

Strategic Contention Resolution with Limited Feedback

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Abstract

In this paper, we study contention resolution protocols from a game-theoretic perspective. We focus on *acknowledgment-based* protocols, where a user gets feedback from the channel only when she attempts transmission. In this case she will learn whether her transmission was successful or not. Users that do not transmit will not receive any feedback. We are interested in equilibrium protocols, where no player has an incentive to deviate.

The limited feedback makes the design of equilibrium protocols a hard task as best response policies usually have to be modeled as Partially Observable Markov Decision Processes, which are hard to analyze. Nevertheless, we show how to circumvent this for the case of two players and present an equilibrium protocol. For many players, we give impossibility results for a large class of acknowledgment-based protocols, namely *age-based* and *backoff* protocols with finite expected finishing time. Finally, we provide an age-based equilibrium protocol, which has infinite expected finishing time, but every player finishes in linear time with high probability.

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1 Introduction

Contention resolution in multiple access channels is one of the most fundamental problems in networking. In a multiple access channel (or broadcast channel) multiple users want to communicate with each other by sending messages into the channel. The channel is not centrally controlled, so two or more users can transmit their messages at the same time. If this happens then the messages collide and the transmission is unsuccessful. Contention resolution protocols specify how to resolve such conflicts, while simultaneously optimizing some performance measure, like channel utilization or average throughput.

In this paper we follow the standard assumption that time is divided into discrete time slots, messages are broken up into fixed sized packets, and one packet fits exactly into one time slot. Moreover, we consider one of the simplest possible scenarios where there are n users, each of them having a single packet that needs to be transmitted through the channel. When exactly one user attempts transmission in a given slot, the transmission is successful. However, if more than one users attempt transmission in the same slot, a collision occurs, their transmission fails and they need to retransmit their packages in later time slots.

Under centralized control of the users, avoiding collisions would be simple: exactly one user would transmit at each time step, alternating in a round-robin fashion. The complexity of the problem stems from the fact that there is no centralized control and therefore channel access has to be managed by a distributed protocol. There is a large body of literature that studies efficient distributed contention resolution protocols (see Section 1.2). However, these protocols work under the assumption that users will obediently follow the algorithm. In this paper we follow [9] by dropping this assumption. We model the situation as a non-cooperative stochastic game, where each user acts as a selfish *player* and tries to minimize the expected time before she transmits successfully. Therefore a player will only obey a protocol if it is in her best interest, given the other players stick to the protocol.

Fiat, Mansour, and Nadav [9] designed an incentive-compatible transmission protocol which guarantees that (with high probability) all players will transmit successfully in time linear in n . Their protocol works for a very simple channel feedback structure, where each player receives feedback of the form $0/1/2^+$ after each time step (*ternary* feedback), indicating whether zero, one, or more than one transmission was attempted. Christodoulou, Ligett and Pyrga [8] designed equilibrium protocols for *multiplicity* feedback, where each player receives as feedback the number of players that attempted transmission¹.

The above protocols fall in the class of *full-sensing* protocols [13] where the channel feedback is broadcasted to all sources. However, in wireless channels, there are situations where full-sensing is not possible because of the *hidden-terminal problem* [27]. In this paper, we focus on *acknowledgment-based* protocols, which use a more limited feedback model – the only feedback that a user gets is whether her transmission was successful or not. A user that does not transmit cannot “listen” to channel and therefore does not get any feedback. In other words, the only information that a user has is the history of her own transmission attempts. Acknowledgment-based protocols have been extensively studied in the literature (see e.g. [13] and references therein). *Age-based* and *backoff* protocols both belong to the class of acknowledgment-based protocols.

Age-based protocols can be described by a sequence of probabilities (one for each time-step) of transmitting in each time step. Those probabilities are given beforehand and do not change based on the transmission history. The well known ALOHA protocol [1] is a special

¹ They also assume non-zero transmission costs, as opposed to [9] and to this work.

age-based protocol, where – except for the first round – users always transmit with the same probability. In contrast, in *backoff* protocols, the probability of transmitting in the next time step only depends on the number of unsuccessful transmissions for the user. Here, a popular representative is the *binary exponential backoff* mechanism, which is also used by the Ethernet protocol [20].

The design and the limitations of acknowledgment-based protocols is well-understood [10, 18] if the users are not strategic. In this paper, we focus on the game-theoretic aspect of those protocols.

1.1 Our Results

We study the design of acknowledgment-based *equilibrium protocols*. A user gets feedback only when she attempts transmission, in which case she either receives an acknowledgment, in case of success, or she realizes that a collision occurred (by the lack of an acknowledgment). This model allows for very limited feedback, as opposed to *full-sensing* protocols studied in [9, 8] where all players, even those who did not attempt transmission receive channel feedback.

The feedback models used in [9, 8] allow players, at each given time, to know exactly the number of pending players. This information is very useful for the design of equilibrium protocols. In our case, we assume that the number of pending players is common knowledge only at the beginning. If a player chooses not to transmit during a time-slot, then she is not sure how many players are still in the game. From this time on, she can only sense the existence of other pending players when she participates in a collision.

The analysis of acknowledgment-based equilibrium protocols requires different techniques. In full-sensing protocols, a best response for a source can be modeled as an optimal policy of a Markov Decision Process (MDP) [9]. For an acknowledgment-based protocol, this is in general no longer possible, due to the uncertainty imposed by a non-transmission. However, the best response policy in this case can be modeled as a *Partially Observable Markov Decision Processes* (POMDP), which are more complicated to analyze.

Lack of information makes the design of equilibrium protocols a hard task. In particular, we show in Section 4 that it is impossible to design an age-based or backoff protocol that is in equilibrium and has finite expected finishing time². These impossibility results contribute to a partial characterisation of such protocols and even hold for the case of two players. This stands in contrast to the full-sensing case for which the authors in [9] give an equilibrium protocol, where the k remaining players transmit with probability $\Theta\left(\frac{1}{\sqrt{k}}\right)$. This protocol finishes in finite but exponential time.

In Section 3, we introduce and analyze an equilibrium protocol for two players. An interesting feature of our protocol is that each player is using only limited information of her own history. More precisely, the probability of transmission in a time-slot, depends only on whether a player attempted transmission in the previous slot. Our proof reduces the POMDP for the best response policy to a *finite* MDP, which we then analyze. This reduction crucially relies on the nature of our protocol. We further show that our equilibrium protocol is the unique stationary equilibrium protocol.

For more than two players, we present an age-based equilibrium protocol. Although it has infinite expected finishing time, every player finishes in linear time with high probability. Our

² Note, that for more than two players, always transmitting is an equilibrium protocol with *infinite* expected finishing time [9].

protocol circumvents the lack of information by maintaining an estimation on the number of pending players, which with high probability is an upper bound on the actual number. The protocol uses a deadline mechanism similar to [9]. Their protocol exploits the existence of their finite time equilibrium protocol mentioned above. For our more restricted model it is not known if such a finite time protocol exists for more than two players. This is the main open question left from our work. We stress that our negative results exclude the possibility that such a protocol can be age-based or backoff.

1.2 Related Work

The ALOHA protocol, introduced by Abramson [1] (and modified by Roberts [25] to its slotted version), is a multiple-access communication protocol, which has been around since the 70's. Many subsequent papers study the efficiency of multiple-access protocols when packets are generated by some stochastic process (see for example [12, 11, 24]), while worst-case scenarios of bursty inputs, were studied in [5]. To model such a worst-case scenario, one needs n nodes, each of which must simultaneously transmit a packet; this is also the model we use in this work.

