On the Beer Index of Convexity and Its Variants*

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Abstract

Let S be a subset of $\mathbb{R}^d$ with finite positive Lebesgue measure. The *Beer index of convexity* $\mathrm{b}(S)$ of S is the probability that two points of S chosen uniformly independently at random see each other in S. The *convexity ratio* $\mathrm{c}(S)$ of S is the Lebesgue measure of the largest convex subset of S divided by the Lebesgue measure of S. We investigate the relationship between these two natural measures of convexity of S.

We show that every set $S \subseteq \mathbb{R}^2$ with simply connected components satisfies $\mathrm{b}(S) \leqslant \alpha\,\mathrm{c}(S)$ for an absolute constant α, provided $\mathrm{b}(S)$ is defined. This implies an affirmative answer to the conjecture of Cabello et al. asserting that this estimate holds for simple polygons.

We also consider higher-order generalizations of $\mathrm{b}(S)$. For $1 \leqslant k \leqslant d$, the *$k$-index of convexity* $\mathrm{b}_k(S)$ of $S \subseteq \mathbb{R}^d$ is the probability that the convex hull of a $(k+1)$-tuple of points chosen uniformly independently at random from S is contained in S. We show that for every $d \geqslant 2$ there is a constant $\beta(d) > 0$ such that every set $S \subseteq \mathbb{R}^d$ satisfies $\mathrm{b}_d(S) \leqslant \beta\,\mathrm{c}(S)$, provided $\mathrm{b}_d(S)$ exists. We provide an almost matching lower bound by showing that there is a constant $\gamma(d) > 0$ such that for every $\varepsilon \in (0, 1]$ there is a set $S \subseteq \mathbb{R}^d$ of Lebesgue measure one satisfying $\mathrm{c}(S) \leqslant \varepsilon$ and $\mathrm{b}_d(S) \geqslant \gamma \frac{\varepsilon}{\log_2 1/\varepsilon} \geqslant \gamma \frac{\mathrm{c}(S)}{\log_2 1/\mathrm{c}(S)}$.

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1 Introduction

For positive integers k and d and a Lebesgue measurable set $S \subseteq \mathbb{R}^d$, we use $\lambda_k(S)$ to denote the k-dimensional Lebesgue measure of S. We omit the subscript k when it is clear from the context. We also write 'measure' instead of 'Lebesgue measure', as we do not use any other measure in the paper.

For a set $S \subseteq \mathbb{R}^d$, let $\mathrm{smc}(S)$ denote the supremum of the measures of convex subsets of S. Since all convex subsets of $\mathbb{R}^d$ are measurable [12], the value of $\mathrm{smc}(S)$ is well defined.

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Moreover, Goodman's result [9] implies that the supremum is achieved on compact sets S, hence it can be replaced by maximum in this case. When S has finite positive measure, let $\mathrm{c}(S)$ be defined as $\mathrm{smc}(S)/\lambda_d(S)$. We call the parameter $\mathrm{c}(S)$ the *convexity ratio* of S.

For two points $A, B \in \mathbb{R}^d$, let $\overline{AB}$ denote the closed line segment with endpoints A and B. Let S be a subset of $\mathbb{R}^d$. We say that points $A, B \in S$ are *visible* one from the other or *see* each other in S if the line segment $\overline{AB}$ is contained in S. For a point $A \in S$, we use $\mathrm{Vis}(A, S)$ to denote the set of points that are visible from A in S. More generally, for a subset T of S, we use $\mathrm{Vis}(T, S)$ to denote the set of points that are visible in S from T. That is, $\mathrm{Vis}(T, S)$ is the set of points $A \in S$ for which there is a point $B \in T$ such that $\overline{AB} \subseteq S$.

Let $\mathrm{Seg}(S)$ denote the set $\{(A, B) \in S \times S \colon \overline{AB} \subseteq S\} \subseteq (\mathbb{R}^d)^2$, which we call the *segment set* of S. For a set $S \subseteq \mathbb{R}^d$ with finite positive measure and with measurable $\mathrm{Seg}(S)$, we define the parameter $\mathrm{b}(S) \in [0, 1]$ by

$$\mathrm{b}(S) := \frac{\lambda_{2d}(\mathrm{Seg}(S))}{\lambda_d(S)^2}.$$

If S is not measurable, or if its measure is not positive and finite, or if $\mathrm{Seg}(S)$ is not measurable, we leave $\mathrm{b}(S)$ undefined. Note that if $\mathrm{b}(S)$ is defined for a set S, then $\mathrm{c}(S)$ is defined as well.

We call $\mathrm{b}(S)$ the *Beer index of convexity* (or just *Beer index*) of S. It can be interpreted as the probability that two points A and B of S chosen uniformly independently at random see each other in S.

1.1 Previous results

The Beer index was introduced in the 1970s by Beer [2, 3, 4], who called it 'the index of convexity'. Beer was motivated by studying the continuity properties of $\lambda(\mathrm{Vis}(A, S))$ as a function of A. For polygonal regions, an equivalent parameter was later independently defined by Stern [19], who called it 'the degree of convexity'. Stern was motivated by the problem of finding a computationally tractable way to quantify how close a given set is to being convex. He showed that the Beer index of a polygon P can be approximated by a Monte Carlo estimation. Later, Rote [17] showed that for a polygonal region P with n edges the Beer index can be evaluated in polynomial time as a sum of $O(n^9)$ closed-form expressions.

Cabello et al. [7] have studied the relationship between the Beer index and the convexity ratio, and applied their results in the analysis of their near-linear-time approximation algorithm for finding the largest convex subset of a polygon. We describe some of their results in more detail in Subsection 1.3.

1.2 Terminology and notation

We assume familiarity with basic topological notions such as path-connectedness, simple connectedness, Jordan curve, etc. The reader can find these definitions, for example, in Prasolov's book [16].

Let ∂S, S°, and $\overline{S}$ denote the boundary, the interior, and the closure of a set S, respectively. For a point $A \in \mathbb{R}^2$ and $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(A)$ denote the open disc centered at A with radius ε. For a set $X \subseteq \mathbb{R}^2$ and $\varepsilon > 0$, let $\mathcal{N}_\varepsilon(X) = \bigcup_{A \in X} \mathcal{N}_\varepsilon(A)$. A *neighborhood* of a point $A \in \mathbb{R}^2$ or a set $X \subseteq \mathbb{R}^2$ is a set of the form $\mathcal{N}_\varepsilon(A)$ or $\mathcal{N}_\varepsilon(X)$, respectively, for some $\varepsilon > 0$.

A closed interval with endpoints a and b is denoted by $[a, b]$. Intervals $[a, b]$ with $a > b$ are considered empty. For a point $A \in \mathbb{R}^2$, we use $x(A)$ and $y(A)$ to denote the x-coordinate and the y-coordinate of A, respectively.

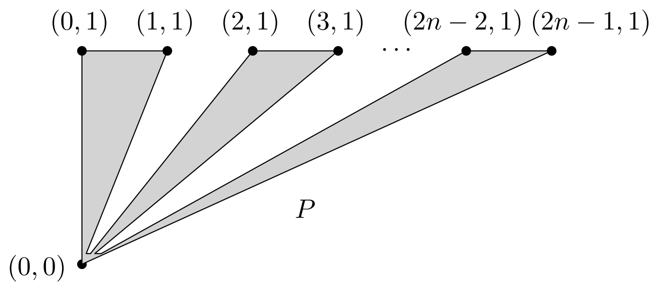

Figure 1 A star-shaped polygon P with $\mathrm{b}(P) \geqslant \frac{1}{n} - \varepsilon$ and $\mathrm{c}(P) \leqslant \frac{1}{n}$. The polygon P with $4n-1$ vertices is a union of n triangles $(0,0)(2i,1)(2i+1,1)$, $i = 0,\dots,n-1$, and of a triangle $(0,0)(0,\delta)((2n-1)\delta,\delta)$, where δ is very small.

A *polygonal curve* Γ in $\mathbb{R}^d$ is a curve specified by a sequence $(A_1,\dots,A_n)$ of points of $\mathbb{R}^d$ such that Γ consists of the line segments connecting the points A_i and A_{i+1} for $i = 1,\dots,n-1$. If $A_1 = A_n$, then the polygonal curve Γ is *closed*. A polygonal curve that is not closed is called a *polygonal line*.

