

Constrained Monotone Function Maximization and the Supermodular Degree*

Moran Feldman¹ and Rani Izsak²

1 School of Computer and Communications, EPFL
Route Cantonale, 1015 Lausanne, Switzerland
moran.feldman@epfl.ch

2 Faculty of Mathematics and Computer Science, Weizmann Institute of Science
234 Herzl Street, Rehovot 7610001, Israel
ran.izsak@weizmann.ac.il

Abstract

The problem of maximizing a constrained monotone set function has many practical applications and generalizes many combinatorial problems such as k -COVERAGE, MAX-SAT, SET PACKING, MAXIMUM INDEPENDENT SET and WELFARE MAXIMIZATION. Unfortunately, it is generally not possible to maximize a monotone set function up to an acceptable approximation ratio, even subject to simple constraints. One highly studied approach to cope with this hardness is to restrict the set function, for example, by requiring it to be submodular. An outstanding disadvantage of imposing such a restriction on the set function is that no result is implied for set functions deviating from the restriction, even slightly. A more flexible approach, studied by Feige and Izsak [ITCS 2013], is to design an approximation algorithm whose approximation ratio depends on the complexity of the instance, as measured by some complexity measure. Specifically, they introduced a complexity measure called *supermodular degree*, measuring deviation from submodularity, and designed an algorithm for the welfare maximization problem with an approximation ratio that depends on this measure.

In this work, we give the first (to the best of our knowledge) algorithm for maximizing an *arbitrary* monotone set function, subject to a k -extendible system. This class of constraints captures, for example, the intersection of k -matroids (note that a single matroid constraint is sufficient to capture the welfare maximization problem). Our approximation ratio deteriorates gracefully with the complexity of the set function and k . Our work can be seen as generalizing both the classic result of Fisher, Nemhauser and Wolsey [Mathematical Programming Study 1978], for maximizing a submodular set function subject to a k -extendible system, and the result of Feige and Izsak for the welfare maximization problem. Moreover, when our algorithm is applied to each one of these simpler cases, it obtains the same approximation ratio as of the respective original work. That is, the generalization does not incur any penalty. Finally, we also consider the less general problem of maximizing a monotone set function subject to a uniform matroid constraint, and give a somewhat better approximation ratio for it.

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1 Introduction

A set function f is a function assigning a non-negative real value to every subset of a given ground set \mathcal{N} . A set function is (non-decreasing) monotone if $f(A) \leq f(B)$ whenever $A \subseteq B \subseteq \mathcal{N}$. Monotone set functions are often used to represent utility/cost functions in economics and algorithmic game theory. From a theoretical perspective, many combinatorial problems such as k -COVERAGE, MAX-SAT, SET PACKING and MAXIMUM INDEPENDENT SET can be represented as constrained maximization of monotone set functions.

Unfortunately, it is generally not possible to maximize a general monotone set function up to an acceptable approximation ratio, even subject to simple constraints. For example, consider the case of a partition matroid constraint, where the ground set is partitioned into subsets of size m , and we are allowed to pick only a single element from each subset. This problem generalizes the well-known WELFARE MAXIMIZATION problem¹, and thus, cannot be generally approximated by a factor of $O(\log m/m)$, in time polynomial in n and m (see Blumrosen and Nisan [2]).²

One highly studied approach to cope with this hardness is to restrict the set function. A common restriction is submodularity. A set function is submodular if the marginal contribution of an element to a set can only decrease as the set increases. More formally, for every two sets $A \subseteq B \subseteq \mathcal{N}$ and element $u \in \mathcal{N} \setminus B$, $f(B \cup \{u\}) - f(B) \leq f(A \cup \{u\}) - f(A)$. Submodular functions are motivated by many real world applications since they represent the principle of economy of scale, and are also induced by many natural combinatorial structures (*e.g.*, the cut function of a graph is submodular). Fortunately, it has been shown that submodular functions can be maximized, up to a constant approximation ratio, subject to various constraints. For example, maximizing a monotone submodular function subject to the partition matroid constraint, considered above, has a $(1 - 1/e)$ -approximation algorithm (see Calinescu, Chekuri, Pal and Vondrák [4]).

An outstanding disadvantage of imposing a restriction on the set function, such as submodularity, is that no result is implied for functions deviating from the restriction, even slightly. A more flexible approach, studied by [12], is to define a complexity measure for set functions, and then design an approximation algorithm whose guarantee depends on this measure. More specifically, [12] introduced a complexity measure called **supermodular degree**. A submodular set function has a supermodular degree of 0. The supermodular degree becomes larger as the function deviates from submodularity. Feige and Izsak [12] designed a $(1/(d+2))$ -approximation algorithm for the welfare maximization problem, where d is the maximum supermodular degree of the bidders' utility functions.

In a classic work, Fisher, Nemhauser and Wolsey [16] introduced a $(1/(k+1))$ -approximation algorithm for maximizing a submodular set function subject to a k -extendible system (in fact, they proved this approximation ratio even for a more general class of constraints called k -systems). In this work, we leverage their work, together with the supermodular degree, and give the first (to the best of our knowledge) algorithm for maximizing an *arbitrary* monotone set function subject to a k -extendible system. Note that k -extendible system generalizes, for example, the intersection of k -matroids (see Section 2 for definitions), and thus, also the welfare maximization problem, which can be captured by a single matroid

¹ The welfare maximization problem consists of a set \mathcal{B} of m bidders and a set \mathcal{N} of n items. Each bidder $b \in \mathcal{B}$ has a monotone utility function $u_b : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$. The objective is to assign a disjoint set $\mathcal{N}_b \subseteq \mathcal{N}$ of items to each bidder in a way maximizing $\sum_{b \in \mathcal{B}} u_b(\mathcal{N}_b)$ (*i.e.*, the “social welfare”).

² This result applies to value oracles, which we use throughout this work.

constraint. As in the works of Fisher, Nemhauser and Wolsey [16] and [12], our algorithm is greedy. Like in [12], the approximation ratio of our algorithm deteriorates gracefully with the complexity of the set function. Interestingly, when our algorithm is applied to the simpler cases studied by [16] and [12], its approximation ratio is exactly the same as that proved by the respective work. That is, we have no penalty for generality, either for handling an arbitrary set function (as opposed to only submodular) or for handling an arbitrary k -extendible system (as opposed to only welfare maximization). We also show a hardness result, depending on k and the supermodular degree of the instance, suggesting the approximation ratio of our algorithm is almost the best possible. Finally, we consider the less general problem of maximizing a monotone set function subject to a **uniform matroid** constraint (see Section 2), and give a somewhat better approximation ratio for it.