A large class of contention resolution protocols explicitly deals with *conflict resolution*; where if $k \geq 2$ users collide (out of a total of n users), then a resolution algorithm is called on to resolve this conflict (by ensuring that all the colliding packets are successfully transmitted), before any other source is allowed to use the channel [7, 6, 15, 28]. There have been many positive and negative results on the efficiency of protocols under various information models (see [13] for an overview of results). When k is known, [10] provides an $O(k + \log k \log n)$ *acknowledgment-based* algorithm, while [18] provides a matching lower bound. For the ternary model, [14] provides a bound of $\Omega(k(\log n / \log k))$ for all deterministic algorithms.

A variety of game theoretic models of slotted ALOHA have also been proposed and studied; see for example [2, 17, 3]. However, much of this work only considers transmission protocols that always transmit with the same fixed probability (perhaps as a function of the number of players in the game). Other game theoretic approaches have considered pricing schemes [29] and cases in which the channel quality changes with time and players must choose their transmission levels accordingly [19, 30, 4]. [16] studied a game-theoretic model that lies between the contention and congestion model, where the decision of *when* to submit is part of the action space of the players. As discussed in the previous section, the most relevant game-theoretic model to our work, is the one studied by Fiat, Mansour, and Nadav [9] and by Christodoulou, Ligett, and Pyrga [8]. In [8], efficient ϵ -equilibrium protocols are designed, but the authors assume non-zero transmission costs, in which case the efficient protocol of [9] does not apply. Their protocols use *multiplicity* feedback (the number of attempted transmissions) which again falls in the class of full-sensing protocols.

2 Model

Game Structure. Let $N = \{1, 2, \dots, n\}$ be the set of agents, each one of which has a single packet that he wants to send through a common channel. All players know n . We assume time is discretized into slots $t = 1, 2, \dots$. The players that have not yet successfully transmitted their packet are called *pending* and initially all n players are pending. At any given time slot t , a pending player i has two available actions, either to *transmit* his packet or to *remain quiet*. In a (*mixed*) *strategy*, a player i transmits his packet at time t with some probability that potentially depends on information that i has gained from the channel based on previous transmission attempts. If exactly one player transmits in a given slot t ,

then his transmission is *successful*, the successful player exits the game (i.e. he is no longer pending), and the game continues with the rest of the players. On the other hand, whenever two or more agents try to access the channel (i.e. transmit) at the same slot, a *collision* occurs and their transmissions fail, in which case the agents remain in the game. Therefore, in case of collision or if the channel is idle (i.e. no player attempts to transmit) the set of pending agents remains unchanged. The game continues until all players have successfully transmitted their packets.

Transmission protocols. Let $X_{i,t}$ be the indicator variable that indicates whether player i attempted transmission at time t . For any $t \geq 1$, we denote by \vec{X}_t the transmission vector at time t , i.e. $\vec{X}_t = (X_{1,t}, X_{2,t}, \dots, X_{n,t})$. An *acknowledgment-based* protocol, uses very limited channel feedback. After each time step t , only players that attempted a transmission receive feedback, and the rest get no information. In fact, the information received by a player i who transmitted during t is whether his transmission was successful (in which case he gets an acknowledgement and exits the game) or whether there was a collision.

Let $\vec{h}_{i,t}$ be the vector of the *personal transmission history* of player i up to time t , i.e. $\vec{h}_{i,t} = (X_{i,1}, X_{i,2}, \dots, X_{i,t})$. We also denote by \vec{h}_t the transmission history of all players up to time t , i.e. $\vec{h}_t = (\vec{h}_{1,t}, \vec{h}_{2,t}, \dots, \vec{h}_{n,t})$. In an acknowledgment-based protocol, the actions of player i at time t depend only (a) on his personal history $\vec{h}_{i,t-1}$ and (b) on whether he is pending or not at t . A *decision rule* $f_{i,t}$ for a pending player i at time t , is a function that maps $\vec{h}_{i,t-1}$ to a probability $\Pr(X_{i,t} = 1 | \vec{h}_{i,t-1})$. For a player $i \in N$, a (*transmission*) *protocol* f_i is a sequence of decision rules $f_i = \{f_{i,t}\}_{t \geq 1} = f_{i,1}, f_{i,2}, \dots$.

A transmission protocol is *anonymous* if and only if the decision rule assigns the same transmission probability to all players with the same personal history. In particular, for any two players $i \neq j$ and any $t \geq 0$, if $\vec{h}_{i,t-1} = \vec{h}_{j,t-1}$, it holds that $f_{i,t}(\vec{h}_{i,t-1}) = f_{j,t}(\vec{h}_{j,t-1})$. In this case, we drop the subscript i in the notation, i.e. we write $f = f_1 = \dots = f_n$.

We call a protocol f_i for player i *age-based* if and only if, for any $t \geq 1$, the transmission probability $\Pr(X_{i,t} = 1 | \vec{h}_{i,t-1})$ depends only (a) on time t and (b) on whether player i is pending or not at t . In this case, we will denote the transmission probability by $p_{i,t} \stackrel{\text{def}}{=} \Pr(X_{i,t} = 1 | \vec{h}_{i,t-1}) = f_{i,t}(\vec{h}_{i,t-1})$.

A protocol is called *backoff* if the decision rule at time t is a function of the number of *unsuccessful* transmissions. We call a transmission protocol f_i *non-blocking* if and only if, for any $t \geq 1$ and any transition history $\vec{h}_{i,t-1}$, the transmission probability $\Pr(X_{i,t} = 1 | \vec{h}_{i,t-1})$ is always smaller than 1. A protocol f_i for player i is a *deadline protocol with deadline* $t_0 \in \{1, 2, \dots\}$ if and only if $f_{i,t}(\vec{h}_{i,t-1}) = 1$, for any player i , any time slot $t \geq t_0$ and any transmission history $\vec{h}_{i,t-1}$. A *persistent player* is one that uses the deadline protocol with deadline 1.

Efficiency. Assume that all n players in the game employ an anonymous protocol f . We will say that f is *efficient* if and only if all players will have successfully transmitted by time $\Theta(n)$ with high probability (i.e. with probability tending to 1, as n goes to infinity).

Individual utility. Let $\vec{f} = (f_1, f_2, \dots, f_n)$ be such that player i uses protocol f_i , $i \in N$. For a given transmission sequence $\vec{X}_1, \vec{X}_2, \dots$, which is consistent with \vec{f} , define the *latency* or *success time* of agent i as $T_i \stackrel{\text{def}}{=} \inf\{t : X_{i,t} = 1, X_{j,t} = 0, \forall j \neq i\}$. That is, T_i is the time at which i successfully transmits. Given a transmission history \vec{h}_t , the n -tuple of protocols \vec{f} induces a probability distribution over sequences of further transmissions. In that case, we write $C_i^{\vec{f}}(\vec{h}_t) \stackrel{\text{def}}{=} \mathbb{E}[T_i | \vec{h}_t, \vec{f}] = \mathbb{E}[T_i | \vec{h}_{i,t}, \vec{f}]$ for the expected latency of agent i incurred by a

sequence of transmissions that starts with \vec{h}_t and then continues based on \vec{f} . For anonymous protocols, i.e. when $f_1 = f_2 = \dots = f_n = f$, we will simply write $C_i^f(\vec{h}_t)$ instead³.

Equilibria. The objective of every agent is to minimize her expected latency. We say that $\vec{f} = \{f_1, f_2, \dots, f_n\}$ is in *equilibrium* if for any transmission history \vec{h}_t the agents cannot decrease their expected latency by unilaterally deviating after t ; that is, for all agents i , for all time slots t , and for all decision rules f'_i for agent i , we have

$$C_i^{\vec{f}}(\vec{h}_t) \leq C_i^{(\vec{f}_{-i}, f'_i)}(\vec{h}_t),$$

where (\vec{f}_{-i}, f'_i) denotes the protocol profile⁴ where every agent $j \neq i$ uses protocol f_j and agent i uses protocol f'_i .