A set $X \subseteq \mathbb{R}^2$ is *polygonally connected*, or *p-connected* for short, if any two points of X can be connected by a polygonal line in X, or equivalently, by a self-avoiding polygonal line in X. For a set X, the relation "A and B can be connected by a polygonal line in X" is an equivalence relation on X, and its equivalence classes are the *p-components* of X. A set S is *p-componentwise simply connected* if every p-component of S is simply connected.

A *line segment* in $\mathbb{R}^d$ is a bounded convex subset of a line. A *closed line segment* includes both endpoints, while an *open line segment* excludes both endpoints. For two points A and B in $\mathbb{R}^d$, we use AB to denote the open line segment with endpoints A and B. A closed line segment with endpoints A and B is denoted by $\overline{AB}$.

We say that a set $S \subseteq \mathbb{R}^d$ is *star-shaped* if there is a point $C \in S$ such that $\mathrm{Vis}(C,S) = S$. That is, a star-shaped set S contains a point which sees the entire S. Similarly, we say that a set S is *weakly star-shaped* if S contains a line segment ℓ such that $\mathrm{Vis}(\ell,S) = S$.

1.3 Results

We start with a few simple observations. Let S be a subset of $\mathbb{R}^2$ such that $\mathrm{Seg}(S)$ is measurable. For every $\varepsilon > 0$, S contains a convex subset K of measure at least $(\mathrm{c}(S)-\varepsilon)\lambda_2(S)$. Two random points of S both belong to K with probability at least $(\mathrm{c}(S)-\varepsilon)^2$, hence $\mathrm{b}(S) \geqslant (\mathrm{c}(S)-\varepsilon)^2$. This yields $\mathrm{b}(S) \geqslant \mathrm{c}(S)^2$. This simple lower bound on $\mathrm{b}(S)$ is tight, as shown by a set S which is a disjoint union of a single large convex component and a large number of small components of negligible size.

It is more challenging to find an upper bound on $\mathrm{b}(S)$ in terms of $\mathrm{c}(S)$, possibly under additional assumptions on the set S. This is the general problem addressed in this paper.

As a motivating example, observe that a set S consisting of n disjoint convex components of the same size satisfies $\mathrm{b}(S) = \mathrm{c}(S) = \frac{1}{n}$. It is easy to modify this example to obtain, for any $\varepsilon > 0$, a simple star-shaped polygon P with $\mathrm{b}(P) \geqslant \frac{1}{n} - \varepsilon$ and $\mathrm{c}(P) \leqslant \frac{1}{n}$, see Figure 1. This shows that $\mathrm{b}(S)$ cannot be bounded from above by a sublinear function of $\mathrm{c}(S)$, even for simple polygons S.

For weakly star-shaped polygons, Cabello et al. [7] showed that the above example is essentially optimal, providing the following linear upper bound on $\mathrm{b}(S)$.

▶ **Theorem 1** ([7, Theorem 5]). *For every weakly star-shaped simple polygon P, we have $\mathrm{b}(P) \leqslant 18\,\mathrm{c}(P)$.*

For polygons that are not weakly star-shaped, Cabello et al. [7] gave a superlinear bound.

▶ **Theorem 2** ([7, Theorem 6]). *Every simple polygon P satisfies*

$$\mathrm{b}(P) \leqslant 12\,\mathrm{c}(P)\left(1 + \log_2 \frac{1}{\mathrm{c}(P)}\right).$$

Moreover, Cabello et al. [7] conjectured that even for a general simple polygon P, $\mathrm{b}(P)$ can be bounded from above by a linear function of $\mathrm{c}(P)$. The next theorem, which is the first main result of this paper, confirms this conjecture. Recall that $\mathrm{b}(S)$ is defined for a set S if and only if S has finite positive measure and $\mathrm{Seg}(S)$ is measurable. Recall also that a set is p-componentwise simply connected if its polygonally-connected components are simply connected. In particular, every simply connected set is p-componentwise simply connected.

▶ **Theorem 3.** *Every p-componentwise simply connected set $S \subseteq \mathbb{R}^2$ whose $\mathrm{b}(S)$ is defined satisfies $\mathrm{b}(S) \leqslant 180\,\mathrm{c}(S)$.*

It is clear that every simple polygon satisfies the assumptions of Theorem 3, hence we directly obtain the following, which confirms the conjecture of Cabello et al. [7].

▶ **Corollary 4.** *Every simple polygon $P \subseteq \mathbb{R}^2$ satisfies $\mathrm{b}(P) \leqslant 180\,\mathrm{c}(P)$.*

The main restriction in Theorem 3 is the assumption that S is p-componentwise simply connected. This assumption cannot be omitted, as shown by the set $S = [0,1]^2 \setminus \mathbb{Q}^2$, where it is easy to verify that $\mathrm{c}(S) = 0$ and $\mathrm{b}(S) = 1$.

A related construction shows that Theorem 3 fails in higher dimensions. To see this, consider again the set $S = [0,1]^2 \setminus \mathbb{Q}^2$, and define a set $S' \subseteq \mathbb{R}^3$ by

$$S' := \{(tx, ty, t) \colon t \in [0,1] \text{ and } (x,y) \in S\}.$$

Again, it is easy to verify that $\mathrm{c}(S') = 0$ and $\mathrm{b}(S') = 1$, although S' is simply connected, even star-shaped.

Despite these examples, we will show that meaningful analogues of Theorem 3 for higher dimensions and for sets that are not p-componentwise simply connected are possible. The key is to use higher-order generalizations of the Beer index, which we introduce now.

For a set $S \subseteq \mathbb{R}^d$, we define the set $\mathrm{Simp}_k(S) \subseteq (\mathbb{R}^d)^{k+1}$ by

$$\mathrm{Simp}_k(S) := \{(A_0, \ldots, A_k) \in S^{k+1} \colon \mathrm{Conv}(\{A_0, \ldots, A_k\}) \subseteq S\},$$

where the operator Conv denotes the convex hull of a set of points. We call $\mathrm{Simp}_k(S)$ the *k-simplex set* of S. Note that $\mathrm{Simp}_1(S) = \mathrm{Seg}(S)$.

For an integer $k \in \{1, 2, \ldots, d\}$ and a set $S \subseteq \mathbb{R}^d$ with finite positive measure and with measurable $\mathrm{Simp}_k(S)$, we define $\mathrm{b}_k(S)$ by

$$\mathrm{b}_k(S) := \frac{\lambda_{(k+1)d}(\mathrm{Simp}_k(S))}{\lambda_d(S)^{k+1}}.$$

Note that $\mathrm{b}_1(S) = \mathrm{b}(S)$. We call $\mathrm{b}_k(S)$ the *k-index of convexity* of S. We again leave $\mathrm{b}_k(S)$ undefined if S or $\mathrm{Simp}_k(S)$ is non-measurable, or if the measure of S is not finite and positive.

We can view $\mathrm{b}_k(S)$ as the probability that the convex hull of $k+1$ points chosen from S uniformly independently at random is contained in S. For any $S \subseteq \mathbb{R}^d$, we have $\mathrm{b}_1(S) \geqslant \mathrm{b}_2(S) \geqslant \cdots \geqslant \mathrm{b}_d(S)$, provided all the $\mathrm{b}_k(S)$ are defined.

We remark that the set $S = [0,1]^d \setminus \mathbb{Q}^d$ satisfies $\mathrm{c}(S) = 0$ and $\mathrm{b}_1(S) = \mathrm{b}_2(S) = \cdots = \mathrm{b}_{d-1}(S) = 1$. Thus, for a general set $S \subseteq \mathbb{R}^d$, only the d-index of convexity can conceivably admit a nontrivial upper bound in terms of $\mathrm{c}(S)$. Our next result shows that such an upper bound on $\mathrm{b}_d(S)$ exists and is linear in $\mathrm{c}(S)$.

▶ **Theorem 5.** *For every $d \geqslant 2$, there is a constant $\beta = \beta(d) > 0$ such that every set $S \subseteq \mathbb{R}^d$ with defined $\mathrm{b}_d(S)$ satisfies $\mathrm{b}_d(S) \leqslant \beta\,\mathrm{c}(S)$.*

We do not know if the linear upper bound in Theorem 5 is best possible. We can, however, construct examples showing that the bound is optimal up to a logarithmic factor. This is our last main result.

