solved in $\mathrm{O}(n^2m)$ time.*

To the best of our knowledge, Corollary 6 is the first polynomially solvable case of the $C_{\leq 6}$-free 2-matching problem. Furthermore, combined with Lemma 8 in § 2, Theorem 3 leads to the following corollary, an extension of Corollary 6.

► **Corollary 7.** *The $C_{\leq k}$-free 2-matching problem in bipartite graphs is solvable in $\mathrm{O}(kn^3 + n^2m)$ time if every cycle of length $2t$ such that $2t \leq k$ has at least $(t-1)(t-2)$ chords.*

Note that Corollary 6 is exactly the case $k = 6$ of Corollary 7.

It is noteworthy that, unlike the literature of $C_{\leq k}$-free 2-matchings and $\mathcal{K}$-covering 2-factors, our assumption that each $G[U]$ is Hamilton-laceable does not depend on the size of the forbidden structures. As stated above, one benefit of this is that our result provides the first polynomially solvable case of the $C_{\leq k}$-free 2-matching problem for $k \geq 5$, and thus has a potential to provide better approximation ratios for the graph-TSP and the minimum 2-edge connected subgraph problem.

We further remark that our framework contains the both cases where multiplicities on edges are forbidden and allowed. That is, in the former case we only deal with simple 2-matchings and one edge can only contribute one to the degree of its endpoints. In the latter case, we can put multiplicity two on some edges. In the literature of the $C_{\leq k}$-free 2-matching problem, these two cases have formed different streams. The aforementioned results are of the former case, and results for the latter case include [2, 7, 24]. To the best of our knowledge, not much connection between these two cases is found. In our framework, forbidding multiplicity on an edge $uv \in E$ corresponds to having $\{u, v\}$ in $\mathcal{U}$, and it is clear that $G[\{u, v\}]$ is Hamilton-laceable if $uv \in E$. While in this paper we mainly keep the former case in mind, we note that our framework can represent both cases.

1.3 Organization of the Paper

The rest of the paper is organized as follows. In § 2, we present some previous work, observations, and examples of Hamilton-laceable graphs. After that, we exhibit our contribution on $\mathcal{U}$-feasible 2-matchings in bipartite graphs where $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. We present a min-max theorem in § 3. Section 4 is devoted to describing a combinatorial algorithm for finding a maximum $\mathcal{U}$-feasible 2-matching, which provides a constructive proof for the min-max theorem. In § 5, we exhibit decomposition theorems corresponding to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions. In § 6, we demonstrate an application of our framework by showing that a regular bipartite graph admit a certain kind of $\mathcal{U}$-feasible 2-factor. Section 7 concludes this paper.

2 Hamilton-Laceable Graph

This section is devoted to a discussion on Hamilton-laceable graphs. We first note that the concept of Hamilton-laceable graphs is a bipartite analogue of that of *Hamilton-connected graphs*, which is well-known in the field of graph theory [3]. A graph is *Hamilton-connected* if it has a Hamilton path between an arbitrary pair of distinct vertices. Thus, a Hamilton-connected graph is nonbipartite if it has at least three vertices.

In what follows, we always assume that $G = (V, E)$ is bipartite. Trivial examples of a Hamilton-laceable graph are a graph of a single vertex, and a graph of two vertices connected by an edge. It is also clear that a complete bipartite graph on $2t$ vertices, denoted by $K_{t,t}$, is Hamilton-laceable. Recall that a special case $K_{2,2}$, a cycle of length four, is an example of a Hamilton-laceable graph.

If $G = (V, E)$ is Hamilton-laceable, a graph $(V, \tilde{E})$ satisfying $\tilde{E} \supseteq E$ is also Hamilton-laceable. Thus, it would be of interest to find Hamilton-laceable graphs with as few edges as possible. Indeed, the concept of Hamilton-laceable graphs was introduced as a generalized property of Hamiltonicity of *d-dimensional rectangular lattices* by Simmons [26]. A d-dimensional rectangular lattice is a graph (V, E) represented by d positive integers $a_1, \ldots, a_d$ as $V = \{x \in \mathbf{Z}^d : 0 \leq x_i \leq a_i,\ i = 1, \ldots, d\}$ and $E = \{xy : x, y \in V,\ \sum_{i=1}^{d} |x_i - y_i| = 1\}$. Simmons [26] proved that all d-dimensional rectangular lattices are Hamilton-laceable except for the two-dimensional lattices of order $2 \times r$ ($r \neq 2$) and $3 \times 2r$. This result provides a class of Hamilton-laceable graphs (V, E) with $|E| \approx d|V|$. For instance, every hypercube is

Hamilton-laceable. The following lemma also provides a sufficient condition for a graph to be Hamilton-laceable.