1.1 Related Work

Extensive work has been conducted in recent years in the area of maximizing monotone submodular set functions subject to various constraints. We mention here the most relevant results. Historically, one of the very first problems examined was maximizing a monotone submodular set function subject to a matroid constraint. Several special cases of matroids and submodular functions were studied in [6, 20, 21, 25, 26], using the greedy approach. Recently, the general problem, with an arbitrary matroid and an arbitrary submodular set function, was given a tight approximation of $(1 - 1/e)$ by Calinescu et al. [4]. A matching lower bound is due to [29, 30].

The problem of maximizing a monotone submodular set function over the intersection of k matroids was considered by Fisher et al. [16], who gave a greedy algorithm with an approximation ratio of $1/(k + 1)$, and stated that their proof extends to the more general class of k -systems using the outline of Jenkyns [25] (the extended proof is explicitly given by Calinescu et al. [4]). For k -intersection systems and k -exchange systems, this result was improved by Lee et al. [27] and Feldman et al. [15], respectively, to $1/(k + \varepsilon)$, for every constant $\varepsilon > 0$. The improvement is based on a local search approach that exploits exchange properties of the underlying combinatorial structure. Ward [34] further improved the approximation ratio for k -exchange systems to $2/(k + 3 + \varepsilon)$ using a non-oblivious local search. However, for maximizing a monotone submodular set function over k -extendible independence systems (and the more general class of k -systems), the current best known approximation is still $1/(k + 1)$ [16].

Other related lines of work deal with maximization of non-monotone submodular set functions (constrained or unconstrained) (see [3, 14, 33] for a few examples) and minimization of submodular set functions [17, 18, 23, 24].

The welfare maximization problem (or combinatorial auction) is unique in the sense that it was studied in the context of many classes of utility (set) functions, including classes generalizing submodular set functions such as sub-additive [10] and fractionally sub-additive valuations [9]. For many of these classes a constant approximation algorithm is known [1, 8, 10, 13, 19] assuming access to a demand oracle, which given a vector of prices returns a set of elements maximizing the welfare of a player given these prices. However, when only a value oracle is available to the algorithm (*i. e.*, the only access the algorithm has to the utility functions is by evaluating them on a chosen set) one cannot get a better than a polynomial approximation ratio, even for fractionally sub-additive valuations [9]. We are not aware of any other maximization subject to a constraint problem that was studied with respect to a non-submodular objective before our work.

2 Preliminaries

In this work, we consider set functions $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ that are (non-decreasing) monotone (*i. e.*, $A \subseteq B \subseteq \mathcal{N}$ implies $f(A) \leq f(B)$) and non-negative. We denote the cardinality of \mathcal{N} by n . For readability, given a set $S \subseteq \mathcal{N}$ and an element $u \in \mathcal{N}$ we use $S + u$ to denote $S \cup \{u\}$ and $S - u$ to denote $S \setminus \{u\}$.

2.1 Independence Systems

Given a ground set \mathcal{N} , a pair $(\mathcal{N}, \mathcal{I})$ is called an **independence system** if $\mathcal{I} \subseteq 2^{\mathcal{N}}$ is hereditary (that is, for every set $S \in \mathcal{I}$, every set $S' \subseteq S$ is also in \mathcal{I}). Independence systems are further divided into a few known classes. The probably most highly researched class of independence systems is the class of matroids.

► **Definition 1 (Matroid).** An independence system is a **matroid** if for every two sets $S, T \in \mathcal{I}$ such that $|S| > |T|$, there exists an element $u \in S \setminus T$, such that $T + u \in \mathcal{I}$. This property is called the *augmentation property* of matroids.

Two important types of matroids are uniform and partition matroids. In a **uniform matroid** a subset is independent if and only if its size is at most k , for some fixed k . In a **partition matroid**, the ground set \mathcal{N} is partitioned into multiple subsets $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_k$, and an independent set is allowed to contain at most a single element from each subset \mathcal{N}_i .

Some classes of independence systems are parametrized by a value $k \in \mathbb{N}$ ($k \geq 1$). The following is a simple example of such a class.

► **Definition 2 (k -intersection).** An independence system $(\mathcal{N}, \mathcal{I})$ is a **k -intersection** if there exist k matroids $(\mathcal{N}, \mathcal{I}_1) \dots (\mathcal{N}, \mathcal{I}_k)$ such that a set $S \subseteq \mathcal{N}$ is in \mathcal{I} if and only if $S \in \bigcap_{i=1}^k \mathcal{I}_i$.

The problem of k -dimensional matching can be represented as maximizing a linear function over a k -intersection independence system. In this problem, one looks for a maximum weight matching in a k -sided hypergraph, *i. e.*, an hypergraph where the nodes can be partitioned into k “sides” and each edge contains exactly one node of each side. The representation of this problem as the intersection of k partition matroids consists of one matroid per “side” of the hypergraph. The ground set of such a matroid is the set of edges, and a subset of edges is independent if and only if no two edges in it share a common vertex of the side in question.

The following definition, introduced by Mestre [28], describes a more general class of independence systems which is central to our work.

► **Definition 3 (k -extendible).** An independence system $(\mathcal{N}, \mathcal{I})$ is a **k -extendible system** if for every two subsets $T \subseteq S \in \mathcal{I}$ and element $u \notin T$ for which $T \cup \{u\} \in \mathcal{I}$, there exists a subset $Y \subseteq S \setminus T$ of cardinality at most k for which $S \setminus Y + u \in \mathcal{I}$.

The problem of maximizing a linear function over a k -extendible system captures the problem of k -set packing.³ In this problem, one is given a weighted collection of subsets of \mathcal{N} , each of cardinality at most k , and seeks a maximum weight sub-collection of pairwise disjoint sets. The corresponding k -extendible system is as follows. The ground set contains the sets as elements. The independent subsets are all subsets of pairwise disjoint sets. Let us explain why this is a k -extendible system. Adding a set S of size k to an independent set I , while respecting disjointness, requires that every elements of S is not contained in any other

³ k -set packing is, in fact, already captured by a smaller class called k -exchange, defined by [15].

set of I . On the other hand, since I is independent, each element is contained in at most one set of I . Therefore, in order to add S , while preserving disjointness, we need to remove up to k sets from I , as required by Definition 3.

The most general class of independence systems considered is given by Definition 5. The following definition is used to define it.