▶ **Theorem 6.** *For every $d \geqslant 2$, there is a constant $\gamma = \gamma(d) > 0$ such that for every $\varepsilon \in (0, 1]$, there is a set $S \subseteq \mathbb{R}^d$ satisfying $\mathrm{c}(S) \leqslant \varepsilon$ and $\mathrm{b}_d(S) \geqslant \gamma \frac{\varepsilon}{\log_2 1/\varepsilon}$, and in particular, we have $\mathrm{b}_d(S) \geqslant \gamma \frac{\mathrm{c}(S)}{\log_2 1/\,\mathrm{c}(S)}$.*

In this extended abstract, some proofs have been omitted due to space constraints. The omitted proofs can be found in the full version of this paper [1].

2 Bounding the mutual visibility in the plane

The goal of this section is to prove Theorem 3. Since the proof is rather long and complicated, let us first present a high-level overview of its main ideas.

We first show that it is sufficient to prove the estimate from Theorem 3 for bounded open simply connected sets. This is formalized by the next lemma, whose proof is omitted.

▶ **Lemma 7.** *Let $\alpha > 0$ be a constant such that every open bounded simply connected set $T \subseteq \mathbb{R}^2$ satisfies $\mathrm{b}(T) \leqslant \alpha\,\mathrm{c}(T)$. It follows that every p-componentwise simply connected set $S \subseteq \mathbb{R}^2$ with defined $\mathrm{b}(S)$ satisfies $\mathrm{b}(S) \leqslant \alpha\,\mathrm{c}(S)$.*

Suppose now that S is a bounded open simply connected set. We seek a bound of the form $\mathrm{b}(S) = O(\mathrm{c}(S))$. This is equivalent to a bound of the form $\lambda_4(\mathrm{Seg}(S)) = O(\mathrm{smc}(S)\lambda_2(S))$. We therefore need a suitable upper bound on $\lambda_4(\mathrm{Seg}(S))$.

We first choose in S a *diagonal* ℓ (i.e., an inclusion-maximal line segment in S), and show that the set $S \setminus \ell$ is a union of two open simply connected sets S_1 and S_2 (Lemma 10). It is not hard to show that the segments in S that cross the diagonal ℓ contribute to $\lambda_4(\mathrm{Seg}(S))$ by at most $O(\mathrm{smc}(S)\lambda_2(S))$ (Lemma 14). Our main task is to bound the measure of $\mathrm{Seg}(S_i \cup \ell)$ for $i = 1, 2$. The two sets $S_i \cup \ell$ are what we call *rooted sets*. Informally, a rooted set is a union of a simply connected open set S' and an open segment $r \subseteq \partial S'$, called the root.

To bound $\lambda_4(\mathrm{Seg}(R))$ for a rooted set R with root r, we partition R into *levels* $L_1, L_2, \ldots,$ where L_k contains the points of R that can be connected to r by a polygonal line with k segments, but not by a polygonal line with $k-1$ segments. Each segment in R is contained in a union $L_i \cup L_{i+1}$ for some $i \geqslant 1$. Thus, a bound of the form $\lambda_4(\mathrm{Seg}(L_i \cup L_{i+1})) = O(\mathrm{smc}(R)\lambda_2(L_i \cup L_{i+1}))$ implies the required bound for $\lambda_4(\mathrm{Seg}(R))$.

We will show that each p-component of $L_i \cup L_{i+1}$ is a rooted set, with the extra property that all its points are reachable from its root by a polygonal line with at most two segments (Lemma 11). To handle such sets, we will generalize the techniques that Cabello et al. [7] have used to handle weakly star-shaped sets in their proof of Theorem 1. We will assign to every point $A \in R$ a set $\mathfrak{T}(A)$ of measure $O(\mathrm{smc}(R))$, such that for every $(A, B) \in \mathrm{Seg}(R)$, we have either $B \in \mathfrak{T}(A)$ or $A \in \mathfrak{T}(B)$ (Lemma 13). From this, Theorem 3 will follow easily.

To proceed with the proof of Theorem 3 for bounded open simply connected sets, we need a few auxiliary lemmas.

▶ **Lemma 8.** *For every positive integer d, if S is an open subset of $\mathbb{R}^d$, then the set $\mathrm{Seg}(S)$ is open and the set $\mathrm{Vis}(A, S)$ is open for every point $A \in S$.*

Proof. Choose a pair of points $(A, B) \in \mathrm{Seg}(S)$. Since S is open and $\overline{AB}$ is compact, there is $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(\overline{AB}) \subseteq S$. Consequently, for any $A' \in \mathcal{N}_\varepsilon(A)$ and $B' \in \mathcal{N}_\varepsilon(B)$, we have $\overline{A'B'} \subseteq S$, that is, $(A', B') \in \mathrm{Seg}(S)$. This shows that the set $\mathrm{Seg}(S)$ is open. If we fix $A' = A$, then it follows that the set $\mathrm{Vis}(A, S)$ is open. ◀

▶ **Lemma 9.** *Let S be a simply connected subset of $\mathbb{R}^2$ and let ℓ and ℓ' be line segments in S. It follows that the set $\mathrm{Vis}(\ell', S) \cap \ell$ is a (possibly empty) subsegment of ℓ.*

Proof. The statement is trivially true if ℓ and ℓ' intersect or have the same supporting line, or if $\mathrm{Vis}(\ell', S) \cap \ell$ is empty. Suppose that these situations do not occur. Let $A, B \in \ell$ and $A', B' \in \ell'$ be such that $\overline{AA'}, \overline{BB'} \subseteq S$. The points A, A', B', B form a (possibly self-intersecting) tetragon Q whose boundary is contained in S. Since S is simply connected, the interior of Q is contained in S. If Q is not self-intersecting, then clearly $\overline{AB} \subseteq \mathrm{Vis}(\ell', S)$. Otherwise, $\overline{AA'}$ and $\overline{BB'}$ have a point D in common, and every point $C \in AB$ is visible in R from the point $C' \in A'B'$ such that $D \in \overline{CC'}$. This shows that $\mathrm{Vis}(\ell', S) \cap \ell$ is a convex subset and hence a subsegment of ℓ. ◀

Now, we define rooted sets and their tree-structured decomposition, and we explain how they arise in the proof of Theorem 3.

A set $S \subseteq \mathbb{R}^2$ is *half-open* if every point $A \in S$ has a neighborhood $\mathcal{N}_\varepsilon(A)$ that satisfies one of the following two conditions:

1. $\mathcal{N}_\varepsilon(A) \subseteq S$,
2. $\mathcal{N}_\varepsilon(A) \cap \partial S$ is a diameter of $\mathcal{N}_\varepsilon(A)$ splitting it into two subsets, one of which (including the diameter) is $\mathcal{N}_\varepsilon(A) \cap S$ and the other (excluding the diameter) is $\mathcal{N}_\varepsilon(A) \setminus S$.

The condition 1 holds for points $A \in S^\circ$, while the condition 2 holds for points $A \in \partial S$. A set $R \subseteq \mathbb{R}^2$ is a *rooted set* if the following conditions are satisfied:

1. R is bounded,
2. R is p-connected and simply connected,
3. R is half-open,
4. $R \cap \partial R$ is an open line segment.

The open line segment $R \cap \partial R$ is called the *root* of R. Every rooted set, as the union of a non-empty open set and an open line segment, is measurable and has positive measure.

A *diagonal* of a set $S \subseteq \mathbb{R}^2$ is a line segment contained in S that is not a proper subset of any other line segment contained in S. Clearly, if S is open, then every diagonal of S is an open line segment. It is easy to see that the root of a rooted set is a diagonal. The following lemma allows us to use a diagonal to split a bounded open simply connected subset of $\mathbb{R}^2$ into two rooted sets. It is intuitively clear, and its formal proof is omitted.

▶ **Lemma 10.** *Let S be a bounded open simply connected subset of $\mathbb{R}^2$, and let ℓ be a diagonal of S. It follows that the set $S \setminus \ell$ has two p-components S_1 and S_2. Moreover, $S_1 \cup \ell$ and $S_2 \cup \ell$ are rooted sets, and ℓ is their common root.*

Let R be a rooted set. For a positive integer k, the *kth level* L_k of R is the set of points of R that can be connected to the root of R by a polygonal line in R consisting of k segments but cannot be connected to the root of R by a polygonal line in R consisting of fewer than k segments. We consider a degenerate one-vertex polygonal line as consisting of one degenerate segment, so the root of R is part of L_1. Thus $L_1 = \mathrm{Vis}(r, R)$, where r denotes the root of R. A *k-body* of R is a p-component of L_k. A *body* of R is a k-body of R for some k. See Figure 2 for an example of a rooted set and its partitioning into levels and bodies.