3 An equilibrium protocol for two players

In this section we show that there is an anonymous acknowledgment-based protocol in equilibrium, when $n = 2$.

We define the protocol f as follows: for any $t \geq 1$, player i and transmission history $\vec{h}_{i,t-1}$,

$$f_{i,t}(\vec{h}_{i,t-1}) = \begin{cases} \frac{2}{3}, & \text{if } X_{i,t-1} = 1 \text{ or } t = 1 \\ 1, & \text{if } X_{i,t-1} = 0. \end{cases} \quad (1)$$

► **Theorem 1.** *There is an anonymous acknowledgment-based equilibrium protocol for two players.*

Proof. We will show that protocol f is in equilibrium. Let Alice and Bob be the two players in the system. We will show that when Bob sticks with playing f , any deviation for Alice, at any possible slot, will be less profitable for her.

Let's denote by $C_i^{f,j}$, for $j \in \{0, 1\}$, the expected success time for a pending player i given that in the last round he attempted transmission ($j = 1$) or not ($j = 0$) i.e., $C_i^{f,j} = \mathbb{E}[T_i | \vec{h}_t, f, X_{i,t} = j]$. The following claim asserts that the expected success time for Alice depends only on whether she attempted a transmission or not in the previous slot. For the proof, we compute the expected time to absorption for the Markov chain \mathcal{M} shown in Figure 1a, starting from states A and B .

► **Claim 2.** $C_{Alice}^{f,j} = 2 + j$, for $j \in \{0, 1\}$.

Proof. The situation from Alice's perspective can be modeled as a Markov chain \mathcal{M} with state space $\{A, B, C, D\}$. A is the initial state where both players are pending (and they both know this). A is reached either in $t = 1$, or when Alice transmitted in the previous time step and there was a collision. State B models the case when both players are pending, but Alice does not know this, because she did not transmit in the previous time step. State C is reached when only Alice is pending; notice that, by definition of the protocol, there is no way for Alice to distinguish with certainty between states B and C if both herself and Bob

³ Abusing notation slightly, we will also write $C_i^{\vec{f}}(\vec{h}_0)$ for the *unconditional* expected latency of player i induced by \vec{f} .

⁴ For an anonymous protocol f , we denote by (f_{-i}, f'_i) the profile where agent $j \neq i$ uses protocol f and agent i uses protocol f'_i .

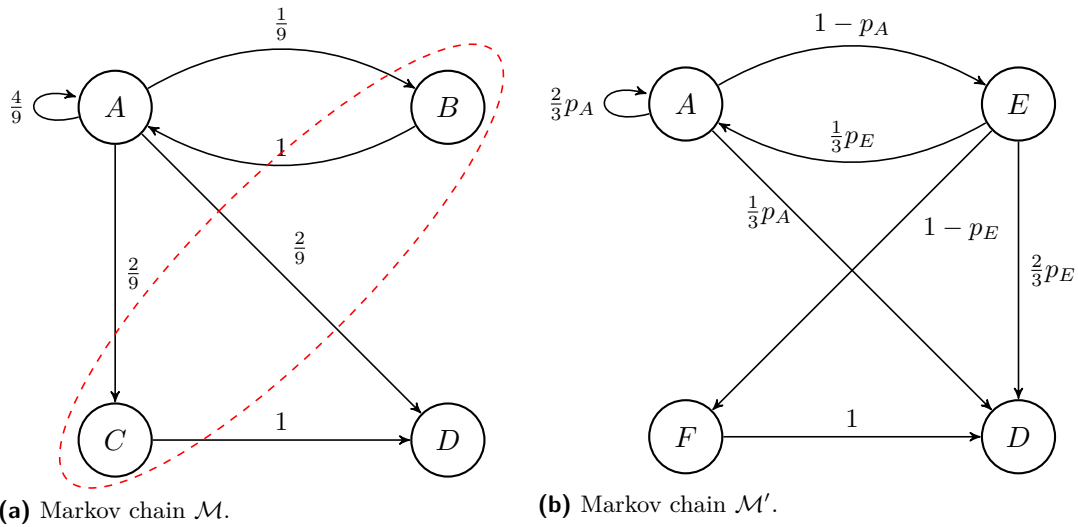


Figure 1 Markov chains used in the analysis.

use f . Finally, D is the state in which Alice has successfully transmitted. The transition graph of \mathcal{M} is shown in Figure 1a.

For example, we can see from the transition graph that the probability that we visit state A at time $t + 1$, given that we are in B at t is given by $\Pr(\mathcal{M}_{t+1} = A | \mathcal{M}_t = B) = 1$. Indeed, if $\mathcal{M}_t = B$, neither player transmitted at t , so both will transmit with probability 1 at $t + 1$, causing a collision, after which Alice (and also Bob) can deduce that all players are still pending. Similarly, $\Pr(\mathcal{M}_{t+1} = D | \mathcal{M}_t = C) = 1$, because, being at C means that only Bob transmitted (successfully) at t and so Alice will transmit (also successfully, being the only pending player) at $t + 1$ with probability 1.

Clearly, $C_{Alice}^{f,1}$ is equal to the expected hitting time k_A^D that \mathcal{M} needs to reach state D , given that we start from A . By definition, we have $k_C^D = 1$, $k_D^D = 0$, and by the Markov property, we get $k_B^D = k_A^D + 1$ and $k_A^D = 1 + \frac{4}{9}k_A^D + \frac{1}{9}k_B^D + \frac{2}{9}k_C^D + \frac{2}{9}k_D^D$. By rearranging terms and making the substitutions, we conclude that $C_{Alice}^{f,1} = 3$.

Calculating $C_{Alice}^{f,0}$ is a bit more tricky, because since Alice did not attempt transmission at the previous slot, she cannot be certain in which state she is, but she knows that is at state B with probability $1/3$ and in C with $2/3$. Therefore $C_{Alice}^{f,0} = \frac{1}{3}k_B^D + \frac{2}{3}k_C^D = 2$. ◀

It remains to be shown that for any transmission history up to any time t , the optimal (best-response) strategy for Alice is to follow f . Notice that this situation from Alice’s point of view can be described by an infinite-horizon, undiscounted *Partially observable Markov Decision Process (POMDP)*, by the direct modification of the Markov chain \mathcal{M} that is described in the proof of Claim 2. This process is partially observable due to the uncertainty created whenever Alice does not attempt transmission. This creates complications in the analysis, as general results about the existence of optimal stationary policies in MDPs [23], do not carry over immediately and also optimal policies are not always well-defined for undiscounted POMDPs with infinite horizon [22]. Fortunately, by exploiting the nature of our specific protocol f , and in particular the fact that a player using f never misses two transmissions in a row, we are able to circumvent this difficulty and model the situation as an MDP.

Following the notation in [21], the *state space* of the MDP is $\mathcal{I} = \{A, E, F, D\}$. The states are interpreted as follows: As in the Markov chain \mathcal{M} , state A describes the situation in which

both players are pending and they both know it (this is reached just after a collision, or at time $t = 1$) and state D corresponds to the state in which Alice successfully transmitted. F is the state in which Alice did not transmit for two consecutive rounds. Since Bob follows f , he will have transmitted in one of these two rounds. Thus, in F Alice is the only pending player and she knows it. Note that in F the unique optimal strategy for Alice is to transmit in the next round. Finally, E is the state in which Alice is uncertain whether she is the only pending player in the system; this happens at t if she did not transmit at $t - 1$, but transmitted at $t - 2$ and there was a collision. State E essentially corresponds to a combination of states B and C in Figure 1a.