► **Definition 4 (Base).** Given an independence system $(\mathcal{N}, \mathcal{I})$ and a set $S \subseteq \mathcal{N}$, we say that a set $B \subseteq S$ is a **base** of S if $B \in \mathcal{I}$ but $B + u \notin \mathcal{I}$ for every element $u \in S \setminus B$. Furthermore, if $S = \mathcal{N}$, then we say that B is a base of the set system itself, or simply, a base.

► **Definition 5 (k -system).** An independence system $(\mathcal{N}, \mathcal{I})$ is a **k -system** if for every set $S \subseteq \mathcal{N}$, the ratio between the sizes of the smallest and largest bases of S is at most k .

An example of a natural problem which can be represented by a k -system, but not by a k -extendible system is given by [4]. The following (strict) inclusions can be shown to hold [4]:

$$\text{matroids} \subset k\text{-intersection} \subset k\text{-extendible systems} \subset k\text{-systems} .$$

2.2 Degrees of Dependency

We use the following standard definition.

► **Definition 6 (Marginal set function).** Let $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ be a set function and let $u \in \mathcal{N}$. The **marginal set function** of f with respect to u , denoted by $f(u | \cdot)$ is defined as $f(u | S) \stackrel{\text{def}}{=} f(S + u) - f(S)$. When the underlying set function f is clear from the context, we sometimes call $f(u | S)$ the **marginal contribution** of u to the set S . For subsets $S, T \subseteq \mathcal{N}$, we also use the notation $f(T | S) \stackrel{\text{def}}{=} f(S \cup T) - f(S)$.

We recall the definitions of the complexity measures used in this work (defined by [12]).

► **Definition 7 (Dependency degree).** The **dependency degree** of an element $u \in \mathcal{N}$ by f is defined as the cardinality of the set $\mathcal{D}_f(u) = \{v \in \mathcal{N} \mid \exists S \subseteq \mathcal{N} f(u | S + v) \neq f(u | S)\}$, containing all elements whose existence in a set might affect the marginal contribution of u . $\mathcal{D}_f(u)$ is called the **dependency set** of u by f . The **dependency degree** of a function f , denoted by \mathcal{D}_f , is simply the maximum dependency degree of any element $u \in \mathcal{N}$. Formally, $\mathcal{D}_f = \max_{u \in \mathcal{N}} |\mathcal{D}_f(u)|$. When the underlying set function is clear from the context, we sometimes omit it from the notations.

Note that $0 \leq \mathcal{D}_f \leq n - 1$ for any set function f . $\mathcal{D}_f = 0$ when f is *linear*, and becomes larger as f deviates from linearity.

► **Definition 8 (Supermodular (dependency) degree).** The **supermodular degree** of an element $u \in \mathcal{N}$ by f is defined as the cardinality of the set $\mathcal{D}_f^+(u) = \{v \in \mathcal{N} \mid \exists S \subseteq \mathcal{N} f(u | S + v) > f(u | S)\}$, containing all elements whose existence in a set might *increase* the marginal contribution of u . $\mathcal{D}_f^+(u)$ is called the **supermodular dependency set** of u by f . The **supermodular degree** of a function f , denoted by \mathcal{D}_f^+ , is simply the maximum supermodular degree of any element $u \in \mathcal{N}$. Formally, $\mathcal{D}_f^+ = \max_{u \in \mathcal{N}} |\mathcal{D}_f^+(u)|$. Again, when the underlying set function is clear from the context, we sometimes omit it from the notations.

Note that $0 \leq \mathcal{D}_f^+ \leq \mathcal{D}_f \leq n - 1$ for any set function f . $\mathcal{D}_f^+ = 0$ when f is *submodular*, and becomes larger as f deviates from submodularity.

2.3 Representing the Input

Generally speaking, a set function might assign 2^n different values for the subsets of a ground set of size n . Thus, one cannot assume that any set function has a succinct (*i. e.*, polynomial in n) representation. Therefore, it is a common practice to assume access to a set function via oracles. That is, an algorithm handling a set function often gets an access to an oracle that answers queries about the function, instead of getting an explicit representation of the function. Arguably, the most basic type of an oracle is the **value oracle**, which given any subset of the ground set, returns the value assigned to it by the set function. Formally:

► **Definition 9** (Value oracle). Value oracle of a set function $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ is the following:

Input: A subset $S \subseteq \mathcal{N}$.

Output: $f(S)$.

Similarly, since in a given independence system the number of independence subsets might be, generally, exponential in the size of the ground set, it is common to use the following type of oracle.

► **Definition 10** (Independence oracle). Independence oracle of an independence system $(\mathcal{N}, \mathcal{I})$ is the following:

Input: A subset $S \subseteq \mathcal{N}$.

Output: A Boolean value indicating whether $S \in \mathcal{I}$.

Our algorithms use the above standard oracles. Additionally, in order to manipulate a function with respect to the dependency/supermodular degree, we need a way to know what are the (supermodular) dependencies of a given element in the ground set. Oracles doing so were introduced by [12], and were used in their algorithms for the welfare maximization problem. Formally:

► **Definition 11** (Dependency and Supermodular oracles). Dependency oracle (Supermodular oracle) of a set function $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ is the following:

Input: An element $u \in \mathcal{N}$.

Output: The set $\mathcal{D}(u)$ ($\mathcal{D}^+(u)$) of the (supermodular) dependencies of u with respect to f .

2.4 Our Results

Our main result is an algorithm for maximizing any monotone set function subject to a k -extendible system, with an approximation ratio that degrades gracefully as the supermodular degree increases. Note that our algorithm achieves the best known approximation ratios also for the more specific problems of welfare maximization [12] and maximizing a monotone submodular function subject to a k -extendible system [16].

► **Theorem 12.** *There exists a $(1/(k(\mathcal{D}_f^+ + 1) + 1))$ -approximation algorithm of $\text{Poly}(|\mathcal{N}|, 2^{\mathcal{D}_f^+})$ time complexity for the problem of maximizing a non-negative monotone set function f subject to a k -extendible system.*

Note that an exponential dependence in \mathcal{D}_f^+ is unavoidable, since, otherwise, we would get a polynomial time $(n - 1)$ -approximation algorithm for maximizing any set function subject to a k -extendible system.⁴

⁴ To see that this cannot be done, consider the problem of maximizing the following family of set functions subject to a uniform $k = n/2$ matroid constraint. Each function in the family has a value of 1 for sets

We show a similar result also for the dependency degree, providing a better approximation ratio when $\mathcal{D}_f = \mathcal{D}_f^+$.