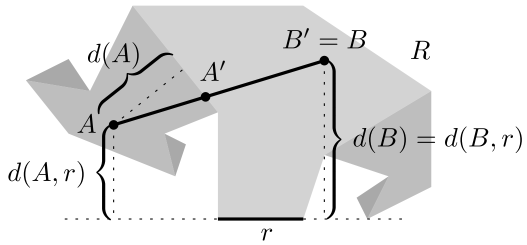

Figure 2 Example of a rooted set R partitioned into six bodies. The three levels of R are distinguished with three shades of gray. The segment $A'B'$ is the base segment of $\overline{AB}$.

We say that a rooted set P is *attached* to a set $Q \subseteq \mathbb{R}^2 \setminus P$ if the root of P is subset of the interior of $P \cup Q$. The following lemma explains the structure of levels and bodies. Although it is intuitively clear, its formal proof requires quite a lot of work and is omitted.

▶ **Lemma 11.** *Let R be a rooted set and $(L_k)_{k \geqslant 1}$ be its partition into levels. It follows that*
1. *$R = \bigcup_{k \geqslant 1} L_k$; consequently, R is the union of all its bodies;*
2. *every body P of R is a rooted set such that $P = \mathrm{Vis}(r, P)$, where r denotes the root of P;*
3. *L_1 is the unique 1-body of R, and the root of L_1 is the root of R;*
4. *every j-body P of R with $j \geqslant 2$ is attached to a unique $(j-1)$-body of R.*

Lemma 11 yields a tree structure on the bodies of R. The root of this tree is the unique 1-body L_1 of R, called the *root body* of R. For a k-body P of R with $k \geqslant 2$, the parent of P in the tree is the unique $(k-1)$-body of R that P is attached to, called the *parent body* of P.

▶ **Lemma 12.** *Let R be a rooted set, $(L_k)_{k \geqslant 1}$ be the partition of R into levels, ℓ be a closed line segment in R, and $k \geqslant 1$ be minimum such that $\ell \cap L_k \neq \emptyset$. It follows that $\ell \subseteq L_k \cup L_{k+1}$, $\ell \cap L_k$ is a subsegment of ℓ contained in a single k-body P of R, and $\ell \cap L_{k+1}$ consists of at most two subsegments of ℓ each contained in a single $(k+1)$-body whose parent body is P.*

Proof. The definition of the levels directly yields $\ell \subseteq L_k \cup L_{k+1}$. The segment ℓ splits into subsegments each contained in a single k-body or $(k+1)$-body of R. By Lemma 11, the bodies of any two consecutive of these subsegments are in the parent-child relation of the body tree. This implies that $\ell \cap L_k$ lies within a single k-body P. By Lemma 9, $\ell \cap L_k$ is a subsegment of ℓ. Consequently, $\ell \cap L_{k+1}$ consists of at most two subsegments. ◀

In the setting of Lemma 12, we call the subsegment $\ell \cap L_k$ of ℓ the *base segment* of ℓ, and we call the body P that contains $\ell \cap L_k$ the *base body* of ℓ. See Figure 2 for an example.

The following lemma is the crucial part of the proof of Theorem 3.

▶ **Lemma 13.** *If R is a rooted set, then every point $A \in R$ can be assigned a measurable set $\mathfrak{T}(A) \subseteq \mathbb{R}^2$ so that the following is satisfied:*
1. *$\lambda_2(\mathfrak{T}(A)) < 87\,\mathrm{smc}(R)$;*
2. *for every line segment $\overline{BC}$ in R, we have either $B \in \mathfrak{T}(C)$ or $C \in \mathfrak{T}(B)$;*
3. *the set $\{(A, B) \colon A \in R \text{ and } B \in \mathfrak{T}(A)\}$ is measurable.*

Proof. Let P be a body of R with the root r. First, we show that P is entirely contained in one closed half-plane defined by the supporting line of r. Let h^- and h^+ be the two open half-planes defined by the supporting line of r. According to the definition of a rooted set, the sets $\{D \in r \colon \exists \varepsilon > 0 \colon \mathcal{N}_\varepsilon(D) \cap h^- = \mathcal{N}_\varepsilon(D) \cap (P \setminus r)\}$ and $\{D \in r \colon \exists \varepsilon > 0 \colon \mathcal{N}_\varepsilon(D) \cap h^+ = \mathcal{N}_\varepsilon(D) \cap (P \setminus r)\}$ are open and partition the entire r, hence one of them must be empty. This

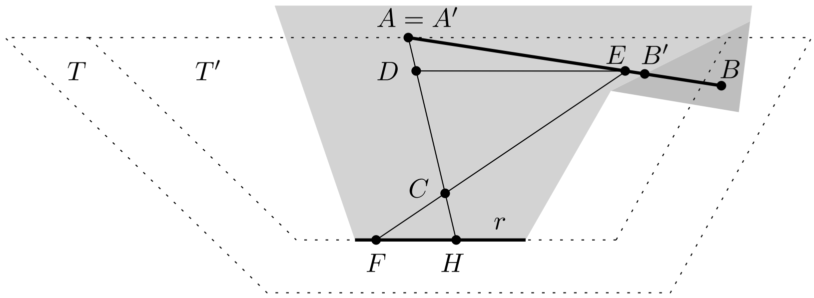

Figure 3 Illustration for the proof of Claim 1 in the proof of Lemma 13.

implies that the segments connecting r to $P \setminus r$ lie all in h^- or all in h^+. Since $P = \mathrm{Vis}(r, P)$, we conclude that $P \subseteq h^-$ or $P \subseteq h^+$.

According to the above, we can rotate and translate the set R so that r lies on the x-axis and P lies in the half-plane $\{B \in \mathbb{R}^2 \colon y(B) \geqslant 0\}$. For a point $A \in R$, we use $d(A, r)$ to denote the y-coordinate of A after such a rotation and translation of R. We use $d(A)$ to denote $d(A, r)$ where r is the root of the body of A. It follows that $d(A) \geqslant 0$ for every $A \in R$.

Let $\gamma \in (0, 1)$ be a fixed constant whose value will be specified at the end of the proof. For a point $A \in R$, we define the sets

$$\mathfrak{V}_1(A) := \{B \in \mathrm{Vis}(A, S) \colon |A'B'| \geqslant \gamma|AB|,\ A \in \mathrm{Vis}(r'', R),\ d(A', r'') \geqslant d(B', r'')\},$$
$$\mathfrak{V}_2(A) := \{B \in \mathrm{Vis}(A, S) \colon |A'B'| \geqslant \gamma|AB|,\ A \notin \mathrm{Vis}(r'', R),\ d(A', r'') \geqslant d(B', r'')\},$$
$$\mathfrak{V}_3(A) := \{B \in \mathrm{Vis}(A, S) \colon |A'B'| < \gamma|AB|,\ |AA'| \geqslant |BB'|\},$$

where r'' denotes the root of the base body of $\overline{AB}$ and A' and B' denote the endpoints of the base segment of $\overline{AB}$ such that $|AA'| < |AB'|$. These sets are pairwise disjoint, and we have $A \in \bigcup_{i=1}^{3} \mathfrak{V}_i(B)$ or $B \in \bigcup_{i=1}^{3} \mathfrak{V}_i(A)$ for every line segment $\overline{AB}$ in R. If for some $B \in \bigcup_{i=1}^{3} \mathfrak{V}_i(A)$ the point A lies on r'', then we have $B \in \mathfrak{V}_1(A)$ and $\mathfrak{V}_1(A) \subseteq r''$.

For the rest of the proof, we fix a point $A \in R$. We show that the union $\bigcup_{i=1}^{3} \mathfrak{V}_i(A)$ is contained in a measurable set $\mathfrak{T}(A) \subseteq \mathbb{R}^2$ with $\lambda_2(\mathfrak{T}(A)) < 87\,\mathrm{smc}(R)$ that is the union of three trapezoids. We let P be the body of A and r be the root of P. If P is a k-body with $k \geqslant 2$, then we use r' to denote the root of the parent body of P.