▶ **Lemma 8** (Simmons [28]). *Deleting fewer than $t-1$ edges from $K_{t,t}$ or $K_{t,t+1}$ maintains Hamilton-laceability.*

Furthermore, Simmons [27] discussed the minimum number l_t of the edges of Hamilton-laceable graphs with $|V^+| = t$. It holds that $3t - \lceil t/3 \rceil \leq l_t \leq 3t-1$ for the case (i) in Definition 2, and $l_t = 3t+1$ for the case (ii) in Definition 2.

The motivation of introducing Hamilton-laceable graph in this paper comes from an analysis in [31], which reveals that cycles of length four in the $C_{\leq 4}$-free 2-matching problem in bipartite graphs serve as factor-critical components for the nonbipartite matching problem: if $U \subseteq V$ induces a cycle of length four in a bipartite graph, for an arbitrary pair $u \in U^+$ and $v \in U^-$, $G[U]$ contains a 2-matching of size three in which only u and v have degree one. Indeed, this is the property which makes it possible to execute the shrinking and expanding procedures in the algorithms in [16, 25], which is shed light on by a decomposition theorem [31] resembling the Edmonds-Gallai decomposition. Observe that the definition of Hamilton-laceable graphs generalizes the above property of C_4. In the following sections we reveal that the property in Definition 2(i) plays a key role to provide a tractable class of restricted 2-matchings in bipartite graphs.

3 Min-Max Theorem

In this section, we describe a min-max theorem for the $\mathcal{U}$-feasible 2-matching problem in bipartite graphs where each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. Our theorem is an extension of Király's min-max theorem [21] for the $C_{\leq 4}$-free 2-matching problem in bipartite graphs. For $X \subseteq V$, let $\bar{X} = V \setminus X$ and $c'(X)$ denote the number of components in $G[X]$ consisting of a single vertex, a single edge, or a single cycle of length four.

▶ **Theorem 9** ([21]). *Let $G = (V, E)$ be a bipartite graph. Then, it holds that*

$$\max\{|F| \colon F \text{ is a } C_{\leq 4}\text{-free 2-matching}\} = \min\{|V| + |X| - c'(\bar{X}) \colon X \subseteq V\}.$$

Observe that every component contributing to $c'(\bar{X})$ is Hamilton-laceable. We now exhibit our theorem extending Theorem 9. For $X \subseteq V$, let $c(X)$ denote the number of components in $G[X]$ whose vertex set belongs to $\mathcal{U}$.

▶ **Theorem 10.** *Let $G = (V, E)$ be a bipartite graph and $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets in G such that $G[U]$ is Hamilton-laceable for each $U \in \mathcal{U}$. Then, it holds that*

$$\max\{|F| \colon F \text{ is a } \mathcal{U}\text{-feasible 2-matching}\} = \min\{|V| + |X| - c(\bar{X}) \colon X \subseteq V\}. \tag{1}$$

It is not difficult to see that Theorem 9 is indeed a special case of Theorem 10 where $\mathcal{U} = \{U \subseteq V \colon |U| \leq 4,\ G[U] \text{ is Hamilton-laceable}\}$.

Before proving Theorem 10, we first show that the inequality max $\leq$ min in (1) holds for an arbitrary G and $\mathcal{U}$, i.e., G may not be bipartite and $G[U]$ may not be Hamilton-laceable for $U \in \mathcal{U}$. For disjoint vertex sets $X, Y \subseteq V$, let $E[X, Y]$ denote the set of edges connecting X and Y, $G[X, Y] = (X \cup Y, E[X, Y])$, and $F[X, Y] = F \cap E[X, Y]$ for $F \subseteq E$.

▶ **Lemma 11.** *Let $G = (V, E)$ be a graph and $\mathcal{U} \subseteq 2^V$ be a family of vertex subsets in G. For an arbitrary $\mathcal{U}$-feasible 2-matching F and $X \subseteq V$, it holds that $|F| \leq |V| + |X| - c(\bar{X})$.*

Proof. Since F is a 2-matching, $2|F[X]| + |F[X, \bar{X}]| \le 2|X|$ follows. Moreover, since F is $\mathcal{U}$-feasible, it holds that $|F[\bar{X}]| \le |\bar{X}| - c(\bar{X})$. Therefore, $|F| = |F[X]| + |F[X, \bar{X}]| + |F[\bar{X}]| \le 2|F[X]| + |F[X, \bar{X}]| + |F[\bar{X}]| \le 2|X| + |\bar{X}| - c(\bar{X}) = |V| + |X| - c(\bar{X})$. ◀

The following lemma directly follows from the proof for Lemma 11. For $F \subseteq E$ and $u \in V$, denote the number of edges in F incident to u by $\deg_F(u)$.