► **Theorem 13.** *There exists a $(1/(k(\mathcal{D}_f + 1)))$ -approximation algorithm of $\text{Poly}(|\mathcal{N}|, 2^{\mathcal{D}_f})$ time complexity for the problem of maximizing a non-negative monotone set function f subject to a k -extendible system.*

On the other hand, we give tight examples for both algorithms guaranteed by Theorems 12 and 13, and show the following hardness result via a reduction from k -dimensional matching.

► **Theorem 14.** *No polynomial time algorithm for maximizing a non-negative monotone set function f subject to a k -intersection independence system has an approximation ratio within $O\left(\frac{\log k + \log \mathcal{D}_f}{k \mathcal{D}_f}\right)$, unless $\mathcal{P} = \mathcal{NP}$. This is true even if k and \mathcal{D}_f are considered constants.*

Note that since $\mathcal{D}_f^+ \leq \mathcal{D}_f$ for any set function f , the hardness claimed in Theorem 14 holds also in terms of \mathcal{D}_f^+ .

Finally, we also consider the special case of a uniform matroid constraint, *i. e.*, where one is allowed to pick an arbitrary subset of \mathcal{N} of size at most k . For this simpler constraint we present an algorithm whose approximation ratio has a somewhat better dependence on \mathcal{D}_f^+ .⁵

► **Theorem 15.** *There exists a $(1 - e^{-1/(\mathcal{D}_f^+ + 1)})$ -approximation algorithm of $\text{Poly}(|\mathcal{N}|, 2^{\mathcal{D}_f^+})$ time complexity for the problem of maximizing a non-negative monotone set function f subject to a uniform matroid constraint.*

► **Theorem 16.** *No polynomial time algorithm for maximizing a non-negative monotone set function f subject to a uniform matroid constraint has a constant approximation ratio, unless SSE (Small-Set Expansion Hypothesis)⁶ is false.*

3 k -Extendible System

In this section we prove Theorems 12 and 14. The proof of Theorem 13 uses similar ideas and is omitted, due to space constraints.

3.1 Algorithm for k -extendible System (Proof of Theorem 12)

We consider in this section Algorithm 1, and prove it fulfills the guarantees of Theorem 12.

First, let us give some intuition. Let APX be an approximate solution and let OPT be an arbitrary optimal solution. Originally, before the algorithm adds any elements to APX , it still can be that it chooses to add all elements of OPT (together) to APX , and get an optimal solution. At each iteration, when adding elements to APX , this possibility might get ruined for some elements of OPT . If we want to keep the invariant that all elements of

strictly larger than k and for a single set A of size k . For all other sets the function assigns the value of 0 (observe that $\mathcal{D}^+(u) = \mathcal{N} - u$ for every element $u \in \mathcal{N}$, hence, the supermodular oracle is useless in this example). Given a random member of the above family, a deterministic algorithm using a polynomial number of oracle queries can determine A only with an exponentially diminishing probability, and thus, will also output a set of value 1 with such an exponentially diminishing probability. Using Yao's principle, this implies a hardness also for randomized algorithms.

⁵ The guarantee of Theorem 15 is indeed an improvement over the guarantee of Theorem 13 for 1-extendible system, because for every $x \geq 1$, $1 - e^{-1/x} \geq 1 - (1 - x^{-1} + x^{-2}/2) = x^{-1}(1 - x^{-1}/2) \geq x^{-1}/(1 + x^{-1}) = 1/(x + 1)$.

⁶ See [31, 32] and Section 4 for definitions and more information about SSE.

Algorithm 1: Extendible System Greedy(f, \mathcal{I})

```

1 Initialize:  $S_0 \leftarrow \emptyset, i \leftarrow 0.$ 
2 while  $S_i$  is not a base do
3    $i \leftarrow i + 1.$ 
4   Let  $u_i \in \mathcal{N} \setminus S_{i-1}$  and  $D_{best}^+(u_i) \subseteq \mathcal{D}^+(u_i)$  be a pair of an element and a set
   maximizing  $f(D_{best}^+(u_i) + u_i \mid S_{i-1})$  among the pairs obeying
    $S_{i-1} \cup D_{best}^+(u_i) + u_i \in \mathcal{I}.$ 
5    $S_i \leftarrow S_{i-1} \cup D_{best}^+(u_i) + u_i.$ 
6 Return  $S_i.$ 

```

OPT can be added to APX , then we might have to discard some elements of OPT . This discard potentially decreases the value of OPT , and therefore, can be seen as the damage incurred by the iteration. Note that by definition of a k -extendible system, we do not have to discard more than k elements for every element we add. That is, at every iteration, only up to $k(\mathcal{D}_F^+ + 1)$ elements must be discarded. Therefore, if we manage to upper bound the damage of discarding a single element by the benefit of the allocation at the same iteration, we get the desired bound.⁷ Recall that the supermodular dependencies of an element are exactly the elements that may increase its marginal value. Therefore, when discarding an element from OPT , the maximum damage is bounded by the marginal value of this element with respect to its supermodular dependencies in OPT . But, as any subset of OPT can be added to APX , the greedy choice of Algorithm 1 explicitly takes into account the possibility of adding this element and its supermodular dependencies to APX . If another option is chosen, it must have at least the same immediate benefit, as wanted.

We now give a formal proof for Theorem 12. Let us begin with the following observation.

► **Observation 17.** *Whenever S_i is not a base, there exists an element $u \in \mathcal{N} \setminus S_i$ for which $S_i \cup \emptyset + u \in \mathcal{I}$ (note that $\emptyset \subseteq \mathcal{D}^+(u)$). Hence, Algorithm 1 always outputs a base.*

Throughout this section, we denote $d = \mathcal{D}_f^+$. Our proof is by a hybrid argument. That is, we have a sequence of hybrid solutions, one per iteration, where the first hybrid contains an optimal solution (and hence, has an optimal value), and the last hybrid is our approximate solution.⁸ Roughly speaking, we show the following:

1. By adding each element to the approximate solution, we do not lose more than k elements of the iteration's hybrid (note that we add to our solution at most $d + 1$ elements at any given iteration). This is formalized in Lemma 18, and the proof is based on Definition 3 (k -extendible system).
2. The damage from losing an element of an iteration's hybrid is bounded by the profit the algorithm gains at that iteration. This is formalized in Lemma 19, and the proof is based on Definition 8 (supermodular degree).

In conclusion, we show that when moving from one hybrid to the next, we lose no more than $k(d + 1)$ times the profit at the respective iteration.

⁷ The other additive 1 in the denominator of the approximation ratio comes from the fact that by OPT 's value we actually mean its marginal contribution to APX . In this sense, addition of elements to APX also might reduce the value of OPT .