▶ Claim 1. *$\mathfrak{V}_1(A)$ is contained in a trapezoid $\mathfrak{T}_1(A)$ with area $6\gamma^{-2}\,\mathrm{smc}(R)$.*

Let H be a point of r such that $\overline{AH} \subseteq R$. Let T' be the r-parallel trapezoid of height $d(A)$ with bases of length $\frac{8\,\mathrm{smc}(R)}{d(A)}$ and $\frac{4\,\mathrm{smc}(R)}{d(A)}$ such that A is the center of the larger base and H is the center of the smaller base. The homothety with center A and ratio γ^{-1} transforms T' into the trapezoid $T := A + \gamma^{-1}(T' - A)$. Since the area of T' is $6\,\mathrm{smc}(R)$, the area of T is $6\gamma^{-2}\,\mathrm{smc}(R)$. We show that $\mathfrak{V}_1(A) \subseteq T$. See Figure 3 for an illustration.

Let B be a point in $\mathfrak{V}_1(A)$. Using similar techniques to the ones used by Cabello et al. [7] in the proof of Theorem 1, we show that $B \in T$. Let $A'B'$ be the base segment of $\overline{AB}$ such that $|AA'| < |AB'|$. Since $B \in \mathfrak{V}_1(A)$, we have $|A'B'| \geqslant \gamma|AB|$, $A \in \mathrm{Vis}(r'', R)$, and $d(B, r'') \leqslant d(A, r'')$, where r'' denotes the root of the base level of $\overline{AB}$. Since A is visible from r'' in R, the base body of $\overline{AB}$ is the body of A and thus $A = A'$ and $r = r''$. As we have observed, every point $C \in \{A\} \cup AB'$ satisfies $d(C, r) = d(C) \geqslant 0$.

Let $\varepsilon > 0$. There is a point $E \in AB'$ such that $|B'E| < \varepsilon$. Since E lies on the base segment of $\overline{AB}$, there is $F \in r$ such that $\overline{EF} \subseteq R$. It is possible to choose F so that $\overline{AH}$ and $\overline{EF}$ have a point C in common where $C \neq F, H$. Let D be a point of $\overline{AH}$ with $d(D) = d(E)$. The point D exists, as $d(H) = 0 \leqslant d(E) \leqslant d(A)$. The points A, E, F, H

form a self-intersecting tetragon Q whose boundary is contained in R. Since R is simply connected, the interior of Q is contained in R and the triangles ACE and CFH have area at most $\operatorname{smc}(R)$.

The triangle ACE is partitioned into triangles ADE and CDE with areas $\frac{1}{2}(d(A) - d(D))|DE|$ and $\frac{1}{2}(d(D) - d(C))|DE|$, respectively. Therefore, we have $\frac{1}{2}(d(A) - d(C))|DE| = \lambda_2(ACE) \leqslant \operatorname{smc}(R)$. This implies

$$|DE| \leqslant \frac{2\operatorname{smc}(R)}{d(A) - d(C)}.$$

For the triangle CFH, we have $\frac{1}{2}d(C)|FH| = \lambda_2(CFH) \leqslant \operatorname{smc}(R)$. By the similarity of the triangles CFH and CDE, we have $|FH| = |DE|d(C)/(d(E) - d(C))$ and therefore

$$|DE| \leqslant \frac{2\operatorname{smc}(R)}{d(C)^2}(d(E) - d(C)).$$

Since the first upper bound on $|DE|$ is increasing in $d(C)$ and the second is decreasing in $d(C)$, the minimum of the two is maximized when they are equal, that is, when $d(C) = d(A)d(E)/(d(A) + d(E))$. Then we obtain $|DE| \leqslant \frac{2\operatorname{smc}(R)}{d(A)^2}(d(A) + d(E))$. This and $0 \leqslant d(E) \leqslant d(A)$ imply $E \in T'$. Since ε can be made arbitrarily small and T' is compact, we have $B' \in T'$. Since $|AB'| \geqslant \gamma|AB|$, we conclude that $B \in T$. This completes the proof of Claim 1.

▶ Claim 2. *$\mathfrak{V}_2(A)$ is contained in a trapezoid $\mathfrak{T}_2(A)$ with area* $3(1-\gamma)^{-2}\gamma^{-2}\operatorname{smc}(R)$.

We assume the point A is not contained in the first level of R, as otherwise $\mathfrak{V}_2(A)$ is empty. Let p be the r'-parallel line that contains the point A and let q be the supporting line of r. Let p^+ and q^+ denote the closed half-planes defined by p and q, respectively, such that $r' \subseteq p^+$ and $A \notin q^+$. Let O be the intersection point of p and q.

Let $T' \subseteq p^+ \cap q^+$ be the trapezoid of height $d(A, r')$ with one base of length $\frac{4\operatorname{smc}(R)}{(1-\gamma)^2 d(A,r')}$ on p, the other base of length $\frac{2\operatorname{smc}(R)}{(1-\gamma)^2 d(A,r')}$ on the supporting line of r', and one lateral side on q. The homothety with center O and ratio γ^{-1} transforms T' into the trapezoid $T := O + \gamma^{-1}(T' - O)$. Since the area of T' is $3(1-\gamma)^{-2}\operatorname{smc}(R)$, the area of T is $3(1-\gamma)^{-2}\gamma^{-2}\operatorname{smc}(R)$. We show that $\mathfrak{V}_2(A) \subseteq T$. See Figure 3 for an illustration.

Let B be a point of $\mathfrak{V}_2(A)$. We use $A'B'$ to denote the base segment of $\overline{AB}$ such that $|AA'| < |AB'|$. By the definition of $\mathfrak{V}_2(A)$, we have $|A'B'| \geqslant \gamma|AB|$, $A \notin \operatorname{Vis}(r'', R)$, and $d(B, r'') \leqslant d(A, r'')$, where r'' denotes the root of the base body of $\overline{AB}$. By Lemma 12 and the fact that $A \notin \operatorname{Vis}(r'', R)$, we have $r' = r''$. The bound $d(A, r') \geqslant d(B, r')$ thus implies $A' \in r \cap p^+$ and $B \in q^+$. We have $d(C, r') = d(C) \geqslant 0$ for every $C \in A'B'$.

Observe that $(1-\gamma)d(A, r') \leqslant d(A', r') \leqslant d(A, r')$. The upper bound is trivial, as $d(B, r') \leqslant d(A, r')$ and A' lies on $\overline{AB}$. For the lower bound, we use the expression $A' = tA + (1-t)B'$ for some $t \in [0, 1]$. This gives us $d(A', r') = td(A, r') + (1-t)d(B', r')$. By the estimate $|A'B'| \geqslant \gamma|AB|$, we have

$$|AA'| + |BB'| \leqslant (1-\gamma)|AB| = (1-\gamma)(|AB'| + |BB'|).$$

This can be rewritten as $|AA'| \leqslant (1-\gamma)|AB'| - \gamma|BB'|$. Consequently, $|BB'| \geqslant 0$ and $\gamma > 0$ imply $|AA'| \leqslant (1-\gamma)|AB'|$. This implies $t \geqslant 1-\gamma$. Applying the bound $d(B', r') \geqslant 0$, we conclude that $d(A', r') \geqslant (1-\gamma)d(A, r')$.

Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of points from $A'B'$ that converges to A'. For every $n \in \mathbb{N}$, there is a point $H_n \in r'$ such that $\overline{G_n H_n} \subseteq R$. Since $\overline{r'}$ is compact, there is a subsequence of $(H_n)_{n\in\mathbb{N}}$ that converges to a point $H_0 \in \overline{r'}$. We claim that $H_0 \in q$. Suppose otherwise, and

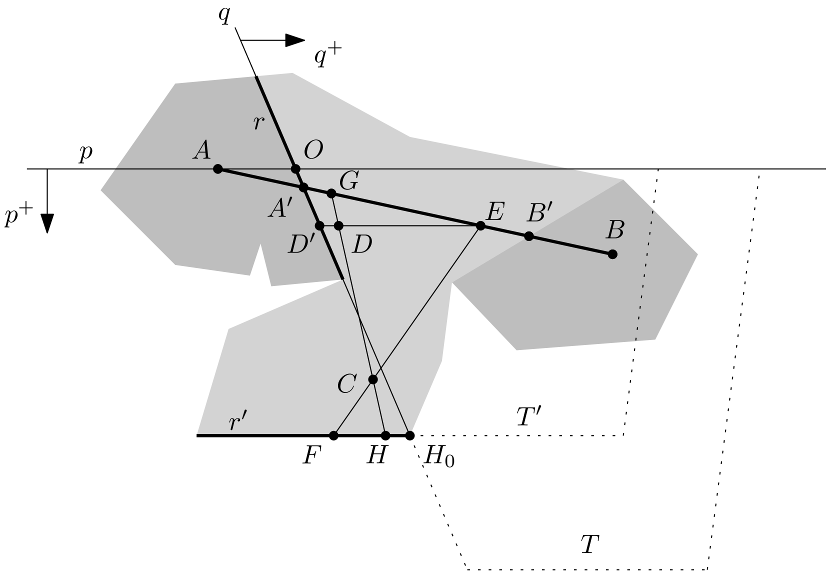

Figure 4 Illustration for the proof of Claim 2 in the proof of Lemma 13.

let $q' \neq q$ be the supporting line of $\overline{A'H_0}$. Let $\varepsilon > 0$ be small enough so that $\mathcal{N}_\varepsilon(A') \subseteq R$. For n large enough, $\overline{G_nH_n}$ is contained in an arbitrarily small neighborhood of q'. Consequently, for n large enough, the supporting line of $\overline{G_nH_n}$ intersects q at a point K_n such that $\overline{G_nK_n} \subseteq \mathcal{N}_\varepsilon(A')$, which implies $K_n \in r \cap \mathrm{Vis}(r', R)$, a contradiction.

Again, let $\varepsilon > 0$. There is a point $E \in A'B'$ such that $|B'E| < \varepsilon$. Let D' be a point of q with $d(D', r') = d(E)$. Let $\delta > 0$. There are points $G \in A'B'$ and $H \in r'$ such that $G \in \mathcal{N}_\delta(A')$ and $\overline{GH} \subseteq R \cap \mathcal{N}_\delta(q)$. If δ is small enough, then $d(E) \leqslant d(A', r') - \delta \leqslant d(G) \leqslant d(A', r')$. Let D be the point of $\overline{GH}$ with $d(D) = d(E)$. The point E lies on $A'B'$ and thus it is visible from a point $F \in r'$. Again, we can choose F so that the line segments $\overline{EF}$ and $\overline{GH}$ have a point C in common where $C \neq F, H$. The points E, F, H, G form a self-intersecting tetragon Q whose boundary is in R. The interior of Q is contained in R, as R is simply connected. Therefore, the area of the triangles CEG and CFH is at most $\mathrm{smc}(R)$. The argument used in the proof of Claim 1 yields $|DE| \leqslant \frac{2\,\mathrm{smc}(R)}{d(G)^2}(d(G)+d(E)) \leqslant \frac{2\,\mathrm{smc}(R)}{(d(A',r')-\delta)^2}(d(A', r')+d(E))$. This and the fact that δ (and consequently $|D'D|$) can be made arbitrarily small yield $|D'E| \leqslant \frac{2\,\mathrm{smc}(R)}{d(A',r')^2}(d(A', r') + d(E))$. This together with $d(A', r') \geqslant (1-\gamma)d(A, r')$ yield $|D'E| \leqslant \frac{2\,\mathrm{smc}(R)}{(1-\gamma)^2 d(A,r')^2}(d(A, r') + d(E))$. This and $0 \leqslant d(E) \leqslant d(A, r')$ imply $E \in T'$. Since ε can be made arbitrarily small and T' is compact, we have $B' \in T'$. Since $|A'B'| \geqslant \gamma|AB| \geqslant \gamma|A'B|$, we conclude that $B \in T$. This completes the proof of Claim 2.

▶ Claim 3. *$\mathfrak{V}_3(A)$ is contained in a trapezoid $\mathfrak{T}_3(A)$ with area* $(4(1-\gamma)^{-2} - 1)\,\mathrm{smc}(R)$.

By Lemma 9, the points of r that are visible from A in R form a subsegment CD of r. The homothety with center A and ratio $2(1-\gamma)^{-1}$ transforms the triagle $T' := ACD$ into the triangle $T'' := A + 2(1-\gamma)^{-1}(T' - A)$. See Figure 5 for an illustration. We claim that $\mathfrak{V}_3(A)$ is a subset of the trapezoid $T := T'' \setminus T'$.

Let B be an arbitrary point of $\mathfrak{V}_3(A)$. Consider the segment $\overline{AB}$ with the base segment $A'B'$ such that $|AA'| < |AB'|$. Since $B \in \mathfrak{V}_3(A)$, we have $|A'B'| < \gamma|AB|$ and $|AA'| \geqslant |BB'|$. This implies $|AA'| \geqslant \frac{1-\gamma}{2}|AB| > 0$ and hence $A \neq A'$ and $B \notin P$. From the definition of C and D, we have $A' \in \overline{CD}$. Since $|AA'| \geqslant \frac{1-\gamma}{2}|AB|$ and $B \notin P$, we have $B \in T$.

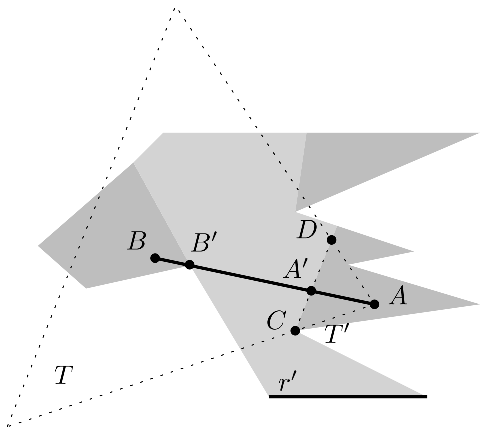

Figure 5 Illustration for the proof of Claim 3 in the proof of Lemma 13.

The area of T is $(4(1-\gamma)^{-2}-1)\lambda_2(T')$. The interior of T' is contained in R, as all points of the open segment CD are visible from A in R. The area of T' is at most $\operatorname{smc}(R)$, as its interior is a convex subset of R. Consequently, the area of T is at most $(4(1-\gamma)^{-2}-1)\operatorname{smc}(R)$. This completes the proof of Claim 3.

To put everything together, we set $\mathfrak{T}(A) := \bigcup_{i=1}^{3} \mathfrak{T}_i(A)$. It follows that $\bigcup_{i=1}^{3} \mathfrak{V}_i(A) \subseteq \mathfrak{T}(A)$ for every $A \in R$. Clearly, the set $\mathfrak{T}(A)$ is measurable. Summing the three estimates on areas of the trapezoids, we obtain

$$\lambda_2(\mathfrak{T}(A)) \leqslant \left(6\gamma^{-2} + 3(1-\gamma)^{-2}\gamma^{-2} + 4(1-\gamma)^{-2} - 1\right)\operatorname{smc}(R)$$

for every point $A \in R$. We choose $\gamma \in (0,1)$ so that the value of the coefficient is minimized. For $x \in (0,1)$, the function $x \mapsto 6x^{-2} + 3(1-x)^{-2}x^{-2} + 4(1-x)^{-2} - 1$ attains its minimum $86.7027 < 87$ at $x \approx 0.5186$. Altogether, we have $\lambda_2(\mathfrak{T}(A)) < 87\operatorname{smc}(R)$ for every $A \in R$.

It remains to show that the set $\{(A,B) \colon A \in R \text{ and } B \in \mathfrak{T}(A)\}$ is measurable. For every body P of R and for $i \in \{1,2,3\}$, the definition of the trapezoid $\mathfrak{T}_i(A)$ in Claim i implies that the set $\{(A,B) \colon A \in P \text{ and } B \in \mathfrak{T}_i(A)\}$ is the intersection of $P \times \mathbb{R}^2$ with a semialgebraic (hence measurable) subset of $(\mathbb{R}^2)^2$ and hence is measurable. There are countably many bodies of R, as each of them has positive measure. Therefore, $\{(A,B) \colon A \in R \text{ and } B \in \mathfrak{T}(A)\}$ is a countable union of measurable sets and hence is measurable. ◀

Let S be a bounded open subset of the plane, and let ℓ be a diagonal of S that lies on the x-axis. For a point $A \in S$, we define the set

$$\mathfrak{S}(A,\ell) := \{B \in \operatorname{Vis}(A,S) \colon AB \cap \ell \neq \emptyset \text{ and } |y(A)| \geqslant |y(B)|\}.$$

The following lemma is a slightly more general version of a result of Cabello et al. [7].