▶ **Lemma 12.** *If a $\mathcal{U}$-feasible 2-matching F and $X \subseteq V$ attain the equality in (1), it holds that*

- $F[X] = \emptyset$,
- $\deg_{F[\{u\}, \bar{X}]}(u) = 2$ *for each* $u \in X$, *and*
- *for each component Q in $G[\bar{X}]$,*

$$|F[V(Q)]| = \begin{cases} |V(Q)| - 1 & \text{if } V(Q) \in \mathcal{U}, \\ |V(Q)| & \text{otherwise.} \end{cases}$$

Proof. By the above proof for Lemma 11, it should hold that $|F[X]| = 0$, $|F[X, \bar{X}]| = 2|X|$, and $|F[\bar{X}]| = |\bar{X}| - c(\bar{X})$ for a $\mathcal{U}$-feasible 2-matching F and $X \subseteq V$ attaining the equality in (1). These respectively lead to the statements in the lemma. ◀

In § 4, we complete a proof of Theorem 10 by establishing an algorithm for finding a $\mathcal{U}$-feasible 2-matching F and $X \subseteq V$ attaining equality in (1). It should be noted that the bipartiteness of G and Hamilton-laceability of $G[U]$ for each $U \in \mathcal{U}$ play an important role in the algorithm, and thus they are key properties to achieving equality in (1) as well.

4 Combinatorial Algorithm

In this section, we describe a combinatorial polynomial-time algorithm for finding a maximum $\mathcal{U}$-feasible 2-matching in bipartite graphs where each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. Our algorithm employs ideas of both of the $C_{\le 4}$-free 2-matching algorithms of Hartvigsen [16] and Pap [25].

4.1 Algorithm Description

Roughly speaking, our algorithm resembles Edmonds' algorithm for nonbipartite matchings [10]. One main feature in our algorithm comes from Pap's algorithm [25]: we shrink $U \in \mathcal{U}$ after we find an alternating path, whereas in Edmonds' and Hartvigsen's algorithms shrinking occurs in the middle of construction of alternating forests. Another feature derives from Hartvigsen's algorithm [16]. A minimizer $X \subseteq V$ of the right-hand side of (1) is basically determined as the set of vertices reachable from the deficient vertices, vertices having at most one incident edge in the optimal solution. In Hartvigsen's and our algorithms, if a vertex resulting from shrinking $U \in \mathcal{U}$ satisfies certain properties, it is regarded as reachable even if it is not reachable.

Before describing the entire algorithm, we present how to shrink and expand $U \in \mathcal{U}$. In order to provide concise notation, in the rest of this section we denote the input of the algorithm by $\hat{G} = (\hat{V}, \hat{E})$ and $\hat{\mathcal{U}} \subseteq 2^{\hat{V}}$, and the graph obtained by repeated shrinkings by $G = (V, E)$. Following standard notation, for a vector $b \in \mathbf{Z}^V$, an edge set $F \subseteq E$ is called a *b-matching* if every vertex $v \in V$ is incident to at most $b(v)$ edges in F. If every vertex $v \in V$ is incident to exactly $b(v)$ edges in F, then F is called a *b-factor*. If $b(v) = t$ for every $v \in V$,

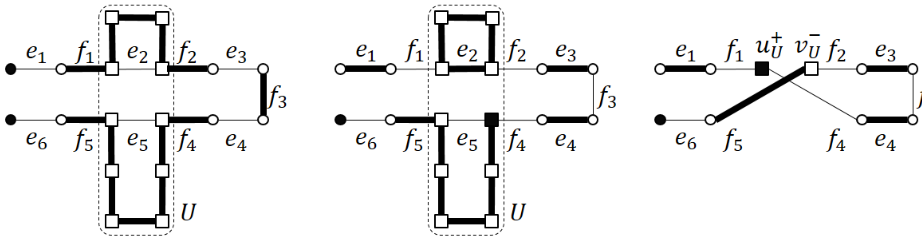

Figure 1 $b_v = 2$ for each v. The thick edges are in F, thin edges in $E \setminus F$, and the vertices in black are in S^+ or S^-. The ten vertices represented by squares form $U \in \mathcal{U}$. In the figure on the left, we have found P consisting of $e_1, f_1, e_2, \ldots, e_5, f_5, e_6$, and $F \triangle (E(P))$ contains a 2-factor in $G[U]$ for $U \in \mathcal{U}$. In this case $i^* = 5$, and the figure in the middle shows $F \triangle (E(P_4))$. The figure on the right shows the graph after $\mathsf{Shrink}(F, P)$.