⁸ Actually, the last hybrid is defined as *containing* our approximate solution, but, as our approximate solution is a base, the hybrid must be exactly equal to it.

Let us formalize the above argument. Let ℓ be the number of iterations performed by Algorithm 1, *i. e.*, ℓ is the final value of i . We recursively define a series of $\ell + 1$ hybrid solutions as follows.

- H_0 is a base containing OPT . By monotonicity, $f(H_0) = f(OPT)$.
- For every $1 \leq i \leq \ell$, H_i is a maximum size independent subset of $H_{i-1} \cup S_i$ containing S_i .

► **Lemma 18.** *For every iteration $1 \leq i \leq \ell$, $|H_{i-1} \setminus H_i| \leq k \cdot |S_i \setminus H_{i-1}| \leq k(d+1)$.*

Proof. Let us denote the elements of $S_i \setminus H_{i-1}$ by v_1, v_2, \dots, v_r . We prove by induction that there exists a collection of sets Y_1, Y_2, \dots, Y_r , each of size at most k , such that: $Y_j \subseteq H_{i-1} \setminus (S_{i-1} \cup \{v_h\}_{h=1}^{j-1})$ and $H_{i-1} \setminus (\cup_{h=1}^j Y_h) \cup \{v_h\}_{h=1}^j \in \mathcal{I}$ for every $0 \leq j \leq r$. For ease of notation, let us denote $Y_1^j = \cup_{h=1}^j Y_h$ and $v_1^j = \{v_h\}_{h=1}^j$. Using this notation, the claim we want to prove can be rephrased as follows: there exists a collection of sets Y_1, Y_2, \dots, Y_r , each of size at most k , such that: $Y_j \subseteq H_{i-1} \setminus (S_{i-1} \cup v_1^{j-1})$ and $(H_{i-1} \setminus Y_1^j) \cup v_1^j \in \mathcal{I}$ for every $0 \leq j \leq r$.

For $j = 0$ the claim is trivial since $H_{i-1} \in \mathcal{I}$. Thus, let us prove the claim for j assuming it holds for $j-1$. By the induction hypothesis, $(H_{i-1} \setminus Y_1^{j-1}) \cup v_1^{j-1} \in \mathcal{I}$. On the other hand, $S_{i-1} \cup v_1^{j-1}$ is a subset of this set which is independent even if we add v_j to it. Since $(\mathcal{N}, \mathcal{I})$ is a k -extendible system, this implies the existence of a set Y_j of size at most k such that:

$$Y_j \subseteq [(H_{i-1} \setminus Y_1^{j-1}) \cup v_1^{j-1}] \setminus [S_{i-1} \cup v_1^{j-1}] \subseteq H_{i-1} \setminus (S_{i-1} \cup v_1^{j-1}) ,$$

and:

$$[(H_{i-1} \setminus Y_1^{j-1}) \cup v_1^{j-1}] \setminus Y_j + v_j \in \mathcal{I} \Rightarrow (H_{i-1} \setminus Y_1^j) \cup v_1^j \in \mathcal{I} ,$$

which completes the induction step. Thus, $(H_{i-1} \setminus Y_1^r) \cup v_1^r \in \mathcal{I}$ is a subset of $H_{i-1} \cup S_i$ which contains S_i and has a size of at least: $|H_{i-1}| - rk + r$. On the other hand, H_i is a maximum size independent subset of $H_{i-1} \cup S_i$, and thus: $|H_i| \geq |H_{i-1}| - rk + r$. Finally, all the elements of H_i belong also to H_{i-1} except, maybe, the elements of $S_i \setminus S_{i-1}$. Hence,

$$|H_{i-1} \setminus H_i| \leq |H_{i-1}| - |H_i| + |S_i \setminus S_{i-1}| \leq |H_{i-1}| - (|H_{i-1}| - rk + r) + r = rk .$$

Lemma 18 now follows, since $r \leq d+1$. ◀

The following lemma upper bounds the loss of moving from one hybrid to the next one.

► **Lemma 19.** *For every iteration $1 \leq i \leq \ell$, $f(H_{i-1}) - f(H_i) \leq k(d+1) \cdot f(D_{best}^+(u_i) + u_i \mid S_{i-1})$, where u_i and $D_{best}^+(u_i)$ are the greedy choices made by Algorithm 1 at iteration i .*

Proof. Order the elements of $H_{i-1} \setminus H_i$ in an arbitrary order v_1, v_2, \dots, v_r , and let $\bar{H}_j = H_{i-1} \setminus \{v_h \mid 1 \leq h \leq j\}$. For every $1 \leq j \leq r$,

$$\begin{aligned} f(\mathcal{D}^+(v_j) \cap \bar{H}_j + v_j \mid S_{i-1}) &= f(v_j \mid (\mathcal{D}^+(v_j) \cap \bar{H}_j) \cup S_{i-1}) + f(\mathcal{D}^+(v_j) \cap \bar{H}_j \mid S_{i-1}) \\ &\geq f(v_j \mid (\mathcal{D}^+(v_j) \cap \bar{H}_j) \cup S_{i-1}) \geq f(v_j \mid \bar{H}_j \cup S_{i-1}) , \end{aligned} \quad (1)$$

where the first inequality follows by monotonicity and the second by Definition 8 (supermodular degree). Specifically, the latter is correct, since the supermodular dependencies of an element are the only ones that can increase its marginal contribution. Therefore, adding

elements of $\bar{H}_j \setminus \mathcal{D}^+(v_j)$ to a set can only decrease the marginal contribution of v_j with respect to this set. Since $\bar{H}_j \cup S_{i-1} = \bar{H}_{j-1} \cup S_{i-1} - v_j$, we get:

$$\begin{aligned} \sum_{j=1}^r f(\mathcal{D}^+(v_j) \cap \bar{H}_j \setminus S_{i-1} + v_j \mid S_{i-1}) &= \sum_{j=1}^r f(\mathcal{D}^+(v_j) \cap \bar{H}_j + v_j \mid S_{i-1}) \\ &\geq \sum_{j=1}^r f(v_j \mid \bar{H}_j \cup S_{i-1}) = f(\bar{H}_0 \cup S_{i-1}) - f(\bar{H}_r \cup S_{i-1}) \geq f(H_{i-1}) - f(H_i) , \end{aligned}$$

where the two equalities follow by Definition 6 (marginal set function); the first inequality follows by (1) and the last inequality holds since $\bar{H}_0 = H_{i-1} \supseteq S_{i-1}$ and $\bar{H}_r \cup S_{i-1} \subseteq H_i$. Lemma 19 now follows by recalling that $r \leq k(d+1)$ (by Lemma 18), and noticing that the pair $(v_j, \mathcal{D}^+(v_j) \cap \bar{H}_j \setminus S_{i-1})$ is a candidate pair that Algorithm 1 can choose on Line 4 for every element $v_j \in H_{i-1} \setminus H_i$. ◀