▶ **Lemma 14.** *Let S be a bounded open simply connected subset of $\mathbb{R}^2$, and let ℓ be its diagonal that lies on the x-axis. It follows that $\lambda_2(\mathfrak{S}(A,\ell)) \leqslant 3\operatorname{smc}(S)$ for every $A \in S$.*

Proof. Using an argument similar to the proof of Lemma 8, we can show that the set $\{B \in \operatorname{Vis}(A,S) \colon AB \cap \ell \neq \emptyset\}$ is open. Therefore, $\mathfrak{S}(A,\ell)$ is the intersection of an open set and the closed half-plane $\{(x,y) \in \mathbb{R}^2 \colon y \leqslant -y(A)\}$ or $\{(x,y) \in \mathbb{R}^2 \colon y \geqslant -y(A)\}$, whichever contains A. Consequently, the set $\mathfrak{S}(A,\ell)$ is measurable for every point $A \in S$.

We clearly have $\lambda_2(\mathfrak{S}(A,\ell)) = 0$ for points $A \in S \setminus \operatorname{Vis}(\ell,S)$. By Lemma 9, the set $\operatorname{Vis}(A,S) \cap \ell$ is an open subsegment CD of ℓ. The interior T° of the triangle $T := ACD$ is contained in S. Since T° is a convex subset of S, we have $\lambda_2(T^\circ) = \frac{1}{2}|CD| \cdot |y(A)| \leqslant \mathrm{smc}(S)$. Therefore, every point $B \in \mathfrak{S}(A, \ell)$ is contained in a trapezoid of height $|y(A)|$ with bases of length $|CD|$ and $2|CD|$. The area of this trapezoid is $\frac{3}{2}|CD| \cdot |y(A)| \leqslant 3\,\mathrm{smc}(S)$. Hence we have $\lambda_2(\mathfrak{S}(A, \ell)) \leqslant 3\,\mathrm{smc}(S)$ for every point $A \in S$. ◀

Proof of Theorem 3. In view of Lemma 7, we can assume without loss of generality that S is an open bounded simply connected set. Let ℓ be a diagonal of S. We can assume without loss of generality that ℓ lies on the x-axis. According to Lemma 10, the set $S \setminus \ell$ has exactly two p-components S_1 and S_2, the sets $S_1 \cup \ell$ and $S_2 \cup \ell$ are rooted sets, and ℓ is their common root. By Lemma 13, for $i \in \{1, 2\}$, every point $A \in S_i \cup \ell$ can be assigned a measurable set $\mathfrak{T}_i(A)$ so that $\lambda_2(\mathfrak{T}_i(A)) < 87\,\mathrm{smc}(S_i \cup \ell) \leqslant 87\,\mathrm{smc}(S)$, every line segment $\overline{BC}$ in $S_i \cup \ell$ satisfies $B \in \mathfrak{T}_i(C)$ or $C \in \mathfrak{T}_i(B)$, and the set $\{(A, B) \colon A \in S_i \cup \ell \text{ and } B \in \mathfrak{T}_i(A)\}$ is measurable. We set $\mathfrak{S}(A) := \mathfrak{T}_i(A) \cup \mathfrak{S}(A, \ell)$ for every point $A \in S_i$ with $i \in \{1, 2\}$. We set $\mathfrak{S}(A) := \mathfrak{T}_1(A) \cup \mathfrak{T}_2(A)$ for every point $A \in \ell = S \setminus (S_1 \cup S_2)$. Let

$$\mathfrak{S} := \{(A, B) \colon A \in S \text{ and } B \in \mathfrak{S}(A)\} \cup \{(B, A) \colon A \in S \text{ and } B \in \mathfrak{S}(A)\} \subseteq (\mathbb{R}^2)^2.$$

It follows that the set $\mathfrak{S}$ is measurable.

Let $\overline{AB}$ be a line segment in S, and suppose $|y(A)| \geqslant |y(B)|$. Then either A and B are in distinct p-components of $S \setminus \ell$ or they both lie in the same component S_i with $i \in \{1, 2\}$. In the first case, we have $B \in \mathfrak{S}(A)$, since AB intersects ℓ and $\mathfrak{S}(A, \ell) \subseteq \mathfrak{S}(A)$. In the second case, we have $B \in \mathfrak{T}_i(A) \subseteq \mathfrak{S}(A)$ or $A \in \mathfrak{T}_i(B) \subseteq \mathfrak{S}(B)$. Therefore, we have $\mathrm{Seg}(S) \subseteq \mathfrak{S}$. Since both $\mathrm{Seg}(S)$ and $\mathfrak{S}$ are measurable, we have

$$\lambda_4(\mathrm{Seg}(S)) \leqslant \lambda_4(\mathfrak{S}) \leqslant 2 \int_{A \in S} \lambda_2(\mathfrak{S}(A)),$$

where the second inequality is implied by Fubini's Theorem. Using the bound $\lambda_2(\mathfrak{S}(A)) \leqslant 90\,\mathrm{smc}(S)$, we obtain

$$\lambda_4(\mathrm{Seg}(S)) \leqslant 2 \int_S 90\,\mathrm{smc}(S) = 180\,\mathrm{smc}(S)\lambda_2(S).$$

Finally, this bound can be rewritten as $\mathrm{b}(S) = \lambda_4(\mathrm{Seg}(S))\lambda_2(S)^{-2} \leqslant 180\,\mathrm{c}(S)$. ◀

3 General dimension

In this section, we sketch the proofs of Theorem 5 and Theorem 6. The detailed proofs can be found in the full version of this paper [1]. In both proofs, we use the operator Aff to denote the affine hull of a set of points.

Sketch of the proof of Theorem 5. Let $T = (B_0, B_1, \ldots, B_d)$ be a $(d+1)$-tuple of distinct affinely independent points of S, ordered in such a way that the following two conditions hold:

1. the segment $\overline{B_0 B_1}$ is the diameter of T, and
2. for $i = 2, \ldots, d-1$, the point B_i has the maximum distance to $\mathrm{Aff}(\{B_0, \ldots, B_{i-1}\})$ among the points $B_i, B_{i+1}, \ldots, B_d$.

For $i = 1, \ldots, d-1$, we define $\mathrm{Box}_i(T)$ inductively as follows:

1. $\mathrm{Box}_1(T) := \overline{B_0 B_1}$,
2. for $i = 2, \ldots, d-1$, $\mathrm{Box}_i(T)$ is the box containing all the points $P \in \mathrm{Aff}(\{B_0, B_1, \ldots, B_i\})$ with the following two properties:

a. the orthogonal projection of P to $\operatorname{Aff}(\{B_0, B_1, \ldots, B_{i-1}\})$ lies in $\operatorname{Box}_{i-1}(T)$, and
b. the distance of P to $\operatorname{Aff}(\{B_0, B_1, \ldots, B_{i-1}\})$ does not exceed the distance of B_i to $\operatorname{Aff}(\{B_0, B_1, \ldots, B_{i-1}\})$,

3. $\operatorname{Box}_d(T)$ is the box containing all the points $P \in \mathbb{R}^d$ such that the orthogonal projection of P to $\operatorname{Aff}(\{B_0, B_1, \ldots, B_{d-1}\})$ lies in $\operatorname{Box}_{d-1}(T)$ and $\lambda_d(\operatorname{Conv}(\{B_0, B_1, \ldots, B_{d-1}, P\})) \leqslant \lambda_d(S)\operatorname{c}(S)$.

