Minimal Paradefinite Logics for Reasoning with Incompleteness and Inconsistency*

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Abstract

Paradefinite ('beyond the definite') logics are logics that can be used for handling contradictory or partial information. As such, paradefinite logics should be both paraconsistent and paracomplete. In this paper we consider the simplest semantic framework for defining paradefinite logics, consisting of four-valued matrices, and study the better accepted logics that are induced by these matrices.

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1 Introduction

Uncertainty in commonsense reasoning and AI involves inconsistent and incomplete information. *Paradefinite logics* (called 'non-alethic' by da Costa and 'paranormal' by Béziau [11]) are logics that successfully handle these two types of indefinite data, and so they have the following two properties:

- *Paraconsistency* [13]: The ability to properly handle contradictory data by rejecting the principle of explosion, in which any proposition can be inferred from an inconsistent set of assumptions.
- *Paracompleteness*: The ability to properly handle incomplete data by rejecting the law of excluded middle, in which for any proposition, either that proposition is 'true' or its negation is 'true'.

Apart of these two primary requirements, a 'decent' logic for reasoning with indefinite data should have some further characteristics, like being expressive enough, faithful to classical logic as much as possible (in the sense that entailments in the logic should also hold in classical logic), and having some maximality properties (which may be intuitively interpreted by the aspiration to retain as much of classical logic as possible, while preserving paraconsistency and paracompleteness).

In this paper we are interested in the 'simplest' paradefinite logics (in terms of the number of the truth values of their semantics) that have the above properties. Obviously, two-valued logics are not adequate for this, as they cannot handle either of the two types of uncertainty. Likewise, three-valued logics can be used for handling just one type of uncertainty (see,

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e.g., [5]), but they cannot simultaneously handle both of them. On the other hand, as shown e.g. in [10] and [2], four truth values *are* enough for reasoning with incompleteness and inconsistency.

This paper is a largely extended study of the work on 4-valued logics mentioned above. Among others, we characterize the 4-valued paradefinite matrices, consider the induced logics, examine them according to the criteria in [3], investigate their relative strengths, and introduce corresponding sound and complete Hilbert-type and Gentzen-type proof systems.

2 Preliminaries

2.1 Propositional Logics

In what follows we denote by $\mathcal{L}$ a propositional language with a set $\mathsf{Atoms}(\mathcal{L}) = \{P_1, P_2, \ldots\}$ of atomic formulas and use p, q, r to vary over this set. The set of the well-formed formulas of $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$ and $\varphi, \psi, \phi, \sigma$ will vary over its elements. The set $\mathsf{Atoms}(\varphi)$ denotes the atomic formulas occurring in φ. Sets of formulas in $\mathcal{W}(\mathcal{L})$ are called *theories* and are denoted by $\mathcal{T}$ or $\mathcal{T}'$. Finite theories are denoted by Γ or Δ. We shall abbreviate $\mathcal{T} \cup \{\psi\}$ by $\mathcal{T}, \psi$ and write $\mathcal{T}, \mathcal{T}'$ instead of $\mathcal{T} \cup \mathcal{T}'$. A *rule* in a language $\mathcal{L}$ is a pair $\langle \Gamma, \psi \rangle$, where $\Gamma \cup \{\psi\}$ is a finite theory. We shall henceforth denote such a rule by Γ/ψ.

▶ **Definition 2.1.** A (propositional) *logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$, such that $\mathcal{L}$ is a propositional language, and $\vdash$ is a binary relation between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, satisfying the following conditions:

- *Reflexivity:* if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
- *Monotonicity:* if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
- *Transitivity:* if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \phi$ then $\mathcal{T}, \mathcal{T}' \vdash \phi$.
- *Structurality:* for every substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\{\theta(\varphi) \mid \varphi \in \mathcal{T}\} \vdash \theta(\psi)$.
- *Non-Triviality:* there is a non-empty theory $\mathcal{T}$ and a formula ψ such that $\mathcal{T} \nvdash \psi$.

A logic $\langle \mathcal{L}, \vdash \rangle$ is *finitary* if for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$ there is a *finite* theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.

Note that the languages that are considered in the sequel are all *propositional*, as this is the heart of every paraconsistent and paracomplete logic ever studied so far. Also, we confine ourselves to paradefinite *logics*, thus no form of non-monotonic reasoning is considered in this paper.

▶ **Definition 2.2.** Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, and let S be a set of rules in $\mathcal{L}$. The *finitary* $\mathbf{L}$*-closure* $C_{\mathbf{L}}(S)$ of S is inductively defined as follows:

- $\langle \theta(\Gamma), \theta(\psi) \rangle \in C_{\mathbf{L}}(S)$, where θ is an $\mathcal{L}$-substitution, Γ is a *finite* theory in $\mathcal{W}(\mathcal{L})$, and either $\Gamma \vdash \psi$ or $\Gamma/\psi \in S$.
- If the pairs $\langle \Gamma_1, \varphi \rangle$ and $\langle \Gamma_2 \cup \{\varphi\}, \psi \rangle$ are both in $C_{\mathbf{L}}(S)$, then so is the pair $\langle \Gamma_1 \cup \Gamma_2, \psi \rangle$.

Next we define what an extension of a logic means.

▶ **Definition 2.3.** Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, and let S be a set of rules in $\mathcal{L}$.

- A logic $\mathbf{L}' = \langle \mathcal{L}, \vdash' \rangle$ is an *extension* of $\mathbf{L}$ (in the same language) if $\vdash \subseteq \vdash'$. We say that $\mathbf{L}'$ is a *proper extension* of $\mathbf{L}$, if $\vdash \subsetneq \vdash'$.
- The *extension of* $\mathbf{L}$ *by* S is the pair $\mathbf{L}^* = \langle \mathcal{L}, \vdash^* \rangle$, where $\vdash^*$ is the binary relation between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, defined by: $\mathcal{T} \vdash^* \psi$ if there is a finite $\Gamma \subseteq \mathcal{T}$ such that $\langle \Gamma, \psi \rangle \in C_{\mathbf{L}}(S)$.
- Extending $\mathbf{L}$ by an axiom schema φ means extending it by the rule $\emptyset/\varphi$.

The usefulness of a logic strongly depends on the question what kind of connectives are available in it. Three particularly important types of connectives are defined next.

▶ **Definition 2.4.** Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic.

- A binary connective $\supset$ of $\mathcal{L}$ is an *implication for* $\mathbf{L}$, if the classical deduction theorem holds for $\supset$ and $\vdash$, that is: $\mathcal{T}, \varphi \vdash \psi$ iff $\mathcal{T} \vdash \varphi \supset \psi$.
- A binary connective $\wedge$ of $\mathcal{L}$ is a *conjunction for* $\mathbf{L}$, if $\mathcal{T} \vdash \psi \wedge \varphi$ iff $\mathcal{T} \vdash \psi$ and $\mathcal{T} \vdash \varphi$.
- A binary connective $\vee$ of $\mathcal{L}$ is a *disjunction for* $\mathbf{L}$, if $\mathcal{T}, \psi \vee \varphi \vdash \sigma$ iff $\mathcal{T}, \psi \vdash \sigma$ and $\mathcal{T}, \varphi \vdash \sigma$.