► **Corollary 20.** *Algorithm 1 is a $1/(k(d+1)+1)$ -approximation algorithm.*

Proof. Adding up Lemma 19 over $1 \leq i \leq \ell$, we get:

$$\begin{aligned} k(d+1) \cdot [f(S_\ell) - f(S_0)] &= k(d+1) \cdot \sum_{i=1}^{\ell} f(D_{best}^+(u_i) + u_i \mid S_{i-1}) \\ &\geq \sum_{i=1}^{\ell} [f(H_{i-1}) - f(H_i)] = f(H_0) - f(H_\ell) . \end{aligned}$$

Notice that $H_\ell = S_\ell$ because S_ℓ is a base, and therefore, every independent set containing S_ℓ must be S_ℓ itself. Recall also that $f(H_0) = f(OPT)$ and $f(S_0) \geq 0$. Plugging these observations into the previous inequality gives:

$$k(d+1) \cdot f(S_\ell) \geq f(OPT) - f(S_\ell) \Rightarrow f(S_\ell) \geq \frac{f(OPT)}{k(d+1)+1} . \quad \blacktriangleleft$$

3.1.1 A Tight Example for Algorithm 1

In this section we present an example showing that our analysis of Algorithm 1 is tight even when the independence system $(\mathcal{N}, \mathcal{I})$ belongs to k -intersection (recall that any independence system that is k -intersection is also k -extendible, but not vice versa).

► **Theorem 21.** *For every $k \geq 1$, $d \geq 0$ and $\varepsilon > 0$, there exists a k -intersection independence system $(\mathcal{N}, \mathcal{I})$ and a function $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ with $\mathcal{D}_f^+ = d$ for which Algorithm 1 produces a $(1 + \varepsilon)/(k(d+1)+1)$ approximation.*

The rest of this section is devoted for constructing the independence system guaranteed by Theorem 21. Let \mathcal{T} be the collection of all sets $T \subseteq \{1, 2, \dots, k+1\} \times \{0, 1, \dots, (d+1)(k+1)-1\}$ obeying the following properties:

- For every $1 \leq i \leq k+1$, there exists exactly one x such that T contains the pair (i, x) .
- At least one pair (i, x) in T has $x \leq d$.
- Let x_{k+1} be such that $(k+1, x) \in T$. Then $x_{k+1} = 0$ or $x_{k+1} > d$.

Intuitively, the first requirement means that we can view a set $T \in \mathcal{T}$ as a point in a $(k+1)$ -dimensional space. The other two requirements make some points illegal. For $k=1$, the space is a $2(d+1) \times 2(d+1)$ grid, and the legal points are the ones that are either in row 0 or in one of the rows $d+1$ to $2(d+1)-1$ and one of the columns 0 to d .

Let \mathcal{N} be the ground set $\{u_T \mid T \in \mathcal{T}\}$. We define k matroids on this ground set as follows. For every $1 \leq i \leq k$, $\mathcal{M}_i = (\mathcal{N}, \mathcal{I}_i)$, where a set $S \subseteq \mathcal{N}$ belongs to \mathcal{I}_i if and only if for every $0 \leq x < (d+1)(k+1)$, $|\{u_T \in S \mid (i, x) \in T\}| \leq 1$. One can easily verify that \mathcal{M}_i is a partition matroid. The independence system we construct is the intersection of these matroids, *i. e.*, it is $(\mathcal{N}, \mathcal{I})$, where $\mathcal{I} = \bigcap_{i=1}^k \mathcal{I}_i$. Next, we define the objective function $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ as follows.

$$f'(S) = \sum_{x=0}^{(d+1)(k+1)-1} \min\{1, |\{u_T \in S \mid (k+1, x) \in T\}|\} .$$

That is, for $k = 1$, f' gains a value of 1 for every row that was “hit” by an element. For every $0 \leq x \leq d$, let $\hat{T}(x) = \{(k+1, 0)\} \cup \{(i, x)\}_{i=1}^k$ (notice that $\hat{T}_x \in \mathcal{T}$).

$$f(S) = \begin{cases} f'(S) + \varepsilon & \text{if } \{u_{\hat{T}(x)}\}_{x=0}^d \subseteq S , \\ f'(S) & \text{otherwise .} \end{cases}$$

One can check that f' is a non-negative monotone submodular function, and thus, $\mathcal{D}_f^+ = d$.

► **Lemma 22.** *Given the above constructed independence system $(\mathcal{N}, \mathcal{I})$ and objective function f , Algorithm 1 outputs a solution of value $1 + \varepsilon$.*

Proof. Consider an arbitrary element $u_T \in \mathcal{N}$. If $T \notin \{\hat{T}(x)\}_{0 \leq x \leq d}$, then $f(u_T \mid S) = f'(u_T \mid S)$ for every set $S \subseteq \mathcal{N}$, and thus, $\mathcal{D}^+(u_T) = \emptyset$ because f' is a submodular function. Hence, for every such u_T we get: $f(\mathcal{D}^+(u_T) + u_T) = 1$. Consider now the case $T \in \{\hat{T}(x)\}_{0 \leq x \leq d}$. In this case, clearly, $\mathcal{D}^+(u_T) = \{u_{\hat{T}(x)}\}_{0 \leq x \leq d} - u_T$, and thus, $f(\mathcal{D}^+(u_T) + u_T) = 1 + \varepsilon$. In conclusion, Algorithm 1 picks exactly the elements of $\{u_{\hat{T}(x)}\}_{0 \leq x \leq d}$ to its solution in the first iteration.

To complete the proof we still need to show that Algorithm 1 cannot increase the value of its solution in the next iterations. Consider an arbitrary element $u_T \in \mathcal{N} \setminus \{u_{\hat{T}(x)}\}_{0 \leq x \leq d}$. By definition, T must contain a pair (i, x) such that $0 \leq x \leq d$. There are two cases:

- If $i \neq k+1$, then u_T cannot coexist in an independent set of \mathcal{M}_i with $u_{\hat{T}(x)}$ because both correspond to sets containing the pair (i, x) .
- If $i = k+1$, then $x = 0$ because $u_T \in \mathcal{N}$.