then a b-matching is simply referred to as a t-matching. Note that this notation is compatible with our definition of 2-matchings.

In the algorithm, we maintain a $\mathcal{U}$-feasible b-matching F in G, where $\mathcal{U} \subsetneq 2^V$ and $b \in \{1,2\}^V$, which can be extended to a $\hat{\mathcal{U}}$-feasible 2-matching in $\hat{G}$. For $b \in \{1,2\}^V$, a b-matching F is *$\mathcal{U}$-feasible* if $F[U]$ is not a b_U-factor in $G[U]$ for every $U \in \mathcal{U}$, where b_U is the restriction of b to U. Initially, $G = \hat{G}$, $\mathcal{U} = \hat{\mathcal{U}}$, $b_v = 2$ for each $v \in V$, and F is an arbitrary $\mathcal{U}$-feasible b-matching, e.g., $F = \emptyset$.

For $F_1, F_2 \subseteq E$, denote the symmetric difference of F_1 and F_2 by $F_1 \triangle F_2$, i.e., $F_1 \triangle F_2 = (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$. Define the set of source vertices by $S^+ = \{u \in V^+ \colon \deg_F(u) < b_u\}$ and sink vertices $S^- = \{v \in V^- \colon \deg_F(v) < b_v\}$. Suppose that we have found an alternating path P with respect to F and $E \setminus F$ such that P starts in S^+ and ends in S^-, and $F \triangle (E(P))$ is not a $\mathcal{U}$-feasible b-matching. We then apply the following shrinking procedure.

Procedure Shrink(F, P). Denote $E(P) = \{e_1, f_1, e_2, \ldots, e_l, f_l, e_{l+1}\}$, where the edges are sorted by the order of appearance in P. Note that $e_j \in E \setminus F$ $(j = 1, \ldots, l+1)$ and $f_j \in F$ $(j = 1, \ldots, l)$. Let P_i be the path consisting of $\bigcup_{j=1}^{i} \{e_j, f_j\}$ for $i = 1, \ldots, l$, P_0 be an empty graph, and $P_{l+1} = P$. Let i^* be the smallest index i such that $F \triangle (E(P_i))$ contains a b-factor in $G[U]$ for some $U \in \mathcal{U}$, and let $F' = F \triangle (E(P_{i^*-1}))$. If more than one such $U \in \mathcal{U}$ exists, choose an arbitrary U. We then update G, b, $\mathcal{U}$, and F as follows. Let u_U^+ and v_U^- be new vertices obtained by contracting the vertices in U^+ and U^-, respectively. Then, reset

$$V := \bar{U} \cup \{u_U, v_U\}, \quad b_v := \begin{cases} 1 & \text{if } v = u_U^+, v_U^-, \\ b_v & \text{otherwise,} \end{cases}$$

$$E := E[\bar{U}] \cup \{u_U^+ v \colon uv \in E, u \in U^+, v \in \bar{U}^-\} \cup \{u v_U^- \colon uv \in E, u \in \bar{U}^+, v \in U^-\},$$

$$F := F'[\bar{U}] \cup \{u_U^+ v \colon uv \in F', u \in U^+, v \in \bar{U}^-\} \cup \{u v_U^- \colon uv \in F', u \in \bar{U}^+, v \in U^-\},$$

$$\mathcal{U} := \{U' \colon U' \in \mathcal{U}, U' \cap U = \emptyset\} \cup \{(U' \setminus U) \cup \{u_U^+, v_U^-\} \colon U' \in \mathcal{U}, U \subsetneq U'\}.$$

See Figure 1 for an illustration. Observe that the update preserves that G is bipartite and F is still a b-matching in G. We then repeat the above procedure.