We say that $\mathbf{L}$ is *semi-normal* if it has (at least) one of the three basic connectives defined above. We say that $\mathbf{L}$ is *normal* if it has *all* these three connectives.

2.2 Many-Valued Matrices

The most standard semantic way of defining many-valued logics is by using the following type of structures (see, e.g., [19, 20, 25]).

▶ **Definition 2.5.** A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ is a function that associates an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ with every n-ary connective $\diamond$ of the language $\mathcal{L}$.

In what follows, we shall denote by $\overline{\mathcal{D}}$ the elements in $\mathcal{V} \setminus \mathcal{D}$. The set $\mathcal{D}$ is used for defining satisfiability and validity as defined below:

▶ **Definition 2.6.** Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}(\mathcal{L}) \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote by $\Lambda_{\mathcal{M}}$ the set of all the $\mathcal{M}$-valuations.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ (alternatively, ν $\mathcal{M}$-*satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory $\mathcal{T}$ are the elements of the set $mod_{\mathcal{M}}(\mathcal{T}) = \cap_{\psi \in \mathcal{T}}\, mod_{\mathcal{M}}(\psi)$.
- A formula ψ is $\mathcal{M}$-*satisfiable* if $mod_{\mathcal{M}}(\psi) \neq \emptyset$. A theory $\mathcal{T}$ is $\mathcal{M}$-satisfiable if $mod_{\mathcal{M}}(\mathcal{T}) \neq \emptyset$.

In the sequel, when it is clear from the context, we shall sometimes omit the subscript '$\mathcal{M}$' and the tilde sign from $\widetilde{\diamond}_{\mathcal{M}}$, and the prefix '$\mathcal{M}$' from the notions above.

▶ **Definition 2.7.** Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$, and let $\mathcal{L} \subseteq \mathcal{L}'$. A matrix $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$ for $\mathcal{L}'$ is called an *expansion* of $\mathcal{M}$ to $\mathcal{L}'$, if $\mathcal{V} = \mathcal{V}'$, $\mathcal{D} = \mathcal{D}'$, and $\mathcal{O}'(\diamond) = \mathcal{O}(\diamond)$ for every connective $\diamond$ of $\mathcal{L}$.

▶ **Definition 2.8.** Given a matrix $\mathcal{M}$, the consequence relation $\vdash_{\mathcal{M}}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by: $\mathcal{T} \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\mathcal{T}) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the consequence relation induced by $\mathcal{M}$.

▶ **Theorem 2.9** ([21, 22])**.** *For every propositional language $\mathcal{L}$ and finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic. If $\mathcal{M}$ is finite, then $\vdash_{\mathcal{M}}$ is also finitary.*

We conclude this section with some simple, easily verified properties of the basic connectives (Definition 2.4).

▶ **Definition 2.10.** Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$ and let $\mathcal{A} \subseteq \mathcal{V}$.

- An n-ary connective $\diamond$ of $\mathcal{L}$ is called $\mathcal{A}$-*closed* if $\tilde{\diamond}(a_1, \ldots, a_n) \in \mathcal{A}$ for every $a_1, \ldots, a_n \in \mathcal{A}$.
- An n-ary connective $\diamond$ of $\mathcal{L}$ is called $\mathcal{A}$-*limited* if for every $a_1, \ldots, a_n \in \mathcal{V}$, if $\tilde{\diamond}(a_1, \ldots, a_n) \in \mathcal{A}$ then $a_1, \ldots, a_n \in \mathcal{A}$.

▶ **Definition 2.11.** Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- A connective $\wedge$ in $\mathcal{L}$ is called an $\mathcal{M}$-*conjunction* if it is $\mathcal{D}$-closed and $\mathcal{D}$-limited, i.e., for every $a, b \in \mathcal{V}$, $a \wedge b \in \mathcal{D}$ iff $a \in \mathcal{D}$ and $b \in \mathcal{D}$.
- A connective $\vee$ in $\mathcal{L}$ is called an $\mathcal{M}$-*disjunction* if it is $\overline{\mathcal{D}}$-closed and $\overline{\mathcal{D}}$-limited, i.e., for every $a, b \in \mathcal{V}$, $a \vee b \in \mathcal{D}$ iff $a \in \mathcal{D}$ or $b \in \mathcal{D}$.
- A connective $\supset$ in $\mathcal{L}$ is called an $\mathcal{M}$-*implication* if for every $a, b \in \mathcal{V}$, $a \supset b \in \mathcal{D}$ iff either $a \notin \mathcal{D}$ or $b \in \mathcal{D}$.

Using the terminologies in Definitions 2.4 and 2.11, we have:

▶ **Theorem 2.12.** *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.*

1. *A connective is an $\mathcal{M}$-conjunction iff it is a conjunction for $\mathbf{L}_{\mathcal{M}}$.*
2. *An $\mathcal{M}$-disjunction is also a disjunction for $\mathbf{L}_{\mathcal{M}}$.*
3. *An $\mathcal{M}$-implication is also an implication for $\mathbf{L}_{\mathcal{M}}$.*

▶ **Corollary 2.13.** *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$, and let $\mathcal{M}'$ be an expansion of $\mathcal{M}$.*

1. *An $\mathcal{M}$-conjunction (respectively: $\mathcal{M}$-disjunction, $\mathcal{M}$-implication) is also a conjunction (respectively: disjunction, implication) for $\mathbf{L}_{\mathcal{M}'}$.*
2. *If $\mathcal{M}$ has either an $\mathcal{M}$-conjunction, or an $\mathcal{M}$-disjunction, or an $\mathcal{M}$-implication, then $\mathbf{L}_{\mathcal{M}'}$ is semi-normal. If $\mathcal{M}$ has all of them then $\mathbf{L}_{\mathcal{M}'}$ is normal.*

3 Paradefinite Logics

In this section we define in precise terms what paradefinite logics are, and consider some related desirable properties.

▶ **Definition 3.1.** Let $\mathcal{L}$ be a propositional language with a unary connective $\neg$, and let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic for $\mathcal{L}$.

- $\mathbf{L}$ is called *pre-$\neg$-paraconsistent* if there are formulas ψ, φ such that $\psi, \neg\psi \nvdash_{\mathbf{L}} \varphi$.
- $\mathbf{L}$ is called *pre-$\neg$-paracomplete* if there is a theory $\mathcal{T}$ and formulas ψ, φ such that $\mathcal{T}, \psi \vdash_{\mathbf{L}} \varphi$ and $\mathcal{T}, \neg\psi \vdash_{\mathbf{L}} \varphi$, but $\mathcal{T} \nvdash_{\mathbf{L}} \varphi$.