From the above analysis, we get that all elements added to the solution after the first iteration contain the pair $(k+1, 0)$ (and thus, no other pair of the form $(k+1, x)$). Hence, they do not increase the value of either f' or f . ◀

To prove Theorem 21, we still need to show that $(\mathcal{N}, \mathcal{I})$ contains an independent set of a high value. Consider the set $S = \{u_{\bar{T}(j)}\}_{j=0}^{k(d+1)}$, where $\bar{T}(j) = \{(i, x) \mid 1 \leq i \leq k+1 \text{ and } x = (i(d+1) - j) \bmod (d+1)(k+1)\}$.

► **Observation 23.** $S \subseteq \mathcal{N}$.

Proof. We need to show that for every $0 \leq j \leq k(d+1)$, $u_{\bar{T}(j)} \in \mathcal{N}$. For $j = 0$, the only pair of the form $(k+1, x)$ in $\bar{T}(0)$ is $(k+1, 0)$, which completes the proof. Thus, we may assume from now on $1 \leq j \leq k(d+1)$, and let $i = \lceil j/(d+1) \rceil$. For shortness, let us denote $h = d+1$. Clearly $1 \leq i \leq k$ and $\bar{T}(j)$ contains the pair (i, x) for:

$$x = (ih - j) \bmod h(k+1) = (\lceil j/h \rceil \cdot h - j) \bmod h(k+1) .$$

To prove the observation, we need to show that $0 \leq x \leq d$. This follows since $\lceil j/h \rceil \cdot h - j \geq (j/h) \cdot h - j = 0$ and $\lceil j/h \rceil \cdot h - j < \lceil j/h + 1 \rceil \cdot h - j = h = d+1$. ◀

► **Observation 24.** For every two values $0 \leq j_1 < j_2 \leq k(d+1)$, $\bar{T}(j_1) \cap \bar{T}(j_2) = \emptyset$. Hence $S \in \mathcal{I}$ and $f(S) \geq f'(S) = |S| = k(d+1) + 1$.

Proof. Assume for the sake of contradiction that $(i, x) \in \bar{T}(j_1) \cap \bar{T}(j_2)$. Then, modulo $(d+1)(k+1)$, the following equivalence must hold:

$$(i(d+1) - j_1) \equiv (i(d+1) - j_2) \Rightarrow j_1 \equiv j_2 ,$$

which is a contradiction since $j_1 \neq j_2$ and they are both in the range $[0, k(d+1)]$. ◀

3.2 Hardness (Proof of Theorem 14)

Before proving Theorem 14 let us state the hardness result of [22] given by Theorem 25. In the r -Dimensional Matching problem one is given an r -sided hypergraph $G = (\bigcup_{i=1}^r V_i, E)$, where every edge $e \in E$ contains exactly one vertex of each set V_i . The objective is to select a maximum size matching $M \subseteq E$, i. e., a subset $M \subseteq E$ of edges which are pairwise disjoint.

► **Theorem 25** (Hazan et al. [22]). *It is NP-hard to approximate r -Dimensional Matching to within $O(\log r/r)$ in polynomial time, even if r is a constant.*

Theorem 14 follows by combining Theorem 25 with the following lemma.

► **Lemma 26.** *Any instance of r -Dimensional Matching can be represented as maximizing a monotone function f with $\mathcal{D}_f^+ = \mathcal{D}_f \leq d$ over a k -intersection set system for every $d \geq 0$ and $k \geq 1$ obeying $r \leq k(d+1)$.*

Proof. For simplicity, assume $r = k(d+1)$. Let $G = (\bigcup_{i=1}^r V_i, E)$ be the graph representing the r -Dimensional Matching instance. We first construct a new graph G' as follows. For every edge $e \in E$ and $1 \leq j \leq d+1$, let $e(j) = e \cap (\bigcup_{i=(j-1)k+1}^{jk} V_i)$, i. e., $e(j)$ is the part of e hitting the vertex sets $V_{(j-1)k+1}, \dots, V_{jk}$. The edges of the new graph $G' = (\bigcup_{i=1}^r V_i, E')$ are then defined as all the edges that can be obtained this way. More formally:

$$E' = \{e(j) \mid e \in E \text{ and } 1 \leq j \leq d+1\} .$$

It is easy to see that the original instance of r -Dimensional Matching is equivalent to the problem of finding a matching in G' maximizing the objective function $f : 2^{E'} \rightarrow \mathbb{R}^+$ defined as follows.

$$f(S) = \sum_{e \in E} \left\lfloor \frac{|S \cap \{e(j) \mid 1 \leq j \leq d+1\}|}{d+1} \right\rfloor .$$

Moreover, $\mathcal{D}_f = \mathcal{D}_f^+ = d$. Thus, to complete the proof we only need to show that the set of all legal matchings of G' can be represented as a k -intersection independence system.

Consider the following partition of the vertices of G' . For every $1 \leq j \leq k$, $V_j' = \bigcup_{i=0}^d V_{j+ki}$. Observe that each edge of G' contains exactly one vertex of V_j' . Hence, the constraint that no two edges intersect on a node of V_j' can be represented by the partition matroid $M_j = (E', \mathcal{I}_j)$ defined as following. A set $S \subseteq E'$ is in \mathcal{I}_j if and only if no two edges of S intersect on a node of V_j' . The set of legal matchings of G' is, then, exactly $\bigcap_{j=1}^k \mathcal{I}_j$. ◀

4 Uniform Matroid Constraint

In this section we prove Theorems 15 and 16.

4.1 Algorithm for Uniform Matroid Constraint (Proof of Theorem 15)

Algorithm 1 given in Section 3 provides a $1/(\mathcal{D}_f^+ + 2)$ approximation for a general k -extendible constraint. In this section, our objective is to improve over this approximation ratio for uniform matroid constraints. Throughout this section we use d to denote \mathcal{D}_f^+ and OPT to denote an arbitrary optimal solution.

A natural greedy approach (similar to Algorithm 1) is to repeatedly select the set $\mathcal{D}^+(u) + u$ with the maximal marginal contribution with respect to the current solution, and add it to the solution. When $d + 1$ divides k , this works well, and can be shown to have an approximation ratio of $1 - e^{1/(d+1)}$. However, when $d + 1$ does not divide k , the algorithm might not be able to perform $\lceil k/(d + 1) \rceil$ iterations, which results in an inferior approximation ratio.

A possible workaround is to add a final iteration in which the algorithm can select any subset of $\mathcal{D}^+(u) + u$ (for any $u \in \mathcal{N}$). This allows the algorithm to perform $\lceil k/(d + 1) \rceil$ iterations on all instances, and it is an open problem to analyse its approximation ratio.