If an alternating path P from S^+ to S^- is found such that $F \triangle (E(P))$ is a $\mathcal{U}$-feasible b-matching, then we reset $F := F \triangle (E(P))$ to augment the current solution, and expand the shrunk vertex sets to return to the original graph $\hat{G}$ as follows. First note that the shrunk vertex sets in $\hat{\mathcal{U}}$ form a laminar family, and it suffices to expand the maximal shrunk vertex sets. Let $\mathcal{U}^* \subseteq 2^{\hat{V}}$ be the family of maximal shrunk vertex sets. For a maximal shrunk

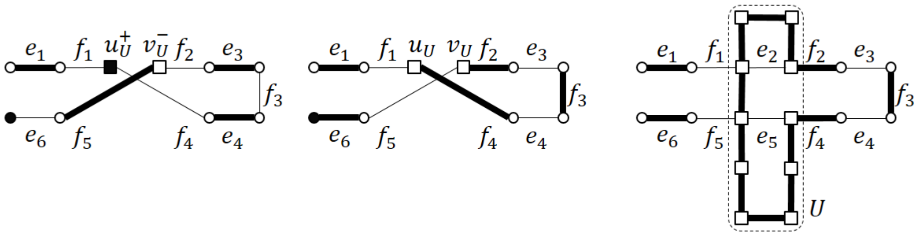

Figure 2 The graph in the middle results from an augmentation in the graph on the left, where the augmenting path P consists of $f_4, e_4, f_3, e_3, f_2, f_5, e_6$. We then expand U, where $\hat{f}_U^+ = f_4$ and $\hat{f}_U^- = f_2$. to obtain the graph on the right.

vertex set $U \subseteq \hat{V}$, denote the unique edge in F incident to u_U^+ by f_U^+, and to v_U^- by f_U^-, if exist. Let $\hat{f}_U^+, \hat{f}_U^- \in \hat{E}$ be the edges corresponding to $f_U^+, f_U^- \in E$, respectively. Denote the vertex in U^+ incident to $\hat{f}_U^+$ by $\hat{u}_U^+$, and that in U^- incident to $\hat{f}_U^-$ by $\hat{v}_U^-$. If f_U^+ (resp., f_U^-) does not exist, let $\hat{u}_U^+$ (resp., $\hat{v}_U^-$) be an arbitrary vertex in U^+ (resp., U^-). Now, since $\hat{G}[U]$ is Hamilton-laceable, $\hat{G}[U]$ has a Hamilton path P_U between $\hat{u}_U^+$ and $\hat{v}_U^-$. In expanding U, we add $E(P_U)$ to F. That is, $\hat{F} := F \cup \bigcup_{U \in \mathcal{U}^*} E(P_U)$. See Figure 2 for an illustration of augmentation and expansion. It is not difficult to see that $\hat{F}$ is a $\mathcal{U}$-feasible 2-matching.

The entire algorithm is described as follows.

Input: A bipartite graph $\hat{G} = (\hat{V}, \hat{E})$ and $\hat{\mathcal{U}} \subseteq 2^{\hat{V}}$ such that $\hat{G}[U]$ is Hamilton-laceable for each $U \in \hat{\mathcal{U}}$.

Output: A maximum $\hat{\mathcal{U}}$-feasible 2-matching $\hat{F}$ in $\hat{G}$.

Step 0: Put $G = \hat{G}$ and $\mathcal{U} = \hat{\mathcal{U}}$. Let F be an arbitrary $\mathcal{U}$-feasible 2-matching in G. Let F be an arbitrary $\mathcal{U}$-feasible 2-matching in G and then go to Step 1.

Step 1: Let $S^+ = \{u \in V^+ \colon \deg_F(u) < b_u\}$ and $S^- = \{v \in V^- \colon \deg_F(v) < b_v\}$. Orient each edge in $E \setminus F$ from V^+ to V^- and each edge in F from V^- to V^+ to obtain a directed graph D. If D has a directed path P from S^+ to S^-, then go to Step 2. Otherwise, go to Step 5.

Step 2: Let $E_P \subseteq E$ be the set of edges corresponding to the directed edges in P. If $F' = F \triangle E_P$ is a $\mathcal{U}$-feasible b-matching, then go to Step 3. Otherwise, go to Step 4.

Step 3 (Augmentation): Reset $F := F'$, expand all maximal shrunk vertex sets, and then go back to Step 1.

Step 4 (Shrinking): Apply $\mathsf{Shrink}(F, P)$, and then go back to Step 1.

Step 5 (Termination): Expand all maximal shrunk vertex sets and return $\hat{F}$.