The first property above intends to capture the idea that a contradictory set of premises should not entail every formula, and the second property indicates that it may happen that a certain statement and its negation do not hold. Both of these intuitions make sense only if the underlying connective $\neg$ somehow represents a 'negation' operation. This is assured by the condition of 'coherence with classical logic', which is defined next. Intuitively, this condition states that a logic that has such a connective should not admit entailments that do not hold in classical logic.

▶ **Definition 3.2.** Let $\mathcal{L}$ be a language with a unary connective $\neg$. A *bivalent $\neg$-interpretation for $\mathcal{L}$* is a function $\mathbf{F}$ that associates a two-valued truth table with each connective of $\mathcal{L}$, such that $\mathbf{F}(\neg)$ is the classical truth table for negation. We denote by $\mathcal{M}_{\mathbf{F}}$ the two-valued matrix for $\mathcal{L}$ induced by $\mathbf{F}$, that is, $\mathcal{M}_{\mathbf{F}} = \langle \{t, f\}, \{t\}, \mathbf{F} \rangle$ (see Definition 2.5).

▶ **Definition 3.3.** Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a propositional logic where $\mathcal{L}$ contains a unary connective $\neg$.

- Let $\mathbf{F}$ be a bivalent $\neg$-interpretation for $\mathcal{L}$. We say that $\mathbf{L}$ is $\mathbf{F}$-*contained in classical logic* if for every $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{W}(\mathcal{L})$, if $\varphi_1, \ldots \varphi_n \vdash_{\mathbf{L}} \psi$ then $\varphi_1, \ldots, \varphi_n \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$.
- $\mathbf{L}$ is $\neg$-*contained in classical logic* [3], if it is $\mathbf{F}$-contained in it for a bivalent $\neg$-interpretation $\mathbf{F}$.
- $\mathbf{L}$ is $\neg$-*coherent with classical logic*, if it has a semi-normal fragment (Definition 2.4) which is $\neg$-contained in classical logic.

▶ **Definition 3.4.** Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a propositional logic where $\mathcal{L}$ contains a unary connective $\neg$. We say that $\neg$ is a *negation* for $\mathbf{L}$, if $\mathbf{L}$ is $\neg$-coherent with classical logic.

▶ Remark. If $\neg$ is a negation for $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, then for every atom $p \in \mathsf{Atoms}(\mathcal{L})$ it holds that $p \nvdash_{\mathbf{L}} \neg p$ and $\neg p \nvdash_{\mathbf{L}} p$.

▶ **Definition 3.5.** Let $\mathcal{L}$ be a language with a unary connective $\neg$, and $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ a logic for $\mathcal{L}$.

- $\mathbf{L}$ is called $\neg$-*paraconsistent* if it is pre-$\neg$-paraconsistent and $\neg$ is a negation of $\mathbf{L}$.
- $\mathbf{L}$ is called $\neg$-*paracomplete* if it is pre-$\neg$-paracomplete and $\neg$ is a negation of $\mathbf{L}$.
- $\mathbf{L}$ is called $\neg$-*paradefinite* if it is $\neg$-paraconsistent and $\neg$-paracomplete.

Henceforth we shall frequently omit the $\neg$ sign (if it is clear from the context), and simply refer to paradefinite [paraconsistent, paracomplete] logics.

4 Four-Valued Paradefinite Matrices

We now show that the availability of at least four different truth values is needed for developing paradefinite logics in the framework of matrices. We then characterize the structure of four-valued paradefinite matrices.

In what follows we suppose that $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a matrix for a language with $\neg$. We say that $\mathcal{M}$ is paradefinite [paraconsistent, paracomplete] if so is $\mathbf{L}_{\mathcal{M}}$ (Definition 2.8).

▶ **Theorem 4.1.**

1. *$\mathcal{M}$ is pre-paraconsistent iff there is an element $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$.*
2. *If $\mathcal{M}$ is paraconsistent then there are three different elements t, f, and $\top$ in $\mathcal{V}$ such that $f = \tilde{\neg}t$, $f \notin \mathcal{D}$, and $\{t, \tilde{\neg}f, \top, \tilde{\neg}\top\} \subseteq \mathcal{D}$.*

Proof. $\mathcal{M}$ is pre-paraconsistent iff $p, \neg p \nvdash_{\mathcal{M}} q$. Obviously, this happens iff $\{p, \neg p\}$ has an $\mathcal{M}$-model. The latter, in turn, is possible iff there is some $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$, as indicated in the first item of the theorem. For the second item we may assume without loss of generality that $\mathcal{M}$ is $\neg$-contained in classical logic. We let $\mathbf{F}$ be a bivalent $\neg$-interpretation such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic. Since $p, \neg\neg p \nvdash_{\mathcal{M}_{\mathbf{F}}} \neg p$, also $p, \neg\neg p \nvdash_{\mathcal{M}} \neg p$, and so there is some $t \in \mathcal{D}$, such that $\tilde{\neg}t \notin \mathcal{D}$, while $\tilde{\neg}\tilde{\neg}t \in \mathcal{D}$. Let $f = \tilde{\neg}t$. Then t and f have the required properties, and together with the first item we are done. ◀

▶ **Theorem 4.2.**

1. *If $\mathcal{M}$ is pre-paracomplete then there is an element $\bot \in \mathcal{V}$ such that $\bot \notin \mathcal{D}$ and $\tilde{\neg}\bot \notin \mathcal{D}$.*
2. *If $\mathcal{M}$ has an $\mathcal{M}$-disjunction and there is an element $\bot \in \mathcal{V}$ such that $\bot \notin \mathcal{D}$ and $\tilde{\neg}\bot \notin \mathcal{D}$, then $\mathcal{M}$ is pre-paracomplete.*

Proof. Suppose first that $\mathcal{M}$ is pre-paracomplete. Then there is a set of formulas Γ and formulas ψ, ϕ, such that (i) $\Gamma, \psi \vdash_{\mathcal{M}} \phi$, (ii) $\Gamma, \neg\psi \vdash_{\mathcal{M}} \phi$, and (iii) $\Gamma \nvdash_{\mathcal{M}} \phi$. From (iii) it follows that there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma) \setminus mod_{\mathcal{M}}(\phi)$. Thus, in order to satisfy conditions (i) and (ii), necessarily $\nu(\psi) \notin \mathcal{D}$ and $\tilde{\neg}\nu(\psi) = \nu(\neg\psi) \notin \mathcal{D}$. Hence $\nu(\psi)$ is the element $\bot$ as required.

For the second item, suppose that its two conditions are satisfied. Let $\vee$ be an $\mathcal{M}$-disjunction. Then by Theorem 2.12, $p \vdash_{\mathcal{M}} \neg p \vee p$ and $\neg p \vdash_{\mathcal{M}} \neg p \vee p$. However, if $\nu(p) = \bot$ then $\nu(\neg p \vee p) \notin \mathcal{D}$ by the definitions of $\bot$ and of an $\mathcal{M}$-disjunction. Hence $\mathcal{M}$ is pre-paracomplete. ◀

By the theorems above, no 2-valued matrix can be paraconsistent or paracomplete, and no 3-valued matrix can be paradefinite. Also, by Theorem 4.1, every paraconsistent (and so every paradefinite) matrix should have at least two designated elements. The structures of the minimally-valued paradefinite matrices is considered next.