Since Alice clearly starts at state A , the *initial distribution* of the MDP is λ , where $\lambda_A = 1$ and $\lambda_E = \lambda_F = \lambda_D = 0$. The set of *actions* for Alice is $\mathcal{A} = [0, 1]$. In particular, if Alice decides to take action $a \in \mathcal{A}$ at time t , then she will transmit with probability a at t . Furthermore, the *cost function* of the MDP is $c(a) = (c_s(a) : s \in \mathcal{I})$ and we have $c_A(a) = c_E(a) = c_F(a) = 1$ and $c_D(a) = 0$ for all $a \in \mathcal{A}$. Finally, for the *transition matrix* of our MDP, notice that, since the MDP describes the situation from Alice's perspective, we calculate transition probabilities by "deferring" the relevant decisions taken by Bob until the time that Alice gets feedback. The transition matrix of our MDP is shown in equation (2) and it is explained in more detailed below.

$$P(a) = \begin{bmatrix} \frac{2a}{3} & 1-a & 0 & \frac{a}{3} \\ \frac{a}{3} & 0 & 1-a & \frac{2a}{3} \\ 0 & 0 & 1-a & a \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (2)$$

In particular, we can see from (2) that the probability to visit state A in one step, given that we are at state E and the action taken is $a \in [0, 1]$, is $P_{E,A}(a) = \frac{a}{3}$. Indeed, this happens at some time t if at time $t - 1$ Alice did not transmit but Bob did not transmit either; therefore, by definition of f , given that we are at E (i.e. Alice did not transmit at time $t - 1$), the probability that we reach A is equal to the probability that Alice transmits at t (which happens with probability a) multiplied by the probability that Bob did not transmit at $t - 1$ (which happens with probability $\frac{1}{3}$). Similarly, the probability that we visit D in one step, given that we are at state E and the action taken is $a \in [0, 1]$, is $P_{E,D}(a) = \frac{2a}{3}$, which is the probability that Alice transmits in the current step and Bob transmitted in the previous one (in which Alice did not transmit, thus Bob was successful).

By Lemma 5.4.2 and Theorem 5.4.3 from [21], there is a stationary policy (i.e. protocol u^*) that is optimal in the sense that it achieves the minimum expected total cost, given that we start at state A . The fact that u^* is stationary significantly reduces the search space of optimal strategies. In particular, this allows us to only consider strategies for which the actions taken by Alice (in the above MDP) depend only on the current state. In fact, we can further reduce the family of optimal strategies considered by noting that in any optimal strategy Alice will transmit with probability 1 when in state F ; indeed, when Alice knows that she is the only pending player, she will decide to transmit with probability 1 in the next time step. Therefore, it only remains to determine the probability of transmission when we are at either state A or E ; denote those by p_A and p_E respectively. Therefore, this leads to a Markov chain \mathcal{M}' with state space $\mathcal{I}' = \mathcal{I}$ and transition probabilities that correspond to actions from the above MDP. The transition graph of \mathcal{M}' is shown in Figure 1b.

Clearly, the expected latency of Alice when she uses protocol u^* and Bob uses protocol f is equal to the expected hitting time $k'_A{}^D$ that \mathcal{M}' needs to reach state D , given that we start from A . By definition, we have $k'_F{}^D = 1$, $k'_D{}^D = 0$, and by the Markov property, we get

$k'_A{}^D = 1 + \frac{2}{3}p_A k'_A{}^D + (1 - p_A)k'_E{}^D$ and $k'_E{}^D = 1 + \frac{1}{3}p_E k'_A{}^D + (1 - p_E)k'_F{}^D$. Rearranging and after substitutions we get $k'_A{}^D = 3$ and $k'_E{}^D = 2$, for any $p_A, p_E \in [0, 1]$. Comparing this to Claim 2, we conclude that if Bob uses f , a best response for Alice is to also follow f . This completes the proof of the Theorem. \blacktriangleleft

3.1 Uniqueness

We will say that a protocol is *stationary* if the decision rule for each player at some time t depends on the information state of the player at t . In particular, the protocol defined in equation (1) is stationary. In this section we show that there are no other stationary equilibria.

► **Theorem 3.** *For two players, the unique stationary anonymous protocol that is in equilibrium is the one defined in equation (1).*

Proof. For the sake of contradiction, assume that there is another stationary protocol that is in equilibrium. As in the analysis of protocol (1) in Section 3, we denote by A the state where both players know they are both pending. Let Alice be one of the two players. Notice that, every time Alice transmits, either there is a collision (in which case Alice returns to state A) or the transmission is successful (so Alice is no longer pending).

For $k = 1, 2, \dots$, let p_k denote the probability that Alice transmits in step k , given that she starts from A at $t = 0$ and she does not transmit in time steps 1 to $k - 1$. Therefore, given that we start from A at time 0, the probability that Alice attempts to transmit for the first time after k steps is $p_k \prod_{k'=1}^{k-1} (1 - p_{k'})$. In particular, in the equilibrium described in the previous section, we had $p_1 = \frac{2}{3}$ and $p_2 = 1$.

First, assume there is another stationary protocol g that is in equilibrium, for which $p_2 = 1$ and $p_1 = p \neq \frac{2}{3}$. Adjusting the transition probabilities in the Markov chain in Figure 1a accordingly, and doing the same analysis we can derive that the expected latency of Alice when both players use protocol g is $k'_A{}^D = \frac{2-p}{2p(1-p)}$. We will show that for all $p \neq \frac{2}{3}$ a player has a profitable deviation. Indeed, first observe that $p > \frac{2}{3}$ implies $k'_A{}^D > 3$. In this case Alice can improve her expected latency by not transmitting for two consecutive time steps and then (successfully) transmitting in the third time step. Second, for the case that $p < \frac{2}{3}$, persistently transmitting in each time step is a deviation which gives the deviator an expected latency of $\frac{1}{1-p}$. For $p < \frac{2}{3}$ this is strictly less than the expected latency $k'_A{}^D = \frac{2-p}{2p(1-p)}$ that Alice has when both players use protocol g . From both cases, we conclude that there is no stationary protocol in equilibrium for which $p_2 = 1$ and $p_1 \neq \frac{2}{3}$.

Now assume that there is another stationary protocol z in equilibrium, for which $p_2 < 1$. Denote α_z the expected latency of Alice when both players use protocol z . Similarly denote $\alpha_{(z')}$ the expected latency of Alice when she unilaterally deviates from z to some other protocol z' . We will consider the following three protocols that Alice can use instead of z : (i) Using protocol $(1z)$, Alice will transmit in the first time step and then continue by following protocol z . (ii) Using protocol $(01z)$, Alice will not transmit in the first time step, but will transmit in the second time step and then follow the protocol z . (iii) Finally, using protocol $(001z)$, Alice will not transmit for the first two time steps, but will transmit in the third time step and then follow the protocol z . The expected latency of Alice when she uses each of those protocols while the other player uses z is given by:

$$\begin{aligned}\alpha_{(1z)} &= 1 + p_1 \alpha_z \\ \alpha_{(01z)} &= 2 + (1 - p_1) p_2 \alpha_z \\ \alpha_{(001z)} &= 3 + (1 - p_1)(1 - p_2) p_3 \alpha_z.\end{aligned}$$

Notice now that all three transmission sequences (1), (0, 1) and (0, 0, 1) are consistent with z . Furthermore, z is acknowledgment-based, so Lemma 4 applies here. Therefore, the above expected latencies must all be equal to α_z . Using the identities $\alpha_z = \alpha_{(1z)} = \alpha_{(01z)}$ we get that $\alpha_z = 2 + p_2 < 3$. But clearly $3 \leq \alpha_{(001z)}$, which is a contradiction to the fact that $\alpha_z = \alpha_{(001z)}$. Thus, there is no equilibrium protocol with $p_2 < 1$. This completes the proof of the theorem. \blacktriangleleft

4 Age-based and backoff protocols

In this section, we focus on two special prominent classes of acknowledgment-based protocols, namely *age-based* and *backoff*, and we show that these cannot be implemented in equilibrium if we insist on finite expected latency.