4.2 Proof for Correctness

At the termination of the algorithm, we have a digraph D in which no directed path from S^+ to S^- exists. Let $R \subseteq V$ denote the set of vertices reachable from S^+ in D, and define $R' \subseteq V$ by

$$\begin{aligned} R' = R &\cup \{v \in (\bar{R})^- \colon v \text{ is not a shrunk vertex}, \deg_{F[R^+, \{v\}]}(v) = 2\} \\ &\cup \{v \in (\bar{R})^- \colon v = v_U^- \text{ for some } U \in \mathcal{U},\, uv \in F \text{ for some } u \in R^+\}. \end{aligned}$$

Finally, define $X \subseteq \hat{V}$ by the set of vertices corresponding to $(\overline{R'})^+ \cup (R')^-$, i.e.,

$$\begin{aligned} X = &\{u \in \hat{V}^+ \colon u \in (\overline{R'})^+ \text{ or } u \in U \text{ for some } U \in \mathcal{U} \text{ with } u_U \in (\overline{R'})^+\} \\ &\cup \{v \in \hat{V}^- \colon v \in (R')^- \text{ or } v \in U \text{ for some } U \in \mathcal{U} \text{ with } v_U \in (R')^-\}. \end{aligned}$$

▶ **Lemma 13.** *The output $\hat{F}$ and X defined above attain the equality in* (1)*.*

Proof. It is not difficult to see that $\hat{F}[X] = \emptyset$. Moreover, since every $v \in X$ satisfies $\deg_{\hat{F}[\{v\},\bar{X}]} = 2$, we have that $|\hat{F}[X, \bar{X}]| = 2|X|$. Finally, since R is defined by reachability from S^+ in D, all edges in $E[\bar{X}]$ belong to F in G. Thus, each edge in $\hat{E}[\bar{X}]$ is in $\hat{F}$ or belongs to $\hat{E}[U]$ for some $U \in \mathcal{U}$ shrunk in G. By the definition of R', it holds that v_U^- has no adjacent edge in $E[\bar{X}]$, which implies that $\hat{G}[U]$ forms a component in $\hat{G}[\bar{X}]$. Thus, it follows that $|\hat{F}[\bar{X}]| = |\bar{X}| - c(\bar{X})$. Therefore, $|\hat{F}| = |\hat{F}[X]| + |\hat{F}[X, \bar{X}]| + |\hat{F}[\bar{X}]| = 2|X| + |\bar{X}| - c(\bar{X}) = |V| + |X| - c(\bar{X})$. ◀

Now Theorem 10 immediately follows from Lemmas 11 and 13. Thus, our algorithm provides a constructive proof for Theorem 10.

4.3 Complexity

Recall that $n = |\hat{V}|$, $m = |\hat{E}|$, and γ is the time for determining if an edge set is $\hat{\mathcal{U}}$-feasible. It is not difficult to see that shrinkings occur $\mathrm{O}(n)$ times between augmentations. Since augmentations occur $\mathrm{O}(n)$ times, shrinkings occur $\mathrm{O}(n^2)$ times in total.

After each shrinking, we search an alternating path, which takes $\mathrm{O}(m)$ time. Moreover, we determine if $F \triangle (E(P_i))$ is $\mathcal{U}$-feasible $\mathrm{O}(n)$ times for each shrinking. The time for this determination is γ in general. If we consider the $C_{\leq k}$-free 2-matching problem, then it suffices to determine if e_i is contained in a cycle of length at least k in $F \triangle (E(P_i))$, which takes $\mathrm{O}(k)$ time. Thus, the time complexity between shrinkings is $\mathrm{O}(n\gamma + m)$ in general, and is $\mathrm{O}(kn + m)$ for the $C_{\leq k}$-free 2-matching case. Therefore, Theorems 3 and 4 are established.

5 Decomposition Theorems

This section is devoted to decomposition theorems for the $\mathcal{U}$-feasible 2-matching problem in bipartite graphs where each $U \in \mathcal{U}$ induces a Hamilton-laceable graph. These theorems correspond to the Dulmage-Mendelsohn and Edmonds-Gallai decompositions, and extend decomposition theorems for the $C_{\leq 4}$-free 2-matchings in bipartite graphs [31]. Proofs for the theorems in this section will appear in a full version of this paper.

Let $X_1 \subseteq V$ be a minimizer of (1) obtained by the algorithm in § 4. By exchanging the roles of V^+ and V^-, i.e., searching alternating paths from S^- to S^+, we obtain another minimizer $X_2 \subseteq V$ of (1). Now partition V into three sets $D, A, C \subseteq V$, where $D = \bar{X}_1^+ \cup \bar{X}_2^-$, $A = X_2^+ \cup X_1^-$, and $C = V \setminus (D \cup A)$.