▶ **Theorem 4.3.** *If $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a $\neg$-paradefinite matrix then there are four elements t,f,$\top$, and $\bot$ in $\mathcal{V}$ such that: (1) $t \in \mathcal{D}$ and $\tilde{\neg}t \notin \mathcal{D}$, (2) $f \notin \mathcal{D}$ and $\tilde{\neg}f \in \mathcal{D}$, (3) $\top \in \mathcal{D}$ and $\tilde{\neg}\top \in \mathcal{D}$, (4) $\bot \notin \mathcal{D}$ and $\tilde{\neg}\bot \notin \mathcal{D}$, (5) $\tilde{\neg}t = f$.*

Proof. This follows from Theorems 4.1 and 4.2. ◀

▶ **Corollary 4.4.** *Let $\mathcal{M}$ be a $\neg$-paradefinite four-valued matrix. Then $\mathcal{M}$ is isomorphic to a matrix of the form $\mathcal{M}' = \langle \{t, f, \top, \bot\}, \{t, \top\}, \mathcal{O} \rangle$, in which $\tilde{\neg}t = f$, $\tilde{\neg}f = t$, $\tilde{\neg}\top \in \{t, \top\}$, and $\tilde{\neg}\bot \in \{f, \bot\}$.*

In the rest of this paper we shall assume that the 4-valued matrices we study have the form described in Corollary 4.4,

5 Dunn-Belnap's Matrix $\mathcal{FOUR}$

Theorem 4.4 leaves exactly four possible interpretations for $\neg$ in four-valued paradefinite matrices. However, the next theorem and its corollary show that the Dunn-Belnap negation ([9, 10, 14, 15]) is by far more natural than the others.[1]

▶ **Theorem 5.1.** *Let $\mathcal{M}$ be a $\neg$-paradefinite 4-valued matrix. Then:*
1. *If $\neg$ is left involutive for $\mathbf{L}_{\mathcal{M}}$ (that is, $\neg\neg p \vdash_{\mathbf{L}_{\mathcal{M}}} p$) then $\neg\bot = \bot$.*
2. *If $\neg$ is right involutive for $\mathbf{L}_{\mathcal{M}}$ (that is, $p \vdash_{\mathbf{L}_{\mathcal{M}}} \neg\neg p$) then $\neg\top = \top$.*

Proof. Suppose that $\neg$ is left involutive. Then $\neg\neg p \vdash_{\mathcal{M}} p$, and so $\neg\bot \neq f$ (otherwise, by Corollary 4.4 $\nu(p) = \bot$ would have been a counter-model). It follows that $\neg\bot = \bot$. Suppose now that $\neg$ is right involutive. Then $p \vdash_{\mathcal{M}} \neg\neg p$, and so $\neg\top \neq t$ (otherwise, by Corollary 4.4 again, $\nu(p) = \top$ would have been a counter-model). Thus $\neg\top = \top$. ◀

▶ **Corollary 5.2.** *The only involutive negation of paradefinite 4-valued logics is Dunn-Belnap negation, defined by $\neg t = f$, $\neg f = t$, $\neg\top = \top$ and $\neg\bot = \bot$.*

Concerning the interpretations of the other connectives, we again follow Belnap's motivation in [9] and [10], where he suggested a four-valued framework for collecting and processing information (this work was later generalized in [7]): Assume a set of sources, each one of them can indicate that an atom p is true (i.e., it assigns p the truth-value 1), false (i.e., it assigns p the truth-value 0), or that it has no knowledge about p. In turn, a mediator assigns

[1] For convenience, we shall denote the interpretation of $\neg$ by $\neg$ as well. A similar convention will be usually used for any other connective.

to an atomic formula p a subset $d(p)$ of $\{0,1\}$ as follows: $1 \in d(p)$ iff some source claims that p is true, and $0 \in d(p)$ iff some source claims that p is false. The mediator's evaluation of complex formulas over $\{\neg, \vee\}$ is then derived as follows:

$0 \in d(\neg\varphi)$ iff $1 \in d(\varphi)$,
$1 \in d(\neg\varphi)$ iff $0 \in d(\varphi)$,
$1 \in d(\varphi \vee \psi)$ iff $1 \in d(\varphi)$ or $1 \in d(\psi)$,
$0 \in d(\varphi \vee \psi)$ iff $0 \in d(\varphi)$ and $0 \in d(\psi)$.

In this model, $\nu(\varphi)=\{0,1\}$ means that φ is known to be true and also known to be false (i.e., the information about φ is inconsistent). $\nu(\varphi)=\{1\}$ means that φ is only known to be true, while $\nu(\varphi)=\{0\}$ means that φ is only known to be false. Finally, $\nu(\varphi)=\emptyset$ means that there is no information about φ. This observation leads to the following identification of the four truth-values with the subsets of $\{0,1\}$: $t = \{1\}, f = \{0\}, \top = \{0,1\}, \bot = \emptyset$.

Accordingly, the truth tables for $\neg$ and $\vee$ that the above principles lead to are the following (where the connective $\wedge$ is defined by: $\varphi \wedge \psi =_{Df} \neg(\neg\varphi \vee \neg\psi)$):

$\tilde{\vee}$	t	f	$\top$	$\bot$
t	t	t	t	t
f	t	f	$\top$	$\bot$
$\top$	t	$\top$	$\top$	t
$\bot$	t	$\bot$	t	$\bot$

$\tilde{\wedge}$	t	f	$\top$	$\bot$
t	t	f	$\top$	$\bot$
f	f	f	f	f
$\top$	$\top$	f	$\top$	f
$\bot$	$\bot$	f	f	$\bot$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$
$\bot$	$\bot$

▶ **Definition 5.3.** The *Dunn-Belnap basic matrix* for the language $\mathcal{L}_{\mathcal{FOUR}} = \{\neg, \vee, \wedge\}$ (or just $\{\neg, \vee\}$) is the matrix $\mathcal{FOUR} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{t, f, \top, \bot\}$, $\mathcal{D} = \{t, \top\}$, and the interpretations of the connectives are given by the truth tables above.

▶ Remark. Another, dual representation of $\mathcal{FOUR}$ uses pairs from $\{1,0\} \times \{1,0\}$. Given such a pair $\langle a, b \rangle$, the first component intuitively represents the information about the truth of a formula, and the second one represents the information about its falsity. According to this representation, we have that $t = \langle 1,0 \rangle$, $f = \langle 0,1 \rangle$, $\top = \langle 1,1 \rangle$, $\bot = \langle 0,0 \rangle$, $\langle a_1, b_1 \rangle \vee \langle a_2, b_2 \rangle = \langle max(a_1, b_1), min(a_2, b_2) \rangle$, $\langle a_1, b_1 \rangle \wedge \langle a_2, b_2 \rangle = \langle min(a_1, b_1), max(a_2, b_2) \rangle$, and $\neg\langle a, b \rangle = \langle b, a \rangle$. This representation is useful for a number of applications (see, e.g., [1, 4, 8, 17]).