In what follows, for any protocol f , any player i that uses f and any time t , we will say that $\vec{h}_{i,t}$ is *consistent with f* if and only if there is a non-zero probability that $\vec{h}_{i,t}$ will occur for player i .

Now we are ready to show in the next Lemma a useful property of all *acknowledgment-based* equilibrium protocols that is essentially an analogue of the property of Nash equilibria for finite games that all pure strategies in the support of a Nash equilibrium are best responses.

► **Lemma 4.** *Let $f \stackrel{\text{def}}{=} \{f_t\}_{t \geq 1}$ be an anonymous acknowledgment-based protocol and let $\pi \stackrel{\text{def}}{=} \pi_1, \pi_2, \dots$ be any 0-1 sequence which is consistent with f . For any (finite) positive integer τ^* , define the protocol*

$$g = g(\tau^*) \stackrel{\text{def}}{=} \begin{cases} \pi_t, & \text{for } 1 \leq t \leq \tau^* \\ f_t, & \text{for } t > \tau^*. \end{cases} \quad (3)$$

We then have that, for any fixed player i , if f is in equilibrium, then

$$C_i^f(\vec{h}_0) = C_i^{(f-i,g)}(\vec{h}_0).$$

Proof. Since we consider acknowledgment-based protocols, for the sake of the analysis, we will assume that players continue to flip coins even after successfully transmitting, so that they eventually find out what their decisions would have been at any time t .⁵

For a fixed player i , we obtain

$$C_i^f(\vec{h}_0) = \mathbb{E}[T_i | \vec{h}_{i,0}, f] = \sum_{\vec{h}_{i,\tau^*}} \mathbb{E}[T_i | \vec{h}_{i,\tau^*}, f] \Pr \left\{ \vec{h}_{i,\tau^*} \text{ happens for } i \right\}. \quad (4)$$

Notice now that, since f is acknowledgment-based, the event $\left\{ \vec{h}_{i,\tau^*} \text{ happens for } i \right\}$ is independent of the transmission sequences of other players. Therefore, $\mathbb{E}[T_i | \vec{h}_{i,\tau^*}, f]$ is equal to the unconditional (i.e. conditioned on $\vec{h}_{i,0}$) expected latency of player i when she uses the protocol defined in equation (3), where the first τ^* terms of π are replaced by $(\pi_1, \dots, \pi_{\tau^*}) = (\vec{X}_{i,1}, \dots, \vec{X}_{i,\tau^*}) = \vec{h}_{i,\tau^*}$.⁶ In particular, we have that $\mathbb{E}[T_i | \vec{h}_{i,\tau^*}, f] = \mathbb{E}[T_i | \vec{h}_{i,0}, (f-i, g)] = C_i^{(f-i,g)}(\vec{h}_0)$.

⁵ In fact, we only need this assumption to hold for any t which is at most some predefined fixed upper bound τ^* .

⁶ Note that this observation is not true for general protocols and different kinds of feedback, which is why the present analysis cannot be used to prove an impossibility result in the case of protocols like those in [9].

Assume now for the sake of contradiction that there is a transmission history \vec{h}_{i,τ^*} for player i such that $\mathbb{E}[T_i|\vec{h}_{i,\tau^*}, f] \neq \mathbb{E}[T_i|\vec{h}_{i,0}, f]$. Clearly, if $\mathbb{E}[T_i|\vec{h}_{i,\tau^*}, f] < \mathbb{E}[T_i|\vec{h}_{i,0}, f]$, then the protocol g is a better protocol for player i , which contradicts the fact that f is in equilibrium. On the other hand, if $\mathbb{E}[T_i|\vec{h}_{i,\tau^*}, f] > \mathbb{E}[T_i|\vec{h}_{i,0}, f]$, then equation (4) implies that there must be another transmission history \vec{h}'_{i,τ^*} for which $\mathbb{E}[T_i|\vec{h}'_{i,\tau^*}, f] < \mathbb{E}[T_i|\vec{h}_{i,0}, f]$.

Therefore, we have that $C_i^{(f-i,g)}(\vec{h}_0) = \mathbb{E}[T_i|\vec{h}_{i,0}, (f_{-i}, g)] = \mathbb{E}[T_i|\vec{h}_{i,\tau^*}, f] = \mathbb{E}[T_i|\vec{h}_{i,0}, f] = C_i^f(\vec{h}_0)$, for any transmission history \vec{h}_{i,τ^*} , and for any finite $\tau^* \geq 1$, thus also for any 0-1 sequence π that is consistent with f . ◀

The next corollary is an interesting consequence of Lemma 4 regarding *non-blocking* anonymous age-based protocols.

► **Corollary 5.** *Let $f \stackrel{\text{def}}{=} \{f_t\}_{t \geq 1}$ be a non-blocking anonymous age-based protocol. If the expected latency of a player using protocol f is finite, i.e. $\mathbb{E}[T_i|\vec{h}_{i,0}, f] < \infty$, then f is not in equilibrium.*

Proof. Assume for the sake of contradiction that f is in equilibrium and let $\tau^* \stackrel{\text{def}}{=} \left\lceil \mathbb{E}[T_i|\vec{h}_{i,0}, f] \right\rceil$ be finite, where i is a fixed player using f . Consider the protocol $g = g(\tau^*)$ as defined in (3), where the first τ^* terms of π are set equal to 0. Clearly, any player using g has expected latency at least $\tau^* + 1$, irrespectively of the transmissions of the other players. Notice also that π is consistent with f up to τ^* , since $\Pr\{\vec{h}_{i,\tau^*} = (0, \dots, 0) | f\} = \prod_{t=1}^{\tau^*} (1 - p_{i,t}) > 0$. Therefore, by Lemma 4 we have that $\tau^* + 1 > \mathbb{E}[T_i|\vec{h}_{i,0}, f] = \mathbb{E}[T_i|\vec{h}_{i,0}, (f_{-i}, g)] \geq \tau^* + 1$, which is a contradiction. We conclude that either f is not in equilibrium, or τ^* is ∞ . ◀

We are now ready to show the main result of this section.

► **Theorem 6.** *There is no anonymous age-based protocol f for $n \geq 2$ players that is in equilibrium and has $\mathbb{E}[T_i|\vec{h}_{i,0}, f] < \infty$, for any player i .*

Proof. For the sake of contradiction, let's assume that $f = \{f_t\}_{t \geq 1}$ is an age-based protocol in equilibrium with finite expected latency, i.e. $\mathbb{E}[T_i|\vec{h}_{i,0}, f] < \infty$. The next claim asserts the existence of a finite positive integer τ^* where the protocol dictates transmission, with certain properties, which will be a useful ingredient for the rest of the proof.