It can be verified that if $T \in \operatorname{Simp}_d(S)$, then $\operatorname{Box}_d(T)$ contains the point B_d. Also, it can be shown that the λ_d-measure of $\operatorname{Box}_d(T)$ is equal to $z := 2^{d-2}d!\operatorname{smc}(S)$, which is independent of T. From this, we can deduce that the measure of $\operatorname{Simp}_d(S)$ is at most $(d+1)\lambda_d(S)^d z$, and hence $\operatorname{b}_d(S)$ is at most $(d+1)z/\lambda_d(S)$, which is of order $\operatorname{c}(S)$. ◀

Sketch of the proof of Theorem 6. To obtain a set S with arbitrarily small convexity ratio $\operatorname{c}(S)$ and with the d-index of convexity $\operatorname{b}_d(S)$ of order $\operatorname{c}(S)/\log_2(1/\operatorname{c}(S))$, we let S be the open d-dimensional box $(0,1)^d$ with n points removed. We show that no matter which n-tuple of points we remove from the box, the d-index of convexity $\operatorname{b}_d(S)$ is still of order $\Omega(\frac{1}{n})$. Moreover, we show that for some constant $\alpha = \alpha(d) > 0$ it is possible to remove $n = \alpha\frac{1}{\varepsilon}\log_2\frac{1}{\varepsilon}$ points from the box such that every convex subset of $(0,1)^d$ with measure at least ε contains a removed point. That is, we obtain $\operatorname{c}(S) \leqslant \varepsilon$ and $\operatorname{b}_d(S) \geqslant \gamma\varepsilon/\log_2(1/\varepsilon)$ for some constant $\gamma = \gamma(d) > 0$. Such an n-tuple of points to be removed is called an *ε-net* for convex subsets of $(0,1)^d$. To find it, we first use John's Lemma [11] to reduce the problem to finding, for a suitably smaller ε', an *ε'-net* for ellipsoids restricted to $(0,1)^d$. Then, we apply a continuous version of the well-known Epsilon Net Theorem for families with bounded Vapnik-Chervonenkis dimension due to Haussler and Welzl [10] (see also [14]). ◀

It is a natural question whether the bound for $\operatorname{b}_d(S)$ in Theorem 6 can be improved to $\operatorname{b}_d(S) = \Omega(\operatorname{c}(S))$. In the plane, this is related to the famous problem of Danzer and Rogers (see [6, 15] and Problem E14 in [8]) which asks whether for given $\varepsilon > 0$ there is a set $N' \subseteq (0,1)^2$ of size $O(\frac{1}{\varepsilon})$ with the property that every convex set of area ε within the unit square contains at least one point from N'.

If this problem was to be answered affirmatively, then we could use such a set N' to stab $(0,1)^2$ in our proof of Theorem 6 which would yield the desired bound for $\operatorname{b}_2(S)$. However it is generally believed that the answer is likely to be nonlinear in $\frac{1}{\varepsilon}$.

4 Other variants and open problems

We have seen in Theorem 3 that a p-componentwise simply connected set $S \subseteq \mathbb{R}^2$ whose $\operatorname{b}(S)$ is defined satisfies $\operatorname{b}(S) \leqslant \alpha\operatorname{c}(S)$, for an absolute constant $\alpha \leqslant 180$. Equivalently, such a set S satisfies $\operatorname{smc}(S) \geqslant \operatorname{b}(S)\lambda_2(S)/180$.

By a result of Blaschke [5] (see also Sas [18]), every convex set $K \subseteq \mathbb{R}^2$ contains a triangle of measure at least $\frac{3\sqrt{3}}{4\pi}\lambda_2(K)$. In view of this, Theorem 3 yields the following consequence.

▶ **Corollary 15.** *There is a constant $\alpha > 0$ such that every p-componentwise simply connected set $S \subseteq \mathbb{R}^2$ whose $\operatorname{b}(S)$ is defined contains a triangle $T \subseteq S$ of measure at least $\alpha\operatorname{b}(S)\lambda_2(S)$.*

A similar argument works in higher dimensions as well. For every $d \geqslant 2$, there is a constant $\beta = \beta(d)$ such that every convex set $K \subseteq \mathbb{R}^d$ contains a simplex of measure at least $\beta\lambda_d(K)$ (see e.g. Lassak [13]). Therefore, Theorem 5 can be rephrased in the following equivalent form.

▶ **Corollary 16.** *For every $d \geqslant 2$, there is a constant $\alpha = \alpha(d) > 0$ such that every set $S \subseteq \mathbb{R}^d$ whose $\mathrm{b}_d(S)$ is defined contains a simplex T of measure at least $\alpha\,\mathrm{b}_d(S)\lambda_d(S)$.*

What can we say about sets $S \subseteq \mathbb{R}^2$ that are not p-componentwise simply connected? First of all, we can consider a weaker form of simple connectivity: we call a set S *p-componentwise simply $\triangle$-connected* if for every triangle T such that $\partial T \subseteq S$ we have $T \subseteq S$. We conjecture that Theorem 3 can be extended to p-componentwise simply $\triangle$-connected sets.

▶ **Conjecture 17.** *There is an absolute constant $\alpha > 0$ such that every p-componentwise simply $\triangle$-connected set $S \subseteq \mathbb{R}^2$ whose $\mathrm{b}(S)$ is defined satisfies $\mathrm{b}(S) \leqslant \alpha\,\mathrm{c}(S)$.*

What does the value of $\mathrm{b}(S)$ say about a planar set S that does not satisfy even a weak form of simple connectivity? Such a set may not contain any convex subset of positive measure, even when $\mathrm{b}(S)$ is equal to 1. However, we conjecture that a large $\mathrm{b}(S)$ implies the existence of a large convex set whose boundary belongs to S.

▶ **Conjecture 18.** *For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $S \subseteq \mathbb{R}^2$ is a set with $\mathrm{b}(S) \geqslant \varepsilon$, then there is a bounded convex set $C \subseteq \mathbb{R}^2$ with $\lambda(C) \geqslant \delta\lambda(S)$ and $\partial C \subseteq S$.*

Theorem 3 shows that Conjecture 18 holds for p-componentwise simply connected sets, with δ being a constant multiple of ε. It is possible that even in the general setting of Conjecture 18, δ can be taken as a constant multiple of ε.

Motivated by Corollary 15, we propose a stronger version of Conjecture 18, where the convex set C is required to be a triangle.

▶ **Conjecture 19.** *For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $S \subseteq \mathbb{R}^2$ is a set with $\mathrm{b}(S) \geqslant \varepsilon$, then there is a triangle $T \subseteq \mathbb{R}^2$ with $\lambda(T) \geqslant \delta\lambda(S)$ and $\partial T \subseteq S$.*

Note that Conjecture 19 holds when restricted to p-componentwise simply connected sets, as implied by Corollary 15.

We can generalise Conjecture 19 to higher dimensions and to higher-order indices of convexity. To state the general conjecture, we introduce the following notation: for a set $X \subseteq \mathbb{R}^d$, let $\binom{X}{k}$ be the set of k-element subsets of X, and let the set $\mathrm{Skel}_k(X)$ be defined by

$$\mathrm{Skel}_k(X) := \bigcup_{Y \in \binom{X}{k+1}} \mathrm{Conv}(Y).$$

If X is the vertex set of a d-dimensional simplex $T = \mathrm{Conv}(X)$, then $\mathrm{Skel}_k(X)$ is often called the *k-dimensional skeleton* of T. Our general conjecture states, roughly speaking, that sets with large k-index of convexity should contain the k-dimensional skeleton of a large simplex. Here is the precise statement.

▶ **Conjecture 20.** *For every $k, d \in \mathbb{N}$ such that $1 \leqslant k \leqslant d$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that if $S \subseteq \mathbb{R}^d$ is a set with $\mathrm{b}_k(S) \geqslant \varepsilon$, then there is a simplex T with vertex set X such that $\lambda_d(T) \geqslant \delta\lambda_d(S)$ and $\mathrm{Skel}_k(X) \subseteq S$.*

Corollary 16 asserts that this conjecture holds in the special case of $k = d \geqslant 2$, since $\mathrm{Skel}_d(X) = \mathrm{Conv}(X) = T$. Corollary 15 shows that the conjecture holds for $k = 1$ and $d = 2$ if S is further assumed to be p-componentwise simply connected. In all these cases, δ can be taken as a constant multiple of ε, with the constant depending on k and d.

Finally, we can ask whether there is a way to generalize Theorem 3 to higher dimensions, by replacing simple connectivity with another topological property. Here is an example of one such possible generalization.

▶ **Conjecture 21.** *For every $d \geqslant 2$, there is a constant $\alpha = \alpha(d) > 0$ such that if $S \subseteq \mathbb{R}^d$ is a set with defined $\mathrm{b}_{d-1}(S)$ whose every p-component is contractible, then $\mathrm{b}_{d-1}(S) \leqslant \alpha\,\mathrm{c}(S)$.*

A modification of the proof of Theorem 5 implies that Conjecture 21 is true for star-shaped sets S.

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