A common way of defining and understanding the disjunction, conjunction and negation of $\mathcal{FOUR}$ is with respect to the partial order $\leq_{\mathsf{t}}$ on $\{t, f, \top, \bot\}$, in which t is the maximal element, f is the minimal element, and $\top, \bot$ are intermediate $\leq_{\mathsf{t}}$-incomparable elements. This order may be intuitively understood as reflecting differences in the amount of *truth* that each element exhibits. Here, $\tilde{\wedge}$ and $\tilde{\vee}$ are the meet and the join (respectively) of $\leq_{\mathsf{t}}$, and $\tilde{\neg}$ is order reversing with respect to $\leq_{\mathsf{t}}$. Note that this interpretation of $\neg$ coincides with that of the unique involutive negation of paradefinite four-valued logics given in Corollary 5.2.

For characterizing the expressive power of the languages of $\mathcal{FOUR}$ it is convenient to order the truth-values in the partial order $\leq_{\mathsf{k}}$ that intuitively reflects differences in the amount of *knowledge* (or *information*) that the truth values convey. According to this relation $\top$ is the maximal element, $\bot$ is the minimal element, and t, f are intermediate $\leq_{\mathsf{k}}$-incomparable elements.

Together, the lattices $\langle \{t, f, \top, \bot\}, \leq_{\mathsf{t}} \rangle$ and $\langle \{t, f, \top, \bot\}, \leq_{\mathsf{k}} \rangle$ form a single four-valued structure (denoted again by $\mathcal{FOUR}$), known as Belnap's bilattice ([9, 10]), which is represented in the double-Hasse diagram of Figure 1.

Following Fitting's notations (see [16]), we shall denote the join and the meet of $\leq_{\mathsf{k}}$ by $\oplus$ and $\otimes$ (respectively). The $\leq_{\mathsf{k}}$-reversing function on $\{t, f, \top, \bot\}$ which is dual to $\tilde{\neg}$ is called *conflation* [16], and the corresponding connective is usually denoted by $-$. The truth tables of these $\leq_{\mathsf{k}}$-connectives are given below.

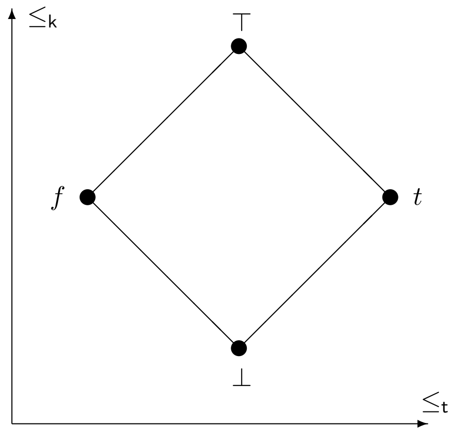

Figure 1 The bilattice $\mathcal{FOUR}$

$\tilde{\oplus}$	t	f	$\top$	$\bot$
t	t	$\top$	$\top$	t
f	$\top$	f	$\top$	f
$\top$	$\top$	$\top$	$\top$	$\top$
$\bot$	t	f	$\top$	$\bot$

$\tilde{\otimes}$	t	f	$\top$	$\bot$
t	t	$\bot$	t	$\bot$
f	$\bot$	f	f	$\bot$
$\top$	t	f	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

$\tilde{-}$	
t	t
f	f
$\top$	$\bot$
$\bot$	$\top$

6 Important Expansions of $\mathcal{FOUR}$

As noted before, the logic $\mathbf{L}_{\mathcal{FOUR}}$ (also denoted **4Basic**), induced by $\mathcal{FOUR}$, has some appealing applications in the context of logics for AI. Also, it has some desirable properties, like being semi-normal (it is easy to verify that $\vee$ is a $\mathcal{FOUR}$-disjunction and that $\wedge$ is a $\mathcal{FOUR}$-conjunction), paradefinite, and $\neg$-contained in classical logic. However, **4Basic** also has some drawbacks, one of which is considered next.

▶ **Theorem 6.1.** **4Basic** *is not normal (since no implication is definable in it).*

Sketch of proof. Note, first, that every n-valued function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ which is representable in the language of $\{\vee, \wedge, \neg\}$ [2] must be $\le_k$-monotonic, i.e., if $a_i \le_k b_i$ for every $1 \le i \le n$ then $g(a_1, \dots, a_n) \le_k g(b_1, \dots, b_n)$ as well. This implies that only $\le_k$-monotonic connectives are definable in this language. Now, suppose for contradiction that $\supset$ is a definable implication for **4Basic**. In particular, one may verify that (i) $\vdash_{\mathbf{4Basic}} p \supset p$, and (ii) $p, p \supset q \vdash_{\mathbf{4Basic}} q$. Now, (i) entails that $\tilde{\supset}(f, f) \in \{t, \top\}$. Therefore, it follows from the $\le_k$-monotonicity of $\supset$ that $\tilde{\supset}(\top, f) \in \{t, \top\}$. This contradicts (ii), since it is refuted by any assignment ν such that $\nu(p) = \top$ and $\nu(q) = f$. ◀

The last theorem, together with the fact that definable functions in the language of $\{\vee, \wedge, \neg\}$ are $\{\bot\}$-closed (and so no tautologies are available in this language), imply that the language of $\mathcal{FOUR}$ is rather limited, even if we add to it propositional constants for

[2] That is, there is a formula ψ in $\{\vee, \wedge, \neg\}$, such that $\mathsf{Atoms}(\psi) \subseteq \{P_1, \dots, P_n\}$, and for every $a_1, \dots, a_n \in \{t, f, \top, \bot\}$ it holds that $g(a_1, \dots, a_n) = \nu(\psi)$, where $\nu \in \Lambda_{\mathcal{FOUR}}$ is defined by $\nu(P_i) = a_i$ for all $1 \le i \le n$.

the two classical truth-values. Therefore, we now introduce several other useful and natural connectives on $\{t, f, \top, \bot\}$ that cannot be defined in the language of $\mathcal{FOUR}$.

- The following connective is an $\mathcal{M}$-implication for every paradefinite four-valued matrix $\mathcal{M}$ (of the form considered in Corollary 4.4), in which it is definable:

$$a \tilde{\supset} b = \begin{cases} b & \text{if } a \in \{t, \top\}, \\ t & \text{if } a \in \{f, \bot\}. \end{cases}$$

- We denote by $\mathsf{t}, \mathsf{f}, \mathsf{c}$ (contradictory) and u (unknown) the propositional constants to be interpreted, respectively, by the truth-values $t, f, \top$, and $\bot$ (thus, for instance, $\forall \nu \in \Lambda_{\mathcal{M}}\ \nu(\mathsf{c}) = \top$).

Using the connectives above, in the following sections we shall consider some important expansions of the matrix $\mathcal{FOUR}$.[3]

6.1 A Maximal Expansion

First, we consider expansions of $\mathcal{FOUR}$ in which all the operations on $\{t, f, \top, \bot\}$ are definable.