► **Claim 7.** *Let f be an anonymous age-based protocol for n players that is in equilibrium and has $\mathbb{E}[T_i|\vec{h}_{i,0}, f] < \infty$, then there is a finite positive integer τ^* such that*

- (a) $f_{\tau^*} = 1$,
- (b) $f_{\tau^*-1} < 1$ and
- (c) there exist $\tau_1 < \dots < \tau_{n-1} < \tau^*$, such that $f_{\tau_j} < 1$, for all $j = 1, \dots, n-1$.

Proof. For any time t , define Z_t^f to be the number of non-blocking probabilities of the protocol f up to t , i.e. $Z_t^f \stackrel{\text{def}}{=} \sum_{t' \leq t} (1 - \lfloor f_{t'} \rfloor)$. Set $\tau' \stackrel{\text{def}}{=} \inf\{t : f_t = 1, Z_t^f \geq n-1\}$. Assume for the sake of contradiction that there does not exist a τ^* with the properties described in the claim. In particular, this means that $\tau' = \infty$. However, the latter can happen if one of the following cases is true:

- (i) There is no finite τ such that $f_\tau = 1$.
- (ii) There exists finite τ such that $f_\tau = 1$, $Z_\tau^f \leq n-2$ and $f_t = 1$, for all $t \geq \tau$.
- (iii) There exists finite τ such that $f_\tau = 1$, $Z_\tau^f \leq n-2$ and $f_t < 1$, for all $t \geq \tau$.

We now prove that in all those cases we get a contradiction. Case (i) comes in contradiction with Corollary 5.

If case (ii) holds, then clearly, if all players use f , at most $n - 2$ players can successfully transmit before τ and the rest will remain pending for ever. But this means that the expected latency of a player i using f is at least

$$\Pr\{i \text{ does not successfully transmit before } \tau | \vec{h}_{i,0}, f\} \cdot \infty = \infty,$$

which leads to a contradiction, since we assumed $\mathbb{E}[T_i | \vec{h}_{i,0}, f] < \infty$.

Suppose now that case (iii) holds. Consider the protocol g defined as follows:

$$g \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } f_t < 1, \text{ for } 1 \leq t \leq \mathbb{E}[T_i | \vec{h}_{i,0}, f] \\ 1, & \text{if } f_t = 1, \text{ for } 1 \leq t \leq \mathbb{E}[T_i | \vec{h}_{i,0}, f] \\ f_t, & \text{for } t > \mathbb{E}[T_i | \vec{h}_{i,0}, f]. \end{cases} \quad (5)$$

Let i be a fixed player (say Alice). Notice that, if all other players use f and Alice uses g , then Alice has expected latency strictly larger than $\mathbb{E}[T_i | \vec{h}_{i,0}, f]$; indeed, for any $t \leq \mathbb{E}[T_i | \vec{h}_{i,0}, f]$, Alice only attempts a transmission when $f_t = 1$ and there is at least one more other pending player using f , and so there is a collision. However, since the initial (deterministic) sequence of $\left\lfloor \mathbb{E}[T_i | \vec{h}_{i,0}, f] \right\rfloor$ transmissions of g is consistent with f , by Lemma 4 we have that $\mathbb{E}[T_i | \vec{h}_{i,0}, f] = \mathbb{E}[T_i | \vec{h}_{i,0}, (f_{-i}, g)] > \mathbb{E}[T_i | \vec{h}_{i,0}, f]$, which is a contradiction. This completes the proof of the claim. \blacktriangleleft

Take a τ^* as described in the above claim and consider the protocol Q defined as follows

$$Q \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } f_t < 1, \text{ for } 1 \leq t \leq \tau^* - 2 \\ 1, & \text{if } f_t = 1, \text{ for } 1 \leq t \leq \tau^* - 2 \\ 1, & \text{for } t = \tau^* - 1 \text{ and } t = \tau^* \\ f_t, & \text{for } t > \tau^*. \end{cases} \quad (6)$$

Notice that, since the initial (deterministic) sequence of transmissions of Q is consistent with f , by Lemma 4 we have that $\mathbb{E}[T_i | \vec{h}_{i,0}, f] = \mathbb{E}[T_i | \vec{h}_{i,0}, (f_{-i}, Q)]$.

Now consider the protocol Q' , which is the same as Q , with the only difference⁷ that $Q'_{\tau^*} = 0$. In fact, we show that, $\mathbb{E}[T_i | \vec{h}_{i,0}, (f_{-i}, Q')] < \mathbb{E}[T_i | \vec{h}_{i,0}, (f_{-i}, Q)]$ which implies $\mathbb{E}[T_i | \vec{h}_{i,0}, (f_{-i}, Q')] < \mathbb{E}[T_i | \vec{h}_{i,0}, f]$, which contradicts the assumption that f is in equilibrium.

Notice now that protocols Q and Q' are identical for any $t \neq \tau^*$, and if there are at least 3 pending players at τ^* (i.e. Alice and at least two others), then there would be a collision at τ^* no matter which of the two protocols Alice uses (i.e. the same players that were pending at τ^* would be pending at the start of time slot $\tau^* + 1$ as well). Therefore, the two protocols behave the same in this case. However, if there are exactly 2 pending players at τ^* (i.e. Alice and exactly one more, say Bob) the two protocols behave differently. Indeed, if Alice uses protocol Q , then there will be a collision at τ^* , leaving exactly 2 pending players at $\tau^* + 1$. However, if Alice uses protocol Q' , then Bob will be able to successfully transmit at τ^* , leaving Alice the only pending player at time $\tau^* + 1$, which implies a strictly smaller expected latency. The proof is completed by noting that, by definition of τ^* , the probability that there will be exactly 2 players pending at τ^* is strictly positive (since there are at least $n - 2$ steps before $\tau^* - 1$ with transmission probability strictly less than 1). \blacktriangleleft

⁷ Note that Q' does not agree with f whenever $f_t = 1$, so Lemma 4 does not apply to Q' .

Now we conclude with the impossibility result for backoff protocols, the proof of which shares similarities to the proof of Corollary 5.

► **Theorem 8.** *There is no anonymous backoff protocol f in equilibrium for $n \geq 2$ players with $\mathbb{E}[T_i | \vec{h}_{i,0}, f] < \infty$, for any player i .*

Proof. Assume for the sake of contradiction that f is in equilibrium and let $\tau^* \stackrel{\text{def}}{=} \left\lceil \mathbb{E}[T_i | \vec{h}_{i,0}, f] \right\rceil$ be finite, where i is a fixed player using f . By definition, we have that $f_i = \{p_{i,k}\}_{k \geq 0}$, where $p_{i,k}$ denotes the transmission probability of player i after k unsuccessful transmissions. Notice also that we may assume without loss of generality that $p_{i,0} \neq 1$. Indeed, suppose there is finite integer $s > 0$, such that $p_{i,k'} = 1$, for all $k' < s$ and $p_{i,s} \neq 1$ (if s is not finite, then clearly f does not have finite expected latency). Then the protocol $f' = \{p'_{i,k}\}_{k \geq 0}$, with $p'_{i,k} = p_{i,k+s}$, for all $k \geq 0$ is also an equilibrium.