Now the following theorems are established. Theorem 14 provides a characterization of D. Note that such a characterization appears in both of the Dulmage-Mendelsohn and Edmonds-Gallai decompositions. Theorem 15 corresponds to the Dulmage-Mendelsohn decomposition, and suggests that X_1 and X_2 are canonical minimizers of (1). Finally, Theorem 16 corresponds to the Edmonds-Gallai decomposition. Figure 3 should help in understanding the statements in Theorem 16.

▶ **Theorem 14.** $D = \{v \colon \exists a$ *maximum* $\mathcal{U}$*-feasible* 2*-matching* F *with* $\deg_F(v) \leq 1\}$.

▶ **Theorem 15.** *For an arbitrary minimizer* $Y \subseteq V$ *of* (1)*, it holds that* $X_2^+ \subseteq Y^+ \subseteq X_1^+$ *and* $X_1^- \subseteq Y^- \subseteq X_2^+$.

▶ **Theorem 16.**

1. *For each* $e \in E[D, A]$*, there exists a maximum* $\mathcal{U}$*-feasible* 2*-matching containing* e.
2. *The vertex set of each component in* $G[D]$ *and* $G[D, C]$ *is a singleton or belongs to* $\mathcal{U}$.

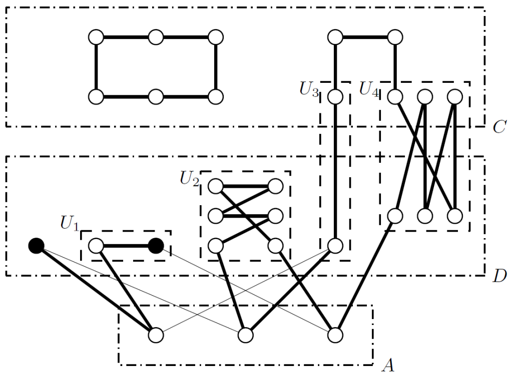

Figure 3 The thick lines are edges in a maximum $\mathcal{U}$-feasible 2-matching F, and the thin lines are edges in $E \setminus F$. The two vertices in black are those at which the degree of F is not two. The vertex sets U_1, U_2, U_3, and U_4 are in $\mathcal{U}$. Some edges in $E \setminus F$ are omitted.

3. *Shrink the components in $G[D]$ and $G[D, C]$ in the manner of* Shrink(F, P) *to obtain a new graph $G' = (V', E')$, denote the vertex subsets of V' corresponding to D, C by D', C', and define $b' \in \{1, 2\}^{D' \cup C'}$ by*

$$b'_v = \begin{cases} 1 & \text{if } v = u_U^+ \text{ or } v = v_U^- \text{ for some } U \in \mathcal{U}, \\ 2 & \text{otherwise.} \end{cases}$$

Then,

   a. *$G'[U']$ has a $b'_{U'}$-factor, and*

   b. *for arbitrary $A' \subseteq A$, it holds that $b'(\Gamma(A') \cap D') > 2|A'|$, where $\Gamma(A')$ is the set of vertices in $V \setminus A'$ adjacent to some vertex in A'.*

4. *An arbitrary maximum $\mathcal{U}$-feasible 2-matching F is composed of the following edges.*

   a. *In $G[D]$ and $G[D, C]$, F contains $|V(Q)| - 1$ edges in $E[V(Q)]$ for each component Q.*

   b. *For $u \in A$, F contains two edges connecting u and distinct components in $G[D]$.*

   c. *In $G[U]$, $F[U]$ corresponds to a $b'_{U'}$-factor in $G'[U']$.*

5. *Both $A \cup C^+$ and $A \cup C^-$ minimize* (1).

6 Applications

One main motivation of the restricted 2-matching problem is its application to designing approximation algorithms for NP-hard problems related to the TSP. Indeed, several recent work [20, 30, 32, 33] provide improved approximation ratios for the graph-TSP and the minimum 2-edge connected subgraph problem in cubic bipartite graphs, and these approximation algorithms are based on a property that ever cubic bipartite graph admits a $C_{\leq 4}$-free 2-factor. More generally, a d-regular bipartite graph with $d \geq 3$ admits a $C_{\leq 4}$-free 2-factor.