▶ **Definition 6.2.** Let $\mathcal{L}_{All} = \{\neg, \vee, \wedge, -, \oplus, \otimes, \supset, \mathsf{f}, \mathsf{t}, \mathsf{c}, \mathsf{u}\}$. The matrix $\mathcal{M}_{All}$ is the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{All}$. The logic that is induced by $\mathcal{M}_{All}$ is denoted by **4All** (or, as before, $\mathbf{L}_{\mathcal{M}_{All}}$).

As the next theorem shows, the set of connectives in $\mathcal{L}_{All}$ (and actually a proper subset of it) is indeed sufficient for defining any operation on $\{t, f, \top, \bot\}$.

▶ **Theorem 6.3.** *The language of* $\{\neg, \vee, \wedge, \supset, \mathsf{c}, \mathsf{u}\}$ *is functionally complete for* $\{t, f, \top, \bot\}$.

▶ Remark. Since $\bot = f \otimes \neg f$ while $\top = f \oplus \neg f$, the language of $\{\neg, \vee, \wedge, \supset, \otimes, \oplus, \mathsf{f}\}$ is also functionally complete for $\{t, f, \top, \bot\}$. The use of this language has a certain advantage of modularity over the use of $\{\neg, \vee, \wedge, \supset, \mathsf{c}, \mathsf{u}\}$, since it has been proved in [6] that if Ξ is a subset of $\{\otimes, \oplus, \mathsf{f}\}$, then a function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\{\neg, \wedge, \supset\} \cup \Xi$ iff it is S-closed for every $S \in \{\{\top\}, \{t, f, \top\}, \{t, f, \bot\}\}$ for which all the (functions that directly correspond to the) connectives in Ξ are S-closed. In other words:

▶ Theorem 6.4.

- *g is representable in $\{\neg, \vee, \wedge, \supset\}$ iff it is $\{\top\}$-closed, $\{t, f, \bot\}$-closed, and $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \mathsf{f}\}$ iff it is $\{t, f, \bot\}$-closed and $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus\}$ iff it is $\{\top\}$-closed and $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \otimes\}$ iff it is $\{\top\}$-closed and $\{t, f, \bot\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \otimes, \mathsf{f}\}$ iff it is $\{t, f, \bot\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus, \otimes\}$ iff it is $\{\top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus, \mathsf{f}\}$ iff it is $\{t, f, \top\}$-closed.*
- *g is representable in $\{\neg, \vee, \wedge, \supset, \oplus, \otimes, \mathsf{f}\}$.*

[3] Due to lack of space, proofs in the rest of this paper are omitted. Complete proofs will be provided in the full version of the paper.

It is also worth noting that it is easy to find examples that show that the eight fragments in the theorem above are different from each other (see [2] and [6]).

Note that $\mathcal{M}_{All}$, like any other 4-valued matrix where the $\leq_{\mathsf{k}}$-meet $\otimes$, the $\leq_{\mathsf{k}}$-join $\oplus$, or either of the propositional constants c and u is definable in its language, is not $\{t, f\}$-closed (indeed, $a \oplus b \notin \{t, f\}$ and $a \otimes b \notin \{t, f\}$ for any $a \neq b \in \{t, f\}$). This implies that **4All** is only $\neg$-*coherent* with classical logic but not $\neg$-*included* in it.

▶ **Theorem 6.5.** *The logic* **4All** *is paradefinite and normal.*

The next theorem follows from the fact that **4All** has no proper extensions (in the same language).

▶ **Theorem 6.6.** *The logic* **4All** *(unlike the logic* **4Basic**!*) is* maximally paraconsistent *in the sense that every proper extension of* **4All** *(Definition 2.3) is not pre-paraconsistent.*

6.2 A Maximal Monotonic Expansion

In [10] Belnap suggested to use the sources-mediator model described previously only for languages with monotonic interpretations of the connectives. The reason was to achieve stability in the sense that the arrival of new data from new sources does not change previous knowledge about truth and falsity. From Belnap's point of view an optimal language for information processing is therefore a language in which it is possible to represent *all* monotonic functions, and only monotonic functions. Next we show that not much should be added to the basic language of $\{\neg, \vee, \wedge\}$ (or just $\{\neg, \vee\}$) in order to obtain such a language.

▶ **Definition 6.7.** Let $\mathcal{L}_{Mon} = \{\neg, \vee, \wedge, \mathsf{c}, \mathsf{u}\}$. We denote by $\mathcal{M}_{Mon}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{Mon}$. The logic that is induced by $\mathcal{M}_{Mon}$ is denoted by **4Mon**.

▶ **Theorem 6.8.** *A function* $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ *is representable in* $\mathcal{L}_{Mon}$ *iff it is* $\leq_{\mathsf{k}}$*-monotonic (i.e., if* $a_i \leq_{\mathsf{k}} b_i$ *for every* $1 \leq i \leq n$ *then* $g(a_1, \ldots, a_n) \leq_{\mathsf{k}} g(b_1, \ldots, b_n)$*).*

▶ **Corollary 6.9.** *The logic* **4Mon** *contains every logic which is induced by a matrix of the form of Corollary 4.4 that employs only monotonic functions.*

▶ **Theorem 6.10.** *The logic* **4Mon** *is paradefinite. It has no proper extensions in its language, and so it is maximally paraconsistent (see Theorem 6.6).*

6.3 A Maximal Classically Closed Expansion

We now examine the maximal expansions of $\mathcal{FOUR}$ by connectives that are $\{t, f\}$-closed.

▶ **Definition 6.11.** Let $\mathcal{L}_{CC} = \{\neg, -, \vee, \wedge, \supset\}$. We denote by $\mathcal{M}_{CC}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{CC}$. The logic that is induced by $\mathcal{M}_{CC}$ is denoted by **4CC**.

▶ **Theorem 6.12.** *A function* $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ *is representable in* $\mathcal{L}_{CC}$ *iff it is* $\{t, f\}$*-closed.*

▶ **Corollary 6.13.** *The logic* **4CC** *contains every logic which is induced by a matrix of the form of Corollary 4.4 and is* $\neg$*-contained in classical logic.*

▶ **Theorem 6.14.** *The logic* **4CC** *is paradefinite and normal. It is* $\neg$*-contained in classical logic and has no proper extensions in its language, thus it is maximally paraconsistent (in the sense described in Theorem 6.6).*

We note, in addition to the properties considered in Proposition 6.14, that **4CC** is also maximal relative to classical logic. This means, intuitively, that any attempt to add to it a tautology of classical logic which is not provable in **4CC** should necessarily end-up with classical logic (see [3] for the exact definition of this property).