An alternative workaround is to reduce into the case of k divisible by $d + 1$. This is the idea behind Algorithm 2. Observe that $d' + 1$ divides $k - r$ (the size of the part of OPT we do not guess). When we say that the algorithm “guesses” values, we simply mean “sequentially tries all possible values”.

Algorithm 2: Guess Greedy(f, k)

- 1 Guess: $d' = \max_{u \in OPT} |\mathcal{D}^+(u) \cap OPT|$, an element u^* for which $|\mathcal{D}^+(u^*) \cap OPT| = d'$ and the set $C = \mathcal{D}^+(u^*) \cap OPT$ itself.
 - 2 Initialize: $r = k \bmod (d' + 1)$, $\ell = (k - r)/(d' + 1)$ and $S_0 \subseteq C$ as an arbitrary subset of size r .
 - 3 **for** $i = 1$ **to** ℓ **do**
 - 4 Let $u_i \in \mathcal{N}$ and $D_{best}^+(u_i) \subseteq \mathcal{D}^+(u_i)$ be a pair of an element and a set of size at most d' maximizing $f(D_{best}^+(u_i) + u_i \mid S_{i-1})$.
 - 5 $S_i \leftarrow S_{i-1} \cup D_{best}^+(u_i) + u_i$.
 - 6 Return S_ℓ .
-

► **Observation 27.** *Algorithm 2 returns a feasible solution and has a polynomial time complexity in n and 2^d .*

Due to space constraints, we only sketch the analysis of the approximation ratio of Algorithm 2. To simplify notation, define $k' \stackrel{\text{def}}{=} k - r$. Observe that $\ell = k'/(d' + 1)$.

► **Lemma 28.** *For every $0 \leq i \leq \ell$, $f(S_i) \geq (1 - 1/k')^i \cdot f(S_0) + [1 - (1 - 1/k')^i] \cdot f(OPT)$.*

Proof Sketch. We prove the lemma by induction. For $i = 0$ the claim is trivial since $f(S_0) = (1 - 1/k')^0 \cdot f(S_0) + [1 - (1 - 1/k')^0] \cdot f(OPT)$. Next, assume the claim holds for $i - 1$, and let us prove it for $i > 0$. Observe that $S_0 \subseteq OPT$, and therefore, $|OPT \setminus S_0| = k'$. Using the same methods used in the proof of Lemma 19, it can be shown that:

$$\sum_{u \in OPT \setminus S_0} f(\mathcal{D}^+(u) \cap OPT + u \mid S_{i-1}) \geq f(OPT) - f(S_{i-1}) .$$

Notice that the pair $(u, \mathcal{D}^+(u) \cap OPT)$ is a candidate pair to be selected as $(u_i, D_{best}^+(u_i))$ for every element $u \in OPT \setminus S_0$. Hence, $f(D_{best}^+(u_i) + u_i \mid S_{i-1}) \geq [f(OPT) - f(S_{i-1})]/k'$.

Thus,

$$\begin{aligned} f(S_i) &\geq [f(OPT) - f(S_{i-1})]/k' + f(S_{i-1}) = f(OPT)/k' + (1 - 1/k') \cdot f(S_{i-1}) \\ &\geq (1 - 1/k')^i \cdot f(S_0) + [1 - (1 - 1/k')^i] \cdot f(OPT) , \end{aligned}$$

where the last inequality follows by induction hypothesis. \blacktriangleleft

► **Corollary 29.** *The approximation ratio of Algorithm 2 is $1 - e^{-1/(d'+1)} \geq 1 - e^{-1/(d+1)}$.*

4.2 Hardness (Proof of Theorem 16)

The GAP SMALL-SET EXPANSION PROBLEM (introduced by [31]) is the following promise problem.

► **Problem 1** (GAP SMALL-SET EXPANSION(η, δ)).

Input: An undirected graph $G = (V, E)$.

Output: Is $\phi_G(\delta) \geq 1 - \eta$ or $\phi_G(\delta) \leq \eta$? ($\phi_G(\delta)$ is the edge expansion of G with respect to subsets of size exactly $\delta|V|$.)

The Small-Set Expansion Hypothesis (SSE), introduced by Raghavendra and Steurer [31] (see, also, [32]) is the following.

► **Hypothesis 1.** *For every $\eta > 0$, there exists δ such that Problem 1 parametrized by η and δ is \mathcal{NP} -hard.*

A hypergraph representation $F = (V_F, E_F, w_F)$ of a set function $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}^+$ (defined by [5, 7]) is the following. The set V_F contains exactly a single vertex for each element of the ground set \mathcal{N} . The set E_F is the set of the hyperedges of the hypergraph. The function $w_F : E_F \rightarrow \mathbb{R}$ assigns a real value for each hyperedge $e \in E_F$, and these values obey the following property: for every set $S \subseteq \mathcal{N}$, the sum of the values of the hyperedges in the hypergraph induced by the vertices representing S is exactly $f(S)$. It is well known that any set function (normalized to have $f(\emptyset) = 0$) can be uniquely represented by a hypergraph representation and vice versa.

Using the above definition, we show that maximizing a monotone set function subject to a uniform matroid constraint captures Problem 1. Let $G = (V, E)$ be an arbitrary instance of Problem 1 with parameters η and δ . We construct from G an hypergraph representation $F = (V_F, E_F, w_F)$. The sets V_F and E_F are chosen as identical to V and E , respectively (*i. e.*, all the hyperedges are of rank 2, hence, the hypergraph is in fact a graph). The value function w_F gives a value of 1 for every edge of E_F . We can now consider the problem of finding a set of size at most $\delta|V|$ maximizing the set function f corresponding to the hypergraph representation F . Theorem 16 follows immediately by the observation that a constant approximation for the last problem implies a constant approximation for Problem 1.

5 Future Research

We view this work as a proof of concept showing that one can obtain interesting results for the problem of maximizing an arbitrary monotone set function subject to non-trivial constraints. We would like to point out two possible directions for future research. The first direction is studying the approximation ratio that can be guaranteed for more general problems as a function of the supermodular degree. Two possible such generalizations are a general k -system constraint and a non-monotone objective. Notice that non-monotone objectives are interesting even in the unconstrained case.

The second direction is determining the guarantees that can be achieved for other complexity measures (with respect to either monotone or non-monotone set functions). Specifically, we would like to draw attention to two complexity measures introduced by [11], namely MPH (for monotone set functions) and PLE (for not necessarily monotone set functions). Both measures are based on fractionally sub-additive functions, a strict super-class of submodular functions, and they generally give lower values to set functions in comparison to the supermodular degree. Thus, it is intriguing to show positive results for either of these measures.

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