Consider now the protocol $g = g(\tau^*)$ defined in equation (3), where the first τ^* terms of π are set to 0. Clearly, any player using g has expected latency at least $\tau^* + 1$. Notice also that π is consistent with f up to τ^* , since $\Pr\{\vec{h}_{i,\tau^*} = (0, \dots, 0) | f\} = (1 - p_{i,0})^{\tau^*} > 0$. Therefore, by Lemma 4 we have that $\tau^* + 1 > \mathbb{E}[T_i | \vec{h}_{i,0}, f] = \mathbb{E}[T_i | \vec{h}_{i,0}, (f_{-i}, g)] \geq \tau^* + 1$, which is a contradiction. But this implies that, either f is not in equilibrium, or τ^* is ∞ . ◀

5 An efficient protocol in equilibrium

In this section we present a deadline protocol for n players that is efficient, i.e. with high probability the latency of any player is $\Theta(n)$. Let $t_0 = t_0(n)$ be an integer, to be determined later and let $\beta \in (0, 1)$ be a fixed constant. We consider the following deadline protocol \mathcal{Q} with deadline t_0 , which is defined as follows: The $t_0 - 1$ time steps before the deadline are partitioned into $k + 1$ consecutive intervals I_1, I_2, \dots, I_{k+1} , where $k = k(n)$ is the unique integer satisfying $\beta^{k+1}n \leq \sqrt{n} < \beta^k n$. For any $j \in \{1, \dots, k + 1\}$, define $n_j = \beta^j n$. For $j \in \{1, \dots, k\}$ the length of interval I_j is $\ell_j = \left\lfloor \frac{e}{\beta} n_j \right\rfloor$. Interval I_{k+1} is special and has length $\ell_{k+1} = n$. In particular, this gives

$$t_0 \stackrel{\text{def}}{=} 1 + \sum_{j=1}^{k+1} \ell_j \leq 1 + n + en \sum_{j=1}^k \beta^{j-1} = 1 + n + en \frac{1 - \beta^{k-1}}{1 - \beta} \leq n \left(1 + \frac{e}{1 - \beta} \right),$$

where the last inequality holds for any constant $\beta \in (0, 1)$ and $n \rightarrow \infty$. For any $t \geq 1$, the decision rule at time t for protocol \mathcal{Q} is given by

$$\mathcal{Q}_t = \begin{cases} \frac{1}{n_j}, & \text{if } t \in I_j, j = 1, 2, \dots, k + 1 \\ 1, & \text{if } t \geq t_0. \end{cases} \quad (7)$$

Notice that, by definition, \mathcal{Q} is an age-based protocol. Furthermore, if at least two out of n players use protocol \mathcal{Q} , then, no matter what protocol the rest of the players use, there is a non-zero probability that there will be no successful transmission until the deadline t_0 , and thus all players will remain pending for ever. In particular, this is at least the probability that the two players using \mathcal{Q} attempt a transmission in every step until t_0 , which happens with probability $\prod_{t=1}^{t_0-1} (\mathcal{Q}_t)^2 \geq \frac{1}{n^{t_0}} > 0$. Therefore, if there are at least two players using \mathcal{Q} , the expected latency of any player is ∞ , hence \mathcal{Q} is in equilibrium, for any $n \geq 3$ and deadline t_0 .

In Theorem 11 we prove that \mathcal{Q} is also efficient; when all players in the system use protocol \mathcal{Q} , then with high probability all players will successfully transmit before the deadline t_0 .

For the proof, we use two elementary Lemmas that formalize the fact that, in each interval, a significant number of players successfully transmit with high probability. For the proofs, we employ standard concentration results from probability theory.

► **Lemma 9.** *Assume that all players in the system use protocol \mathcal{Q} . For any $j \in \{1, \dots, k\}$, if the number of pending players before interval I_j is at most n_j , then after I_j , with probability at least $1 - \exp(-\frac{1}{3}\beta^{j+2}n)$ there will be at most n_{j+1} pending players.*

Proof. Fix $j \in \{1, \dots, k\}$ and assume that the precondition of the lemma is fulfilled, i.e., before interval I_j there are at most n_j pending players. Let r_t denote the number of pending players at time t . In particular, for any $t \in I_j$, if the preconditions of the lemma is fulfilled, we have $r_t \leq n_j$. Therefore the probability of a successful transmission in round $t \in I_j$ is given by

$$r_t \mathcal{Q}_t (1 - \mathcal{Q}_t)^{r_t - 1} \geq r_t \mathcal{Q}_t (1 - \mathcal{Q}_t)^{n_j - 1} = r_t \frac{1}{n_j} \left(1 - \frac{1}{n_j}\right)^{n_j - 1} \geq \frac{1}{e} \frac{r_t}{n_j},$$

where in the last inequality we used the fact that $(1 - \frac{1}{x})^{x-1} \geq \frac{1}{e}$, for any $x > 1$. Therefore, for any round $t \in I_j$, either we already have $r_t \leq n_{j+1} = \beta n_j$ pending players, or the probability of a successful transmission in round t is at least $a \stackrel{\text{def}}{=} \frac{1}{e} \frac{n_{j+1}}{n_j} = \frac{\beta}{e}$.

Let now X_j be the random variable counting the number of successful transmissions in interval I_j . Notice that, by the above discussion, given that at the start of interval I_j there are at least n_{j+1} pending players, X_j stochastically dominates a Binomial random variable $Y_j \sim \text{Bin}(\ell_j, a)$, with mean value $\ell_j \cdot a$. Therefore, by a Chernoff bound (see [26]), we get

$$\Pr(X_j < (1 - \beta)\ell_j \cdot a) \leq \Pr(Y_j < (1 - \beta)\ell_j \cdot a) \leq \exp\left(-\frac{1}{2}\beta^2 \ell_j \cdot a\right) \leq \exp\left(-\frac{1}{3}\beta^2 n_j\right),$$

where in the last inequality we used the fact that, by definition, $n_j \geq \sqrt{n}$, for all $j \leq k$, thus $\ell_j \cdot \frac{\beta}{e} \geq \frac{2}{3}n_j$. This directly implies the lemma. ◀

► **Lemma 10.** *If the number of pending players at the start of interval I_{k+1} is at most n_{k+1} , then after interval I_{k+1} , with probability at least $1 - \exp(-\frac{1}{3}n_{k+1})$ all players will have successfully transmitted.*

Proof. Consider a fixed player (say Alice) that is pending at the start of interval I_{k+1} . Given that there are at most $n_{k+1} = \beta^{k+1}n$ pending players at any time step $t \in I_{k+1}$, the probability that Alice successfully transmits during t is at least

$$\mathcal{Q}_t (1 - \mathcal{Q}_t)^{n_{k+1} - 1} = \frac{1}{n_{k+1}} \left(1 - \frac{1}{n_{k+1}}\right)^{n_{k+1} - 1}.$$

Therefore, since $|I_{k+1}| = \ell_{k+1} = n$, the probability that Alice is still pending after interval I_{k+1} is at most

$$\left(1 - \frac{1}{n_{k+1}} \left(1 - \frac{1}{n_{k+1}}\right)^{n_{k+1} - 1}\right)^n \leq \exp\left(-\frac{n}{n_{k+1}} \left(1 - \frac{1}{n_{k+1}}\right)^{n_{k+1} - 1}\right). \quad (8)$$

Recall that, by definition, k is the (unique) smallest integer satisfying $n_{k+1} \leq \sqrt{n} < n_k$. In particular, this implies that $n_{k+1} > \beta\sqrt{n}$, therefore n_{k+1} goes to ∞ as $n \rightarrow \infty$. Additionally, we have that $\frac{n}{n_{k+1}} \geq n_{k+1}$. Therefore, using the fact that $(1 - \frac{1}{x})^{x-1} \geq \frac{1}{e}$, for any $x > 1$, the right hand side of (8) is at most $\exp(-\frac{1}{e}n_{k+1})$.

By the union bound, given that there are at most n_{k+1} pending players at the start of interval I_{k+1} , the probability that there is at least one pending player after I_{k+1} is at most $n_{k+1} \exp(-\frac{1}{e}n_{k+1}) \leq \exp(-\frac{1}{3}n_{k+1})$, as stated in the Lemma. ◀

We are now ready to prove our main Theorem.