6.4 A Maximal Non-Exploding Expansion

Next, we consider the maximal expansions of $\mathcal{FOUR}$ which are non-exploding in the following sense:

▶ **Definition 6.15.** A logic $\langle \mathcal{L}, \vdash \rangle$ is *non-exploding*, if for every theory $\mathcal{T}$ in $\mathcal{L}$ such that $\mathsf{Atoms}(\mathcal{T}) \neq \mathsf{Atoms}(\mathcal{L})$ there is a formula ψ in $\mathcal{L}$ such that $\mathcal{T} \nvdash \psi$.

▶ **Definition 6.16.** Let $\mathcal{L}_{Nex} = \{\neg, \vee, \wedge, \supset, \oplus, \otimes\}$. We denote by $\mathcal{M}_{Nex}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{Nex}$. The logic that is induced by $\mathcal{M}_{Nex}$ is denoted by **4Nex**.

▶ **Theorem 6.17.** *A function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{Nex}$ iff it is $\{\top\}$-closed.*

▶ **Corollary 6.18.** *The logic* **4Nex** *contains every logic which is induced by a matrix of the form of Corollary 4.4 and is non-exploding.*

▶ **Theorem 6.19.** *The logic* **4Nex** *is paradefinite. It is non-exploding but not $\neg$-contained in classical logic. Also,* **4Nex** *has no proper extensions in its language, thus it is maximally paraconsistent.*

6.5 A Maximal Flexible Expansion

The combination of $\{t, f, \top\}$-closure and $\{t, f, \bot\}$-closure is a very desirable property, since it allows flexibility in the use of the four basic truth-values. Obviously, there is no point in using c in case no contradiction is expected, while in the dual case there is no point in using u. The use of connectives which have both of the above properties ensures that one can easily switch from the use of the four-valued framework to the use of the appropriate 3-valued framework. Also, this combination is a natural strengthening of the condition of classical closure. These considerations motivate the four-valued logic introduced next.

▶ **Definition 6.20.** A function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is called *flexible* iff it is both $\{t, f, \top\}$-closed and $\{t, f, \bot\}$-closed.

Obviously, every flexible function is classically closed, but the converse is not true.

▶ **Definition 6.21.** Let $\mathcal{L}_{Flex} = \{\neg, \vee, \wedge, \supset, \mathsf{f}\}$. We denote by $\mathcal{M}_{Flex}$ the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{Flex}$. The logic that is induced by $\mathcal{M}_{Flex}$ is denoted by **4Flex**.

▶ **Theorem 6.22.** *A function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{Flex}$ iff it is flexible.*

▶ **Corollary 6.23.** *The logic* **4Flex** *contains every logic that is induced by a matrix of the form of Corollary 4.4 and employs only flexible connectives.*

▶ **Theorem 6.24.** *The logic* **4Flex** *is a paradefinite and normal. It is $\neg$-contained in classical logic and not non-exploding. This logic is neither maximally paraconsistent nor maximally paraconsistent relative to classical logic.*

6.6 The Classical Expansion

The last expansion of $\mathcal{FOUR}$ we present is the maximal one which is *both* non-explosive and flexible.

▶ **Definition 6.25.** Let $\mathcal{L}_{CL} = \{\neg, \vee, \wedge, \supset\}$. We denote by $\mathcal{M}_{4CL}$ is the expansion of $\mathcal{FOUR}$ to $\mathcal{L}_{CL}$. The logic that is induced by $\mathcal{M}_{4CL}$ is denoted by **4CL**.

▶ **Theorem 6.26.** *A function $g : \{t, f, \top, \bot\}^n \to \{t, f, \top, \bot\}$ is representable in $\mathcal{L}_{CL}$ iff it is flexible and $\{\top\}$-closed.*

▶ **Corollary 6.27.** *The logic* **4CL** *contains every non-exploding logic which is induced by a matrix of the form of Corollary 4.4 and employs only flexible connectives.*

▶ **Theorem 6.28.** *The logic* **4CL** *is paradefinite and normal. It is $\neg$-contained in classical logic and non-exploding. This logic is neither maximally paraconsistent nor maximally paraconsistent relative to classical logic.*

7 Proof Theory

We conclude by considering proof systems for the $\neg$-paradefinite logics presented in this paper.

7.1 Gentzen-type Systems

First, we consider Gentzen-type systems [18]. We show that each of the logics considered here has a corresponding cut-free, sound and complete sequent calculus, which is a fragment of the sequent calculus $G_{\mathbf{4All}}$, presented in Figure 2.

For each $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4Mon}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}, \mathbf{4Basic}\}$ we denote by $G_{\mathbf{L}}$ the restriction of $G_{\mathbf{4All}}$ to the language of $\mathbf{L}$ (i.e., the Gentzen-type system in the language of $\mathbf{L}$ whose axioms and rules are the axioms and rules of $G_{\mathbf{4All}}$ which are relevant to that language). Also, we denote by $\vdash_{G_{\mathbf{L}}}$ the consequence relation induced by $G_{\mathbf{L}}$, that is: $\mathcal{T} \vdash_{G_{\mathbf{L}}} \varphi$, if there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \varphi$ is provable in $G_{\mathbf{L}}$ from the empty set of sequents (see, e.g., [23] and [24]).

▶ **Theorem 7.1.** *For each $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4Mon}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}, \mathbf{4Basic}\}$ $G_{\mathbf{L}}$ is sound and complete for $\mathbf{L}$: $\mathcal{T} \vdash_{G_{\mathbf{L}}} \psi$ iff $\mathcal{T} \vdash_{\mathbf{L}} \psi$. Moreover, $G_{\mathbf{L}}$ admits cut-elimination.*

7.2 Hilbert-type Systems

Next, we consider sound and complete Hilbert-type systems for $\neg$-paradefinite logics which have an implication connective. Again, we show that these are fragments of the same proof system, which has Modus Ponens [MP] as its sole rule of inference.

Consider the proof system $H_{\mathbf{4All}}$ in Figure 3. For $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}\}$ we denote by $H_{\mathbf{L}}$ the restriction of $H_{\mathbf{4All}}$ to the language of $\mathbf{L}$ (i.e., the Hilbert-type system in the language of $\mathbf{L}$ whose axioms and rules are the axioms and rules of $H_{\mathbf{4All}}$ which are relevant to that language). We denote by $\vdash_{H_{\mathbf{L}}}$ the consequence relation induced by $H_{\mathbf{L}}$.

▶ **Theorem 7.2.** *For every $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}\}$ we have that $\vdash_{H_{\mathbf{L}}} = \vdash_{G_{\mathbf{L}}}$.*

By Theorems 7.1 and 7.2 we also have the following result.