▶ **Theorem 17** ([30], see also [20, 33]). *Every d-regular bipartite graph such that $d \geq 3$ has a $C_{\leq 4}$-free 2-factor.*

Here we exhibit an extension of Theorem 17 by utilizing our min-max theorem (Theorem 10) for $\mathcal{U}$-feasible 2-matchings. That is, we prove that every regular bipartite graph admits a 2-factor which excludes not only every C_4 but also the longer cycles inducing Hamilton-laceable graphs. For a graph $G = (V, E)$ and a positive even integer k, define

$\mathcal{U}_{\leq k} \subseteq 2^V$ by $\mathcal{U}_{\leq k} = \{U \subseteq V : |U| \leq k,\ |U| \text{ is even},\ G[U] \text{ is Hamilton-laceable}\}$. We now establish the following theorem.

▶ **Theorem 18.** *Let k be a positive even integer and G be a d-regular bipartite graph such that $d \geq k/2 + 1$. Then G has a $\mathcal{U}_{\leq k}$-feasible 2-factor and it can be found in $\mathrm{O}(kn^3 + n^2m)$ time.*

Proof. By (1) in Theorem 10, it suffices to show that $|X| \geq c(\bar{X})$ holds for an arbitrary $X \subseteq V$. The time-complexity is straightforward from Theorem 3.

For $\ell = 1, \ldots, k/2$, denote by $c_\ell(\bar{X})$ the number of components in $G[\bar{X}]$ whose vertex set U satisfies that $U \in \mathcal{U}_k$ and $|U| = 2\ell$. Also denote the number of isolated vertices in $G[\bar{X}]$ by $c_0(\bar{X})$. Note that $c(\bar{X}) = \sum_{\ell=0}^{k/2} d \cdot c_\ell(\bar{X})$. Then, for $\ell = 1, \ldots, k/2$, a component in $G[\bar{X}]$ contributing to $c_\ell(\bar{X})$ has at least $2\ell(d - \ell)$ incident edges in $E[X, \bar{X}]$, and it follows that $2\ell(d - \ell) \geq 2(d - 1) \geq d$ from $d \geq k/2 + 1$. Therefore, we have that

$$|E[X, \bar{X}]| \geq d \cdot c_0(\bar{X}) + \sum_{\ell=1}^{k/2} 2\ell(d - \ell)c_\ell(\bar{X}) \geq d \cdot c_0(\bar{X}) + \sum_{\ell=1}^{k/2} d \cdot c_\ell(\bar{X}) = d \cdot c(\bar{X}).$$

Since G is d-regular, it also follows that $|E[X, \bar{X}]| \leq d|X|$. We thus conclude $|X| \geq c(\bar{X})$. ◀

It is notable that the $\mathcal{U}_{\leq k}$-feasibility of a 2-matching is a relaxed condition of $C_{\leq k}$-freeness, and they coincide when $k = 4$. Thus, Theorem 17 is a special case of Theorem 18 where $k = 4$. For the case $k = 6$, while determining whether a bipartite graph admits a $C_{\leq k}$-free 2-factor is NP-complete [13], Theorem 18 provides a sufficient condition for the existence of a 2-factor obeying a relaxed property.

The following corollary on $C_{\leq 6}$-free 2-factors is a special case $k = 6$ of Theorem 18.

▶ **Corollary 19.** *In every d-regular bipartite graph such that $d \geq 4$, there exists a 2-factor excluding any cycle of length six and with at least two chords and any cycle of length four, and such a 2-factor can be found in $\mathrm{O}(n^2m)$ time.*

7 Conclusion

We have introduced the concept of $\mathcal{U}$-feasible 2-matchings, which is a new framework of restricted 2-matchings. This concept includes those of $C_{\leq k}$-free 2-matchings and 2-factors covering prescribed edge cuts. We then extended the theory of $C_{\leq 4}$-free 2-matchings in bipartite graphs: a min-max theorem (Theorem 10), a polynomial combinatorial algorithm (Theorems 3 and 4), and decomposition theorems (Theorems 14, 15, and 16). Immediate consequences of these theorems are Corollaries 5, 6, and 7, which are, to the best of our knowledge, the first positive results on the $C_{\leq k}$-free 2-matching problem for $k \geq 6$. We have further provided an application of Theorem 10 to prove the existence of a certain kind of $\mathcal{U}$-feasible 2-factor in regular bipartite graphs (Theorem 18). Further direction of research shall include more applications of the theory established here, in particular to designing approximation algorithms for NP-hard problems related to the TSP.

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