► **Theorem 11.** *Protocol \mathcal{Q} is efficient. In particular, for any constant $\beta \in (0, 1)$, when all players use \mathcal{Q} , the probability that there is a pending player after time $t_0 \leq n \left(1 + \frac{e}{1-\beta}\right)$ is at most $\exp(-\Theta(\sqrt{n}))$.*

Proof. It suffices to show that with high probability every player will have successfully transmitted before t_0 . Note that, the probability that there are still pending players at $t_0 = \Theta(n)$ is upper bounded by the probability that (a) there exists $j \in \{1, 2, \dots, k\}$ such that, at the end of interval I_j there are more than n_{j+1} pending players, or (b) there are still pending players after interval I_{k+1} .

Therefore, by Lemma 9 and Lemma 10 and the union bound, the probability that not all players successfully transmit before t_0 is at most

$$\exp\left(-\frac{1}{3}n_{k+1}\right) + \sum_{j=1}^k \exp\left(-\frac{1}{3}\beta^2 n_j\right). \quad (9)$$

Since $n_j \geq n_{k+1} \geq \beta\sqrt{n}$, for any $j \in \{1, 2, \dots, k\}$, the above upper bound becomes $(k+1)\exp(-\Theta(\sqrt{n}))$. The proof is concluded by noting that, by definition of k , we have $k = \Theta(\log n)$. ◀

We note that, in our analysis, $\beta \in (0, 1)$ can be any constant arbitrarily close to 0, therefore, by Theorem 11, the upper bound on the latency of protocol \mathcal{Q} can be as small as $(1+e)n + o(n)$ with high probability.

References

- 1 N. Abramson. The ALOHA system: Another alternative for computer communications. In *Proceedings of the November 17-19, 1970, fall joint computer conference*, pages 281–285. ACM New York, NY, USA, 1970.
- 2 E. Altman, R. El Azouzi, and T. Jiménez. Slotted aloha as a game with partial information. *Comput. Netw.*, 45(6):701–713, 2004. doi:10.1016/j.comnet.2004.02.013.
- 3 E. Altman, D. Barman, A. Benslimane, and R. El Azouzi. Slotted aloha with priorities and random power. In *Proc. IEEE Infocom*, 2005.
- 4 V. Auletta, L. Moscardelli, P. Penna, and G. Persiano. Interference games in wireless networks. In *WINE*, pages 278–285, 2008.
- 5 M. Bender, M. Farach-Colton, S He, B. Kuszmaul, and C. Leiserson. Adversarial contention resolution for simple channels. In *SPAA '05*, pages 325–332. ACM, 2005. doi:10.1145/1073970.1074023.
- 6 J. Capetanakis. Generalized tdma: The multi-accessing tree protocol. *IEEE Transactions on Communications*, 27(10):1476–1484, 1979.
- 7 J. Capetanakis. Tree algorithms for packet broadcast channels. *IEEE Transactions on Information Theory*, 25(5):505–515, 1979.
- 8 George Christodoulou, Katrina Ligett, and Evangelia Pyrga. Contention resolution under selfishness. *Algorithmica*, 70(4):675–693, 2014.
- 9 A. Fiat, Y. Mansour, and U. Nadav. Efficient contention resolution protocols for selfish agents. In *SODA '07*, pages 179–188, Philadelphia, PA, USA, 2007. SIAM.

- 10 Mihály Geréb-Graus and Thanasis Tsantilas. Efficient optical communication in parallel computers. In *SPAA '92*, pages 41–48, New York, NY, USA, 1992. ACM. doi:10.1145/140901.140906.
- 11 L. A. Goldberg and P. D. MacKenzie. Analysis of practical backoff protocols for contention resolution with multiple servers. *J. Comput. Syst. Sci.*, 58(1):232–258, 1999. doi:10.1006/jcss.1998.1590.
- 12 L. A. Goldberg, P. D. Mackenzie, M. Paterson, and A. Srinivasan. Contention resolution with constant expected delay. *J. ACM*, 47(6):1048–1096, 2000. doi:10.1145/355541.355567.
- 13 Leslie Ann Goldberg. Notes on contention resolution. <http://www.cs.ox.ac.uk/people/leslieann.goldberg/contention.html>, 2002.
- 14 A. Greenberg and S. Winograd. A lower bound on the time needed in the worst case to resolve conflicts deterministically in multiple access channels. *J. ACM*, 32(3):589–596, 1985. doi:10.1145/3828.214125.
- 15 Hayes J. An adaptive technique for local distribution. *IEEE Transactions on Communications*, 26(8):1178–1186, 1978.
- 16 Elias Koutsoupias and Katia Papakonstantinou. Contention issues in congestion games. In *Automata, Languages, and Programming - 39th International Colloquium, ICALP 2012, Warwick, UK, July 9-13, 2012, Proceedings, Part II*, pages 623–635, 2012. doi:10.1007/978-3-642-31585-5_55.
- 17 R.T. Ma, V. Misra, and D. Rubenstein. Modeling and analysis of generalized slotted-ahloha mac protocols in cooperative, competitive and adversarial environments. In *ICDCS'06*, page 62, Washington, DC, USA, 2006. IEEE. doi:10.1109/ICDCS.2006.56.
- 18 P. D. MacKenzie, C. G. Plaxton, and R. Rajaraman. On contention resolution protocols and associated probabilistic phenomena. *J. ACM*, 45(2):324–378, 1998. doi:10.1145/274787.274816.
- 19 I. Menache and N. Shimkin. Efficient rate-constrained nash equilibrium in collision channels with state information. In *INFOCOM 2008.*, pages 403–411, 2008.
- 20 R. Metcalfe and D. Boggs. Distributed packet switching for local computer networks. *Communications of the ACM*, 19:395–404, 1976.
- 21 J.R. Norris. *Markov Chains*. Cambridge University Press, 1998.
- 22 Loren K. Platzman. Optimal infinite-horizon undiscounted control of finite probabilistic systems. *SIAM Journal on Control and Optimization*, 18(4):362–380, 1980.
- 23 Martin L. Puterman. *Markov Decision Processes: Discrete Stochastic Dynamic Programming*. John Wiley & Sons, Inc., 1994.
- 24 P. Raghavan and E. Upfal. Stochastic contention resolution with short delays. Technical report, Weizmann Science Press of Israel, Jerusalem, Israel, Israel, 1995.
- 25 L. Roberts. Aloha packet system with and without slots and capture. *SIGCOMM Comput. Commun. Rev.*, 5(2):28–42, April 1975. doi:10.1145/1024916.1024920.
- 26 Sheldon R. Ross. *A First Course in Probability*. Pearson, 2012.
- 27 F.A. Tobagi and L. Kleinrock. Packet switching in radio channels: Part ii—the hidden terminal problem in carrier sense multiple-access and the busy-tone solution. *IEEE Transactions on Communications*, 23(12):1417–1433, 1975.
- 28 B. S. Tsybakov and V. A. Mikhailov. Free synchronous packet access in a broadcast channel with feedback. *Problems of Information Transmission*, 14(4):259–280, 1978.
- 29 D. Wang, C. Comaniciu, and U. Tureli. Cooperation and fairness for slotted aloha. *Wirel. Pers. Commun.*, 43(1):13–27, 2007. doi:10.1007/s11277-006-9240-5.
- 30 D. Zheng, W. Ge, and J. Zhang. Distributed opportunistic scheduling for ad-hoc communications: an optimal stopping approach. In *MobiHoc'07*, pages 1–10. ACM, 2007. doi:10.1145/1288107.1288109.