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening: $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$ Cut: $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

Logical Rules:

$[\wedge\Rightarrow]$	$\dfrac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}$	$[\Rightarrow\wedge]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$
$[\vee\Rightarrow]$	$\dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$	$[\Rightarrow\vee]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$
$[\supset\Rightarrow]$	$\dfrac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$	$[\Rightarrow\supset]$	$\dfrac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$
$[\otimes\Rightarrow]$	$\dfrac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \otimes \phi \Rightarrow \Delta}$	$[\Rightarrow\otimes]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \otimes \phi}$
$[\oplus\Rightarrow]$	$\dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \oplus \phi \Rightarrow \Delta}$	$[\Rightarrow\oplus]$	$\dfrac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \oplus \phi}$
$[\neg\neg\Rightarrow]$	$\dfrac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$	$[\Rightarrow\neg\neg]$	$\dfrac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$
$[-\Rightarrow]$	$\dfrac{\Gamma, \Rightarrow \Delta, \neg\psi}{\Gamma, -\psi \Rightarrow \Delta}$	$[\Rightarrow-]$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\psi}$
$[\neg-\Rightarrow]$	$\dfrac{\Gamma, \Rightarrow \Delta, \psi}{\Gamma, \neg-\psi \Rightarrow \Delta}$	$[\Rightarrow\neg-]$	$\dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg-\psi}$
$[\neg\wedge\Rightarrow]$	$\dfrac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\wedge]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$
$[\neg\vee\Rightarrow]$	$\dfrac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\vee]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$
$[\neg\supset\Rightarrow]$	$\dfrac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta}$	$[\Rightarrow\neg\supset]$	$\dfrac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \supset \psi), \Delta}$
$[\neg\otimes\Rightarrow]$	$\dfrac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \otimes \phi) \Rightarrow \Delta}$	$[\Rightarrow\neg\otimes]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \otimes \phi)}$
$[\neg\oplus\Rightarrow]$	$\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \oplus \phi) \Rightarrow \Delta}$	$[\Rightarrow\neg\oplus]$	$\dfrac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \oplus \phi)}$
$[\mathsf{f}\Rightarrow]$	$\Gamma, \mathsf{f} \Rightarrow \Delta$	$[\Rightarrow\neg\mathsf{f}]$	$\Gamma \Rightarrow \Delta, \neg\mathsf{f}$
$[\Rightarrow\mathsf{c}]$	$\Gamma \Rightarrow \Delta, \mathsf{c}$	$[\Rightarrow\neg\mathsf{c}]$	$\Gamma \Rightarrow \Delta, \neg\mathsf{c}$
$[\mathsf{u}\Rightarrow]$	$\Gamma, \mathsf{u} \Rightarrow \Delta$	$[\neg\mathsf{u}\Rightarrow]$	$\Gamma, \neg\mathsf{u} \Rightarrow \Delta$

Figure 2 The proof system $G_{\mathbf{4All}}$.

▶ **Corollary 7.3.** *For every* $\mathbf{L} \in \{\mathbf{4All}, \mathbf{4CC}, \mathbf{4Nex}, \mathbf{4Flex}, \mathbf{4CL}\}$*,* $H_{\mathbf{L}}$ *is sound and complete for* $\mathbf{L}$*.*

▶ Remark. Other proof systems for paradefinite logics have been considered in the literature, and in many cases it is possible to show that they are equivalent to some of the proof systems considered here. For instance, Bou and Rivieccio's Hilbert-style proof system introduced in [12] has 23 rules for the language of $\{\neg, \vee, \wedge, \otimes, \oplus\}$, and no axioms. In [12] it is shown that this system is equivalent to the corresponding fragment of $G_{\mathbf{4All}}$, and it is not difficult to see that it is obtained by a straightforward translation of that system.

Inference Rule: [MP] $\dfrac{\psi \quad \psi \supset \varphi}{\varphi}$

Axioms:

$[\Rightarrow\supset 1]$	$\psi \supset (\varphi \supset \psi)$		
$[\Rightarrow\supset 2]$	$(\psi \supset (\varphi \supset \tau)) \supset ((\psi \supset \varphi) \supset (\psi \supset \tau))$		
$[\Rightarrow\supset 3]$	$((\psi \supset \varphi) \supset \psi) \supset \psi$		
$[\Rightarrow\wedge\supset]$	$\psi \wedge \varphi \supset \psi, \quad \psi \wedge \varphi \supset \varphi$	$[\Rightarrow\supset\wedge]$	$\psi \supset (\varphi \supset \psi \wedge \varphi)$
$[\Rightarrow\supset\vee]$	$\psi \supset \psi \vee \varphi, \quad \varphi \supset \psi \vee \varphi$	$[\Rightarrow\vee\supset]$	$(\psi \supset \tau) \supset ((\varphi \supset \tau) \supset (\psi \vee \varphi \supset \tau))$
$[\neg\neg\Rightarrow]$	$\neg\neg\varphi \supset \varphi$	$[\Rightarrow\neg\neg]$	$\varphi \supset \neg\neg\varphi$
$[\neg\supset\Rightarrow 1]$	$\neg(\varphi \supset \psi) \supset \varphi$	$[\neg\supset\Rightarrow 2]$	$\neg(\varphi \supset \psi) \supset \neg\psi$
$[\Rightarrow\neg\supset]$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$		
$[\neg\vee\Rightarrow 1]$	$\neg(\varphi \vee \psi) \supset \neg\varphi$	$[\neg\vee\Rightarrow 2]$	$\neg(\varphi \vee \psi) \supset \neg\psi$
$[\Rightarrow\neg\vee]$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$		
$[\neg\wedge\Rightarrow]$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$		
$[\Rightarrow\neg\wedge 1]$	$\neg\varphi \supset \neg(\varphi \wedge \psi)$	$[\Rightarrow\neg\wedge 2]$	$\neg\psi \supset \neg(\varphi \wedge \psi)$
$[\Rightarrow\otimes]$	$\psi \supset \varphi \supset \psi \otimes \varphi$	$[\otimes\Rightarrow]$	$\psi \otimes \varphi \supset \psi, \quad \psi \otimes \varphi \supset \varphi$
$[\Rightarrow\oplus]$	$\psi \supset \psi \oplus \varphi, \quad \varphi \supset \psi \oplus \varphi$	$[\oplus\Rightarrow]$	$(\psi \supset \tau) \supset (\varphi \supset \tau) \supset (\psi \oplus \varphi \supset \tau)$
$[\Rightarrow\neg\oplus]$	$\neg\psi \oplus \neg\varphi \supset \neg(\psi \oplus \varphi)$	$[\neg\oplus\Rightarrow]$	$\neg(\psi \oplus \varphi) \supset \neg\psi \oplus \neg\varphi$
$[\Rightarrow\neg\otimes]$	$\neg\psi \otimes \neg\varphi \supset \neg(\psi \otimes \varphi)$	$[\neg\otimes\Rightarrow]$	$\neg(\psi \otimes \varphi) \supset \neg\psi \otimes \neg\varphi$
$[\mathsf{f}\Rightarrow]$	$\mathsf{f} \supset \psi$		
$[\mathsf{u}\Rightarrow]$	$\mathsf{u} \supset \psi$	$[\Rightarrow\mathsf{c}]$	$\psi \supset \mathsf{c}$
$[\neg\mathsf{u}\Rightarrow]$	$\neg\mathsf{u} \supset \psi$	$[\Rightarrow\neg\mathsf{c}]$	$\psi \supset \neg\mathsf{c}$

Figure 3 The proof system $H_{\mathbf{4All}}$.

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